PCF: THE ADVANCED PCF THEOREMS
E69

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ABSTRACT. This is a revised version of [She96, §6].

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§ 0. Introduction

§ 1. On pcf

This is a revised version of [She96, §6] more self-contained, large part done according to lectures in the Hebrew University Fall 2003
Recall

Definition 1.1. Let $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle, f_\alpha \in {}^\kappa\mathrm{Ord}, I$ an ideal on κ.
1) We say that $f \in {}^\kappa\mathrm{Ord}$ is a $\le_I$-l.u.b. of $\bar{f}$ when:

- (a) $\alpha < \delta \Rightarrow f_\alpha \le_I f$
- (b) if $f' \in {}^\kappa\mathrm{Ord}$ and $(\forall \alpha < \delta)(f_\alpha \le_I f')$ then $f \le_I f'$.

2) We say that f is a $\le_I$-e.u.b. of $\bar{f}$ when

- (a) $\alpha < \delta \Rightarrow f_\alpha \le_I f$
- (b) if $f' \in {}^\kappa\mathrm{Ord}$ and $f' <_I \mathrm{Max}\{f, 1_\kappa\}$ then $f' <_I \mathrm{Max}\{f_\alpha, 1_\kappa\}$ for some $\alpha < \delta$.

3) $\bar{f}$ is $\le_I$-increasing if $\alpha < \beta \Rightarrow f_\alpha \le_I f_\beta$, similarly $<_I$-increasing. We say $\bar{f}$ is eventually $<_I$-increasing: it is $\le_I$-increasing and $(\forall \alpha < \delta)(\exists \beta < \delta)(f_\alpha <_I f_\beta)$.
4) We may replace I by the dual ideal on κ.

Remark 1.2. For $\kappa, I, \bar{f}$ as in Definition 1.1, if $\bar{f}$ is a $\le_I$-e.u.b. of $\bar{f}$ then f is a $\le_I$-l.u.b. of $\bar{f}$.

Definition 1.3. 1) We say that $\bar{s}$ witness or exemplifies $\bar{f}$ is $(< \sigma)$-chaotic for D when, for some κ

- (a) $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle$ is a sequence of members of ${}^\kappa\mathrm{Ord}$
- (b) D is a filter on κ (or an ideal on κ)
- (c) $\bar{f}$ is $<_D$-increasing
- (d) $\bar{s} = \langle s_i : i < \kappa \rangle, s_i$ a non-empty set of $< \sigma$ ordinals
- (e) for every $\alpha < \delta$ for some $\beta \in (\alpha, \delta)$ and $g \in \prod\limits_{i<\kappa} s_i$ we have $f_\alpha \le_D g \le_D f_\beta$.

2) Instead "$(< \sigma^+)$-chaotic" we may say "σ-chaotic".

Claim 1.4. *Assume*

- (a) *I an ideal on κ*
- (b) *$\bar{f} = \langle f_\alpha : \alpha < \delta \rangle$ is $<_I$-increasing, $f_\alpha \in {}^\kappa\mathrm{Ord}$*
- (c) *$J \supseteq I$ is an ideal on κ and $\bar{s}$ witnesses $\bar{f}$ is $(< \sigma)$-chaotic for J.*

Then $\bar{f}$ has no $\le_I$-e.u.b. f such that $\{i < \kappa : \mathrm{cf}(f(i)) \ge \sigma\} \in J$.

Discussion 1.5. What is the aim of clause (c) of 1.4? For $\le_I$-increasing sequence $\bar{f}, \langle f_\alpha : \alpha < \delta \rangle$ in ${}^\kappa\mathrm{Ord}$ we are interested whether it has an appropriate $\le_I$-e.u.b. Of course, I may be a maximal ideal on κ and $\langle f_t : t \in \mathrm{cf}((\omega, <)^\kappa/D)\rangle$ is $<_I$-increasing cofinal in $(\omega, <)^\kappa/D$, so it has an $<_I$-e.u.b. the sequence $\omega_\kappa = \langle \omega : i < \kappa \rangle$, but this is not what interests us now; we like to have a $\le_I$-e.u.b. g such that $(\forall i)(\mathrm{cf}(g(i)) > \kappa)$.

Proof. Toward contradiction assume that $f \in {}^\kappa\mathrm{Ord}$ is a $\le_I$-e.u.b. of $\bar{f}$ and $A_1 := \{i < \kappa : \mathrm{cf}(f(i)) \ge \sigma\} \notin I$ hence $A \notin I$.

We define a function $f' \in {}^\kappa\mathrm{Ord}$ as follows:

$\circledast$ (a) if $i \in A$ then $f'(i) = \sup(s_i \cap f(i)) + 1$

(b) if $i \in \kappa \backslash A$ then $f'(i) = 0$.

Now that $i \in A \Rightarrow \mathrm{cf}(g(i)) \geq \sigma > |s_i| \Rightarrow f'(i) < f(i) \leq \mathrm{Max}\{g(i), 1\}$ and $i \in \kappa \backslash A \Rightarrow f'(i) = 0 \Rightarrow f'(i) < \mathrm{Max}\{f(i), 1\}$. So by clause (b) of Definition 1.1(2) we know that for some $\alpha < \delta$ we have $f' <_I \mathrm{Max}\{f_\alpha, 1\}$. But "$\bar{s}$ witness that $\bar{f}$ is $(< \sigma)$-chaotic" hence we can find $g \in \prod\limits_{i<\kappa} s_i$ and $\beta \in (\alpha, \delta)$ such that $f_\alpha \leq_I g \leq_I f_\beta$ and as $\bar{f}$ is $<_I$-increasing without loss of generality $g <_I f_\beta$.

So $A_2 := \{i < \kappa : f_\alpha(i) \leq g(i) < f_\beta(i) \leq f(i)$ and $f'(i) < \mathrm{Max}\{f_\alpha(i), 1\} = \kappa\}$ mod I hence $A := A_1 \cap A_2 \neq \emptyset$ mod I hence $A \neq \emptyset$. So for any $i \in A$ we have $f_\alpha(i) \leq g(i) < f_\beta(i) \leq f(i)$ and $f(i) \in s_i$ hence $g(i) < f'(i) := \sup(s_i \cap f(i)) + 1$ and so $f'(i) \geq 1$.

Also $f'(i) < \mathrm{Max}\{f_\alpha(i, 1)\}$ hence $f'(i) < f_\alpha(i)$. Together $f'(i) < f_\alpha(i) \leq g(i) < f'(i)$, contradiction. $\square_{1.4}$

Lemma 1.6. *Suppose* $\mathrm{cf}(\delta) > \kappa^+, I$ *an ideal on* κ *and* $f_\alpha \in {}^\kappa\mathrm{Ord}$ *for* $\alpha < \delta$ *is* $\leq_I$*-increasing.* *Then there are* $\bar{J}, \bar{s}, \bar{f}'$ *satisfying:*

(A) $\bar{s} = \langle s_i : i < \kappa \rangle$, *each* s_i *a set of* $\leq \kappa$ *ordinals,*

(B) $\sup\{f_\alpha(i) : \alpha < \delta\} \in s_i$; *moreover is* $\max(s_i)$

(C) $\bar{f}' = \langle f'_\alpha : \alpha < \delta \rangle$ *where* $f'_\alpha \in \prod\limits_{i<\kappa} s_i$ *is defined by* $f'_\alpha(i) = \mathrm{Min}\{s_i \backslash f_\alpha(i)\}$, *(similar to rounding!)*

(D) $\mathrm{cf}[f'_\alpha(i)] \leq \kappa$ *(e.g.* $f'_\alpha(i)$ *is a successor ordinal) implies* $f'_\alpha(i) = f_\alpha(i)$

(E) $\bar{J} = \langle J_\alpha : \alpha < \delta \rangle, J_\alpha$ *is an ideal on* κ *extending* I *(for* $\alpha < \delta$*), decreasing with* α *(in fact for some* $a_{\alpha,\beta} \subseteq \kappa$ *(for* $\alpha < \beta < \kappa$*) we have* $a_{\alpha,\beta}/I$ *decreases with* β*, increases with* α *and* J_α *is the ideal generated by* $I \cup \{a_{\alpha,\beta} : \beta$ *belongs to* $(\alpha, \lambda)\}$*) so possibly* $J_\alpha = \mathscr{P}(\kappa)$ *and possibly* $J_\alpha = I$

such that:

(F) *if* D *is an ultrafilter on* κ *disjoint to* J_α *then* f'_α/D *is a* $<_D$*-l.u.b and even* $<_D$*-e.u.b. of* $\langle f_\beta/D : \beta < \alpha \rangle$ *which is eventually* $<_D$*-increasing and* $\{i < \kappa : \mathrm{cf}[f'_\alpha(i))] > \kappa\} \in D$.

Moreover

$(F)^+$ *if* $\kappa \notin J_\alpha$ *then* f'_α *is an* $<_{J_\alpha}$*-e.u.b (= exact upper bound) of* $\langle f_\beta : \beta < \delta \rangle$ *and* $\beta \in (\alpha, \delta) \Rightarrow f'_\beta =_{J_\alpha} f'_\alpha$

(G) *if* D *is an ultrafilter on* κ *disjoint to* I *but for every* α *not disjoint to* J_α *then* $\bar{s}$ *exemplifies* $\langle f_\alpha : \alpha < \delta \rangle$ *is* κ *chaotic for* D *as exemplified by* $\bar{s}$ *(see Definition 1.3), i.e., for some club* E *of* $\delta, \beta < \gamma \in E \Rightarrow f_\beta \leq_D f'_\beta <_D f_\gamma$

(H) *if* $\mathrm{cf}(\delta) > 2^\kappa$ *then* $\langle f_\alpha : \alpha < \delta \rangle$ *has a* $\leq_I$*-l.u.b. and even* $\leq_I$*-e.u.b. and for every large enough* α *we have* $I_\alpha = I$

(I) *if* $b_\alpha =: \{i : f'_\alpha(i)$ *has cofinality* $\leq \kappa$ *(e.g., is a successor)*$\} \notin J_\alpha$ *then: for every* $\beta \in (\alpha, \delta)$ *we have* $f'_\alpha \restriction b_\alpha = f_\beta \restriction b_\alpha \mod J_\alpha$.

Remark 1.7. Compare with [She97b].

Proof. Let $\alpha^* = \cup\{f_\alpha(i) + 1 : \alpha < \delta, i < \kappa\}$ and $S = \{j < \alpha^* : j$ has cofinality $\leq \kappa\}, \bar{e} = \langle e_j : j \in S \rangle$ be such that

(a) $e_j \subseteq j, |e_j| \leq \kappa$ for every $j \in S$
(b) if $j = i+1$ then $e_j = \{i\}$
(c) if j is limit, then $j = \sup(e_j)$ and $j' \in S \cap e_j \Rightarrow e_{j'} \subseteq e_j$.

For a set $a \subseteq \alpha^*$ let $c\ell_{\bar{e}}(a) = a \cup \bigcup\limits_{j \in a \cap S} e_j$ hence by clause (c) clearly $c\ell_{\bar{e}}(c\ell_{\bar{e}}(a)) = c\ell_{\bar{e}}(a)$ and $[a \subseteq b \Rightarrow c\ell_{\bar{e}}(a) \subseteq c\ell_{\bar{e}}(b)]$ and $|c\ell_{\bar{e}}(a)| \leq |a| + \kappa$. We try to choose by induction on $\zeta < \kappa^+$, the following objects: α_ζ, D_ζ, g_ζ, $\bar{s}_\zeta = \langle s_{\zeta,i} : i < \kappa\rangle, \langle f_{\zeta,\alpha} : \alpha < \delta\rangle$ such that:

$\boxtimes$ (a) $g_\zeta \in {}^\kappa\mathrm{Ord}$ and $g_\zeta(i) \leq \cup\{f_\alpha(i) : \alpha < \delta\}$
(b) $s_{\zeta,i} = c\ell_{\bar{e}}[\{g_\epsilon(i) : \epsilon < \zeta\} \cup \{\sup_{\alpha<\delta} f_\alpha(i)\}]$ so it is a set of $\leq \kappa$ ordinals increasing with ζ and $\sup_{\alpha<\delta} f_\alpha(i) \in s_{\zeta,i}$, moreover $\sup_{\alpha<\delta} f_\alpha(i) = \max(s_{\zeta,i})$
(c) $f_{\zeta,\alpha} \in {}^\kappa\mathrm{Ord}$ is defined by $f_{\zeta,\alpha}(i) = \mathrm{Min}\{s_{\zeta,i} \backslash f_\alpha(i)\}$,
(d) D_ζ is an ultrafilter on κ disjoint to I
(e) $f_\alpha \leq_{D_\zeta} g_\zeta$ for $\alpha < \delta$
(f) α_ζ is an ordinal $< \delta$
(g) $\alpha_\zeta \leq \alpha < \delta \Rightarrow g_\zeta <_{D_\zeta} f_{\zeta,\alpha}$.

If we succeed, let $\alpha(*) = \sup\{\alpha_\zeta : \zeta < \kappa^+\}$, so as $\mathrm{cf}(\delta) > \kappa^+$ clearly $\alpha(*) < \delta$. Now let $i < \kappa$ and look at $\langle f_{\zeta,\alpha(*)}(i) : \zeta < \kappa^+\rangle$; by its definition (see clause (c)), $f_{\zeta,\alpha(*)}(i)$ is the minimal member of the set $s_{\zeta,i} \backslash f_{\alpha(*)}(i)$. This set increases with ζ, so $f_{\zeta,\alpha(*)}(i)$ decreases with ζ (though not necessarily strictly), hence is eventually constant; so for some $\xi_i < \kappa^+$ we have $\zeta \in [\xi_i, \kappa^+) \Rightarrow f_{\zeta,\alpha(*)}(i) = f_{\xi_i,\alpha(*)}(i)$. Let $\xi(*) = \sup_{i<\kappa} \xi_i$, so $\xi(*) < \kappa^+$, hence

$\bigodot_1$ $\zeta \in [\xi(*), \kappa^+) and i < \kappa \Rightarrow f_{\zeta,\alpha(*)}(i) = f_{\xi(*),\alpha(*)}(i)$.

By clauses (e) + (g) of $\boxtimes$ we know that $f_{\alpha(*)} \leq_{D_{\xi(*)}} g_{\xi(*)} <_{D_{\xi(*)}} f_{\xi(*),\alpha(*)}$ hence for some $i < \kappa$ we have $f_{\alpha(*)}(i) \leq g_{\xi(*)}(i) < f_{\xi(*),\alpha(*)}(i)$. But $g_{\xi(*)}(i) \in s_{\xi(*)+1,i}$ by clause (b) of $\boxtimes$ hence recalling the definition of $f_{\xi(*)+1,\alpha(*)}(i)$ in clause (c) of $\boxtimes$ and the previous sentence $f_{\xi(*)+1,\alpha(*)}(i) \leq g_{\xi(*)}(i) < f_{\xi(*),\alpha(*)}(i)$, contradicting the statement $\odot_1$.

So necessarily we are stuck in the induction process. Let $\zeta < \kappa^+$ be the first ordinal that breaks the induction. Clearly $s_{\zeta,i}(i < \kappa), f_{\zeta,\alpha}(\alpha < \delta)$ are well defined.

Let $s_i =: s_{\zeta,i}$ (for $i < \kappa$) and $f'_\alpha = f_{\zeta,\alpha}$ (for $\alpha < \delta$), as defined in $\boxtimes$, clearly they are well defined. Clearly s_i is a set of $\leq \kappa$ ordinals and:

$(*)_1$ $f_\alpha \leq f'_\alpha$
$(*)_2$ $\alpha < \beta \Rightarrow f'_\alpha \leq_I f'_\beta$
$(*)_3$ if $b = \{i : f'_\alpha(i) < f'_\beta(i)\} \notin I$ and $\alpha < \beta < \delta$ then $f'_\alpha \restriction b <_I f_\beta \restriction b$.

We let for $\alpha < \delta$

$\bigodot_2$ $J_\alpha = \big\{b \subseteq \kappa : b \in I$ or $b \notin I$ and for every $\beta \in (\alpha, \delta)$ we have: $f'_\alpha \restriction (\kappa \setminus b) =_I f'_\beta \restriction (\kappa \setminus b)\big\}$
$\bigodot_3$ for $\alpha < \beta < \delta$ we let $a_{\alpha,\beta} =: \{i < \kappa : f'_\alpha(i) < f'_\beta(i)\}$.

<u>Then</u> as $\langle f'_\alpha : \alpha < \delta\rangle$ is $\leq_I$-increasing (i.e., $(*)_2$):

$(*)_4$ $a_{\alpha,\beta}/I$ increases with β, decreases with α, J_α increases with α

$(*)_5$ J_α is an ideal on κ extending I, in fact is the ideal generated by $I \cup \{a_{\alpha,\beta} : \beta \in (\alpha,\delta)\}$

$(*)_6$ if D is an ultrafilter on κ disjoint to J_α, then f'_α/D is a $<_D$-lub of $\{f_\beta/D : \beta < \delta\}$.

[Why? We know that $\beta \in (\alpha,\delta) \Rightarrow a_{\alpha,\beta} = \emptyset \mod D$, so $f_\beta \le f'_\beta =_D f'_\alpha$ for $\beta \in (\alpha,\delta)$, so f'_α/D is an $\le_D$-upper bound. If it is not a least upper bound then for some $g \in {}^\kappa\mathrm{Ord}$, for every $\beta < \delta$ we have $f_\beta \le_D g <_D f'_\alpha$ and we can get a contradiction to the choice of $\zeta, \bar{s}, f'_\beta$ because: (D,g,α) could serve as $D_\zeta, g_\zeta, \alpha_\zeta$.]

$(*)_7$ If D is an ultrafilter on κ disjoint to I but not to J_α for every $\alpha < \delta$ then $\bar{s}$ exemplifies that $\langle f_\alpha : \alpha < \delta\rangle$ is κ^+-chaotic for D, see Definition 1.3.

[Why? For every $\alpha < \delta$ for some $\beta \in (\alpha,\delta)$ we have $a_{\alpha,\beta} \in D$, i.e., $\{i < \kappa : f'_\alpha(i) < f'_\beta(i)\} \in D$, so $\langle f'_\alpha/D : \alpha < \delta\rangle$ is not eventually constant, so if $\alpha < \beta, f'_\alpha <_D f'_\beta$ then $f'_\alpha <_D f_\beta$ (by $(*)_3$) and $f_\alpha \le_D f'_\alpha$ (by (c)). So $f_\alpha \le_D f'_\alpha <_D f_\beta$ as required.]

$(*)_8$ if $\kappa \notin J_\alpha$ then f'_α is an $\le_{J_\alpha}$-e.u.b. of $\langle f_\beta : \beta < \delta\rangle$.

[Why? By $(*)_6$, f'_α is a $\le_{J_\alpha}$-upper bound of $\langle f_\beta : \beta < \delta\rangle$; so assume that it is not a $\le_{J_\alpha}$-e.u.b. of $\langle f_\beta : \beta < \delta\rangle$, hence there is a function g with domain κ, such that $g <_{J_\alpha} \mathrm{Max}\{1, f'_\alpha\}$, but for no $\beta < \delta$ do we have

$$c_\beta =: \{i < \kappa : g(i) < \mathrm{Max}\{1, f_\beta(i)\}\} = \kappa \mod J_\alpha.$$

Clearly $\langle c_\beta : \beta < \delta\rangle$ is increasing modulo J_α so there is an ultrafilter D on κ disjoint to $J_\alpha \cup \{c_\beta : \beta < \delta\}$. So $\beta < \delta \Rightarrow f_\beta \le_D g \le_D f'_\alpha$, so we get a contradiction to $(*)_6$ except when $g =_D f'_\alpha$ and then $f'_\alpha =_D 0_\kappa$ (as $g(i) < 1 \vee g(i) < f'_\alpha(i)$). If we can demand $c^* = \{i : f'_\alpha(i) = 0\} \notin D$ we are done, but easily $c^* \setminus c_\beta \in J_\alpha$ so we finish.]

$(*)_9$ If $\mathrm{cf}[f'_\alpha(i)] \le \kappa$ then $f'_\alpha(i) = f_\alpha(i)$ so clause (D) of the lemma holds.

[Why? By the definition of $s_\zeta = c\ell_{\bar{e}}[\ldots]$ and the choice of $\bar{e}$, and of $f'_\alpha(i)$.]

$(*)_{10}$ Clause (I) of the conclusion holds.

[Why? As $f_\alpha \le_{J_\alpha} f_\beta \le_{J_\alpha} f'_\alpha$ and $f_\alpha \restriction b_\alpha =_{J_\alpha} f'_\alpha \restriction b_\alpha$ by $(*)_9$.]

$(*)_{11}$ if $\alpha < \beta < \delta$ then $f'_\alpha = f'_\beta \mod J_\alpha$, so clause $(\mathrm{F})^+$ holds.

[Why? First, $\bar{f}$ is $\le_I$-increasing hence it is $\le_{J_\alpha}$-increasing. Second, $\beta \le \alpha \Rightarrow f_\beta \le_I f_\alpha \le f'_\alpha \Rightarrow f_\beta \le_{J_\alpha} f'_\alpha$. Third, if $\beta \in (\alpha,\delta)$ then $a_{\alpha,\beta} = \{i < \kappa : f'_\alpha(i) < f'_\beta(i)\} \in J_\alpha$, hence $f'_\beta \le_{J_\alpha} f'_\alpha$ but as $f_\alpha \le_I f_\beta$ clearly $f'_\alpha \le_I f'_\beta$ hence $f'_\alpha \le_{J_\alpha} f'_\beta$, so together $f'_\alpha =_{J_\alpha} f'_\beta$.]

$(*)_{12}$ if $\mathrm{cf}(\delta) > 2^\kappa$ then for some $\alpha(*)$, $J_{\alpha(*)} = I$ (hence $\bar{f}$ has a $\le_I$-e.u.b.)

[Why? As $\langle J_\alpha : \alpha < \delta\rangle$ is a $\supseteq$-decreasing sequence of subsets of $\mathscr{P}(\kappa)$ it is eventually constant, say, i.e., there is $\alpha(*) < \delta$ such that $\alpha(*) \le \alpha < \delta \Rightarrow J_\alpha = J_{\alpha(*)}$. Also $I \subseteq J_{\alpha(*)}$, but if $I \ne J_{\alpha(*)}$ then there is an ultrafilter D of κ disjoint to I but not to $J_{\alpha(*)}$ hence $\langle s_i : i < \kappa\rangle$ witness being κ-chaotic. But this implies $\mathrm{cf}(\delta) \le \prod\limits_{i<\kappa} |s_i| \le \kappa^\kappa = 2^\kappa$, contradiction.]

