INNER PRODUCT SPACE WITH NO ORTHO-NORMAL BASIS WITHOUT CHOICE
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ABSTRACT. We prove in ZF that there is an inner product space, in fact, nicely definable with no orthonormal basis.

§ 1.

The theorem below is known in ZFC, but probably not in ZF; really we use the simple black box (see [She]).

Theorem 1.1. *(ZF) There is an inner-product space V over $\mathbb{R}$ with no ortho-normal basis.*

Remark 1.2. In fact, nicely definable one, here - Borel.

Proof. Stage A:

Let V_1 be the Hilbert space over $\mathbb{R}$ with orthonormal basis $\{x_\eta : \eta \in {}^{\omega\geq}\omega\}$, so an element x has a unique representation as $x = \Sigma\{a_{x,\eta} x_\eta : \eta \in {}^{\omega>}\omega\}$ with $a_{x,\eta} \in \mathbb{R}$ and norm $< \infty$ so $\mathrm{supp}_1(x) := \{\eta : a_{x,\eta} \neq 0\}$ is countable and $\mathrm{supp}^1_k(x) := \{\eta : |a_\eta| \geq \frac{1}{k+1}\}$ finite for every $k < \omega$ where the norm is $\Sigma\{a^2_{x,\eta} : \eta \in {}^{\omega>}\omega\}$. The inner product is $((\Sigma a_\eta x_\eta), (\Sigma a'_\eta x_\eta)) = \Sigma\{a_\eta a'_\eta : \eta \in {}^{\omega\geq}\omega\} \in \mathbb{R}$.

For $\eta \in {}^\omega\omega$ let $y_\eta = x_\eta + \sum\limits_{n<\omega} \frac{1}{2^n} x_{\eta\restriction n}$.

Let V be the subspace of V_1 generated by $\{x_\eta : \eta \in {}^{\omega>}\omega\} \cup \{y_\eta : \eta \in {}^\omega\omega\}$ so as a vector space it is $\bigoplus\limits_{\eta \in {}^{\omega>}\omega} \mathbb{R}x_\eta \oplus \bigoplus\limits_{\eta \in {}^\omega\omega} \mathbb{R}y_\eta$ and it "inherits" the innder product from V_1.

Toward contradiction assume that $\{z_s : s \in S\}$ is an ortho-normal basis of V. So every $x \in V$ has the unique representation $\sum\limits_{s\in S} b_{x,s} z_s$, where $b_{s,n} \in \mathbb{R}$ and for $k \in [1, \omega)$ and $x \in V$ let $\mathrm{supp}^2_k(x) := \{s \in S : |b_{x,s}| \geq \frac{1}{2k+1}\}$, so finite and $\mathrm{supp}_2(x) := \{s \in S : b_{x,s} \neq 0\}$ so countable.

Stage B:

We choose η_n by induction on n such that:

$\boxplus_1$ (a) $\eta_\eta \in {}^n\omega$

(b) $\eta_m = \eta_n \restriction m$ if $m < n$

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(c) if $n = m + 1$ then $\eta_n = \eta_m{}^\frown\langle i\rangle$ with $i < \omega$ minimal such that: if $\ell \le m, s \in \text{supp}^2_m(x_{\eta_\ell})$ and $\nu \in \text{supp}^1_m(z_s) \subseteq {}^{\omega>}\omega$ then $\neg(\eta_m{}^\frown\langle i\rangle \trianglelefteq \nu)$.

This is well defined as in clause (c), $\text{supp}^2_m(x_{\eta_\ell})$ is a finite subset of S and for each $s \in \text{supp}^2_m(x_{\eta_\ell})$, the set $\text{supp}^1_m(z_s)$ is a finite subset of ${}^{\omega>}\omega$.

Lastly, let

$\boxplus_2$ (a) $\eta_\omega := \cup\{\eta_n : n < \omega\} \in {}^\omega\omega$
(b) $S_1 = \cup\{\text{supp}_2(x_\rho) : \rho \triangleleft \eta_\omega\}$
(c) $S_2 = S\backslash S_1$
(d) X_ℓ is the closure inside V of $\oplus\{\mathbb{R}z_s : s \in S_\ell\}$ for $\ell = 1, 2$
(e) $S_{1,n} := \cup\{\text{sup}^2_m(x_{\eta_\ell}) : m, \ell \le n\}$.

Note

$\boxplus_3$ $V = X_1 \oplus X_2$, i.e. X_1, X_2 are orthogonal but $X_1 + X_2$ is V

$\boxplus_4$ $S_1 = \cup\{\text{supp}^2_m(x_{\eta_n}) : n < \omega, m < n\} = \cup\{S_{1,n} : n < \omega\}$

$\boxplus_5$ $\eta_n \in S_1$ for $n < \omega$.

Stage C: As $y_{\eta_\omega} \in V$ see Stage A and $\boxplus_2(a)$ of Stage B, recalling $\boxplus_3$

$\otimes_1$ there are $y^1 \in X_1, y^2 \in X_2$ such that $y_{\eta_\omega} = y^1 + y^2$.

Also

$\otimes_2$ $\{\rho : \eta_{n+1} \trianglelefteq \rho \in {}^{\omega\ge}\omega\}$ is disjoint to $\cup\{\text{supp}^1_m(z_s) : s \in S_{1,n}\}$ for every $n < \omega$.

[Why? By the choice of η_{n+1} in $\boxplus_1(c)$.]

$\otimes_3$ $\eta_\omega \notin \text{supp}_1(z_s) = \cup\{\text{supp}^1_m(z_s) : m < omega\}$ for every $s \in S_1$.

[Why? The $\notin$ by $\otimes_2$.]

Hence by $\boxplus_3$

$\otimes_4$ if $s \in S_1$ then y_{η_ω}, z_s are orthogonal (in V_1).

But

$\otimes_5$ $(y_{\eta_\omega}, x_{\eta_n}) = \frac{1}{2^n}$.

[Why? By the choice of y_η is stage N.]

By $\boxplus_5 + \otimes_4 + \otimes_5$ we get contradiction. $\square_{1.1}$

References

[She] Saharon Shelah, *Black Boxes*, arXiv: 0812.0656 Ch. IV of The Non-Structure Theory" book [Sh:e].

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