

The number of models of a fixed Scott rank, for a counterexample to the analytic Vaught conjecture

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October 5, 2020

Abstract

We show that if $\gamma \in \omega \cup \{\aleph_0\}$ and \mathcal{A} is a counterexample to the analytic Vaught conjecture having exactly γ many models of Scott rank ω_1 , then there exists a club $C \subseteq \omega_1$ such that \mathcal{A} has exactly γ many models of Scott rank α , for each $\alpha \in C$.

Throughout this note τ represents a countable relational vocabulary. The set of τ -structures with domain ω is naturally seen as a Polish space X_τ , where a basic open set is given by the set of structures in which $R(i_0, \dots, i_{n-1})$ holds, for R an n -ary relation symbol from τ and $i_0, \dots, i_{n-1} \in \omega$ (see Section 11.3 of [1], for instance). Given a sentence $\phi \in \mathcal{L}_{\aleph_1, \aleph_0}(\tau)$, the set of models of ϕ with domain ω is a Borel subset of X_τ . By a theorem of Lopez-Escobar [4], every Borel subset of X_τ which is closed under isomorphism is also the set of models (with domain ω) of some $\mathcal{L}_{\aleph_1, \aleph_0}(\tau)$ sentence.

We call the following (false) statement the *analytic Vaught conjecture*: for every countable relational vocabulary τ , every analytic subset of X_τ which is closed under isomorphism and contains uncountably many nonisomorphic structures contains a perfect set of nonisomorphic structures. Steel [11] presents two counterexamples to this statement, one due to H. Friedman and the other to K. Kunen.

Given a τ -structure M , we let $\text{SP}_\alpha(M)$ denote the Scott process of M of length α , as defined in [3] (this is essentially the same as the standard definition appearing in [9, 5, 6]; we assume some familiarity with [3] in the arguments below, but expect that familiarity with the classical Scott analysis will suffice). Scott's Isomorphism Theorem [9] (rephrased) says that if α is a (necessarily countable) ordinal and M and N are countable τ -structures of Scott rank at most α , then M and N are isomorphic if and only if $\text{SP}_{\alpha+1}(M) = \text{SP}_{\alpha+1}(N)$.

Given a set $\mathcal{A} \subseteq X_\tau$, we let \mathcal{A}^* denote the class of (ground model, but possibly uncountable) τ -structures M which are isomorphic to an element of the reinterpretation of \mathcal{A} in any (equivalently, every, by Σ_1^1 -absoluteness) outer model in which M is countable. If \mathcal{A} is the set of τ -structures on ω satisfying a sentence ϕ of $\mathcal{L}_{\aleph_1, \aleph_0}(\tau)$, then \mathcal{A}^* as defined above is simply the class of models of ϕ .

For an ordinal α , we let $\text{SP}_\alpha(\mathcal{A})$ denote the set of the Scott processes of length α for structures in \mathcal{A}^* . If \mathcal{A} is a counterexample to the analytic Vaught conjecture, then $|\text{SP}_\alpha(\mathcal{A})| \leq |\alpha|$ (for $\alpha < \omega_1$ this follows

*Supported in part by NSF Grant DMS-1201494 and DMS-1764320.

†Research partially support by NSF grant no: DMS 1101597, and German-Israeli Foundation for Scientific Research & Development grant no. 963-98.6/2007.

by an induction argument using the Perfect Set Property for analytic sets; considering of a forcing extension via $\text{Col}(\omega, \alpha)$ completes the argument for $\alpha \geq \omega_1$.

We also let \mathcal{A}_α denote respectively the class of structures \mathcal{A}^* of Scott rank α . The following well-known fact (slightly restated here) appears as Corollary 10.2 in [3].

Fact 0.1. *Suppose that \mathcal{A} is a counterexample to the analytic Vaught conjecture, and let $x \subseteq \omega$ be such that \mathcal{A} is Σ_1^1 in x . Let M be a member of the reinterpreted version of \mathcal{A} in a forcing extension of V , and let α be an ordinal. Then $\text{SP}_\alpha(M) \in L[x]$.*

The proof of Fact 0.1 given in [3] shows the following. Similar arguments appear in Section 1 of [2] and Chapter 32 of [7].