The reader can check the rest. $\square_{1.6}$

Example 1.8. 1) We show that l.u.b and e.u.b are not the same. Let I be an ideal on $\kappa, \kappa^+ < \lambda = \mathrm{cf}(\lambda), \bar{a} = \langle a_\alpha : \alpha < \lambda\rangle$ be a sequence of subsets of κ, (strictly) increasing modulo I, $\kappa \backslash a_\alpha \notin I$ but there is no $b \in \mathscr{P}(\kappa)\backslash I$ such that $\bigwedge_\alpha b \cap a_\alpha \in I$. [Does this occur? E.g., for $I = [\kappa]^{<\kappa}$, the existence of such $\bar{a}$ is known to be consistent; e.g., MA $and \kappa = \aleph_0 and \lambda = 2^{\aleph_0}$. Moreover, for any κ and $\kappa^+ < \lambda = \mathrm{cf}(\lambda) \le 2^\kappa$ we can find $a_\alpha \subseteq \kappa$ for $\alpha < \lambda$ such that, e.g., any Boolean combination of the a_α's has cardinality κ (less needed). Let I_0 be the ideal on κ generated by $[\kappa]^{<\kappa} \cup \{a_\alpha \backslash a_\beta : \alpha < \beta < \lambda\}$, and let I be maximal in $\{J : J$ an ideal on $\kappa, I_0 \subseteq J$ and $[\alpha < \beta < \lambda \Rightarrow a_\beta \backslash a_\alpha \notin J]\}$. So if G.C.H. fails, we have examples.]

For $\alpha < \lambda$, we let $f_\alpha : \kappa \to \mathrm{Ord}$ be:

$$f_\alpha(i) = \begin{cases} \alpha & \text{if } i \in \kappa \setminus a_\alpha, \\ \lambda + \alpha & \text{if } i \in a_\alpha. \end{cases}$$

Now the constant function $f \in {}^\kappa\mathrm{Ord}, f(i) = \lambda + \lambda$ is a l.u.b of $\langle f_\alpha : \alpha < \lambda\rangle$ but not an e.u.b. (both $\mod I$) (no e.u.b. is exemplified by $g \in {}^\kappa\mathrm{Ord}$ which is constantly λ).

2) Why do we require "$\mathrm{cf}(\delta) > \kappa^+$" rather than "$\mathrm{cf}(\delta) > \kappa$"? As we have to, by Kojman-Shelah [KS00].

Recall (see [She97b, 2.3(2)])

Definition 1.9. We say that $\bar{f} = \langle f_\alpha : \alpha < \delta\rangle$ obeys $\langle u_\alpha : \alpha \in S\rangle$ when

- (a) $f_\alpha : w \to \mathrm{Ord}$ for some fixed set w
- (b) S a set of ordinals
- (c) $u_\alpha \subseteq \alpha$
- (d) if $\alpha \in S \cap \delta$ and $\beta \in u_\alpha$ then $t \in w \Rightarrow f_\beta(t) \le f_\alpha(t)$.

Claim 1.10. *Assume I is an ideal on $\kappa, \bar{f} = \langle f_\alpha : \alpha < \delta\rangle$ is $\le_I$-increasing and obeys $\bar{u} = \langle u_\alpha : \alpha \in S\rangle$. The sequence $\bar{f}$ has a $\le_I$-e.u.b. when for some S^+ we have $\circledast_1$ or $\circledast_2$ where*

- $\circledast_1$ (a) *$S^+ \subseteq \{\alpha < \delta : \mathrm{cf}(\alpha) > \kappa\}$*
  - (b) *S^+ is a stationary subset of δ*
  - (c) *for each $\alpha \in S^+$ there are unbounded subsets u, v of α for which $\beta \in v \Rightarrow u \cap \beta \subseteq u_\beta$.*
- $\circledast_2$ *$S^+ = \{\delta\}$ and for δ clause (c) of $\circledast_1$ holds.*

Proof. By [She97b]. $\square_{1.10}$

Remark 1.11. 1) Connected to $\check{I}[\lambda]$, see [She97b].

Claim 1.12. *Suppose J a σ-complete ideal on $\delta^*, \mu > \kappa = \mathrm{cf}(\mu), \mu = \mathrm{tlim}_J\langle\lambda_i : i < \delta\rangle, \delta^* < \mu, \lambda_i = \mathrm{cf}(\lambda_i) > \delta^*$ for $i < \delta^*$ and $\lambda = \mathrm{tcf}(\prod_{i<\delta^*} \lambda_i/J)$, and $\langle f_\alpha : \alpha < \lambda\rangle$ exemplifies this.*

<u>Then</u> *we have*

- $(*)$ *if $\langle u_\beta : \beta < \lambda\rangle$ is a sequence of pairwise disjoint non-empty subsets of λ, each of cardinality $\le \sigma$ (not $< \sigma$!) and $\alpha^* < \mu^+$, then we can find $B \subseteq \lambda$ such that:*

(*a*) $\mathrm{otp}(B) = \alpha^*$,
(*b*) *if* $\beta \in B, \gamma \in B$ *and* $\beta < \gamma$ *then* $\sup(u_\beta) < \min(u_\gamma)$,
(*c*) *we can find* $s_\zeta \in J$ *for* $\zeta \in \bigcup_{i \in B} u_i$ *such that: if* $\zeta \in \bigcup_{\beta \in B} u_\beta, \xi \in \bigcup_{\beta \in B} u_\beta, \zeta < \xi$ *and* $i \in \delta \backslash (s_\zeta \cup s_\xi)$, *then* $f_\zeta(i) < f_\xi(i)$.

Proof. First assume $\alpha^* < \mu$. For each regular $\theta < \mu$, as $\theta^+ < \lambda = \mathrm{cf}(\lambda)$ there is a stationary $S_\theta \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta < \delta\}$ which is in $\check{I}[\lambda]$ (see [She93a, 1.5]) which is equivalent (see [She93a, 1.2(1)]) to:

($*$) there is $\bar{C}^\theta = \langle C^\theta_\alpha : \alpha < \lambda \rangle$
 (α) C^θ_α a subset of α, with no accumulation points (in C^θ_α),
 (β) $[\alpha \in \mathrm{nacc}(C^\theta_\beta) \Rightarrow C^\theta_\alpha = C^\theta_\beta \cap \alpha]$,
 (γ) for some club E^0_θ of λ,
 $[\delta \in S_\theta \cap E^0_\theta \Rightarrow \mathrm{cf}(\delta) = \theta < \delta \wedge \delta = \sup(C^\theta_\delta) \wedge \mathrm{otp}(C^\theta_\delta) = \theta]$.

Without loss of generality $S_\theta \subseteq E^0_\theta$, and $\bigwedge_{\alpha<\delta} \mathrm{otp}(C^\theta_\alpha) \leq \theta$. By [She94g, 2.3,Def.1.3] for some club E_θ of λ, $\langle g\ell(C^\theta_\alpha, E_\theta) : \alpha \in S_\theta \rangle$ guess clubs (i.e., for every club $E \subseteq E_\theta$ of λ, for stationarily many $\zeta \in S_\theta$, $g\ell(C^\theta_\zeta, E_\theta) \subseteq E$) (remember $g\ell(C^\theta_\delta, E_\theta) = \{\sup(\gamma \cap E_\theta) : \gamma \in C^\theta_\delta ; \gamma > \mathrm{Min}(E_\theta)\}$). Let $C^{\theta,*}_\alpha = \{\gamma \in C^\theta_\alpha : \gamma = \mathrm{Min}(C^\theta_\alpha \backslash \sup(\gamma \cap E_\theta))\}$, they have all the properties of the C^θ_α's and guess clubs in a weak sense: for every club E of λ for some $\alpha \in S_\theta \cap E$, if $\gamma_1 < \gamma_2$ are successive members of E then $|(\gamma_1, \gamma_2] \cap C^{\theta,*}_\alpha| \leq 1$; moreover, the function $\gamma \mapsto \sup(E \cap \gamma)$ is one to one on $C^{\theta,*}_\alpha$.

Now we define by induction on $\zeta < \lambda$, an ordinal α_ζ and functions $g^\zeta_\theta \in \prod_{i<\delta^*} \lambda_i$ (for each $\theta \in \Theta =: \{\theta : \theta < \mu, \theta \text{ regular uncountable}\}$).

For given ζ, let $\alpha_\zeta < \lambda$ be minimal such that:

$$\xi < \zeta \Rightarrow \alpha_\xi < \alpha_\zeta$$

$$\xi < \zeta \wedge \theta \in \Theta \Rightarrow g^\xi_\theta < f_{\alpha_\zeta} \mod J.$$

Now α_ζ exists as $\langle f_\alpha : \alpha < \lambda \rangle$ is $<_J$-increasing cofinal in $\prod_{i<\delta^*} \lambda_i / J$. Now for each $\theta \in \Theta$ we define g^ζ_θ as follows:

for $i < \delta^*, g^\zeta_\theta(i)$ is $\sup[\{g^\xi_\theta(i) + 1 : \xi \in C^\theta_\zeta\} \cup \{f_{\alpha_\zeta}(i) + 1\}]$ if this number is $< \lambda_i$, and $f_{\alpha_\zeta}(i) + 1$ otherwise.

Having made the definition we prove the assertion. We are given $\langle u_\beta : \beta < \lambda \rangle$, a sequence of pairwise disjoint non-empty subsets of λ, each of cardinality $\leq \sigma$ and $\alpha^* < \mu$. We should find B as promised; let $\theta =: (|\alpha^*| + |\delta^*|)^+$ so $\theta < \mu$ is regular $> |\delta^*|$. Let $E = \{\delta \in E_\theta : (\forall \zeta)[\zeta < \delta \Leftrightarrow \sup(u_\zeta) < \delta \Leftrightarrow u_\zeta \subseteq \delta \Leftrightarrow \alpha_\zeta < \delta]\}$. Choose $\alpha \in S_\theta \cap \mathrm{acc}(E)$ such that $g\ell(C^\theta_\zeta, E_\theta) \subseteq E$; hence letting $C^{\theta,*}_\alpha = \{\gamma_i : i < \theta\}$ (increasing), $\gamma(i) = \gamma_i$, we know that $i < \delta^* \Rightarrow (\gamma_i, \gamma_{i+1}) \cap E \neq \emptyset$. Now let $B =: \{\gamma_{5i+3} : i < \alpha^*\}$ we shall prove that B is as required. For $\alpha \in u_{\gamma(5\zeta+3)}, \zeta < \alpha^*$, let $s^o_\alpha = \{i < \delta^* : g^{\gamma(5\zeta+1)}_\theta(i) < f_\alpha(i) < g^{\gamma(5\zeta+4)}_\theta(i)\}$, for each $\zeta < \alpha^*$ let $\langle \alpha_{\zeta,\varepsilon} : \varepsilon < |u_{\gamma(5\zeta+3)}| \rangle$ enumerate $u_{\gamma(5\zeta+3)}$ and let

$$s^1_{\alpha_{\zeta,\varepsilon}} = \{i : \text{ for every } \xi < \epsilon, f_{\alpha_{\zeta,\xi}}(i) < f_{\alpha_{\zeta,\epsilon}}(i) \quad \Leftrightarrow \alpha_{\zeta,\xi} < \alpha_{\zeta,\epsilon} \\ \Leftrightarrow f_{\alpha_{\zeta,\xi}}(i) \leq f_{\alpha_{\zeta,\epsilon}}(i)\}.$$

Lastly, for $\alpha \in \bigcup\limits_{\zeta<\alpha^*} u_{5\zeta+3}$ let $s_\alpha = s^o_\alpha \cup s^1_\alpha$ and it is enough to check that $\langle \zeta_\alpha : \alpha \in B\rangle$ witness that B is as required. Also we have to consider $\alpha^* \in [\mu, \mu^+)$, we prove this by induction on α^* and in the induction step we use $\theta = (\mathrm{cf}(\alpha^*) + |\delta^*|)^+$ using a similar proof. $\square_{1.12}$

Remark 1.13. In 1.12:
1) We can avoid guessing clubs.
2) Assume $\sigma < \theta_1 < \theta_2 < \mu$ are regular and there is $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta_1\}$ from $I[\lambda]$ such that for every $\zeta < \lambda$ (or at least a club) of cofinality θ_2, $S \cap \zeta$ is stationary and $\langle f_\alpha : \alpha < \lambda\rangle$ obey suitable $\bar{C}^\theta$ (see [She94c, §2]). Then for some $A \subseteq \lambda$ unbounded, for every $\langle u_\beta : \beta < \theta_2\rangle$ sequence of pairwise disjoint non-empty subsets of A, each of cardinality $< \sigma$ with $[\min u_\beta, \sup u_\beta]$ pairwise disjoint we have: for every $B_0 \subseteq A$ of order type θ_2, for some $B \subseteq B_0$, $|B| = \theta_1$, (c) of $(*)$ of 1.12 holds.
3) In $(*)$ of 1.12, "$\alpha^* < \mu$" can be replaced by "$\alpha^* < \mu^+$" (prove by induction on α^*).

Observation 1.14. *Assume $\lambda < \lambda^{<\lambda}, \mu = \mathrm{Min}\{\tau : 2^\tau > \lambda\}$. Then there are δ, χ and $\mathscr{T}$, satisfying the condition $(*)$ below for $\chi = 2^\mu$ or at least arbitrarily large regular $\chi < 2^\mu$*

- $(*)$ *$\mathscr{T}$ a tree with δ levels, (where $\delta \leq \mu$) with a set X of $\geq \chi$ δ-branches, and for $\alpha < \delta$, $\bigcup\limits_{\beta<\alpha} |\mathscr{T}_\beta| < \lambda$.*

Proof. So let $\chi \leq 2^\mu$ be regular, $\chi > \lambda$.

Case 1: $\bigwedge\limits_{\alpha<\mu} 2^{|\alpha|} < \lambda$. Then $\mathscr{T} = {}^{\mu>}2, \mathscr{T}_\alpha = {}^\alpha 2$ are O.K. (the set of branches ${}^\mu 2$ has cardinality 2^μ).

Case 2: Not Case 1. So for some $\theta < \mu$, $2^\theta \geq \lambda$, but by the choice of $\mu, 2^\theta \leq \lambda$, so $2^\theta = \lambda, \theta < \mu$ and so $\theta \leq \alpha < \mu \Rightarrow 2^{|\alpha|} = 2^\theta$. Note $|{}^{\mu>}2| = \lambda$ as $\mu \leq \lambda$. Note also that $\mu = \mathrm{cf}(\mu)$ in this case (by the Bukovsky-Hechler theorem).

Subcase 2A: $\mathrm{cf}(\lambda) \neq \mu = \mathrm{cf}(\mu)$.

Let ${}^{\mu>}2 = \bigcup\limits_{j<\lambda} B_j, B_j$ increasing with $j, |B_j| < \lambda$. For each $\eta \in {}^\mu 2$, (as $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\mu)$) for some $j_\eta < \lambda$,

$$\mu = \sup\{\zeta < \mu : \eta \restriction \zeta \in B_{j_\eta}\}.$$

So as $\mathrm{cf}(\chi) \neq \mu$, for some ordinal $j^* < \lambda$ we have

$$\{\eta \in {}^\mu 2 : j_\eta \leq j^*\} \text{ has cardinality } \geq \chi.$$

As $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\mu)$ and $\mu \leq \lambda$ (by its definition) clearly $\mu < \lambda$, hence $|B_{j^*}| \times \mu < \lambda$.

Let

$$\mathscr{T} = \{\eta \restriction \epsilon : \epsilon < \ell g(\eta) \text{ and } \eta \in B_{j^*}\}.$$

It is as required.

Subcase 2B: Not 2A so $\mathrm{cf}(\lambda) = \mu = \mathrm{cf}(\mu)$.

If $\lambda = \mu$ we get $\lambda = \lambda^{<\lambda}$ contradicting an assumption.

So $\lambda > \mu$, so λ singular. Now if $\alpha < \mu, \mu < \sigma_i = \mathrm{cf}(\sigma_i) < \lambda$ for $i < \alpha$ then (see [She94e, ?, 1.3(10)]) $\max \mathrm{pcf}\{\sigma_i : i < \alpha\} \le \prod\limits_{i<\alpha} \sigma_i \le \lambda^{|\alpha|} \le (2^\theta)^{|\alpha|} \le 2^{<\mu} = \lambda$, but as λ is singular and $\max \mathrm{pcf}\{\sigma_i : i < \alpha\}$ is regular (see [She94c, 1.9]), clearly the inequality is strict, i.e., $\max \mathrm{pcf}\{\sigma_i : i < \alpha\} < \lambda$. So let $\langle \sigma_i : i < \mu\rangle$ be a strictly increasing sequence of regulars in (μ, λ) with limit λ, and by [She94b, 3.4] there is $\mathscr{T} \subseteq \prod\limits_{i<\mu} \sigma_i$ satisfying $|\{\nu \restriction i : \nu \in \mathscr{T}\}| \le \max \mathrm{pcf}\{\sigma_j : j < i\} < \lambda$, and number of μ-branches $> \lambda$. In fact we can get any regular cardinal in $(\lambda, \mathrm{pp}^+(\lambda))$ in the same way.

Let $\lambda^* = \min\{\lambda' : \mu < \lambda' \le \lambda, \mathrm{cf}(\lambda') = \mu \text{ and } \mathrm{pp}(\lambda') > \lambda\}$, so (by [She94b, 2.3]), also λ^* has those properties and $\mathrm{pp}(\lambda^*) \ge \mathrm{pp}(\lambda)$. So if $\mathrm{pp}^+(\lambda^*) = (2^\mu)^+$ or $\mathrm{pp}(\lambda^*) = 2^\mu$ is singular, we are done. So assume this fails.

If $\mu > \aleph_0$, then (as in [She96, 3.4]) $\alpha < 2^\mu \Rightarrow \mathrm{cov}(\alpha, \mu^+, \mu^+, \mu) < 2^\mu$ and we can finish as in subcase 2A (actually $\mathrm{cov}(2^{<\mu}, \mu^+, \mu^+, \mu) < 2^\mu$ suffices which holds by the previous sentence and [She94b, 5.4]). If $\mu = \aleph_0$ all is easy. $\square_{1.14}$

Claim 1.15. *Assume* $\mathfrak{b}_0 \subseteq \ldots \subseteq \mathfrak{b}_k \subseteq \mathfrak{b}_{k+1} \subseteq \cdots$ *for* $k < \omega, \mathfrak{a} = \bigcup\limits_{k<\omega} \mathfrak{b}_k$ *(and* $|\mathfrak{a}|^+ < \mathrm{Min}(\mathfrak{a})$*) and* $\lambda \in \mathrm{pcf}(\mathfrak{a}) \backslash \bigcup\limits_{k<\omega} \mathrm{pcf}(\mathfrak{b}_k)$.

1) We can find finite $\mathfrak{d}_k \subseteq \mathrm{pcf}(\mathfrak{b}_k \backslash \mathfrak{b}_{k-1})$ *(stipulating* $\mathfrak{b}_{-1} = \emptyset$*) such that* $\lambda \in \mathrm{pcf}(\cup\{\mathfrak{d}_k : k < \omega\})$.

2) Moreover, we can demand $\mathfrak{d}_k \subseteq \mathrm{pcf}(\mathfrak{b}_k) \backslash (\mathrm{pcf}(\mathfrak{b}_{k-1}))$.

Proof. We start to repeat the proof of [She94a, 1.5] for $\kappa = \omega$. But there we apply [She94a, 1.4] to $\langle \mathfrak{b}_\zeta : \zeta < \kappa\rangle$ and get $\langle\langle \mathfrak{c}_{\zeta,\ell} : \ell \le n(\zeta)\rangle : \zeta < \kappa\rangle$ and let $\lambda_{\zeta,\ell} = \max \mathrm{pcf}(\mathfrak{c}_{\zeta,\ell})$. Here we apply the same claim ([She94a, 1.4]) to $\langle \mathfrak{b}_k \backslash \mathfrak{b}_{k-1} : k < \omega\rangle$ to get part (1). As for part (2), in the proof of [She94a, 1.5] we let $\delta = |\mathfrak{a}|^+ + \aleph_2$ choose $\langle N_i : i < \delta\rangle$, but now we have to adapt the proof of [She94a, 1.4] (applied to $\mathfrak{a}$, $\langle \mathfrak{b}_k : k < \omega\rangle$, $\langle N_i : i < \delta\rangle$); we have gotten there, toward the end, $\alpha < \delta$ such that $E_\alpha \subseteq E$. Let $E_\alpha = \{i_k : k < \omega\}, i_k < i_{k+1}$. But now instead of applying [She94a, 1.3] to each $\mathfrak{b}_\ell$ separately, we try to choose $\langle \mathfrak{c}_{\zeta,\ell} : \ell \le n(\zeta)\rangle$ by induction on $\zeta < \omega$. For $\zeta = 0$ we apply [?, 1.3]. For $\zeta > 0$, we apply [She94a, 1.3] to $\mathfrak{b}_\zeta$ but there defining by induction on $\ell, \mathfrak{c}_\ell = \mathfrak{c}_{\zeta,\ell} \subseteq \mathfrak{a}$ such that $\max(\mathrm{pcf}(\mathfrak{a} \backslash \mathfrak{c}_{\zeta,0} \backslash \cdots \backslash \mathfrak{c}_{\zeta,\ell-1}) \cap \mathrm{pcf}(\mathfrak{b}_\zeta)$ is strictly decreasing with ℓ. $\square$

We use:

Observation 1.16. *If* $|\mathfrak{a}_i| < \mathrm{Min}(\mathfrak{a}_i)$ *for* $i < i^*$*, then* $\mathfrak{c} = \bigcap\limits_{i<i^*} \mathrm{pcf}(\mathfrak{a}_i)$ *has a last element or is empty.*

Proof. By renaming without loss of generality $\langle |\mathfrak{a}_i| : i < i^*\rangle$ is non-decreasing. By [She94f, 1.12]

$(*)_1$ $\mathfrak{d} \subseteq \mathfrak{c} and |\mathfrak{d}| < \mathrm{Min}(\mathfrak{d}) \Rightarrow \mathrm{pcf}(\mathfrak{d}) \subseteq \mathfrak{c}$.

By [She94a, 2.6] or 2.7(2)

$(*)_2$ if $\lambda \in \mathrm{pcf}(\mathfrak{d}), \mathfrak{d} \subseteq \mathfrak{c}, |\mathfrak{d}| < \mathrm{Min}(\mathfrak{d})$ then for some $\geq \subseteq \mathfrak{d}$ we have $|\geq| \leq \mathrm{Min}(\mathfrak{a}_0), \lambda \in \mathrm{pcf}(\geq)$.

Now choose by induction on $\zeta < |\mathfrak{a}_0|^+, \theta_\zeta \in \mathfrak{c}$, satisfying $\theta_\zeta > \max \mathrm{pcf}\{\theta_\epsilon : \epsilon < \zeta\}$. If we are stuck in ζ, $\max \mathrm{pcf}\{\theta_\epsilon : \epsilon < \zeta\}$ is the desired maximum by $(*)_1$. If we succeed the cardinal $\theta = \max \mathrm{pcf}\{\theta_\epsilon : \epsilon < |\mathfrak{a}_0|^+\}$ is in $\mathrm{pcf}\{\theta_\epsilon : \epsilon < \zeta\}$ for some $\zeta < |\mathfrak{a}_0|^+$ by $(*)_2$; easy contradiction. $\square_{1.16}$

Conclusion 1.17. *Assume $\aleph_0 = \mathrm{cf}(\mu) \leq \kappa \leq \mu_0 < \mu, [\mu' \in (\mu_0, \mu) and \mathrm{cf}(\mu') \leq \kappa \Rightarrow \mathrm{pp}_\kappa(\mu') < \lambda]$ and $\mathrm{pp}^+_\kappa(\mu) > \lambda = \mathrm{cf}(\lambda) > \mu$. Then we can find λ_n for $n < \omega, \mu_0 < \lambda_n < \lambda_{n+1} < \mu$, $\mu = \bigcup_{n<\omega} \lambda_n$ and $\lambda = \mathrm{tcf}(\prod_{n<\omega} \lambda_n / J)$ for some ideal J on ω (extending J^{bd}_ω).*

Proof. Let $\mathfrak{a} \subseteq (\mu_0, \mu) \cap \mathrm{Reg}, |\mathfrak{a}| \leq \kappa, \lambda \in \mathrm{pcf}(\mathfrak{a})$. Without loss of generality $\lambda = \max \mathrm{pcf}(\mathfrak{a})$, let $\mu = \bigcup_{n<\omega} \mu^0_n$, $\mu_0 \leq \mu^0_n < \mu^0_{n+1} < \mu$, let $\mu^1_n = \mu^0_n + \sup\{\mathrm{pp}_\kappa(\mu') : \mu_0 < \mu' \leq \mu^0_n$ and $\mathrm{cf}(\mu') \leq \kappa\}$, by [She94b, 2.3] $\mu^1_n < \mu, \mu^1_n = \mu^0_n + \sup\{\mathrm{pp}_\kappa(\mu') : \mu_0 < \mu' < \mu^1_n$ and $\mathrm{cf}(\mu') \leq \kappa\}$ and obviously $\mu^1_n \leq \mu^1_{n+1}$; by replacing by a subsequence without loss of generality $\mu^1_n < \mu^1_{n+1}$. Now let $\mathfrak{b}_n = \mathfrak{a} \cap \mu^1_n$ and apply the previous claim 1.15: to $\mathfrak{b}_k =: \mathfrak{a} \cap (\mu^1_n)^+$, note:

$$\max \mathrm{pcf}(\mathfrak{b}_k) \leq \mu^1_k < \mathrm{Min}(\mathfrak{b}_{k+1} \backslash \mathfrak{b}_k).$$

$\square_{1.17}$

Claim 1.18. *1) Assume $\aleph_0 < \mathrm{cf}(\mu) = \kappa < \mu_0 < \mu, 2^\kappa < \mu$ and $[\mu_0 \leq \mu' < \mu and \mathrm{cf}(\mu') \leq \kappa \Rightarrow \mathrm{pp}_\kappa(\mu') < \mu]$. If $\mu < \lambda = \mathrm{cf}(\lambda) < \mathrm{pp}^+(\mu)$ then there is a tree $\mathscr{T}$ with κ levels, each level of cardinality $< \mu$, $\mathscr{T}$ has exactly λ κ-branches.*
2) Suppose $\langle \lambda_i : i < \kappa \rangle$ is a strictly increasing sequence of regular cardinals, $2^\kappa < \lambda_0, \mathfrak{a} =: \{\lambda_i : i < \kappa\}, \lambda = \max \mathrm{pcf}(\mathfrak{a}), \lambda_j > \max \mathrm{pcf}\{\lambda_i : i < j\}$ for each $j < \kappa$ (or at least $\sum_{i<j} \lambda_i > \max \mathrm{pcf}\{\lambda_i : i < j\}$) and $\mathfrak{a} \notin J$ where $J = \{\mathfrak{b} \subseteq \mathfrak{a} : \mathfrak{b}$ is the union of countably many members of $J_{<\lambda}[\mathfrak{a}]\}$ (so $J \supseteq J^{\mathrm{bd}}_{\mathfrak{a}}$ and $cf(\kappa) > \aleph_0$). Then the conclusion of (1) holds with $\mu = \sum_{i<\kappa} \lambda_i$.

Proof. 1) By (2) and [She94a, §1] (or can use the conclusion of [She94e, AG,5.7]).
2) For each $\mathfrak{b} \subseteq \mathfrak{a}$ define the function $g_{\mathfrak{b}} : \kappa \to \mathrm{Reg}$ by

$$g_{\mathfrak{b}}(i) = \max \mathrm{pcf}[\mathfrak{b} \cap \{\lambda_j : j < i\}].$$

Clearly $[\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \Rightarrow g_{\mathfrak{b}_1} \leq g_{\mathfrak{b}_2}]$. As $\mathrm{cf}(\kappa) > \aleph_0, J$ is $\aleph_1$-complete, there is $\mathfrak{b} \subseteq \mathfrak{a}, \mathfrak{b} \notin J$ such that:

$$\mathfrak{c} \subseteq \mathfrak{b} and \mathfrak{c} \notin J \Rightarrow \neg g_{\mathfrak{c}} <_J g_{\mathfrak{b}}.$$

Let $\lambda^*_i = \max \mathrm{pcf}(\mathfrak{b} \cap \{\lambda_j : j < i\})$. For each i let $\mathfrak{b}_i = \mathfrak{b} \cap \{\lambda_j : j < i\}$ and $\langle \langle f^{\mathfrak{b}}_{\lambda,\alpha} : \alpha < \lambda \rangle : \lambda \in \mathrm{pcf}(\mathfrak{b}) \rangle$ be as in [She94a, §1].

Let

$$\mathscr{T}^0_i = \{\operatorname*{Max}_{0<\ell<n} f^{\mathfrak{b}}_{\lambda_\ell,\alpha_\ell} \restriction \mathfrak{b}_i : \lambda_\ell \in \mathrm{pcf}(\mathfrak{b}_i), \alpha_\ell < \lambda_\ell, n < \omega\}.$$

Let $\mathscr{T}_i = \{f \in \mathscr{T}_i^0 :$ for every $j < i, f \restriction \mathfrak{b}_j \in \mathscr{T}_j^0$ moreover for some $f' \in \prod\limits_{j<\kappa} \lambda_j$, for every j, $f' \restriction \mathfrak{b}_j \in \mathscr{T}_j^0$ and $f \subseteq f'\}$, and $\mathscr{T} = \bigcup\limits_{i<\kappa} \mathscr{T}_i$, clearly it is a tree, $\mathscr{T}_i$ its ith level (or empty), $|\mathscr{T}_i| \le \lambda_i^*$. By [She94a, 1.3,1.4] for every $g \in \prod \mathfrak{b}$ for some $f \in \prod \mathfrak{b}$, $\bigwedge\limits_{i<\kappa} f \restriction \mathfrak{b}_i \in \mathscr{T}_i^0$ hence $\bigwedge\limits_{i<\kappa} f \restriction \mathfrak{b}_i \in \mathscr{T}_i$. So $|\mathscr{T}_i| = \lambda_i^*$, and $\mathscr{T}$ has $\ge \lambda$ κ-branches. By the observation below we can finish (apply it essentially to $\mathscr{F} = \{\eta$: for some $f \in \prod \mathfrak{b}$ for $i < \kappa$ we have $\eta(i) = f \restriction \mathfrak{b}_i$ and for every $i < \kappa, f \restriction \mathfrak{b}_i \in \mathscr{T}_i^0\}$), then find $A \subseteq \kappa, \kappa \setminus A \in J$ and $g^* \in \prod\limits_{i<\kappa} (\lambda_i + 1)$ such that $Y' =: \{f \in F : f \restriction A < g^* \restriction A\}$ has cardinality λ and then the tree will be $\mathscr{T}'$ where $\mathscr{T}_i' =: \{f \restriction \mathfrak{b}_i : f \in Y'\}$ and $\mathscr{T}' = \bigcup\limits_{i<\kappa} \mathscr{T}_i'$. (So actually this proves that if we have such a tree with $\ge \theta(\mathrm{cf}(\theta) > 2^\kappa)$ κ-branches then there is one with exactly θ κ-branches.) $\square_{1.18}$

Observation 1.19. *If $\mathscr{F} \subseteq \prod\limits_{i<\kappa} \lambda_i$, J an $\aleph_1$-complete ideal on κ, and $[f \ne g \in \mathscr{F} \Rightarrow f \ne_J g]$ and $|\mathscr{F}| \ge \theta, \mathrm{cf}(\theta) > 2^\kappa$, then for some $g^* \in \prod\limits_{i<\kappa} (\lambda_i + 1)$ we have:*

(a) *$Y = \{f \in \mathscr{F} : f <_J g^*\}$ has cardinality θ,*
(b) *for $f' <_J g^*$, we have $|\{f \in \mathscr{F} : f \le_J f'\}| < \theta$,*
(c) *there* [1] *are $f_\alpha \in Y$ for $\alpha < \theta$ such that: $f_\alpha <_J g^*, [\alpha < \beta < \theta \Rightarrow \neg f_\beta <_J f_\alpha]$.*

(Also in [She06, §1]*).*

Proof. Let $Z =: \{g : g \in \prod\limits_{i<\kappa} (\lambda_i + 1)$ and $Y_g =: \{f \in \mathscr{F} : f \le_J g\}$ has cardinality $\ge \theta\}$. Clearly $\langle \lambda_i : i < \kappa \rangle \in Z$ so there is $g^* \in Z$ such that: $[g' \in Z \Rightarrow \neg g' <_J g^*]$; so clause (b) holds. Let $Y = \{f \in \mathscr{F} : f <_J g^*\}$, easily $Y \subseteq Y_{g^*}$ and $|Y_{g^*} \setminus Y| \le 2^\kappa$ hence $|Y| \ge \theta$, also clearly $[f_1 \ne f_2 \in \mathscr{F} and f_1 \le_J f_2 \Rightarrow f_1 <_J f_2]$. If (a) fails, necessarily by the previous sentence $|Y| > \theta$. For each $f \in Y$ let $Y_f = \{h \in Y : h \le_J f\}$, so by clause (b) we have $|Y_f| < \theta$ hence by the Hajnal free subset theorem for some $Z' \subseteq Z$, $|Z'| = \lambda^+$, and $f_1 \ne f_2 \in Z' \Rightarrow f_1 \notin Y_{f_2}$ so $[f_1 \ne f_2 \in Z' \Rightarrow \neg f_1 <_J f_2]$. But there is no such Z' of cardinality $> 2^\kappa$ ([She86, 2.2,p.264]) so clause (a) holds. As for clause (c): choose $f_\alpha \in \mathscr{F}$ by induction on α, such that $f_\alpha \in Y \setminus \bigcup_{\beta<\alpha} Y_{f_\beta}$; it exists by cardinality considerations and $\langle f_\alpha : \alpha < \theta \rangle$ is as required (in (c)). $\square_{1.19}$

Observation 1.20. *Let $\kappa < \lambda$ be regular uncountable, $2^\kappa < \mu_i < \lambda$ (for $i < \kappa$), μ_i increasing in i. The following are equivalent:*

(A) *there is $\mathscr{F} \subseteq {}^\kappa\lambda$ such that:*
  (i) *$|\mathscr{F}| = \lambda$,*
  (ii) *$|\{f \restriction i : f \in \mathscr{F}\}| \le \mu_i$,*
  (iii) *$[f \ne g \in \mathscr{F} \Rightarrow f \ne_{J_\kappa^{\mathrm{bd}}} g]$;*
(B) *there be a sequence $\langle \lambda_i : i < \kappa \rangle$ such that:*
  (i) *$2^\kappa < \lambda_i = \mathrm{cf}(\lambda_i) \le \mu_i$,*

[1] Or straightening clause (i) see the proof of 1.20

(ii) $\max \mathrm{pcf}\{\lambda_i : i < \kappa\} = \lambda$,

(iii) *for* $j < \kappa, \mu_j \geq \max \mathrm{pcf}\{\lambda_i : i < j\}$*;*

(C) *there is an increasing sequence* $\langle \mathfrak{a}_i : i < \kappa \rangle$ *such that* $\lambda \in \mathrm{pcf}(\bigcup\limits_{i<\kappa} \mathfrak{a}_i), \mathrm{pcf}(\mathfrak{a}_i) \subseteq \mu_i$ *(so* $\mathrm{Min}(\bigcup\limits_{i<\kappa} \mathfrak{a}_i) > |\bigcup\limits_{i<\kappa} \mathfrak{a}_i|$*).*

Proof. $(B) \Rightarrow (A)$: By [She94b, 3.4].
$(A) \Rightarrow (B)$: If $(\forall \theta)[\theta \geq 2^\kappa \Rightarrow \theta^\kappa \leq \theta^+]$ we can directly prove (B) if for a club of $i < \kappa, \mu_i > \bigcup\limits_{j<i} \mu_j$, and contradict (A) if this fails. Otherwise every normal filter D on κ is nice (see [She94d, §1]). Let $\mathscr{F}$ exemplify (A).

Let $K = \{(D, g) : D$ a normal filter on κ, $g \in {}^\kappa(\lambda+1), \lambda = |\{f \in \mathscr{F} : f <_D g\}|\}$. Clearly K is not empty (let g be constantly λ) so by [She94d] we can find $(D, g) \in K$ such that:

$(*)_1$ if $A \subseteq \kappa, A \neq \emptyset \mod D, g_1 <_{D+A} g$ then $\lambda > |\{f \in \mathscr{F} : f <_{D+A} g_1\}|$.

Let $\mathscr{F}^* = \{f \in \mathscr{F} : f <_D g\}$, so (as in the proof of 1.18) $|\mathscr{F}^*| = \lambda$.

We claim:

$(*)_2$ if $h \in \mathscr{F}^*$ then $\{f \in \mathscr{F}^* : \neg h \leq_D f\}$ has cardinality $< \lambda$.

[Why? Otherwise for some $h \in \mathscr{F}^*, \mathscr{F}' =: \{f \in \mathscr{F}^* : \neg h \leq_D f\}$ has cardinality λ, for $A \subseteq \kappa$ let $\mathscr{F}'_A = \{f \in \mathscr{F}^* : f \restriction A \leq h \restriction A\}$ so $\mathscr{F}' = \bigcup\{\mathscr{F}'_A : A \subseteq \kappa, A \neq \emptyset \mod D\}$, hence (recall that $2^\kappa < \lambda$) for some $A \subseteq \kappa, A \neq \emptyset \mod D$ and $|\mathscr{F}'_A| = \lambda$; now $(D + A, h)$ contradicts $(*)_1$].

By $(*)_2$ we can choose by induction on $\alpha < \lambda$, a function $f_\alpha \in F^*$ such that $\bigwedge\limits_{\beta<\alpha} f_\beta <_D f_\alpha$. By [She94b, 1.2A(3)] $\langle f_\alpha : \alpha < \lambda \rangle$ has an e.u.b. f^*. Let $\lambda_i = \mathrm{cf}(f^*(i))$, clearly $\{i < \kappa : \lambda_i \leq 2^\kappa\} = \emptyset \mod D$, so without loss of generality $\bigwedge\limits_{i<\kappa} \mathrm{cf}(f^*(i)) > 2^\kappa$ so λ_i is regular $\in (2^\kappa, \lambda]$, and $\lambda = \mathrm{tcf}(\prod\limits_{i<\kappa} \lambda_i / D)$. Let $J_i = \{A \subseteq i : \max \mathrm{pcf}\{\lambda_j : j \in A\} \leq \mu_i\}$; so (remembering (ii) of (A)) we can find $h_i \in \prod\limits_{j<i} f^*(i)$ such that:

$(*)_3$ if $\{j : j < i\} \notin J_i$, then for every $f \in \mathscr{F}, f \restriction i <_{J_i} h_i$.

Let $h \in \prod\limits_{i<\kappa} f^*(i)$ be defined by:
$h(i) = \sup\{h_j(i) : j \in (i, \kappa)$ and $\{j : j < i\} \notin J_i\}$. As $\bigwedge\limits_i \mathrm{cf}[f^*(i)] > 2^\kappa$, clearly $h < f^*$ hence by the choice of f^* for some $\alpha(*) < \lambda$ we have: $h <_D f_{\alpha(*)}$ and let $A =: \{i < \kappa : h(i) < f_{\alpha(*)}(i)\}$, so $A \in D$. Define λ'_i as follows: λ'_i is λ_i if $i \in A$, and is $(2^\kappa)^+$ if $i \in \kappa \backslash A$. Now $\langle \lambda'_i : i < \kappa \rangle$ is as required in (B).
$(B) \Rightarrow (C)$: Straightforward.
$(C) \Rightarrow (B)$: By [She94a, §1]. $\square_{1.20}$

Claim 1.21. *If* $\mathscr{F} \subseteq {}^\kappa\mathrm{Ord}, 2^\kappa < \theta = \mathrm{cf}(\theta) \leq |\mathscr{F}|$ *<u>then</u> we can find* $g^* \in {}^\kappa\mathrm{Ord}$ *and a proper ideal* I *on* κ *and* $A \subseteq \kappa, A \in I$ *such that:*

(a) $\prod\limits_{i<\kappa} g^*(i)/I$ *has true cofinality* θ*, and for each* $i \in \kappa \setminus A$ *we have* $\mathrm{cf}[g^*(i)] > 2^\kappa$*,*

(b) *for every $g \in {}^{\kappa}\mathrm{Ord}$ satisfying $g \restriction A = g^* \restriction A$, $g \restriction (\kappa \backslash A) < g^* \restriction (\kappa \backslash A)$ we can find $f \in \mathscr{F}$ such that: $f \restriction A = g^* \restriction A, g \restriction (\kappa \backslash A) < f \restriction (\kappa \backslash A) < g^* \restriction (\kappa \backslash A)$.*

Proof. As in [She93b, 3.7], proof of $(A) \Rightarrow (B)$. (In short let $f_\alpha \in \mathscr{F}$ for $\alpha < \theta$ be distinct, χ large enough, $\langle N_i : i < (2^\kappa)^+ \rangle$ as there, $\delta_i =: \sup(\theta \cap N_i), g_i \in {}^{\kappa}\mathrm{Ord}, g_i(\zeta) =: \mathrm{Min}[N \cap \mathrm{Ord} \backslash f_{\delta_i}(\zeta)], A \subseteq \kappa$ and $S \subseteq \{i < (2^\kappa)^+ : \mathrm{cf}(i) = \kappa^+\}$ stationary, $[i \in S \Rightarrow g_i = g^*]$, $[\zeta < \alpha and i \in S \Rightarrow [f_{\delta_i}(\zeta) = g^*(\zeta) \equiv \zeta \in A]$ and for some $i(*) < (2^\kappa)^+$, $g^* \in N_{i(*)}$, so $[\zeta \in \kappa \setminus A \Rightarrow \mathrm{cf}(g^*(\zeta)) > 2^\kappa]$. $\square_{1.21}$

Claim 1.22. *Suppose D is a σ-complete filter on $\theta = \mathrm{cf}(\theta), \kappa$ an infinite cardinal, $\theta > |\alpha|^\kappa$ for $\alpha < \sigma$, and for each $\alpha < \theta$, $\bar{\beta} = \langle \beta^\alpha_\epsilon : \epsilon < \kappa \rangle$ is a sequence of ordinals. Then for every $X \subseteq \theta, X \neq \emptyset \mod D$ there is $\langle \beta^*_\epsilon : \epsilon < \kappa \rangle$ (a sequence of ordinals) and $w \subseteq \kappa$ such that:*

(a) $\epsilon \in \kappa \backslash w \Rightarrow \sigma \leq \mathrm{cf}(\beta^*_\epsilon) \leq \theta$,
(b) *if $\beta'_\epsilon \leq \beta^*_\epsilon$ and $[\epsilon \in w \equiv \beta'_\epsilon = \beta^*_\epsilon]$, then $\{\alpha \in X$: for every $\epsilon < \kappa$ we have $\beta'_\epsilon \leq \beta^\alpha_\epsilon \leq \beta^*_\epsilon$ and $[\epsilon \in w \equiv \beta^\alpha_\epsilon = \beta^*_\epsilon]\} \neq \emptyset \mod D$.*

Proof. Essentially by the same proof as 1.21 (replacing δ_i by $\mathrm{Min}\{\alpha \in X$: for every $Y \in N_i \cap D$ we have $\alpha \in Y\}$). See more [She02, §6]. (See [She99, §7]). $\square_{1.22}$

Remark 1.23. We can rephrase the conclusion as:

(a) $B =: \{\alpha \in X$: if $\epsilon \in w$ then $\beta^\alpha_\epsilon = \beta^*_\epsilon$, and: if $\epsilon \in \kappa \setminus w$ then β^α_ϵ is $< \beta^*_\epsilon$ but $> \sup\{\beta^*_\zeta : \zeta < \epsilon, \beta^\alpha_\zeta < \beta^*_\epsilon\}\}$ is $\neq \emptyset \mod D$
(b) If $\beta'_\epsilon < \beta^*_\epsilon$ for $\epsilon \in \kappa \setminus w$ then $\{\alpha \in B$: if $\epsilon \in \kappa \setminus w$ then $\beta^\alpha_\epsilon > \beta'_\epsilon\} \neq \emptyset \mod D$
(c) $\epsilon \in \kappa \setminus w \Rightarrow \mathrm{cf}(\beta'_\epsilon)$ is $\leq \theta$ but $\geq \sigma$.

Remark 1.24. If $|\mathfrak{a}| < \min(\mathfrak{a}), \mathscr{F} \subseteq \Pi\mathfrak{a}$, $|\mathscr{F}| = \theta = \mathrm{cf}(\theta) \notin \mathrm{pcf}(\mathfrak{a})$ and even $\theta > \sigma = \sup(\theta^+ \cap \mathrm{pcf}(\mathfrak{a}))$ then for some $g \in \Pi\mathfrak{a}$, the set $\{f \in \mathscr{F} : f < g\}$ is unbounded in θ (or use a σ-complete D as in 1.23). (This is as $\Pi\mathfrak{a}/J_{<\theta}[\mathfrak{a}]$ is $\min(\mathrm{pcf}(\mathfrak{a}) \setminus \theta)$-directed as the ideal $J_{<\theta}[\mathfrak{a}]$ is generated by $\leq \sigma$ sets; this is discussed in [She02, §6].)