Theorem 0.1. *Suppose that \mathcal{A} is a counterexample to the analytic Vaught conjecture, and let $x \subseteq \omega$ be such that \mathcal{A} is Σ_1^1 in x . Let Y be a countable elementary submodel of $H((2^{\aleph_1})^+)$ with $x \in Y$, let $\delta = Y \cap \omega_1$ and let P be the transitive collapse of Y . Then $\text{SP}_{\delta+1}(\mathcal{A}) = \text{SP}_{\delta+1}(\mathcal{A})^P$.*

Proof. Since \mathcal{A} is a counterexample to the analytic Vaught conjecture, $\text{SP}_\alpha(\mathcal{A})^P$ is a countable set in P , for each $\alpha < \delta$. It follows that $\text{SP}_\alpha(\mathcal{A})^P = \text{SP}_\alpha(\mathcal{A})$ for each such α , since P is correct about the Σ_1^1 statement asserting that some object satisfying the conditions for membership in $\text{SP}_\alpha(\mathcal{A})$ is unequal to all the members of the countable set $\text{SP}_\alpha(\mathcal{A})^P$. Letting g be P -generic for $\text{Col}(\omega, \delta)$ (the partial order of finite partial functions from ω to δ , ordered by inclusion), the same argument applies to show first that $\text{SP}_\delta(\mathcal{A}) = \text{SP}_\delta(\mathcal{A})^{P[g]}$ and then that $\text{SP}_{\delta+1}(\mathcal{A}) = \text{SP}_{\delta+1}(\mathcal{A})^{P[g]}$. However, each member of $\text{SP}_\delta(\mathcal{A})$ in $P[g]$ must be in P , since otherwise there is a $\text{Col}(\omega, \delta)$ -name for an element not in P , and one can find perfectly many generic filters for P giving distinct realizations of this name. The same argument again shows that each member of $\text{SP}_{\delta+1}(\mathcal{A})$ in $P[g]$ must be in P . \square

Theorem 0.2 below can also be proved using material from [10].

Theorem 0.2. *Suppose that \mathcal{A} is a counterexample to the analytic Vaught Conjecture and $\gamma \in \omega \cup \{\aleph_0\}$ is such that there are up to isomorphism exactly γ many elements of \mathcal{A}^* of Scott rank ω_1 . Then for club many $\alpha < \omega_1$ there are exactly γ many models in \mathcal{A} of Scott rank α , up to isomorphism.*

Proof. Let $\mathcal{M} = \{M_n : n \leq \gamma\}$ be pairwise nonisomorphic elements of \mathcal{A}_{ω_1} such that every element of \mathcal{A}_{ω_1} is isomorphic to some element of \mathcal{M} . Let \mathcal{Y} be the set of countable elementary substructures of $H((2^{\aleph_1})^+)$ containing (as elements) \mathcal{M} and a (fixed) code for \mathcal{A} . We show that for each $Y \in \mathcal{Y}$, letting \mathcal{M}_Y be the image of \mathcal{M} under the transitive collapse of Y , every element of $\mathcal{A}_{Y \cap \omega_1}$ is isomorphic to an element of \mathcal{M}_Y . As the members of \mathcal{M}_Y will be nonisomorphic, this will establish the theorem.

Fix $Y \in \mathcal{Y}$, let $\alpha = Y \cap \omega_1$ and let P be the transitive collapse of Y . By Theorem 0.1, $\text{SP}_{\alpha+1}(\mathcal{A}) = \text{SP}_{\alpha+1}(\mathcal{A})^P$. Suppose toward a contradiction that there exists an $N \in \mathcal{A}_\alpha \setminus \mathcal{M}_Y$. Then $\text{SP}_{\alpha+1}(N) \in P$. Proposition 5.19 of [3] then implies that the δ -th level of $\text{SP}_{\delta+1}(N)$ amalgamates, as defined in Definition 5.16 of [3]. Since amalgamation is a first order property it is witnessed in P . It follows from Proposition 7.10 of [3] that there is a model of $\text{SP}_{\delta+1}(N)$ in P , contradicting the elementarity of the collapse and the assumed property of \mathcal{M} . \square

The proofs of Theorems 0.1 and 0.2 can be used to prove the following variation, which we leave to the interested reader : there is fragment T of ZFC such that, if $x \subseteq \omega$ is a code for an analytic class \mathcal{A} of τ -structures then for any transitive model P of T containing x and any ordinal α in P , if $\text{SP}_\alpha(\mathcal{A})$ is countable then $\text{SP}_\alpha(\mathcal{A}) = \text{SP}_\alpha(\mathcal{A})^P$, and, if in addition $\alpha < \omega_2^P$, then every structure in \mathcal{A}^* of Scott rank α is isomorphic to one in P .

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