Remark 1.25. It is useful to note that 1.22 is useful to use [She97a, §4,5.14]: e.g., for if $n < \omega$, $\theta_0 < \theta_1 < \cdots < \theta_n$, satisfying $(*)$ below, for any $\beta'_\epsilon \leq \beta^*_\epsilon$ satisfying $[\epsilon \in w \equiv \beta'_\epsilon < \beta^*_\epsilon]$ we can find $\alpha < \gamma$ in X such that:

$$\epsilon \in w \equiv \beta^\alpha_\epsilon = \beta^*_\epsilon,$$

$$\{\epsilon, \zeta\} \subseteq \kappa \setminus w and \{\mathrm{cf}(\beta^*_\varepsilon), \mathrm{cf}(\beta^*_\zeta)\} \subseteq [\theta_\ell, \theta_{\ell+1})) and \ell \text{ even } \Rightarrow \beta^\alpha_\epsilon < \beta^\gamma_\zeta,$$

$$\{\epsilon, \zeta\} \subseteq \kappa \setminus w and \{\mathrm{cf}(\beta^*_\varepsilon), \mathrm{cf}(\beta^*_\zeta)\} \subseteq [\theta_\ell, \theta_{\ell+1}) and \ell \text{ odd } \Rightarrow \beta^\gamma_\epsilon < \beta^\alpha_\zeta$$

where

$(*)$ (a) $\epsilon \in \kappa \setminus w \Rightarrow \mathrm{cf}(\beta^*_\epsilon) \in [\theta_0, \theta_n)$, and
(b) $\max \mathrm{pcf}[\{\mathrm{cf}(\beta^*_\epsilon) : \epsilon \in \kappa \backslash w\} \cap \theta_\ell] \leq \theta_\ell$ (which holds if $\theta_\ell = \sigma^+_\ell, \sigma^\kappa_\ell = \sigma_\ell$ for $\ell \in \{\ell, \ldots, n\}$).

§ 2. Nice generating sequences

Claim 2.1. *For any $\mathfrak{a}$, $|\mathfrak{a}| < \mathrm{Min}(\mathfrak{a})$, we can find $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{a}\rangle$ such that:*

(α) *$\bar{\mathfrak{b}}$ is a generating sequence, i.e.*

$$\lambda \in \mathfrak{a} \Rightarrow J_{\leq\lambda}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda,$$

(β) *$\bar{\mathfrak{b}}$ is smooth, i.e., for $\theta < \lambda$ in $\mathfrak{a}$,*

$$\theta \in \mathfrak{b}_\lambda \Rightarrow \mathfrak{b}_\theta \subseteq \mathfrak{b}_\lambda,$$

(γ) *$\bar{\mathfrak{b}}$ is closed, i.e., for $\lambda \in \mathfrak{a}$ we have $\mathfrak{b}_\lambda = \mathfrak{a} \cap \mathrm{pcf}(\mathfrak{b}_\lambda)$.*

Definition 2.2. 1) For a set a and set $\mathfrak{a}$ of regular cardinals let $\mathrm{Ch}^{\mathfrak{a}}_a$ be the function with domain $a \cap \mathfrak{a}$ defined by $\mathrm{Ch}^{\mathfrak{a}}_a(\theta) = \sup(a \cap \theta)$.
2) We may write N instead of $|N|$, where N is a model (usually an elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi)$ for some reasonable χ.

Observation 2.3. *If $\mathfrak{a} \subseteq a$ and $|a| < \mathrm{Min}(\mathfrak{a})$ then $\mathrm{ch}^{\mathfrak{a}}_a \in \Pi\mathfrak{a}$.*

Proof. Let $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ be as in [She94a, 2.6] or Definition [She97b, 2.12]. For $\lambda \in \mathfrak{a}$, let $\bar{f}^{\mathfrak{a},\lambda} = \langle f^{\mathfrak{a},\lambda}_\alpha : \alpha < \lambda\rangle$ be a $<_{J_{<\lambda}[\mathfrak{a}]}$-increasing cofinal sequence of members of $\prod \mathfrak{a}$, satisfying:

$(*)_1$ if $\delta < \lambda, |\mathfrak{a}| < \mathrm{cf}(\delta) < \mathrm{Min}(\mathfrak{a})$ and $\theta \in \mathfrak{a}$ then:

$$f^{\mathfrak{a},\lambda}_\delta(\theta) = \mathrm{Min}\{\bigcup_{\alpha \in C} f^{\mathfrak{a},\lambda}_\alpha(\theta) : C \text{ a club of } \delta\}$$

[exists by [She94c, Def.3.3,$(2)^b$ + Fact 3.4(1)]].

Let $\chi = \beth_\omega(\sup(\mathfrak{a}))^+$ and κ satisfies $|\mathfrak{a}| < \kappa = \mathrm{cf}(\kappa) < \mathrm{Min}(\mathfrak{a})$ (without loss of generality there is such κ) and let $\bar{N} = \langle N_i : i < \kappa\rangle$ be an increasing continuous sequence of elementary submodels of $(\mathscr{H}(\chi), \in, <^*_\chi)$, $N_i \cap \kappa$ an ordinal, $\bar{N} \restriction (i+1) \in N_{i+1}$, $\|N_i\| < \kappa$, and $\mathfrak{a}, \langle \bar{f}^{\mathfrak{a},\lambda} : \lambda \in \mathfrak{a}\rangle$ and κ belong to N_0. Let $N_\kappa = \bigcup_{i<\kappa} N_i$. Clearly by 2.3

$(*)_2$ $\mathrm{Ch}^{\mathfrak{a}}_{N_i} \in \Pi\mathfrak{a}$ for $i \leq \kappa$.

Now for every $\lambda \in \mathfrak{a}$ the sequence $\langle \mathrm{Ch}^{\mathfrak{a}}_{N_i}(\lambda) : i \leq \kappa\rangle$ is increasing continuous (note that $\lambda \in N_0 \subseteq N_i \subseteq N_{i+1}$ and $N_i, \lambda \in N_{i+1}$ hence $\sup(N_i \cap \lambda) \in N_{i+1} \cap \lambda$ hence $\mathrm{Ch}^{\mathfrak{a}}_{N_i}(\lambda)$ is $< \sup(N_{i+1} \cap \lambda)$). Hence $\{\mathrm{Ch}^{\mathfrak{a}}_{N_i}(\lambda) : i < \kappa\}$ is a club of $\mathrm{Ch}^{\mathfrak{a}}_{N_\kappa}(\lambda)$; moreover, for every club E of κ the set $\{\mathrm{Ch}^{\mathfrak{a}}_{N_i}(\lambda) : i \in E\}$ is a club of $\mathrm{Ch}^{\mathfrak{a}}_{N_\kappa}(\lambda)$. Hence by $(*)_1$, for every $\lambda \in \mathfrak{a}$, for some club E_λ of κ,

$(*)_3$ (α) if $\theta \in \mathfrak{a}$ and $E \subseteq E_\lambda$ is a club of κ then $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa \cap \lambda)}(\theta) = \bigcup_{\alpha \in E} f^{\mathfrak{a},\lambda}_{\sup(N_\alpha \cap \lambda)}(\theta)$

(β) $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa \cap \lambda)}(\theta) \in c\ell(\theta \cap N_\kappa)$, (i.e., the closure as a set of ordinals).

Let $E = \bigcap_{\lambda \in \mathfrak{a}} E_\lambda$, so E is a club of κ. For any $i < j < \kappa$ let

$$\mathfrak{b}^{i,j}_\lambda = \{\theta \in \mathfrak{a} : \mathrm{Ch}^{\mathfrak{a}}_{N_i}(\theta) < f^{\mathfrak{a},\lambda}_{\sup(N_j \cap \lambda)}(\theta)\}.$$

$(*)_4$ for $i < j < \kappa$ and $\lambda \in \mathfrak{a}$, we have:

(α) $J_{\leq\lambda}[\mathfrak{a}] = J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}^{i,j}_\lambda$ (hence $\mathfrak{b}^{i,j}_\lambda = \mathfrak{b}_\lambda[\bar{\mathfrak{a}}] \mod J_{<\lambda}[\mathfrak{a}]$),

(β) $\mathfrak{b}^{i,j}_\lambda \subseteq \lambda^+ \cap \mathfrak{a}$,

(γ) $\langle \mathfrak{b}^{i,j}_\lambda : \lambda \in \mathfrak{a}\rangle \in N_{j+1}$,

(δ) $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa\cap\lambda)} \leq \mathrm{Ch}^{\mathfrak{a}}_{N_\kappa} = \langle \sup(N_\kappa \cap \theta) : \theta \in \mathfrak{a}\rangle$.

[Why?

Clause (α): First as $\mathrm{Ch}^{\mathfrak{a}}_{N_i} \in \Pi\mathfrak{a}$ (by 2.3) there is $\gamma < \lambda$ such that $\mathrm{Ch}^{\mathfrak{a}}_{N_i} <_{J_{=\lambda}[\mathfrak{a}]} f^{\mathfrak{a},\lambda}_\gamma$ and as $\mathfrak{a} \cup \{\mathfrak{a}, N_i\} \subseteq \mathrm{Ch}^{\mathfrak{a}}_{N_{i+1}}$ clearly $\mathrm{Ch}^{\mathfrak{a}}_{N_i} \in N_{i+1}$ hence without loss of generality $\gamma \in \lambda \cap N_{i+1}$ but $i+1 \leq j$ hence $N_{i+1} \subseteq N_j$ hence $\gamma \in N_j$ hence $\gamma < \sup(N_j \cap \lambda)$ hence $f^{\mathfrak{a},\lambda}_\gamma <_{J_{=\lambda}[\mathfrak{a}]} f^{\mathfrak{a},\lambda}_{\sup(N_j\cap\lambda)}$. Together $\mathrm{Ch}^{\mathfrak{a}}_{N_i} <_{J_{=\lambda}[\mathfrak{a}]} f^{\mathfrak{a},\lambda}_{\sup(N_j\cap\lambda)}$ hence by the definition of $\mathfrak{b}^{i,j}_\lambda$ we have $\mathfrak{a}\backslash\mathfrak{b}^{i,j}_\lambda \in J_{=\lambda}[\mathfrak{a}]$ hence $\lambda \notin \mathrm{pcf}(\mathfrak{a}\backslash\mathfrak{b}^{i,j}_\lambda)$ so $J_{\leq\lambda}[\mathfrak{a}] \subseteq J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}^{i,j}_\lambda$.

Second, $(\Pi\mathfrak{a}, <_{J_{\leq\lambda}[\mathfrak{a}]})$ is λ^+-directed hence there is $g \in \Pi\mathfrak{a}$ such that $\alpha < \lambda \Rightarrow f^{\mathfrak{a},\lambda}_\alpha <_{J_{\leq\lambda}[\mathfrak{a}]} g$. As $\bar{\bar{f}}^{\mathfrak{a},\lambda} \in N_0$ without loss of generality $g \in N_0$ hence $g \in N_i$ so $g < \mathrm{Ch}^{\mathfrak{a}}_{N_i}$. By the choice of g, $f^{\mathfrak{a},\lambda}_{\sup(N_j\cap\lambda)} <_{J_{\leq\lambda}[\mathfrak{a}]} g$ so together $f^{\mathfrak{a},\lambda}_{\sup(N_j\cap\lambda)} <_{J_{\leq\lambda}[\mathfrak{a}]} \mathrm{Ch}^{\mathfrak{a}}_{N_i}$ hence $\mathfrak{b}^{i,j}_\lambda \in J_{\leq\lambda}[\mathfrak{a}]$. As $J_{<\lambda}[\mathfrak{a}] \subseteq J_{\leq\lambda}[\mathfrak{a}]$ clearly $J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}^{i,j}_\lambda \subseteq J_{\leq\lambda}[\mathfrak{a}]$. Together we are done.

Clause (β): Because $\Pi(\mathfrak{a}\backslash\lambda^+)$ is λ^+-directed we have $\theta \in \mathfrak{a}\backslash\lambda^+ \Rightarrow \{\theta\} \notin J_{\leq\lambda}[\mathfrak{a}]$.

Clause (γ): As $\mathrm{Ch}^{\mathfrak{a}}_{N_i}, f^{\lambda,\mathfrak{a}}_{\sup(N_j\cap\lambda)}, \bar{f}$ belongs to N_{j+1}.

Clause (δ): For $\theta \in \mathfrak{a}(\subseteq N_0)$ we have $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa\cap\lambda)}(\theta) = \cup\{f^{\mathfrak{a},\lambda}_{\sup(N_\varepsilon\cap\lambda)}(\theta) : \varepsilon \in E_\lambda\} \leq \sup(N_\kappa \cap \theta)$.

So we have proved $(*)_4$.]

$(*)_5$ $\varepsilon(*) < \kappa$ when $\varepsilon(*) = \cup\{\varepsilon_{\lambda,\theta} : \theta < \lambda$ are from $\mathfrak{a}\}$ where $\varepsilon_{\lambda,\theta} = \mathrm{Min}\{\varepsilon < \kappa$: if $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa\cap\lambda)}(\theta) < \sup(N_\kappa \cap \theta)$ then $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa\cap\lambda)}(\theta) < \sup(N_\varepsilon \cap \theta)\}$.

[Why? Obvious.]

$(*)_6$ $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa\cap\lambda)} \restriction \mathfrak{b}^{i,j}_\lambda = \mathrm{Ch}^{\mathfrak{a}}_{N_\kappa} \restriction \mathfrak{b}^{i,j}_\lambda$ when $i < j$ are from $E\backslash\varepsilon(*)$.

[Why? Let $\theta \in \mathfrak{b}^{i,j}_\lambda$, so by $(*)_3(\beta)$ we know that $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa\cap\lambda)}(\theta) \leq \mathrm{Ch}^{\mathfrak{a}}_{N_\kappa}(\theta)$. If the inequality is strict then there is $\beta \in N_\kappa \cap \theta$ such that $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa\cap\lambda)}(\theta) \leq \beta < \mathrm{Ch}^{\mathfrak{a}}_{N_\kappa}(\theta)$ hence for some $\varepsilon < \kappa, \beta \in N_\varepsilon$ hence $\zeta \in (\varepsilon, \kappa) \Rightarrow f^{\mathfrak{a},\lambda}_{\sup(N_\kappa\cap\lambda)}(\theta) < \mathrm{Ch}^{\mathfrak{a}}_{N_\zeta}(\theta)$ hence (as "$i \geq \varepsilon_{\lambda,\theta}$" holds) we have $f^{\mathfrak{a},\lambda}_{\sup(N_\kappa\cap\lambda)}(\theta) < \mathrm{Ch}^{\mathfrak{a}}_{N_i}(\theta)$ so $f^{\mathfrak{a},\lambda}_{\sup(N_j\cap\lambda)}(\theta) \leq f^{\mathfrak{a},\lambda}_{\sup(N_\kappa\cap\lambda)}(\theta) < \mathrm{Ch}^{\mathfrak{a}}_{N_i(\theta)}$, (the first inequality holds as $j \in E_\lambda$). But by the definition of $\mathfrak{b}^{i,j}_\lambda$ this contradicts $\theta \in \mathfrak{b}^{i,j}_\lambda$.]

We now define by induction on $\epsilon < |\mathfrak{a}|^+$, for $\lambda \in \mathfrak{a}$ (and $i < j < \kappa$), the set $\mathfrak{b}^{i,j,\epsilon}_\lambda$:

$(*)_7$ (α) $\mathfrak{b}^{i,j,0}_\lambda = \mathfrak{b}^{i,j}_\lambda$

(β) $\mathfrak{b}^{i,j,\epsilon+1}_j = \mathfrak{b}^{i,j,\epsilon}_\lambda \cup \bigcup\{\mathfrak{b}^{i,j,\epsilon}_\theta : \theta \in \mathfrak{b}^{i,j,\epsilon}_\lambda\} \cup \{\theta \in \mathfrak{a} : \theta \in \mathrm{pcf}(\mathfrak{b}^{i,j,\epsilon}_\lambda)\}$,

(γ) $\mathfrak{b}^{i,j,\epsilon}_\lambda = \bigcup\limits_{\zeta<\epsilon} \mathfrak{b}^{i,j,\zeta}_\lambda$ for $\epsilon < |\mathfrak{a}|^+$ limit.

Clearly for $\lambda \in \mathfrak{a}$, $\langle \mathfrak{b}_\lambda^{i,j,\epsilon} : \epsilon < |\mathfrak{a}|^+\rangle$ belongs to N_{j+1} and is a non-decreasing sequence of subsets of $\mathfrak{a}$, hence for some $\epsilon(i,j,\lambda) < |\mathfrak{a}|^+$, we have

$$[\epsilon \in (\epsilon(i,j,\lambda), |\mathfrak{a}|^+) \Rightarrow \mathfrak{b}_\lambda^{i,j,\epsilon} = \mathfrak{b}_\lambda^{i,j,\epsilon(i,j,\lambda)}].$$

So letting $\epsilon(i,j) = \sup_{\lambda\in\mathfrak{a}} \epsilon(i,j,\lambda) < |\mathfrak{a}|^+$ we have:

$(*)_8$ $\epsilon(i,j) \le \epsilon < |\mathfrak{a}|^+ \Rightarrow \bigwedge\limits_{\lambda\in\mathfrak{a}} \mathfrak{b}_\lambda^{i,j,\epsilon(i,j)} = \mathfrak{b}_\lambda^{i,j,\epsilon}$.

We restrict ourselves to the case $i < j$ are from $E \backslash \varepsilon(*)$. Which of the properties required from $\langle \mathfrak{b}_\lambda : \lambda \in \mathfrak{a}\rangle$ are satisfied by $\langle \mathfrak{b}_\lambda^{i,j,\epsilon(i,j)} : \lambda \in \mathfrak{a}\rangle$? In the conclusion of 2.1 properties (β), (γ) hold by the inductive definition of $\mathfrak{b}_\lambda^{i,j,\epsilon}$ (and the choice of $\epsilon(i,j)$). As for property (α), one half, $J_{\le\lambda}[\mathfrak{a}] \subseteq J_{<\lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda^{i,j,\epsilon(i,j)}$ hold by $(*)_4(\alpha)$ (and $\mathfrak{b}_\lambda^{i,j} = \mathfrak{b}_\lambda^{i,j,0} \subseteq \mathfrak{b}_\lambda^{i,j,\epsilon(i,j)}$), so it is enough to prove (for $\lambda \in \mathfrak{a}$):

$(*)_9$ $\mathfrak{b}_\lambda^{i,j,\epsilon(i,j)} \in J_{\le\lambda}[\mathfrak{a}]$.

For this end we define by induction on $\epsilon < |\mathfrak{a}|^+$ functions $f_\alpha^{\mathfrak{a},\lambda,\epsilon}$ with domain $\mathfrak{b}_\lambda^{i,j,\epsilon}$ for every pair (α,λ) satisfying $\alpha < \lambda \in \mathfrak{a}$, such that $\zeta < \epsilon \Rightarrow f_\alpha^{\mathfrak{a},\lambda,\zeta} \subseteq f_\alpha^{\mathfrak{a},\lambda,\epsilon}$, so the domain increases with ϵ.

We let $f_\alpha^{\mathfrak{a},\lambda,0} = f_\alpha^{\mathfrak{a},\lambda} \restriction \mathfrak{b}_\lambda^{i,j}$, $f_\alpha^{\mathfrak{a},\lambda,\varepsilon} = \bigcup\limits_{\zeta<\epsilon} f_\alpha^{\mathfrak{a},\lambda,\zeta}$ for limit $\epsilon < |\mathfrak{a}|^+$ and $f_\alpha^{\mathfrak{a},\lambda,\epsilon+1}$ is defined by defining each $f_\alpha^{\mathfrak{a},\lambda,\epsilon+1}(\theta)$ as follows:

<u>Case 1</u>: If $\theta \in \mathfrak{b}_\lambda^{i,j,\epsilon}$ then $f_\alpha^{\mathfrak{a},\lambda,\varepsilon+1}(\theta) = f_\alpha^{\mathfrak{a},\lambda,\epsilon}(\theta)$.

<u>Case 2</u>: If $\mu \in \mathfrak{b}_\lambda^{i,j,\epsilon}, \theta \in \mathfrak{b}_\mu^{i,j,\epsilon}$ and not Case 1 and μ minimal under those conditions, then $f_\alpha^{a,\lambda,\varepsilon+1}(\theta) = f_\beta^{\mathfrak{a},\mu,\epsilon}(\theta$ where we choose $\beta = f_\alpha^{\mathfrak{a},\lambda,\epsilon}(\mu)$.

<u>Case 3</u>: If $\theta \in \mathfrak{a} \cap \mathrm{pcf}(\mathfrak{b}_\lambda^{i,j,\epsilon})$ and neither Case 1 nor Case 2, then

$$f_\alpha^{\mathfrak{a},\lambda,\epsilon+1}(\theta) = \mathrm{Min}\{\gamma < \theta : f_\alpha^{\mathfrak{a},\lambda,\epsilon} \restriction \mathfrak{b}_\theta[\mathfrak{a}] \le_{J_{<\theta}[\mathfrak{a}]} f_\gamma^{\mathfrak{a},\theta,\epsilon}\}.$$

Now $\langle\langle \mathfrak{b}_\lambda^{i,j,\epsilon} : \lambda \in \mathfrak{a}\rangle : \epsilon < |\mathfrak{a}|^+\rangle$ can be computed from $\mathfrak{a}$ and $\langle \mathfrak{b}_\lambda^{i,j} : \lambda \in \mathfrak{a}\rangle$. But the latter belongs to N_{j+1} by $(*)_4(\gamma)$, so the former belongs to N_{j+1} and as $\langle\langle \mathfrak{b}_\lambda^{i,j,\epsilon} : \lambda \in \mathfrak{a}\rangle : \epsilon < |\mathfrak{a}|^+\rangle$ is eventually constant, also each member of the sequence belongs to N_{j+1}. As also $\langle\langle f_\alpha^{\mathfrak{a},\lambda} : \alpha < \lambda\rangle : \lambda \in \mathrm{pcf}(\mathfrak{a})\rangle$ belongs to N_{j+1} we clearly get that

$$\big\langle\langle\langle f_\alpha^{\mathfrak{a},\lambda,\epsilon} : \epsilon < |\mathfrak{a}|^+\rangle : \alpha < \lambda\rangle : \lambda \in \mathfrak{a}\big\rangle$$

belongs to N_{j+1}. Next we prove by induction on ϵ that, for $\lambda \in \mathfrak{a}$, we have:

$\otimes_1$ $\theta \in \mathfrak{b}_\lambda^{i,j,\epsilon} and \lambda \in \mathfrak{a} \Rightarrow f_{\sup(N_\kappa\cap\lambda)}^{\mathfrak{a},\lambda,\epsilon}(\theta) = \sup(N_\kappa \cap \theta)$.

For $\epsilon = 0$ this holds by $(*)_6$. For ϵ limit this holds by the induction hypothesis and the definition of $f_\alpha^{\mathfrak{a},\lambda,\epsilon}$ (as union of earlier ones). For $\epsilon + 1$, we check $f_{\sup(N_\kappa\cap\lambda)}^{\mathfrak{a},\lambda,\epsilon+1}(\theta)$ according to the case in its definition; for Case 1 use the induction hypothesis applied to $f_{\sup(N_\kappa\cap\lambda)}^{\mathfrak{a},\lambda,\epsilon}$. For Case 2 (with μ), by the induction hypothesis applied to $f_{\sup(N_\kappa\cap\mu)}^{\mathfrak{a},\mu,\epsilon}$.

Lastly, for Case 3 (with θ) we should note:

(i) $\mathfrak{b}^{i,j,\epsilon}_\lambda \cap \mathfrak{b}_\theta[\mathfrak{a}] \notin J_{<\theta}[\mathfrak{a}]$.

[Why? By the case's assumption $\mathfrak{b}^{i,j,\varepsilon}_\lambda \in (J_\theta[\mathfrak{a}])^+$ and $(*)_4(\alpha)$ above.]

(ii) $f^{\mathfrak{a},\lambda,\epsilon}_{\sup(N_\kappa\cap\lambda)} \restriction (\mathfrak{b}^{i,j,\epsilon}_\lambda \cap \mathfrak{b}^{i,j,\epsilon}_\theta) \subseteq f^{\mathfrak{a},\theta,\epsilon}_{\sup(N_\kappa\cap\theta)}$.

[Why? By the induction hypothesis for ϵ, used concerning λ and θ.]
Hence (by the definition in case 3 and (i) + (ii)),

(iii) $f^{\mathfrak{a},\lambda,\epsilon+1}_{\sup(N_\kappa\cap\lambda)}(\theta) \le \sup(N_\kappa \cap \theta)$.

Now if $\gamma < \sup(N_\kappa \cap \theta)$ then for some $\gamma(1)$ we have $\gamma < \gamma(1) \in N_\kappa \cap \theta$, so letting $\mathfrak{b} =: \mathfrak{b}^{i,j,\epsilon}_\lambda \cap \mathfrak{b}_\theta[\mathfrak{a}] \cap \mathfrak{b}^{i,j,\epsilon}_\theta$, it belongs to $J_{\le\theta}[\mathfrak{a}] \setminus J_{<\theta}[\mathfrak{a}]$ and we have

$$f^{\mathfrak{a},\theta}_\gamma \restriction \mathfrak{b} <_{J_{<\theta}[\mathfrak{a}]} f^{\mathfrak{a},\theta}_{\gamma(1)} \restriction \mathfrak{b} \le f^{\mathfrak{a},\theta,\epsilon}_{\sup(N_\kappa\cap\theta)}$$

hence $f^{\mathfrak{a},\lambda,\epsilon+1}_{\sup(N_\kappa\cap\lambda)}(\theta) > \gamma$; as this holds for every $\gamma < \sup(N_\kappa \cap \theta)$ we have obtained

(iv) $f^{\mathfrak{a},\lambda,\epsilon+1}_{\sup(N_\kappa\cap\lambda)}(\theta) \ge \sup(N_\kappa \cap \theta)$;

together we have finished proving the inductive step for $\epsilon+1$, hence we have proved $\otimes_1$.

This is enough for proving $\mathfrak{b}^{i,j,\epsilon}_\lambda \in J_{\le\lambda}[\mathfrak{a}]$.

Why? If it fails, as $\mathfrak{b}^{i,j,\epsilon}_\lambda \in N_{j+1}$ and $\langle f^{\mathfrak{a},\lambda,\epsilon}_\alpha : \alpha < \lambda\rangle$ belongs to N_{j+1}, there is $g \in \prod \mathfrak{b}^{i,j,\epsilon}_\lambda$ such that

$(*)$ $\alpha < \lambda \Rightarrow f^{\mathfrak{a},\lambda,\epsilon}_\alpha \restriction \mathfrak{b}^{i,j,\epsilon} < g \mod J_{\le\lambda}[\mathfrak{a}]$.

Without loss of generality $g \in N_{j+1}$; by $(*)$, $f^{\mathfrak{a},\lambda,\epsilon}_{\sup(N_\kappa\cap\lambda)} < g \mod J_{\le\lambda}[\mathfrak{a}]$. But $g < \langle \sup(N_\kappa \cap \theta) : \theta \in \mathfrak{b}^{i,j,\epsilon}_\lambda\rangle$. Together this contradicts $\otimes_1$!

This ends the proof of 2.1. $\square_{2.1}$

If $|\mathrm{pcf}(\mathfrak{a})| < \mathrm{Min}(\mathfrak{a})$ then 2.1 is fine and helpful. But as we do not know this, we shall use the following substitute.

Claim 2.4. *Assume $|\mathfrak{a}| < \kappa = \mathrm{cf}(\kappa) < \mathrm{Min}(\mathfrak{a})$ and σ is an infinite ordinal satisfying $|\sigma|^+ < \kappa$. Let $\bar{f}$, $\bar{N} = \langle N_i : i < \kappa\rangle$, N_κ be as in the proof of 2.1. Then we can find $\bar{i} = \langle i_\alpha : \alpha \le \sigma\rangle$, $\bar{\mathfrak{a}} = \langle \mathfrak{a}_\alpha : \alpha < \sigma\rangle$ and $\langle\langle \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_\beta\rangle : \beta < \sigma\rangle$ such that:*

(a) *$\bar{i}$ is a strictly increasing continuous sequence of ordinals $< \kappa$,*
(b) *for $\beta < \sigma$ we have $\langle i_\alpha : \alpha \le \beta\rangle \in N_{i_{\beta+1}}$ hence $\langle N_{i_\alpha} : \alpha \le \beta\rangle \in N_{i_{\beta+1}}$ and $\langle \mathfrak{b}^\gamma_\lambda[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_\gamma$ and $\gamma \le \beta\rangle \in N_{i_{\beta+1}}$, we can get $\bar{i} \restriction (\beta+1) \in N_{i_\beta+1}$ if κ succesor of regular (we just need a suitable partial square)*
(c) *$\mathfrak{a}_\beta = N_{i_\beta} \cap \mathrm{pcf}(\mathfrak{a})$, so $\mathfrak{a}_\beta$ is increasing continuous with $\beta, \mathfrak{a} \subseteq \mathfrak{a}_\beta \subseteq \mathrm{pcf}(\mathfrak{a})$ and $|\mathfrak{a}_\beta| < \kappa$,*
(d) *$\mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}] \subseteq \mathfrak{a}_\beta$ (for $\lambda \in \mathfrak{a}_\beta$),*
(e) *$J_{\le\lambda}[\mathfrak{a}_\beta] = J_{<\lambda}[\mathfrak{a}_\beta] + \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$ (so $\lambda \in \mathfrak{b}^\beta_\lambda[\mathfrak{a}]$ and $\mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}] \subseteq \lambda^+$),*
(f) *if $\mu < \lambda$ are from $\mathfrak{a}_\beta$ and $\mu \in \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$ then $\mathfrak{b}^\beta_\mu[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$ (i.e., smoothness),*
(g) *$\mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}] = \mathfrak{a}_\beta \cap \mathrm{pcf}(\mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}])$ (i.e., closedness),*

(h) *if* $\mathfrak{c} \subseteq \mathfrak{a}_\beta, \beta < \sigma$ *and* $\mathfrak{c} \in N_{i_{\beta+1}}$ *then for some finite* $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+1} \cap \mathrm{pcf}(\mathfrak{c})$*, we have* $\mathfrak{c} \subseteq \bigcup_{\mu \in \mathfrak{d}} \mathfrak{b}^{\beta+1}_\mu[\bar{\mathfrak{a}}]$*;*

more generally (note that in $(h)^+$ *if* $\theta = \aleph_0$ *then we get (h)).*

$(h)^+$ *if* $\mathfrak{c} \subseteq \mathfrak{a}_\beta, \beta < \sigma, \mathfrak{c} \in N_{i_{\beta+1}}, \theta = \mathrm{cf}(\theta) \in N_{i_{\beta+1}}$*, then for some* $\mathfrak{d} \in N_{i_{\beta+1}}, \mathfrak{d} \subseteq \mathfrak{a}_{\beta+1} \cap \mathrm{pcf}_{\theta-complete}(\mathfrak{c})$ *we have* $\mathfrak{c} \subseteq \bigcup_{\mu \in \mathfrak{d}} \mathfrak{b}^{\beta+1}_\mu[\bar{\mathfrak{a}}]$ *and* $|\mathfrak{d}| < \theta$*,*

(i) $\mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$ *increases with* β*.*

This will be proved below.

Claim 2.5. *In 2.4 we can also have:*

1. *if we let* $\mathfrak{b}_\lambda[\bar{\mathfrak{a}}] = \mathfrak{b}^\sigma_\lambda[\mathfrak{a}] = \bigcup_{\beta<\sigma} \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$, $\mathfrak{a}_\sigma = \bigcup_{\beta<\sigma} \mathfrak{a}_\beta$ *then also for* $\beta = \sigma$ *we have* (b) *(use* $N_{i_\beta+1}$*)*, (c), (d), (f), (i)
2. *If* $\sigma = \mathrm{cf}(\sigma) > |\mathfrak{a}|$ *then for* $\beta = \sigma$ *also* (e), (g)
3. *If* $\mathrm{cf}(\sigma) > |\mathfrak{a}|, \mathfrak{c} \in N_{i_\sigma}$, $\mathfrak{c} \subseteq \mathfrak{a}_\sigma$ *(hence* $|\mathfrak{c}| < \mathrm{Min}(\mathfrak{c})$ *and* $\mathfrak{c} \subseteq \mathfrak{a}_\sigma$*), then for some finite* $\mathfrak{d} \subseteq (\mathrm{pcf}(\mathfrak{c})) \cap \mathfrak{a}_\sigma$ *we have* $\mathfrak{c} \subseteq \bigcup_{\mu \in \mathfrak{d}} \mathfrak{b}_\mu[\bar{\mathfrak{a}}]$*. Similarly for* θ*-complete,* $\theta < \mathrm{cf}(\sigma)$ *(i.e., we have clauses* (h), $(\mathrm{h})^+$ *for* $\beta = \sigma$*).*
4. *We can have continuity in* $\delta \leq \sigma$ *when* $\mathrm{cf}(\delta) > |\mathfrak{a}|$*, i.e.,* $\mathfrak{b}^\delta_\lambda[\bar{\mathfrak{a}}] = \bigcup_{\beta<\delta} \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$*.*

We shall prove 2.5 after proving 2.4.

Remark 2.6. 1) If we would like to use length κ, use $\bar{N}$ as produced in [She93a, L2.6] so $\sigma = \kappa$.
2) Concerning 2.5, in 2.6(1) for a club E of $\sigma = \kappa$, we have $\alpha \in E \Rightarrow \mathfrak{b}^\alpha_\lambda[\bar{\mathfrak{a}}] = \mathfrak{b}_\lambda[\bar{\mathfrak{a}}] \cap \mathfrak{a}_\alpha$.
3) We can also use 2.4,2.5 to give an alternative proof of part of the localization theorems similar to the one given in the Spring '89 lectures.

For example:

Claim 2.7. *1) If* $|\mathfrak{a}| < \theta = \mathrm{cf}(\theta) < \mathrm{Min}(\mathfrak{a})$*, for no sequence* $\langle \lambda_i : i < \theta \rangle$ *of members of* $\mathrm{pcf}(\mathfrak{a})$*, do we have* $\bigwedge_{\alpha<\theta} [\lambda_\alpha > \max \mathrm{pcf}\{\lambda_i : i < \alpha\}]$*.*
2) If $|\mathfrak{a}| < \mathrm{Min}(\mathfrak{a}), |\mathfrak{b}| < \mathrm{Min}(\mathfrak{b}), \mathfrak{b} \subseteq \mathrm{pcf}(\mathfrak{a})$ *and* $\lambda \in \mathrm{pcf}(\mathfrak{a})$*, then for some* $\mathfrak{c} \subseteq \mathfrak{b}$ *we have* $|\mathfrak{c}| \leq |\mathfrak{a}|$ *and* $\lambda \in \mathrm{pcf}(\mathfrak{c})$*.*

Proof. Relying on 2.4:
1) Without loss of generality $\mathrm{Min}(\mathfrak{a}) > \theta^{+3}$, let $\kappa = \theta^{+2}$, let $\bar{N}$, N_κ, $\bar{\mathfrak{a}}$, $\mathfrak{b}$ (as a function), $\langle i_\alpha : \alpha \leq \sigma =: |\mathfrak{a}|^+ \rangle$ be as in 2.4 but we in addition assume that $\langle \lambda_i : i < \theta \rangle \in N_0$. So for $j < \theta$, $\mathfrak{c}_j =: \{\lambda_i : i < j\} \in N_0$ (so $\mathfrak{c}_j \subseteq \mathrm{pcf}(\mathfrak{a}) \cap N_0 = \mathfrak{a}_0$) hence (by clause (h) of 2.4), for some finite $\mathfrak{d}_j \subseteq \mathfrak{a}_1 \cap \mathrm{pcf}(\mathfrak{c}_j) = N_{i_1} \cap \mathrm{pcf}(\mathfrak{a}) \cap \mathrm{pcf}(\mathfrak{c}_j)$ we have $\mathfrak{c}_j \subseteq \bigcup_{\lambda \in \mathfrak{d}_j} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]$. Assume $j(1) < j(2) < \theta$. Now if $\mu \in \mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_{j(1)}} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]$ then for some $\mu_0 \in \mathfrak{d}_{j(1)}$ we have $\mu \in \mathfrak{b}^1_{\mu_0}[\bar{\mathfrak{a}}]$; now $\mu_0 \in \mathfrak{d}_{j(1)} \subseteq \mathrm{pcf}(\mathfrak{c}_{j(1)}) \subseteq \mathrm{pcf}(\mathfrak{c}_{j(2)}) \subseteq \mathrm{pcf}(\bigcup_{\lambda \in \mathfrak{d}_{j(2)}} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]) = \bigcup_{\lambda \in \mathfrak{d}_{j(2)}} (\mathrm{pcf}(\mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}])$ hence (by clause (g) of 2.4 as $\mu_0 \in \mathfrak{d}_{j(0)} \subseteq N_1$) for some $\mu_1 \in \mathfrak{d}_{j(2)}, \mu_0 \in \mathfrak{b}^1_{\mu_1}[\bar{\mathfrak{a}}]$. So by clause (f) of 2.4 we have $\mathfrak{b}^1_{\mu_0}[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}^1_{\mu_1}[\bar{\mathfrak{a}}]$ hence remembering $\mu \in \mathfrak{b}^1_{\mu_0}[\bar{\mathfrak{a}}]$, we have $\mu \in \mathfrak{b}^1_{\mu_1}[\bar{\mathfrak{a}}]$. Remembering μ was any member of

$\mathfrak{a}\cap \bigcup\limits_{\lambda\in\mathfrak{d}_{j(1)}} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]$, we have $\mathfrak{a}\cap \bigcup\limits_{\lambda\in\mathfrak{d}j(1)} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}] \subseteq \mathfrak{a}\cap \bigcup\limits_{\lambda\in\mathfrak{d}j(2)} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]$ (holds also without "$\mathfrak{a}\cap$" but not used). So $\langle \mathfrak{a}\cap \bigcup\limits_{\lambda\in\mathfrak{d}_j} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}] : j<\theta\rangle$ is a $\subseteq$-increasing sequence of subsets of $\mathfrak{a}$, but $\mathrm{cf}(\theta) > |\mathfrak{a}|$, so the sequence is eventually constant, say for $j\geq j(*)$. But

$$\begin{aligned}\max\mathrm{pcf}(\mathfrak{a}\cap \bigcup_{\lambda\in\mathfrak{d}_j} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]) &\leq \max\mathrm{pcf}(\bigcup_{\lambda\in\mathfrak{d}_j} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}])\\ &= \max_{\lambda\in\mathfrak{d}_j}(\max\mathrm{pcf}(\mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}]))\\ &= \max_{\lambda\in\mathfrak{d}_j}\lambda \leq \max\mathrm{pcf}\{\lambda_i : i<j\} < \lambda_j\\ &= \max\mathrm{pcf}(\mathfrak{a}\cap \bigcup_{\lambda\in\mathfrak{d}_{j+1}} \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}])\end{aligned}$$

(last equality as $\mathfrak{b}_{\lambda_j}[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}^1_\lambda[\bar{\mathfrak{a}}] \mod J_{<\lambda}[\mathfrak{a}_1]$). Contradiction.
2) (Like [She94a, §3]): If this fails choose a counterexample $\mathfrak{b}$ with $|\mathfrak{b}|$ minimal, and among those with $\max\mathrm{pcf}(\mathfrak{b})$ minimal and among those with $\bigcup\{\mu^+ : \mu\in\lambda\cap\mathrm{pcf}(\mathfrak{b})\}$ minimal. So by the pcf theorem

- $(*)_1$ $\mathrm{pcf}(\mathfrak{b})\cap\lambda$ has no last member
- $(*)_2$ $\mu=\sup[\lambda\cap\mathrm{pcf}(\mathfrak{b})]$ is not in $\mathrm{pcf}(\mathfrak{b})$ or $\mu=\lambda$.
- $(*)_3$ $\max\mathrm{pcf}(\mathfrak{b})=\lambda$.

Try to choose by induction on $i<|\mathfrak{a}|^+$, $\lambda_i\in\lambda\cap\mathrm{pcf}(\mathfrak{b})$, $\lambda_i>\max\mathrm{pcf}\{\lambda_j : j<i\}$. Clearly by part (1), we will be stuck at some i. Now $\mathrm{pcf}\{\lambda_j : j<i\}$ has a last member and is included in $\mathrm{pcf}(\mathfrak{b})$, hence by $(*)_3$ and being stuck at necessarily $\mathrm{pcf}(\{\lambda_j : j<i\}) \nsubseteq \lambda$ but it is $\subseteq \mathrm{pcf}(\mathfrak{b}) \subseteq \lambda^+$, so $\lambda=\max\mathrm{pcf}\{\lambda_j : j<i\}$. For each j, by the choice of "minimal counterexample" for some $\mathfrak{b}_j\subseteq\mathfrak{b}$, we have $|\mathfrak{b}_j|\leq|\mathfrak{a}|$, $\lambda_j\in\mathrm{pcf}(\mathfrak{b}_j)$. So $\lambda\in\mathrm{pcf}\{\lambda_j : j<i\}\subseteq\mathrm{pcf}(\bigcup\limits_{j<i}\mathfrak{b}_j)$ but $\bigcup\limits_{j<i}\mathrm{frb}_j$ is a subset of $\mathfrak{b}$ of cardinality $\leq |i|\times|\mathfrak{a}|=|\mathfrak{a}|$, so we are done. $\square_{2.7}$

Proof. Without loss of generality $\sigma=\omega\sigma$ (as we can use $\omega^\omega\sigma$ so $|\omega^\omega\sigma|=|\sigma|$). Let $\bar{f}^{\mathfrak{a}}=\langle \bar{f}^{\mathfrak{a},\lambda}=\langle\langle f^{\mathfrak{a},\lambda}_\alpha : \alpha<\lambda\rangle : \lambda\in\mathrm{pcf}(\mathfrak{a})\rangle$ and $\langle N_i : i\leq\kappa\rangle$ be chosen as in the proof of 2.1 and without loss of generality $\bar{f}^{\mathfrak{a}}$ belongs to N_0. For $\zeta<\kappa$ we define $\mathfrak{a}^\zeta =: N_\zeta\cap\mathrm{pcf}(\mathfrak{a})$; we also define ${}^\zeta\bar{f}$ as $\langle\langle f^{\mathfrak{a}^\zeta,\lambda}_\alpha : \alpha<\lambda\rangle : \lambda\in\mathrm{pcf}(\mathfrak{a})\rangle$ where $f^{\mathfrak{a}^\zeta,\lambda}_\alpha \in \prod\mathfrak{a}^\zeta$ is defined as follows:

(a) if $\theta\in\mathfrak{a}$, $f^{\mathfrak{a}^\zeta,\lambda}_\alpha(\theta)=f^{\mathfrak{a},\lambda}_\alpha(\theta)$,
(b) if $\theta\in\mathfrak{a}^\zeta\backslash\mathfrak{a}$ and $\mathrm{cf}(\alpha)\notin(|\mathfrak{a}^\zeta|,\mathrm{Min}(\mathfrak{a}))$, then

$$f^{\mathfrak{a}^\zeta,\lambda}_\alpha(\theta)=\mathrm{Min}\{\gamma<\theta : f^{\mathfrak{a},\lambda}_\alpha \restriction \mathfrak{b}_\theta[\mathfrak{a}] \leq_{J_{<\theta}[\mathfrak{b}_\theta[\mathfrak{a}]]} f^{\mathfrak{a},\theta}_\gamma \restriction \mathfrak{b}_\theta[\mathfrak{a}]\},$$

(c) if $\theta\in\mathfrak{a}^\zeta\backslash\mathfrak{a}$ and $\mathrm{cf}(\alpha)\in(|\mathfrak{a}^\zeta|,\mathrm{Min}(\mathfrak{a}))$, define $f^{\mathfrak{a}^\zeta,\lambda}_\alpha(\theta)$ so as to satisfy $(*)_1$ in the proof of 2.1.

Now ${}^\zeta\bar{f}$ is legitimate except that we have only

$$\beta<\gamma<\lambda\in\mathrm{pcf}(\mathfrak{a}) \Rightarrow f^{\mathfrak{a}^\zeta,\lambda}_\beta \leq f^{\mathfrak{a}^\zeta,\lambda}_\gamma \mod J_{<\lambda}[\mathfrak{a}^\zeta]$$

(instead of strict inequality) however we still have $\bigwedge_{\beta<\lambda}\bigvee_{\gamma<\lambda}[f^{\mathfrak{a}^\zeta,\lambda}_\beta < f^{\mathfrak{a}^\zeta,\lambda}_\gamma \mod J_{<\lambda}[\mathfrak{a}^\zeta]]$, but this suffices. (The first statement is actually proved in [She94a, 3.2A], the second in [She94a, 3.2B]; by it also ${}^\zeta\bar{f}$ is cofinal in the required sense.)

For every $\zeta<\kappa$ we can apply the proof of 2.1 with $(N_\zeta\cap\mathrm{pcf}(\mathfrak{a}))$, ${}^\zeta\bar{f}$ and $\langle N_{\zeta+1+i}: i<\kappa\rangle$ here standing for $\mathfrak{a}$, $\bar{f}$, $\bar{N}$ there. In the proof of 2.1 get a club E^ζ of κ (corresponding to E there and without loss of generality $\zeta+\mathrm{Min}(E^\zeta)=\mathrm{Min}(E^\zeta)$ so any $i<j$ from E^ζ are O.K.). Now we can define for $\zeta<\kappa$ and $i<j$ from E^ζ, ${}^\zeta\mathfrak{b}^{i,j}_\lambda$ and $\langle {}^\zeta\mathfrak{b}^{i,j,\epsilon}_\lambda:\epsilon<|\mathfrak{a}^\zeta|^+\rangle$, $\langle\epsilon^\zeta(i,j,\lambda):\lambda\in\mathfrak{a}^\zeta\rangle$,$\epsilon^\zeta(i,j)$, as well as in the proof of 2.1.

Let:

$$E=\{i<\kappa:\ i \text{ is a limit ordinal } (\forall j<i)(j+j<i and j\times j<i) \text{ and } \bigwedge_{j<i} i\in E^j\}.$$

So by [She93a, §1] we can find $\bar{C}=\langle C_\delta:\delta\in S\rangle, S\subseteq\{\delta<\kappa:\mathrm{cf}(\delta)=\mathrm{cf}(\sigma)\}$ stationary, C_δ a club of $\delta,\mathrm{otp}(C_\delta)=\sigma$ such that:

(1) for each $\alpha<\lambda,\{C_\delta\cap\alpha:\alpha\in\mathrm{nacc}(C_\delta)\}$ has cardinality $<\kappa$. If κ is successor of regular, then we can get $[\gamma\in C_\alpha\cap C_\beta\Rightarrow C_\alpha\cap\gamma=C_\beta\cap\gamma]$ and
(2) for every club E' of κ for stationarily many $\delta\in S, C_\delta\subseteq E'$.

Without loss of generality $\bar{C}\in N_0$. For some $\delta^*, C_{\delta^*}\subseteq E$, and let $\{j_\zeta:\zeta\le\omega^2\sigma\}$ enumerate $C_{\delta^*}\cup\{\delta^*\}$. So $\langle j_\zeta:\zeta\le\omega^2\sigma\rangle$ is a strictly increasing continuous sequence of ordinals from $E\subseteq\kappa$ such that $\langle j_\epsilon:\epsilon\le\zeta\rangle\in N_{j_{\zeta+1}}$ and if, e.g., κ is a successor of regulars then $\langle j_\varepsilon:\varepsilon\le\zeta\rangle\in N_{j_\zeta+1}$. Let $j(\zeta)=j_\zeta$ and for $\ell\in\{0,2\}$ let $i_\ell(\zeta)=i^\ell_\zeta=: j_{\omega^\ell(1+\zeta)},\mathfrak{a}_\zeta=N^\ell_{i_\zeta}\cap\mathrm{pcf}(\mathfrak{a})$, and $\bar{\mathfrak{a}}^\ell=:\langle\mathfrak{a}^\ell_\zeta:\zeta<\sigma\rangle, {}^\ell\mathfrak{b}^\zeta_\lambda[\bar{\mathfrak{a}}]=: {}^{i_\ell(\zeta)}\mathfrak{b}_\lambda^{j(\omega^\ell\zeta+1),j(\omega^\ell\zeta+2),\epsilon^\zeta(j(\omega^\ell\zeta+1),j(\omega^\ell\zeta+2))}$. Recall that $\sigma=\omega\sigma$ so $\sigma=\omega^2\sigma$; if the value of ℓ does not matter we omit it. Most of the requirements follow immediately by the proof of 2.1, as

⊛ for each $\zeta<\sigma$, we have $\mathfrak{b}_\zeta$, $\langle\mathfrak{b}^\zeta_\lambda[\bar{\mathfrak{a}}]:\lambda\in\mathfrak{a}_\zeta\rangle$ are as in the proof (hence conclusion of 2.1) and belongs to $N_{i_\beta+3}\subseteq N_{i_{\beta+1}}$.

We are left (for proving 2.4) with proving clauses (h)$^+$ and (i) (remember that (h) is a special case of (h)$^+$ choosing $\theta=\aleph_0$).

For proving clause (i) note that for $\zeta<\xi<\kappa$, $f^{\mathfrak{a}^\zeta,\lambda}_\alpha\subseteq f^{\mathfrak{a}^\xi,\lambda}_\alpha$ hence ${}^\zeta\mathfrak{b}^{i,j}_\lambda\subseteq{}^\xi\mathfrak{b}^{i,j}_\lambda$. Now we can prove by induction on ϵ that ${}^\zeta\mathfrak{b}^{i,j,\epsilon}_\lambda\subseteq{}^\xi\mathfrak{b}^{i,j,\epsilon}_\lambda$ for every $\lambda\in\mathfrak{a}_\zeta$ (check the definition in $(*)_7$ in the proof of 2.1) and the conclusion follows.

Instead of proving (h)$^+$ we prove an apparently weaker version (h)$'$ below, but having $(h)'$ for the case $\ell=0$ gives $(h)^+$ for $\ell=2$ so this is enough [[then note that $\bar{i}'=\langle i_{\omega^2\zeta}:\zeta<\sigma\rangle$, $\bar{\mathfrak{a}}'=\langle\mathfrak{a}_{\omega^2\zeta}:\zeta<\sigma\rangle$, $\langle N_{i(\omega^2\zeta)}:\zeta<\sigma\rangle$, $\langle\mathfrak{b}^{\omega^2\zeta}_\lambda[\bar{\mathfrak{a}}']:\zeta<\sigma,\lambda\in\mathfrak{a}'_\zeta=\mathfrak{a}_{\omega^2\zeta}\rangle$ will exemplify the conclusion]] where:

$(h)'$ if $\mathfrak{c}\subseteq\mathfrak{a}_\beta$, $\beta<\sigma$, $\mathfrak{c}\in N_{i_{\beta+1}},\theta=\mathrm{cf}(\theta)\in N_{i_{\beta+1}}$ <u>then</u> for some frd $\in N_{i_{\beta+\omega+1}+1}$ satisfying $\mathfrak{d}\subseteq\mathfrak{a}_{\beta+\omega}\cap\mathrm{pcf}_{\theta-\mathrm{complete}}(\mathfrak{c})$ we have $\mathfrak{c}\subseteq\bigcup_{\mu\in\mathfrak{d}}\mathfrak{b}^{\beta+\omega}_\mu[\bar{\mathfrak{a}}]$ and $|\mathfrak{d}|<\theta$.

$\square$

Proof. Proof of $(h)'$

So let $\theta, \beta, \mathfrak{c}$ be given; let $\langle \mathfrak{b}_\mu[\bar{\mathfrak{a}}] : \mu \in \mathrm{pcf}(\mathfrak{c})\rangle (\in N_{i_{\beta+1}})$ be a generating sequence. We define by induction on $n < \omega$, A_n, $\langle (\mathfrak{c}_\eta, \lambda_\eta) : \eta \in A_n\rangle$ such that:

(a) $A_0 = \{\langle\rangle\}, \mathfrak{c}_{\langle\rangle} = \mathfrak{c}$, $\lambda_{\langle\rangle} = \max \mathrm{pcf}(\mathfrak{c})$,

(b) $A_n \subseteq {}^n\theta, |A_n| < \theta$,

(c) if $\eta \in A_{n+1}$ then $\eta \restriction n \in A_n$, $\mathfrak{c}_\eta \subseteq \mathfrak{c}_{\eta\restriction n}$, $\lambda_\eta < \lambda_{\eta\restriction n}$ and $\lambda_\eta = \max \mathrm{pcf}(\mathfrak{c}_\eta)$,

(d) $A_n, \langle (\mathfrak{c}_\eta, \lambda_\eta) : \eta \in A_n\rangle$ belongs to $N_{i_{\beta+1+n}}$ hence $\lambda_\eta \in N_{i_{\beta+1+n}}$,

(e) if $\eta \in A_n$ and $\lambda_\eta \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{c}_\eta)$ and $\mathfrak{c}_\eta \not\subseteq \mathfrak{b}^{\beta+1+n}_{\lambda_\eta}[\bar{\mathfrak{a}}]$ then $(\forall \nu)[\nu \in A_{n+1} and \eta \subseteq \nu \Leftrightarrow \nu = \eta\hat{\ }\langle 0\rangle]$ and $\mathfrak{c}_{\eta\hat{\ }\langle 0\rangle} = \mathfrak{c}_\eta \backslash \mathfrak{b}^{\beta+1+n}_{\lambda_\eta}[\bar{\mathfrak{a}}]$ (so $\lambda_{\eta\hat{\ }\langle 0\rangle} = \max \mathrm{pcf}(\mathfrak{c}_{\eta\hat{\ }\langle 0\rangle}) < \lambda_\eta = \max \mathrm{pcf}(\mathfrak{c}_\eta)$,

(f) if $\eta \in A_n$ and $\lambda_\eta \notin \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{c}_\eta)$ then

$$\mathfrak{c}_\eta = \bigcup \{\mathfrak{b}_{\lambda_{\gamma\hat{\ }\langle i\rangle}}[\mathfrak{c}] : i < i_n < \theta, \eta\hat{\ }\langle i\rangle \in A_{n+1}\},$$

and if $\nu = \eta\hat{\ }\langle i\rangle \in A_{n+1}$ then $\mathfrak{c}_\nu = \mathfrak{b}_{\lambda_\nu}[\mathfrak{c}]$,

(g) if $\eta \in A_n$, and $\lambda_\eta \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{c}_\eta)$ but $\mathfrak{c}_\eta \subseteq \mathfrak{b}^{\beta+1-n}_{\lambda_n}[\bar{\mathfrak{a}}]$, then $\neg(\exists \nu)[\eta \triangleleft \nu \in A_{n+1}]$.

There is no problem to carry the definition (we use 2.8(1), the point is that $\mathfrak{c} \in N_{i_{\beta+1+n}}$ implies $\langle \mathfrak{b}_\lambda(\mathfrak{c}) : \lambda \in \mathrm{pcf}_\theta[\mathfrak{c}]\rangle \in N_{i_{\beta+1+n}}$ and as there is $\mathfrak{d}$ as in 2.8(1), there is one in $N_{i_{\beta+1+n+1}}$ so $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+1+n+1})$.

Now let

$$\mathfrak{d}_n =: \{\lambda_\eta : \eta \in A_n \text{ and } \lambda_\eta \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{c}_\eta)\}$$

and $\mathfrak{d} =: \bigcup\limits_{n<\omega} \mathfrak{d}_n$; we shall show that it is as required.

The main point is $\mathfrak{c} \subseteq \bigcup\limits_{\lambda \in \mathfrak{d}} \mathfrak{b}^{\beta+\omega}_\lambda[\bar{\mathfrak{a}}]$; note that

$$[\lambda_\eta \in \mathfrak{d}, \eta \in A_n \Rightarrow \mathfrak{b}^{\beta+1+n}_{\lambda_\eta}[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}^{\beta+\omega}_{\lambda_\eta}[\bar{\mathfrak{a}}]]$$

hence it suffices to show $\mathfrak{c} \subseteq \bigcup\limits_{n<\omega} \bigcup\limits_{\lambda \in \mathfrak{d}_n} \mathfrak{b}^{\beta+1+n}_\lambda[\bar{\mathfrak{a}}]$, so assume $\theta \in \mathfrak{c} \backslash \bigcup\limits_{n<\omega} \bigcup\limits_{\lambda \in \mathfrak{d}_n} \mathfrak{b}^{\beta+1+n}_\lambda[\bar{\mathfrak{a}}]$, and we choose by induction on n, $\eta_n \in A_n$ such that $\eta_0 =<>$, $\eta_{n+1} \restriction n = \eta_n$ and $\theta \in \mathfrak{c}_\eta$; by clauses (e) + (f) above this is possible and $\langle \max \mathrm{pcf}(\mathfrak{c}_{\eta_n}) : n < \omega\rangle$ is (strictly) decreasing, contradiction.

The minor point is $|\mathfrak{d}| < \theta$; if $\theta > \aleph_0$ note that $\bigwedge\limits_n |A_n| < \theta$ and $\theta = \mathrm{cf}(\theta)$ clearly $|\mathfrak{d}| \le |\bigcup_n A_n| < \theta + \aleph_1 = \theta$.

If $\theta = \aleph_0$ (i.e. clause (h)) we should show that $\bigcup\limits_n A_n$ finite; the proof is as above noting that the clause (f) is vacuous now. So $n < \omega \Rightarrow |A_n| = 1$ and for some $n \bigvee\limits_n A_n = \emptyset$, so $\bigcup\limits_n A_n$ is finite. Another minor point is $\mathfrak{d} \in N_{i_{\beta+\omega+1}}$; this holds as the construction is unique from $\mathfrak{c}, \langle \mathfrak{b}_\mu[\mathfrak{c}] : \mu \in \mathrm{pcf}(\mathfrak{c})\rangle, \langle N_j : j < i_{\beta+\omega}\rangle, \langle i_j : j \le \beta + \omega\rangle$, $\langle (\mathfrak{a}_{i(\zeta)}, \langle \mathfrak{b}^\zeta_\lambda[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_{i(\zeta)}\rangle) : \zeta \le \beta + \omega\rangle$; no "outside" information is used so $\langle (A_n, \langle (c_\eta, \lambda_\eta) : \eta \in A_n\rangle) : n < \omega\rangle \in N_{i_{\beta+\omega+1}}$, so (using a choice function) really $\mathfrak{d} \in N_{i_{\beta+\omega+1}}$. $\square_{2.4}$

Proof. Let $\mathfrak{b}_\lambda[\bar{\mathfrak{a}}] = \mathfrak{b}^\sigma_\lambda = \bigcup\limits_{\beta<\sigma} \mathfrak{b}^\beta_\lambda[\mathfrak{a}_\beta]$ and $\mathfrak{a}_\sigma = \bigcup\limits_{\zeta<\sigma} \mathfrak{a}_\zeta$. Part (1) is straightforward. For part (2), for clause (g), for $\beta = \sigma$, the inclusion "$\subseteq$" is straightforward; so assume $\mu \in \mathfrak{a}_\beta \cap \mathrm{pcf}(\mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}])$. Then by 2.4(c) for some $\beta_0 < \beta$, we have $\mu \in \mathfrak{a}_{\beta_0}$, and by 2.7 (which depends on 2.4 only) for some $\beta_1 < \beta$, $\mu \in \mathrm{pcf}(\mathfrak{b}^{\beta_1}_\lambda[\bar{\mathfrak{a}}])$; by monotonicity without loss of generality $\beta_0 = \beta_1$, by clause (g) of 2.4 applied to β_0, $\mu \in \mathfrak{b}^{\beta_0}_\lambda[\bar{\mathfrak{a}}]$. Hence by clause (i) of 2.4, $\mu \in \mathfrak{b}^\beta_\lambda[\bar{\mathfrak{a}}]$, thus proving the other inclusion.

The proof of clause (e) (for 2.5(2)) is similar, and also 2.5(3). For **??**(B)(4) for $\delta < \sigma, \mathrm{cf}(\delta) > |\mathfrak{a}|$ redefine $\mathfrak{b}^\delta_\lambda[\bar{\mathfrak{a}}]$ as $\bigcup\limits_{\beta<\delta} \mathfrak{b}^{\beta+1}_\lambda[\bar{\mathfrak{a}}]$. $\square_{2.5}$

Claim 2.8. *Let θ be regular.*
0) If $\alpha < \theta, \mathrm{pcf}_{\theta\text{-complete}}(\bigcup\limits_{i<\alpha} \mathfrak{a}_i) = \bigcup\limits_{i<\alpha} \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{a}_i)$.
1) If $\langle \mathfrak{b}_\partial[\mathfrak{a}] : \partial \in \mathrm{pcf}(\mathfrak{a})\rangle$ is a generating sequence for $\mathfrak{a}$, $\mathfrak{c} \subseteq \mathfrak{a}$, then for some $\mathfrak{d} \subseteq \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ we have: $|\mathfrak{d}| < \theta$ and $\mathfrak{c} \subseteq \bigcup\limits_{\theta\in\mathfrak{a}} \mathfrak{b}_\theta[\mathfrak{a}]$.
2) If $|\mathfrak{a}\cup\mathfrak{c}| < \mathrm{Min}(\mathfrak{a}), \mathfrak{c} \subseteq \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{a}), \lambda \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ then $\lambda \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{a})$.
3) In (2) we can weaken $|\mathfrak{a} \cup \mathfrak{c}| < \mathrm{Min}(\mathfrak{a})$ to $|\mathfrak{a}| < \mathrm{Min}(\mathfrak{a}), |\mathfrak{c}| < \mathrm{Min}(\mathfrak{c})$.

Proof. (0) and (1): Left to the reader.
2) See [She94f, 1.10–1.12].
3) Similarly. $\square_{2.8}$

Claim 2.9. *1) Let θ be regular $\le |\mathfrak{a}|$. We cannot find $\lambda_\alpha \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{a})$ for $\alpha < |\mathfrak{a}|^+$ such that $\lambda_i > \sup \mathrm{pcf}_{\theta\text{-complete}}(\{\lambda_j : j < i\})$.*
2) Assume $\theta \le |\mathfrak{a}|, \mathfrak{c} \subseteq \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{a})$ (and $|\mathfrak{c}| < \mathrm{Min}(\mathfrak{c})$; of course $|\mathfrak{a}| < \mathrm{Min}(\mathfrak{a})$). If $\lambda \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ then for some $\mathfrak{d} \subseteq \mathfrak{c}$ we have $|\mathfrak{d}| \le |\mathfrak{a}|$ and $\lambda \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{d})$.

Proof. 1) If $\theta = \aleph_0$ we already know it (see 2.7), so assume $\theta > \aleph_0$. We use 2.4 with $\{\theta, \langle \lambda_i : i < |\mathfrak{a}|^+\rangle\} \in N_0$, $\sigma = |\mathfrak{a}|^+$, $\kappa = |\mathfrak{a}|^{+3}$ where, without loss of generality, $\kappa < \mathrm{Min}(\mathfrak{a})$. For each $\alpha < |\mathfrak{a}|^+$ by (h)$^+$ of 2.4 there is $\mathfrak{a}_\alpha \in N_{i_1}, \mathfrak{d}_\alpha \subseteq \mathrm{pcf}_{\theta\text{-complete}}(\{\lambda_i : i < \alpha\})$, $|\mathfrak{d}_\alpha| < \theta$ such that $\{\lambda_i : i < \alpha\} \subseteq \bigcup\limits_{\theta\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\theta[\bar{\mathfrak{a}}]$; hence by clause (g) of 2.4 and part (0) Claim 2.8 we have $\mathfrak{a}_1 \cap \mathrm{pcf}_{\theta\text{-complete}}(\{\lambda_i : i < \alpha\}) \subseteq \bigcup\limits_{\theta\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\theta[\bar{\mathfrak{a}}]$. So for $\alpha < \beta < |\mathfrak{a}|^+$, $\mathfrak{d}_\alpha \subseteq \mathfrak{a}_1 \cap \mathrm{pcf}_{\theta\text{-complete}}\{\lambda_i : i < \alpha\} \subseteq \mathfrak{a}_1 \cap \mathrm{pcf}_{\theta\text{-complete}}\{\lambda_i : i < \beta\} \subseteq \bigcup\limits_{\theta\in\mathfrak{d}_\beta} \mathfrak{b}^1_\theta[\bar{\mathfrak{a}}]$. As the sequence is smooth (i.e., clause (f) of 2.4) clearly $\alpha < \beta \Rightarrow \bigcup\limits_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \subseteq \bigcup\limits_{\mu\in\mathfrak{d}_\beta} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}]$.

So $\langle \bigcup\limits_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a} : \alpha < |\mathfrak{a}|^+\rangle$ is a non-decreasing sequence of subsets of $\mathfrak{a}$ of length $|\mathfrak{a}|^+$, hence for some $\alpha(*) < |\mathfrak{a}|^+$ we have:

$(*)_1$ $\alpha(*) \le \alpha < |\mathfrak{a}|^+ \Rightarrow \bigcup\limits_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a} = \bigcup\limits_{\mu\in\mathfrak{d}_{\alpha(*)}} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}$.

If $\tau \in \mathfrak{a}_1 \cap \mathrm{pcf}_{\theta\text{-complete}}(\{\lambda_i : i < \alpha\})$ then $\tau \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{a})$ (by parts (2),(3) of Claim 2.8), and $\tau \in \mathfrak{b}^1_{\mu_\tau}[\bar{\mathfrak{a}}]$ for some $\mu_\tau \in \mathfrak{d}_\alpha$ so $\mathfrak{b}^1_\tau[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}^1_{\mu_\tau}[\bar{\mathfrak{a}}]$, also $\tau \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{b}^1_\tau[\bar{\mathfrak{a}}] \cap \mathfrak{a})$ (by clause (e) of 2.4), hence

$$\begin{aligned}\tau \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{b}^1_\tau[\bar{\mathfrak{a}}] \cap \mathfrak{a}) &\subseteq \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{b}^1_{\mu_\tau}[\bar{\mathfrak{a}}] \cap \mathfrak{a}) \\ &\subseteq \mathrm{pcf}_{\theta\text{-complete}}(\bigcup_{\mu\in\mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a}).\end{aligned}$$

So $\mathfrak{a}_1 \cap \mathrm{pcf}_{\theta\text{-complete}}(\{\lambda_i : i < \alpha\}) \subseteq \mathrm{pcf}_{\theta\text{-complete}}(\bigcup\limits_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a})$. But for each $\alpha < |\mathfrak{a}|^+$ we have $\lambda_\alpha > \sup \mathrm{pcf}_{\theta\text{-complete}}(\{\lambda_i : i < \alpha\})$, whereas $\mathfrak{d}_\alpha \subseteq \mathrm{pcf}_{\sigma\text{-complete}}\{\lambda_i : i < \alpha\}$, hence $\lambda_\alpha > \sup \mathfrak{d}_\alpha$ hence

$(*)_2$ $\lambda_\alpha > \sup_{\mu \in \mathfrak{d}_\alpha} \max \mathrm{pcf}(\mathfrak{b}^1_\mu[\bar{\mathfrak{a}}]) \geq \sup \mathrm{pcf}_{\theta\text{-complete}}(\bigcup\limits_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a})$.

On the other hand,

$(*)_3$ $\lambda_\alpha \in \mathrm{pcf}_{\theta\text{-complete}}\{\lambda_i : i < \alpha + 1\} \subseteq \mathrm{pcf}_{\theta\text{-complete}}(\bigcup\limits_{\mu \in \mathfrak{d}_{\alpha+1}} \mathfrak{b}^1_\mu[\bar{\mathfrak{a}}] \cap \mathfrak{a})$.

For $\alpha = \alpha(*)$ we get contradiction by $(*)_1 + (*)_2 + (*)_3$.
2) Assume $\mathfrak{a}, \mathfrak{c}, \lambda$ form a counterexample with λ minimal. Without loss of generality $|\mathfrak{a}|^{+3} < \mathrm{Min}(\mathfrak{a})$ and $\lambda = \max \mathrm{pcf}(\mathfrak{a})$ and $\lambda = \max \mathrm{pcf}(\mathfrak{c})$ (just let $\mathfrak{a}' =: \mathfrak{b}_\lambda[\mathfrak{a}], \mathfrak{c}' =: \mathfrak{c} \cap \mathrm{pcf}_\theta[\mathfrak{a}']$; if $\lambda \notin \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{c}')$ then necessarily $\lambda \in \mathrm{pcf}(\mathfrak{c} \backslash \mathfrak{c}')$ (by 2.8(0)) and similarly $\mathfrak{c} \backslash \mathfrak{c}' \subseteq \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{a} \backslash \mathfrak{a}')$ hence by parts (2),(3) of Claim 2.8 we have $\lambda \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{a} \backslash \mathfrak{a}')$, contradiction).

Also without loss of generality $\lambda \notin \mathfrak{c}$. Let $\kappa, \sigma, \bar{N}, \langle i_\alpha = i(\alpha) : \alpha \leq \sigma \rangle, \bar{\mathfrak{a}} = \langle \mathfrak{a}_i : i \leq \sigma \rangle$ be as in 2.4 with $\mathfrak{a} \in N_0, \mathfrak{c} \in N_0, \lambda \in N_0, \sigma = |\mathfrak{a}|^+, \kappa = |\mathfrak{a}|^{+3} < \mathrm{Min}(\mathfrak{a})$. We choose by induction on $\epsilon < |\mathfrak{a}|^+, \lambda_\epsilon, \mathfrak{d}_\epsilon$ such that:

(a) " $\lambda_\epsilon \in \mathfrak{a}_{\omega^2\epsilon+\omega+1}, \mathfrak{d}_\epsilon \in N_{i(\omega^2\epsilon+\omega+1)}$,
(b) $\lambda_\epsilon \in \mathfrak{c}$,
(c) $\mathfrak{d}_\epsilon \subseteq \mathfrak{a}_{\omega^2\epsilon+\omega+1} \cap \mathrm{pcf}_{\theta\text{-complete}}(\{\lambda_\zeta : \zeta < \epsilon\})$,
(d) $|\mathfrak{d}_\epsilon| < \theta$,
(e) $\{\lambda_\zeta : \zeta < \epsilon\} \subseteq \bigcup\limits_{\theta \in \mathfrak{d}_\epsilon} \mathfrak{b}^{\omega^2\epsilon+\omega+1}_\theta[\bar{\mathfrak{a}}]$,
(f) $\lambda_\epsilon \notin \mathrm{pcf}_{\theta\text{-complete}}(\bigcup\limits_{\theta \in \mathfrak{d}_\epsilon} \mathfrak{b}^{\omega^2\epsilon+\omega+1}_\theta[\bar{\mathfrak{a}}])$.

For every $\epsilon < |\mathfrak{a}|^+$ we first choose $\mathfrak{d}_\epsilon$ as the $<^*_\chi$-first element satisfying (c) + (d) + (e) and then if possible λ_ϵ as the $<^*_\chi$-first element satisfying (b) + (f). It is easy to check the requirements and in fact $\langle \lambda_\zeta : \zeta < \epsilon \rangle \in N_{\omega^2\epsilon+1}, \langle \mathfrak{d}_\zeta : \zeta < \epsilon \rangle \in N_{\omega^2\epsilon+1}$ (so clause (a) will hold). But why can we choose at all? Now $\lambda \notin \mathrm{pcf}_{\theta\text{-complete}}\{\lambda_\zeta : \zeta < \epsilon\}$ as $\mathfrak{a}, \mathfrak{c}, \lambda$ form a counterexample with λ minimal and $\epsilon < |\mathfrak{a}|^+$ (by 2.8(3)). As $\lambda = \max \mathrm{pcf}(\mathfrak{a})$ necessarily $\mathrm{pcf}_{\theta\text{-complete}}(\{\lambda_\zeta : \zeta < \epsilon\}) \subseteq \lambda$ hence $\mathfrak{d}_\epsilon \subseteq \lambda$ (by clause (c)). By part (0) of Claim 2.8 (and clause (a)) we know:

$$\begin{aligned} \mathrm{pcf}_{\theta\text{-complete}}[\bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}^{\omega^2\epsilon+\omega+1}_\mu[\bar{\mathfrak{a}}]] &= \bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathrm{pcf}_{\theta\text{-complete}}[\mathfrak{b}^{\omega^2+\omega+1}_\mu[\bar{\mathfrak{a}}]] \\ &\subseteq \bigcup_{\mu \in \mathfrak{d}_\epsilon} (\mu + 1) \subseteq \lambda \end{aligned}$$

(note $\mu = \max \mathrm{pcf}(\mathfrak{b}^\beta_\mu[\bar{\mathfrak{a}}])$). So $\lambda \notin \mathrm{pcf}_{\theta\text{-complete}}(\bigcup\limits_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}^{\omega^2\epsilon+\omega+1}_\mu[\bar{\mathfrak{a}}])$ hence by part (0) of Claim 2.8 $\mathfrak{c} \nsubseteq \bigcup\limits_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}^{\omega^2\epsilon+\omega+1}_\mu[\bar{\mathfrak{a}}]$ so λ_ϵ exists. Now $\mathfrak{d}_\epsilon$ exists by 2.4 clause $(\mathrm{h})^+$.

Now clearly $\left\langle \mathfrak{a} \cap \bigcup\limits_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}^{\omega^2\epsilon+\omega+1}_\mu[\bar{\mathfrak{a}}] : \epsilon < |\mathfrak{a}|^+ \right\rangle$ is non-decreasing (as in the earlier proof) hence eventually constant, say for $\epsilon \geq \epsilon(*)$ (where $\epsilon(*) < |\mathfrak{a}|^+$).

But

(α) $\lambda_\epsilon \in \bigcup\limits_{\mu \in \mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ [clause (e) in the choice of $\lambda_\epsilon, \mathfrak{d}_\epsilon$],

(β) $\mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}] \subseteq \bigcup_{\mu \in \mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ [by clause (f) of 2.4 and (α) alone],

(γ) $\lambda_\epsilon \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{a})$ [as $\lambda_\epsilon \in \mathfrak{c}$ and a hypothesis],

(δ) $\lambda_\epsilon \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}])$ [by (γ) above and clause (e) of 2.4],

(ϵ) $\lambda_\epsilon \notin \mathrm{pcf}(\mathfrak{a} \setminus \mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1})$,

(ζ) $\lambda_\epsilon \in \mathrm{pcf}_{\theta\text{-complete}}(\mathfrak{a} \cap \bigcup\limits_{\mu \in \mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}])$ [by (δ) + (ϵ) + (β)].

But for $\epsilon = \epsilon(*)$, the statement (ζ) contradicts the choice of $\epsilon(*)$ and clause (f) above. $\square_{2.9}$

§ 3.

Definition 3.1. 1) For J an ideal on κ (or any set, $\mathrm{Dom}(J)$-does not matter) and singular μ (usually $\mathrm{cf}(\mu) \leq \kappa$, otherwise the result is 0)

(a) we define $\mathrm{pp}_J(\mu)$ as

$$\sup\{\mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_J) : \begin{array}{l} \lambda_i \in \mathrm{Reg} \cap \mu \backslash \kappa^+ \text{ for } i < \kappa \\ \text{and } \mu = \lim_J \langle \lambda_i : i < \kappa \rangle, \text{ see 3.2(1) and} \\ (\prod_{i<\kappa} \lambda_i, <_J) \text{ has true cofinality}\} \end{array}$$

(b) we define $\mathrm{pp}^+_J(\mu)$ as

$$\sup\{(\mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_J))^+ : \begin{array}{l} \lambda_i \in \mathrm{Reg} \cap \mu \backslash \kappa^+ \text{ for } i < \kappa \\ \text{and } \mu = \lim_J (\langle \lambda_i : i < \kappa \rangle), \text{ see 3.2(1) below and} \\ (\prod_{i<\kappa} \lambda_i, <_J) \text{ has true cofinality}\}. \end{array}$$

2) For $\mathbf{J}$ a family of ideals on (usually but not necessarily on the same set) and singular μ let $\mathrm{pp}_{\mathbf{J}}(\mu) = \sup\{\mathrm{pp}_J(\mu) : J \in \mathbf{J}\}$ and $\mathrm{pp}^+_J(\mu) = \sup\{\mathrm{pp}^+_J(\mu) : J \in \mathbf{J}\}$.
3) For a set $\mathfrak{a}$ of regular cardinals let $\mathrm{pcf}_J(\mathfrak{a}) = \{\mathrm{tcf}(\prod_{t \in \mathrm{Dom}(J)} \lambda_t, <_J) : \lambda_t \in \mathfrak{a}$ for $t \in \mathrm{Dom}(J)\}$; similarly $\mathrm{pcf}_{\mathbf{J}}(\mathfrak{a})$.

Remark 3.2. 1) Recall that $\mu = \lim_J \langle \lambda_t : t \in \mathrm{Dom}(J) \rangle$, where J is an ideal on $\mathrm{Dom}(J)$ mean that for every $\mu_1 < \mu$ the set $\{t \in \mathrm{Dom}(J) : \lambda_t \notin (\mu_1, \mu]\}$ belongs to J.
2) On $\mathrm{pcf}_J(\mathfrak{a})$: check consistency of notation by [She94e].

Observation 3.3. *1) For μ, J as in clause (a) 3.1, the following are equivalent*

(a) *$pp_J(\mu) > 0$*
(b) *the sup is on a non-empty set*
(c) *there is an increasing sequence of length* $\mathrm{cf}(\mu)$ *of member of J whose union is κ*
(d) $\mathrm{pp}_J(\mu) > \mu$
(e) *every cardinal appearing in the* sup *is regular $> \mu$ and the set of those appearing is* $\mathrm{Reg} \cap [\mu^+, \mathrm{pp}^+_J(\mu))$ *and is non-empty.*

Definition 3.4. 1) Assume J is an ideal on $\kappa, \sigma = \mathrm{cf}(\sigma) \leq \kappa, f \in {}^\kappa\mathrm{Ord}$ <u>then</u> we let

$$\mathbf{W}_{J,\sigma}(f^*, < \mu) = \mathrm{Min}\{|\mathscr{P}| : \begin{array}{l} \mathscr{P} \text{ is a family of subsets of } \sup \mathrm{Rang}(f^*) + 1 \\ \text{each of cardinality } < \mu \text{ and for every } f \leq f^*, \\ \mathrm{Rang}(f) \text{ is the union of } < \sigma \\ \text{sets of the form} \\ \{i < \kappa : f(i) \in A\}, A \in \mathscr{P}\}. \end{array}$$

2) If f^* is constantly λ we write λ if $\mu = \lambda$ we can omit $< \mu$.

Remark 3.5. 1) See $\mathrm{cov}(\lambda,\mu,\theta,\sigma) = \mathbf{W}_{[\theta]^{<\sigma},\sigma}(\langle \lambda : i < \theta\rangle,\mu)$.
2) On the case of normal ideals, i.e. prc see [She93b, §1] and more generally prd see [She93b].

We may use several families of ideals.

Definition 3.6. Let

(a) $\mathrm{com}_{\theta,\sigma} = \{J : J \text{ is a } \sigma\text{-complete ideal on } \theta\}$
(b) $\mathrm{nor}_\kappa = \{J : J \text{ a normal ideal on } \kappa\}$
(c) $\mathrm{com}_{I,\sigma} = \{J : J \text{ is a } \sigma\text{-complete ideal on } \mathrm{Dom}(I) \text{ extending the ideal } I\}$
(d) $\mathrm{nor}_I = \{J : J \text{ is a normal ideal on } \mathrm{Dom}(I) \text{ extending the ideal } I\}$.

Claim 3.7. *The $(< \aleph_1)$-covering lemma.*
Assume $\aleph_1 \le \sigma \le \mathrm{cf}(\mu) \le \kappa < \mu$ and I is a σ-complete ideal on κ.
Then

(a) $\mathbf{W}_{I,\sigma}(\mu) = \mathrm{pp}_{\mathrm{com}_\sigma(I)}(\mu)$
(b) *except when $\circledast_{\mu,I,\sigma}$ below holds, we can strengthen the equality in clause (a) to: i.e., if $\mathrm{pp}_{\mathrm{com}_\sigma(I)}$ is a regular cardinal (so $> \mu$) then the* sup *in 3.1(1) is obtained*

$\circledast_{\mu,I,\sigma}$ (a) $\lambda =: \mathrm{pp}_{\mathrm{com}_\sigma(I)}(\mu)$ *is (weakly) inaccessible, the* sup *is not obtained and for some set $\mathfrak{a} \subseteq \mathrm{Reg} \cap \mu$, $|\mathfrak{a}| + \kappa < \mathrm{Min}(\mathfrak{a})$ and $\lambda = \sup(\mathrm{pcf}_{I,\sigma}(\mathfrak{a}))$; recalling $\mathrm{pcf}_{\mathrm{com}_\sigma(I)}(\mathfrak{a}) = \{\prod\limits_{i<\kappa} \lambda_i, <_J : J \in \mathrm{com}_\sigma(I), \lambda_i \in \mathfrak{a} \text{ for } i < \kappa\}$.*

Remark 3.8. 1) This is [She02, 6.13].
In a reasonable case the result $\mathrm{cov}(|\mathfrak{a}|, \kappa^+, \kappa^+, \sigma)$.

Conclusion 3.9. *In 3.7 if $\kappa < \mu_* \le \mu$ then*

(a) $\mathbf{W}_{I,\sigma}(\mu, < \mu_*) = \sup\{\mathrm{pp}_{\mathrm{com}_\sigma(I)}(\mu)' : \mu_* \le \mu' \le \mu, \mathrm{cf}(\mu') \le \kappa\}$
(b) *if in (a) the left side is a regular cardinal then the* sup *is obtained for some sequence $\langle \lambda_i : i < \kappa\rangle$ of regular cardinality and $J \in \mathrm{com}_\sigma(I)$ such that $\lim_J \langle \lambda_i : i < \kappa\rangle$ is well defined and $\in [\mu_*, \mu]$ except possibly when*

$\circledast_{\mu,I,\sigma,\mu_*}$ *as in $\circledast_{\mu,I,\sigma}$ above but $|\mathfrak{a}| < \mu_*$.*

Proof. The inequality $\ge$:
So assume J is a σ-complete ideal on κ extending $I, \lambda_i \in \mathrm{Reg} \cap \mu \backslash \kappa^+$ and $\mu = \lim_J(\langle \lambda_i : i < \kappa\rangle)$ and $\lambda = \mathrm{tcf}(\prod\limits_{i<\kappa} \lambda_i, <_J)$ is well defined and we shall note that $\mathbf{W}_{I,\sigma}(\mu) \ge \lambda$, this clearly suffices, and let $\langle f_\alpha : \alpha < \lambda\rangle$ be $<_J$-increasing cofinal in $(\prod\limits_{i<\kappa} \lambda_i, <_J)$. Now let $|\mathscr{P}| < \lambda$, $\mathscr{P}$ be a family of sets of ordinals each of cardinality $< \mu$. For each $u \in \mathscr{P}$ let $g_u \in \prod\limits_{i<\kappa} \lambda_i$ be defined by $g_u(i) = \sup(u \cap \lambda_i)$ if $|u| < \lambda_i$ and $g_u(i) = 0$ otherwise.

Hence for some $\alpha(u) < \lambda, g_u <_J f_{\alpha(u)}$ and so $\alpha(*) = \cup\{\alpha(u) + 1 : u \in \mathscr{P}\} < \lambda$ and $f_{\alpha(*)}$ exemplifies the failure of $\mathscr{P}$ to exemplify $\lambda > W_{I,\sigma}(\mu)$.

The inequality $\le$:
Assume that λ is regular $\ge \mathrm{pp}^+_{I,\sigma}(\mu)$ and we shall prove that $\mathbf{W}_{I,\sigma}(\mu) < \lambda$, this clearly suffices. Let χ be large enough, and $\mathfrak{B}$ be an elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi)$ of cardinality $< \lambda$ such that $\{I, \sigma, \mu, \lambda\} \subseteq \mathfrak{B}$ and $\lambda \cap \mathfrak{B}$ is an ordinal

which we shall call $\delta_{\mathfrak{B}}$. Let $\mathscr{P} =: [\mu]^{<\mu} \cap \mathfrak{B}$ so $|\mathscr{P}| < \lambda$. Hence it is enough to prove that $\mathbf{W}_{I,\sigma}(\mu) \leq |\mathscr{P}|$ and for this it is enough to praove that $\mathscr{P}$ is as required in Definition 3.3(1). Let $\bar{e} = \langle e_\alpha : \alpha < \mu \rangle \in \mathfrak{B}$ be such that e_α is a club of α of order type $\mathrm{cf}(\alpha)$ so $e_{\alpha+1} = \{\alpha\}, e_0 = \emptyset$.

So let $f_* \in {}^\kappa\mu$ and let $\langle \mu_\varepsilon : \varepsilon < \mathrm{cf}(\mu) \rangle \in \mathfrak{B}$ be an increasing continuous sequence of cardinals from (κ, μ) with limit μ. Now by induction on $n < \omega$ we choose $\varepsilon_n, A_n, g_n, \mathscr{T}_n, \bar{S}_n, \bar{B}_n$ such that

$\circledast_n$ $(A)(a)$ $A_n \in [\mu]^{\leq\kappa}, A_0 = \{\mu_\varepsilon : \varepsilon < \mathrm{cf}(\mu)\}$

(b) g_n is a function from κ to A_n

(c) $f_* \leq g_n$

(d) if $n = m+1$ and $i < \kappa$ then $g_m(i) > f_*(i) \Rightarrow g_n(i) > g_m(i)$

(e) $\mathscr{T}_n \subseteq {}^n\sigma$ has cardinality $< \sigma$

(f) $\mathscr{T}_0 = \{<>\}$

(g) if $n = m+1$ and $\eta \in \mathscr{T}_n$ then $\eta \restriction m \in \mathscr{T}_m$

(h) $\bar{S}_n = \langle S_\eta : \eta \in \mathscr{T}_n \rangle$

(i) $\bar{B}_n = \langle B_\eta : \eta \in \mathscr{T}_n \rangle$

(j) $\varepsilon_n < \mathrm{cf}(\mu)$ and $n = m+1 \Rightarrow \varepsilon_n \geq \varepsilon_m$

(B) for each $\eta \in \mathscr{T}_n$:

(a) $S_\eta \subseteq \kappa, S_\eta \notin \mathscr{T}_n$

(b) if $n = m+1$ then $S_{\eta \restriction m} \supseteq S_\eta$

(c) $B_\eta \in \mathfrak{B}$ is a subset of μ of cardinality $< \mu_{\varepsilon(n)}$

(d) $\{g_n(i) : i \in S_\eta\}$ is included in B_η

$(C)(a)$ if $n = m+1$ and $\eta \in \mathscr{T}_m$ then the set
$S^*_\eta := \{i \in S_\eta : g_m(i) > f_*(i)\} \backslash \cup \{S_{\eta\hat{}<j>} : \eta\hat{}\langle j \rangle \in \mathscr{T}_n\}$
belongs to I.

<u>It is enough to Carry the definition:</u>

Why? As then $\{B_\eta : \eta \in \mathscr{T}_n$ for some $n < \omega\}$ is a family of members of $\mathscr{P}$ (by (B)(c)), its cardinality is $< \sigma$ (as $\sigma = \mathrm{cf}(\sigma) > \aleph_0$ and for each $n < \omega, |\mathscr{T}_n| < \sigma$ by (A)(e)).

Similarly as I is σ-complete the set $S^* = \cup\{S^*_\eta : \eta \in \mathscr{T}_n$ for some $n < \omega\}$ belongs to I. Now for every $i \in \kappa \backslash S^*$, we try to choose $\eta_n \in \mathscr{T}_n$ by induction on $n < \omega$ such that $i \in S_{\eta_n}$ and $n = m+1 \Rightarrow \eta_m = \eta_n \restriction m$ and $g_m(i) > f_*(i)$. For $n = 0$ let $\eta =<>$ so $i \in \kappa = A_0$. For $n = m+1$, as $i \notin S^*_{\eta_m}$, see (C)(a) clearly η_n as required exists. Now if $n = m+1$ again as $i \notin S^*_{\eta_m}$ we get $g_m(i) > f_*(i)$ and by (A)(d) we have $g_m(i) > g_n(i)$. But there is no decreasing ω-sequence of ordinals. So for some $m, g_m(i) \leq f_*(i)$ so by (A)(c), $g_m(i) = f_*(i)$ but $g_n(i) \in B_{\eta_n}$.

<u>Carrying the induction:</u>

<u>Case $n = 0$:</u>

Let $\mathscr{T}_0 = \{<>\}, A_{<>} = \{\mu_\varepsilon : \varepsilon < \mathrm{cf}(\mu)\}$ which has cardinality $\leq \kappa$ as $\mathrm{cf}(\mu) \leq \kappa$ by assumption. Further, let g_0 be defined as the function with domain κ and $g_0(i) = \min\{\mu_\varepsilon : \mu_\varepsilon > f_*(i)\}$, let $S_{<>} = \kappa$ and $B_{<>} = A_0$ which $\in \mathfrak{B}$ as $\langle \mu_\varepsilon : \varepsilon < \mathrm{cf}(\mu) \rangle \in \mathfrak{B}$ (and has cardinality $|A_0| = \mathrm{cf}(\mu) \leq \kappa$).

<u>Case $n = m+1$:</u>

Let $\eta \in \mathscr{T}_m$ and define $S'_\eta = \{i \in S_\eta : g_n(i) > f_*(i)\}$. If $S'_\eta \in I$ then we decide that $j < n \Rightarrow \eta^\frown\langle j\rangle \notin \mathscr{T}_n$, so we have nothing more to do so assume $S'_\eta \notin I$.

Let $\mathfrak{a}_\eta = \{\mathrm{cf}(\alpha) : \alpha \in B_\eta$ and $\mathrm{cf}(\alpha) > |B_\eta| + \kappa\}$ and let

$$\mathfrak{c}_\eta = \{\mathrm{tcf}(\prod_{i \in S'_\eta} \mathrm{cf}(g_n(i)), <_J) : \begin{array}{l} J \text{ is an } \sigma\text{-complete ideal on} \\ S'_\eta \text{ extending } I \restriction S'_\eta \text{ such that } \mu = \lim_J \langle \mathrm{cf}(g_n(i)) : i \in S'_\eta\rangle \\ \text{and } \prod_{i \in S'_\eta} \mathrm{cf}(g_n(i)), <_J) \text{ has true cofinality}\} \end{array}$$

Clearly $\kappa + |\mathfrak{a}_\eta| < \min(\mathfrak{a}_\eta)$ and $\mathfrak{c}_\eta \subseteq \mathrm{pcf}_{I,\sigma}(\mathfrak{a}_\eta) \subseteq \lambda \cap \mathrm{Reg}$ and by $\neg\circledast_{\mu,I,\sigma}$ we know that $\mathrm{pcf}_{I,\sigma}(\mathfrak{a}_\eta)$ is a bounded subset of λ. But $B_\eta \in \mathfrak{B}$ hence $\mathfrak{a}_\eta \in \mathfrak{B}$ hence $\mathrm{pcf}_{I,\sigma}(\mathfrak{a}_\eta) \in \mathfrak{B}$ so as $\mathfrak{B} \cap \lambda = \delta_{\mathfrak{B}} < \lambda$, clearly $\mathrm{pcf}_{I,\sigma}(\mathfrak{a}_\eta) \subseteq \mathfrak{B}$ hence $\theta \in \mathfrak{c}_\eta \Rightarrow \theta < \delta_{\mathfrak{B}}$. Using pcf basic properties let $J_{\eta,\lambda}$ be the σ-complete ideal on $\mathfrak{a}_\eta$ generated by $J_{=\lambda}[\mathfrak{a}_\eta]$ and so $\bar{\mathfrak{a}}_\eta, J_{\eta,\lambda} \in \mathfrak{B}$ and there is a $<_{J_{\eta,\lambda}}$-increasing cofinal sequence $\bar{f}_{\eta,\lambda} = \langle f_{\eta,\lambda,\zeta} : \zeta < \lambda\rangle$ of members of $\Pi\mathfrak{a}_\eta$ such that $f_{\eta,\lambda,\zeta}$ is the $<_{J_{\eta,\lambda}}$-e.u.b. of $\bar{f}_{\eta,\lambda} \restriction \zeta$ when there is such $<_{J_{\eta,\lambda}}$-e.u.b. Without loss of generality $\bar{f}_{\eta,\lambda} \in \mathfrak{B}$ hence $\{f_{\eta,\lambda,\zeta} : \zeta < \lambda\} \subseteq \mathfrak{B}$.

Let $\mathfrak{a}_m = \cup\{\mathfrak{a}_\eta : \eta \in \mathscr{T}_m\}$ and define a $h_m \in \Pi\mathfrak{a}_m$ by $h_m(\theta) = \sup\{\mathrm{otp}(e_{g_m(i)} \cap f_*(i)) : i < \kappa$ and $f_*(i) < g_m(i)\}$. Clearly it is $< \theta$ as $\theta = \mathrm{cf}(\theta) > \mu_{\varepsilon(m)} \geq |B_\eta| + \kappa$ when $\theta \in \mathfrak{a}_\eta$. For each $\eta \in \mathscr{T}_m$ and $\lambda \in \mathfrak{c}_\eta$ let $\zeta_{\eta,\lambda} < \lambda$ be such that $h_m \restriction \mathfrak{a}_\eta < f_{\eta,\lambda,\zeta_{\eta,\lambda}} \mod J_{\eta,\lambda}$, and let

$$S^1_{\eta,\lambda} = \{i \in S_\eta : h_m(\mathrm{cf}(g_i(\theta)) < f_{\eta,\lambda,\zeta_{\eta,\lambda}}(\mathrm{cf}(g_m(i))\}$$

$\odot$ for some subset $\mathfrak{c}'_\eta$ of $\mathfrak{c}_\eta$ of cardinality $< \sigma$ the set $\{i \in S_\eta : i \notin S^1_{\eta,\lambda}$ for every $\lambda \in \mathfrak{C}'_\eta\}$ belongs to I.

[Why? Otherwise, let J be the σ-complete ideal on S_η generated by $I \cup \{S^1_{\eta,\lambda} : \lambda \in \mathfrak{c}_\eta\}$, so $\kappa \notin J$ hence for some $S^* \in J^+$ we know that $(\prod_{i \in S^*} \mathrm{cf}(g_m(i), <_{J \restriction S^*})$ has true cofinaltiy, call it λ^*. Necessarily $\lambda^* \in \mathfrak{c}_\eta$ and easily get a contradiction.]

<u>Case A</u>: $|\cup\{\mathfrak{c}_\eta : \eta \in \mathscr{T}_m\}| < \mu$.

Let $\langle \lambda_{\eta,j} : j < j_\eta\rangle$ list $\mathfrak{c}'_\eta$. Let $\mathfrak{a}'_n = \mathfrak{a}_n \backslash |\bigcup_\eta \mathfrak{c}_\eta|^+$. Now by induction on $k < \omega$ we choose $h_{n,k}, \zeta_{\eta,j,k}$ for $j < j_\eta, \eta \in \mathscr{T}_m$ such that

$\circledast$ (a) $h_{m,k} \in \Pi\mathfrak{a}'_m$

(b) $h_{m,k} < h_{m,k+1}$

(c) $h_{m,0} = h_m$

(d) $\zeta_{\eta,j,k} < \lambda_{\eta,j}$

(e) $\zeta_{\eta,j,k} < \zeta_{\eta,j,k+1}$

(f) $\zeta_{\eta,j,0} = \zeta_{\eta,j}$

(g) $h_{m,k+1}(\theta) = \sup[\{f_{\eta,\lambda_{\eta,j},\zeta_{\eta,j,k}}(\theta) : \eta \in \mathscr{T}_n, \theta \in \mathfrak{a}_\eta\} \cup \{h_{m,k}(\theta)\}]$

(h) $\zeta_{\eta,j,k+1} = \mathrm{Min}\{\zeta < \lambda_{\eta,j} : \zeta > \zeta_{\eta,j,k}$ and $h_{m,k+1} \restriction \mathfrak{a}_\eta < f_{\eta,\lambda_{\eta,j,\zeta}} \mod J_{\eta,\lambda_{\eta,j}}\}$.

There is no problem to carry the induction. Let $h_{m,\omega} \in \Pi\mathfrak{a}_m$ be defined by $h_{m,\omega}(\theta) = \cup\{h_{m,k}(\theta) : k < \omega\}$. Let $S'_{\eta,j} = \{i \in S_\eta : f_*(i)$ is $<$ the $h_{m,\omega}(\mathrm{cf}(g_m(i)))$-ith member of $e_{g_m(i)}\}$.

Now

$\boxtimes$ for some $\mathfrak{c}''_\eta \subseteq \mathfrak{c}_\eta, |\mathfrak{c}''_\eta| < \sigma$ for $\eta \in \mathscr{T}_m$ we have $S_n \backslash \cup \{S_{\eta,j} : \lambda_j \in \mathfrak{c}'_\eta\} \in I$.

Now continue. $\square_{3.7}$

<u>Case B</u>: C not Case A.

Use §2.

* * *

Discussion 3.10. Lemma 3.7 leaves us in a strange situation: clause (a) is fine, but concerning the exception in clause (b); it may well be impossible and $\mathrm{pcf}(\mathfrak{a})$ is always not "so large". We do not know this, we try to clarify the case for reasonable $\mathbf{J}_i$, i.e., closed under products of two.

Observation 3.11. *1) There is $\mu_* < \mu$ such that $(\forall \mu')(\mu_* < \mu' \leq \mu \wedge \mathrm{cf}(\mu') \leq \kappa < \mu') \Rightarrow \mathrm{pp}^+_{\mathbf{J}}(\mu') \leq \mathrm{pp}^+_{\mathbf{J}}(\mu)$ when:*

$\circledast$ (a) $\mathrm{cf}(\mu) \leq \kappa < \mu$

(b) *$\mathbf{J}$ is a set of σ-complete ideals*

(c) $J \in \mathbf{J} \Rightarrow |\mathrm{Dom}(J)| \leq \kappa$

(d) *if $J_\varepsilon \in \mathbf{J}$ for $\varepsilon < \mathrm{cf}(\mu)$ then for some σ-complete ideal I on $\mathrm{cf}(\mu)$, the ideal $J = \Sigma_I \langle J_\varepsilon : \varepsilon < \mathrm{cf}(\mu)\rangle$ belongs to $\mathbf{J}$ (or is just $\leq_{\mathrm{RK}}$ from some $J' \in \mathbf{J}$).*

Proof. Let $\Lambda = \{\mu' : \mu'$ is a cardinal $< \mu$ but $> \kappa$, of cofinality $\leq \kappa$ such that $\mathrm{pp}^+_{\mathbf{J}}(\mu') > \mathrm{pp}_{\mathbf{J}}(\mu)\}$, and assume toward contradiction that $\mu = \sup(\Lambda)$. So we can choose an increasing sequence $\langle \mu_\varepsilon : \varepsilon < \mathrm{cf}(\mu)\rangle$ of members of Λ with limit μ. For each $\varepsilon < \mathrm{cf}(\mu)$ let $J_\varepsilon \in \mathbf{J}$ witnesses $\mu_\varepsilon \in \Lambda$. Without loss of generality $\kappa_\varepsilon = \mathrm{Dom}(J) \leq \kappa$ so we can find $\langle \lambda_{\varepsilon,i} : i < \kappa_\varepsilon\rangle$ witnessing this. In particular $(\prod\limits_{i<\kappa_\varepsilon} \lambda_{\varepsilon,i}, <_{J_\varepsilon}))$ has true cofinality $\lambda_\varepsilon = \mathrm{cf}(\lambda_\varepsilon) \geq \mathrm{pp}^+_{\mathbf{J}}(\mu)$. Let I, J be as in cluase (d) of $\circledast$. $\square_{3.11}$

* *

A dual kind of measure to Definition 3.1 is

Definition 3.12. 1) Assume J is an ideal say on κ and $f^* : \kappa \to \mathrm{Ord}$ and μ cardinal. Then $\mathbf{U}_J(f^*, <\mu) = \mathrm{Min}\{|\mathscr{P}| : \mathscr{P}$ a family of subsets of $\sup \mathrm{Rang}(f) + 1$ each of cardinality $< \mu$ such that for every $f \leq f^*$ (i.e., $f \in \prod\limits_{i<\kappa} (f^*(i)+1)$) there is $A \in \mathscr{P}$ such that $\{i < \kappa : f(i) \in A\} \notin J\}$.

2) If above we write $\mathbf{J}$ instead of J this means $\mathbf{J}$ is a family of ideals on κ and the $\mathscr{P}$ should serve all the $J \in \mathbf{J}$ simultaneously.

Claim 3.13. *We have $\mathbf{U}_{J^{\mathrm{bd}}_\kappa}(\mu, <\mu) = \lambda_*$ if we assume*

$\circledast$ (a) $\mu > \kappa = \mathrm{cf}(\mu) > \aleph_0$

(*b*) $([\kappa]^\kappa, \supseteq)$ *satisfies the* μ*-c.c. or just* μ^+*-c.c. which means that: if* $\mathscr{A} \subseteq [\kappa]^\kappa$ *and* $A \neq B \in \mathscr{A} \Rightarrow |A \cap B| < \kappa$ *then* $|\mathscr{A}| \leq \mu$

(*c*) $\lambda_* = \mathrm{pp}_{J^{\mathrm{bd}}_\kappa}(\mu) = \sup\{\mathrm{tcf}(\prod\limits_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\kappa}) : \lambda_i < \mu$ *is increasing with limit* μ *and* $(\prod\limits_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\kappa})$ *has true cofinality*$\}$.

Claim 3.14. *We can in 3.13 replace* J^{bd}_κ *by any* $\aleph_1$*-complete filter* J *(?) on* κ *(so (b) becomes "*$(J^+, \supseteq)$ *satisfies the* μ^+*-c.c."*

Remark 3.15. If in clause (b) of $\otimes$ of 3.13, we use the μ-c.c. the proof is simpler, using $\mathscr{T}_n \subseteq {}^n(\mu_{\varepsilon_n}), \varepsilon_n \leq \varepsilon_{n+1}$.

Proof. Let

$(*)$ (a) $\bar\mu = \langle \mu_i : i < \kappa \rangle$ is an increasing continuous sequence of singular cardinals $> \kappa$ with limit μ.

Let χ be large enough, $<^*_\chi$ a well ordering of $(\mathscr{H}(\chi), \in)$ and $\mathscr{B}$ an elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi)$ of cardinality λ_* such that $\lambda_* + 1 \subseteq gB$ and $\bar\mu \in \mathfrak{B}$ and let $\mathscr{A} = [\mu]^{<\mu} \cap \mathfrak{B}$.

So $\mathscr{A}$ is a family of sets of the right form and has cardinality $\leq \lambda_*$. It remains to prove the major point: assume S is an unbounded subset of κ, $f^* \in \prod\limits_{i \in S} [\mu_i, \mu_{i+1}]$ we should prove that $(\exists A \in \mathscr{A})(\exists^\kappa i \in S)(f(i) \in A)$.

Let $\bar e = \langle e_\alpha : \alpha < \mu \rangle \in \mathfrak{B}$ be such that e_α is a club of α of order type $\mathrm{cf}(\alpha)$ so $e_{\alpha+1} = \{\alpha\}, e_0 = \emptyset$. Let $\langle \beta_{\alpha,\varepsilon} : \varepsilon < \mathrm{cf}(\alpha) \rangle$ be an increasing enumeration of e_α.

We choose $\varepsilon_n, g_n, A_n, I_n, \langle S_\eta, B_\eta : \eta \in \mathscr{T}_n \rangle$ such that

$\circledast_n$ $(A)(a)$ $\mathscr{T}_n \subseteq {}^n\mu, \mathscr{T}_0 = \{<>\}, [n = m+1 \wedge \eta \in \mathscr{T}_n \Rightarrow \eta \restriction m \in \mathscr{T}_n]$

- (b) $A_n \subseteq \mu$ has cardinality $\leq \kappa$
- (c) $g_n : \kappa \to A_n$
- (d) $i < \kappa \Rightarrow f^*(i) \leq g_n(i)$
- (e) $n = m+1 \Rightarrow g_n \leq g_m$
- (f) $\varepsilon_n < \kappa$ and $n = m+1 \Rightarrow \varepsilon_m < \varepsilon_n$
- (g) if $n = m+1, i \in (\varepsilon_n, \kappa)$ and $g_m(i) > f^*(i)$ then $g_m(i) > g_n(i)$

(B) for $\eta \in \mathscr{T}_n$

- (a) $S_\eta \subseteq \kappa$ has cardinality κ
- (b) $S_\eta \in [\kappa]^\kappa$ and $\nu \triangleleft \eta \Rightarrow S_\eta \subseteq S_\nu$
- (c) $B_\eta \in \mathfrak{B}$ is a subset of μ of cardinality $< \mu_{\varepsilon(n)}$ where $\varepsilon(n) = \mathrm{Min}\{\varepsilon < \kappa : \eta \in {}^n(\mu_\varepsilon)$ and $\varepsilon \geq \varepsilon_n\}$
  - (d) $\{g_n(i) : i \in S_\eta\} \subseteq B_\eta$.

For $n = 0$ let $\varepsilon_0 = 0, A_{<>} = \{\mu_i : i < \kappa\}, \mathscr{T}_0 = \{<>\}, S_{<>} = \kappa, g_m$ is the function with domain κ such that $g_{<>} = \mathrm{Min}\{\alpha \in A_{<>} : f^*(i) < \alpha\}$. Assume $n = m+1$ and we have defined for m.

Let

$$\begin{aligned}\mathfrak{c}_n = \big\{\theta : \quad & \text{there is an increasing sequence } \langle \lambda_i : i < \kappa\rangle \\ & \text{of regular cardinals } \in (\kappa,\mu) \text{ with limit } \mu \text{ such that} \\ & \theta = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\kappa}) \text{ and} \\ & \{\lambda_i : i < \kappa\} \subseteq \{\mathrm{cf}(\alpha) : \alpha \in A_m, \mathrm{cf}(\alpha) > \kappa\big\}.\end{aligned}$$

Of course, $\mathfrak{c}_n \subseteq \mathrm{Reg}\backslash\mu$. Now for each $\theta \in \mathfrak{c}_n$ let $\langle \lambda^\theta_i : i < \kappa\rangle$ exemplifies it so $\{\{\lambda^\theta_i : i < \kappa\} : \theta \in \mathfrak{c}_n\}$ is a family of subsets of $\{\mathrm{cf}(\alpha) : \alpha \in A_m, \mathrm{cf}(\alpha) > \kappa)\}$ each of cardinality κ and the intersection of any two has cardinality $< \kappa$.

As $|A_m| \le \kappa$, by assumption (d) of the claim we know that $|\mathfrak{c}_n| \le \mu$ and let $\langle \lambda_\beta : \beta \le \mu\rangle$ list them.

For each $\eta \in \mathscr{T}_m$ and $\varepsilon < \kappa$ let

$$\mathfrak{a}_{\eta,\varepsilon} = \{\mathrm{cf}(\delta) : \delta \in B_\eta \text{ and } \mathrm{cf}(\delta) > \mu_\varepsilon + |B_\eta|\}$$

so

$$|\mathfrak{a}_{\eta,\varepsilon}| \le |B_\eta| < \min(\mathfrak{a}_\eta).$$

Let $W = \{(\eta,\varepsilon,\beta) : \eta \in \mathscr{T}_m, \varepsilon < \kappa, \beta < \mu_\varepsilon\}$. Clearly $\mathfrak{a}_{\eta,\varepsilon} \in \mathfrak{B}, \lambda_\beta \in \mathfrak{B}$ hence $J_{\eta,\varepsilon,\beta}$ = the κ-complete ideal generated by $J_{=\lambda_\beta}[\mathfrak{a}_{\eta,\varepsilon}]$ belongs to $\mathfrak{B}$ and some $<_{J_{\eta,\varepsilon,\beta}}$-increasing and cofinal sequence $\langle f_{\eta,\varepsilon,\beta,\zeta} : \zeta < \lambda_\beta\rangle$ belongs to $\mathfrak{B}$ and $f_{\eta,\varepsilon,\beta,\zeta}$ is an $<_{J_{\eta,\varepsilon,\beta}}$-e.u.b. of $\langle f_{\eta,\varepsilon,\beta,\xi} : \xi < \zeta\rangle$ when there is one.

We now define a function h_m

$$\mathrm{Dom}(h_m) = \mathfrak{a}^*_m = \cup\{\mathfrak{a}_{\eta,\varepsilon} : \eta \in \mathscr{T}_m \text{ and } \varepsilon < \kappa\}$$

so

$$\theta \in \mathrm{Dom}(h_m) \Rightarrow \kappa < \theta < \mu \wedge \theta \in \mathrm{Reg}$$

(in fact we do not exclude the case $\mathfrak{a}^*_m = \mathrm{Reg} \cap \mu\backslash\kappa^+$) and

$$h_m(\theta) = \sup\{e_{g_n(i)} \cap f * (i) : i < \kappa \text{ and } \mathrm{cf}(g_n(i)) = \theta\}.$$

As $\theta = \mathrm{cf}(\theta) > \kappa$ clearly

$$\theta \in \mathrm{Dom}(h_m) \Rightarrow h_m(\theta) < \theta.$$

We choose now by induction on $k < \omega, h_{m,k}, \langle \zeta^k_{\eta,\varepsilon,\beta} : (\eta,\varepsilon,\beta) \in W\rangle$ such that

$\boxtimes$ (a) $h_{m,k} \in \Pi\mathfrak{a}^*_m$

(b) $h_{m,0} = h_m$

(c) $h_{m,k} \le h_{m,k+1}$

(d) $\zeta^k_{\eta,\varepsilon,\beta} = \mathrm{Min}\{\zeta : h_{m,k} \restriction \mathfrak{a}_{\eta,\varepsilon} <_{J_{\eta,\varepsilon,\beta}} f_{\eta,\varepsilon,\beta,\zeta} \text{ and } \ell < k \Rightarrow \zeta^\ell_{\eta,\varepsilon,\beta} < \zeta\}$

(e) $h_{m,k+1}(\theta) = \sup[\{h_{m,k}(\theta)\} \cup \{f^k_{\eta,\beta,\varepsilon,\zeta^k_{\eta,\varepsilon,\eta}}(\theta)$: the triple $(\eta,\beta,\varepsilon) \in W$ satisfies $(\exists\varepsilon)(\beta < \mu_\varepsilon < \theta)$ and $\theta \in \mathfrak{a}_{\eta,\varepsilon}\}]$.

Note that $h_{m,k+1}(\theta) < \theta$ as the sup is over a set of $< \theta$ ordinals.

So we have carried the definition, and let $h^*_{m,w} \in \Pi\mathfrak{a}_m$ be defined by $h_{m,\omega}(\theta) = \sup\{h_{m,k}(\theta) : k < \omega\}$ and $\zeta_{\eta,\varepsilon,\beta} = \zeta(\eta,\varepsilon,\beta) = \sup\{\zeta^k_{\eta,\varepsilon,\beta} : k < \omega\}$. Now for each $(\eta,\varepsilon,\beta) \in W$ we have $k < \omega \Rightarrow h_{m,k} \restriction \mathfrak{a}_{\eta,\varepsilon} <_{J_{\eta,\varepsilon,\beta}} f^k_{\eta,\varepsilon,\beta,\zeta(\eta,\varepsilon,\beta)}) < h_{m,k+1} \restriction \mathfrak{a}_{\eta,\varepsilon}$. By the choice of $\bar{f}_{\eta,\varepsilon,\beta}$ as $J_{\eta,\varepsilon,\beta}$ is $\aleph_1$-complete it follows that $h_{m,w} \restriction \mathfrak{a}_{\eta,\varepsilon} = f_{\eta,\varepsilon,\beta,\zeta_{\eta,\varepsilon,\beta}} \mod J_{\eta,\varepsilon,\beta}$.

Let

$$\begin{aligned} A_n =: \{\alpha' : \quad & \text{for some } \alpha \in A_n, \mathrm{cf}(\alpha) \in \mathfrak{a}_n \text{ and } \alpha' \\ & \text{is the } h_{m,\omega}(\mathrm{cf}(\alpha))\text{-th member of } e_\alpha\}. \end{aligned}$$

$$\begin{aligned} g_n(i) \quad & \text{is } \alpha' \text{ when } \alpha' \text{ is the } h_{m,\omega}(\mathrm{cf}(g_m(i))\text{-th member of} \\ & e_{g_m(i)} \text{ and zero otherwise.} \end{aligned}$$

The main point is why $\sigma_n \in (\varepsilon_m, \kappa)$ exists.

To finish the induction step on n, let

$$B_{\eta,\varepsilon,\beta} = \mathrm{Rang}(f_{\eta,\varepsilon,\eta,\zeta_{\eta,\varepsilon,\beta}})$$

$$B'_{\eta,\varepsilon} = B_{\eta,\varepsilon,\beta} \cup \{e_\alpha : \alpha \in B_{\eta,\varepsilon} \text{ and } \mathrm{cf}(\alpha) \leq \mu_{\varepsilon(n)}\}$$

and we choose $\langle B_\rho : \rho \in \mathscr{T}_n, \rho \restriction m \in B = \eta$ to list them enumerates $\{B_{\eta,\varepsilon,\beta} : \varepsilon,\beta\}$ are such that $(\eta,\varepsilon,\beta) \in W_m \cup \{B'_{\eta,\varepsilon}\}$ in a way consistent with the induction hypothesis.

Having carried the induction on n, note that

$\circledast_1$ for some $n, u_n = \{i < \kappa : f^*(i) = g_n(i)\} \in [\kappa]^\kappa$

We now choose by induction on $m \leq n$ a sequence $\eta_m \in \mathscr{T}_m$ such that $\eta_0 =<>$ $, m = \ell + 1 \Rightarrow \eta_\ell \triangleleft \eta_m$ and $S_\eta \cap u_n \in [\kappa]^\kappa$. For $m = n$ by

$\circledast(*)$ $u' = u \cap S_{\eta_n} \in [\kappa]^\kappa$ and $\mathrm{Rang}(f^* \cap u') \subseteq B_\eta \in \mathscr{P}$ so we are done.

$\square$

Discussion 3.16. 1) Can we consider "$\mathbf{c}([\mu]^\mu, \supseteq) \leq \mu^+$"? We should look again at §2.
2) More hopeful is to replace $\mathbf{U}_{J^{\mathrm{bd}}_\kappa}(\mu)$ by $\mathbf{U}_{\text{non-stationary}_\kappa}(\mu)$.
3) By 3.11 and **??** we should have the prd version (for which $\mathbf{J}$ and closure, see [She93b].

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