Infinitary Logics and A.E.C.

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Abstract

We prove that every a.e.c. with LST number $\leqslant \kappa$ and vocabulary τ of cardinality $\leqslant \kappa$ can be defined in the logic $\mathbb{L}_{\beth_2(\kappa)^{+++},\kappa^+}(\tau)$. In this logic an a.e.c. is therefore an EC class rather than merely a PC class. This constitutes a major improvement on the level of definability previously given by the Presentation Theorem. As part of our proof, we define the *canonical tree* $\mathcal{S} = \mathcal{S}_{\mathcal{K}}$ of an a.e.c. $\mathcal{K}$. This turns out to be an interesting combinatorial object of the class, beyond the aim of our theorem. Furthermore, we study a connection between the sentences defining an a.e.c. and the relatively new infinitary logic L^1_λ.

Introduction

Given an abstract elementary class (a.e.c.) $\mathcal{K}$, in vocabulary τ of size $\leqslant \kappa = \text{LST}(\mathcal{K})$, we do two main things:

- We provide an infinitary sentence *in the same vocabulary* τ of the a.e.c. that axiomatizes $\mathcal{K}$.

- We also provide a version of the "Tarski-Vaught-criterion," adapted to a.e.c.'s: when $M_1 \subseteq M_2$, for $M_1, M_2 \in \mathcal{K}$, we will provide necessary and sufficient *syntactic* conditions for $M_1 \prec_{\mathcal{K}} M_2$. These will depend on a certain sentence holding *only* in M_2.

Furthermore, on the way and as part of the proofs of the two main results, we build a "canonical tree" for an a.e.c. $\mathcal{K}$. This will be a well-founded tree

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of models in $\mathcal{K}$, all of them of cardinality equal to $\mathrm{LST}(\mathcal{K})$ and will play a role generalizing in a rather striking way the role Scott models played for $\mathbb{L}_{\omega_1,\omega}$, but now for arbitrary a.e.c.'s. Finally, we connect our sentences with sentences in logics close to the first author's logic L^1_κ and other logics similar to L^1_κ.

The Presentation Theorem [5] is central to the development of stability for abstract elementary classes: notably, it enables Ehrenfeucht-Mostowski techniques for classes that have large enough models. This has as an almost immediate consequence stability below a categoricity cardinal and opens the possibility of a relatively advanced classification/stability theory in that wider setting.

The Presentation Theorem had provided a way to capture an a.e.c. as a PC-class: by expanding its vocabulary with infinitely many function symbols, an a.e.c. may be axiomatized by an infinitary formula. Although for the stability-theoretical applications mentioned this expansion is quite useful, the question as to whether it is possible to axiomatize an a.e.c. with an infinitary sentence *in the same vocabulary* of the a.e.c. is natural. Here we provide a positive solution: given an a.e.c $\mathcal{K}$ we provide an infinitary sentence *in the same original vocabulary* $\varphi_\mathcal{K}$ whose models are exactly those in $\mathcal{K}$. Therefore, unlike the situation in the Presentation Theorem, here the class turns out to be an EC Class, not a PC class.

The main idea is that a "canonical tree of models", each of size the LST-number of the class, the tree of height ω ends up providing enough tools; the sentence essentially describes all possible maps from elements of this tree into arbitrary potential models in the class. A combinatorial device (a partition theorem theorem on well-founded trees due to Komjath and the first author [3]) is necessary for our proof.

The two main theorems:

Theorem (Theorem 2.1). *(Axiomatization of an a.e.c. in τ by an infinitary sentence in τ.)* *Let $\mathcal{K}$ be an a.e.c. in vocabulary τ of size $\leqslant \mathrm{LST}(\mathcal{K})$ and let $\lambda = \beth_2(\kappa)^{++}$, where $\kappa = \mathrm{LST}(\mathcal{K})$. Then there is a sentence $\psi_\mathcal{K}$ in the logic $\mathbb{L}_{\lambda^+,\kappa^+}(\tau)$ such that $\mathcal{K} = \mathrm{Mod}(\psi_\mathcal{K})$.*

Theorem (From Theorem 3.1, the main point). *(Syntactic Tarski-Vaught-like criterion for $\prec_\mathcal{K}$-elementarity.)* *If $M_1 \subseteq M_2$ are $\tau = \tau_\mathcal{K}$-structures, then the following are equivalent:*

- $M_1 \prec_\mathcal{K} M_2$

- *if $\bar{a} \in {}^{\kappa \geqslant}(M_1)$ then there exist a tuple $\bar{b}$, a τ-structure N and a map f such that*

1. $\bar{b} \in {}^{\kappa \geqslant}(M_1)$ *and* $N \in \mathcal{S}_1$ *(level 1 of the canonical tree of the class; see page 3)*

2. $\mathrm{Rang}(\bar{a}) \subseteq \mathrm{Rang}(\bar{b})$

3. f *is an isomorphism from* N *onto* $M_1 \upharpoonright \mathrm{Rang}(\bar{b})$

4. $M_2 \models \varphi_{N,\lambda+1,1}[\langle f(a_\alpha^*) \mid \alpha < \kappa \rangle]$.

The second part of the previous theorem, the characterization of being $\prec_{\mathcal{K}}$-elementary, amounts to the following: for every tuple in M_1, the model M_2 satisfies a formula describing the fact that the tuple may be covered by another tuple that has the "eventual tree extendibility" property described by a formula we will call $\varphi_{N,\lambda+1,1}$ in the next section.

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1 Canonical trees and sentences for a.e.c.'s

Fix $\mathcal{K}$ for the remainder of this paper an a.e.c. with vocabulary $\tau = \tau(\mathcal{K})$ and $\mathrm{LST}(\mathcal{K}) = \kappa \geqslant |\tau|$. Let λ be the cardinal $\beth_2(\kappa)^{++}$. Without loss of generality we may assume that all models in $\mathcal{K}$ are of cardinality $\geqslant \kappa$. Furthermore, we will use for the sake of convenience an "empty model" called M_{empt} with the property that $M_{\mathrm{empt}} \prec_{\mathcal{K}} M$ for all $M \in \mathcal{K}$.

1.1 The canonical tree of an a.e.c.

We now build a canonical object for our abstract elementary class $\mathcal{K}$, $\mathcal{S} = \mathcal{S}_{\mathcal{K}}$. This will be a tree with ω-many levels, consisting of models in $\mathcal{K}$ of size κ, organized in a way we now describe. To prove our results, we will use the tree $\mathcal{S}_{\mathcal{K}}$ to "test" membership in $\mathcal{K}$ and "depths" of possible extensions.

Notation 1.1. *We fix the following notation for the rest of this paper.*

- *We first fix a sequence of (different) elements* $(a_\alpha^* \mid \alpha < \kappa \cdot \omega)$ *in some model in* $\mathcal{K}$.

- $\bar{x}_n := \langle x_\alpha \mid \alpha < \kappa \cdot n \rangle$,

- $\bar{x}_{=n} := \langle x_\alpha \mid \alpha \in \{\kappa \cdot n + \zeta \mid \zeta < \kappa\} \rangle$.

We now define the **canonical tree** of $\mathcal{K}$:

- $\mathcal{S}_n := \big\{ M \in \mathcal{K} \mid M$ has universe $(a_\alpha^*)_{\alpha < \kappa \cdot n}$ and $m < n$ implies $M \upharpoonright (a_\alpha^*)_{\alpha < \kappa \cdot m} \prec_{\mathcal{K}} M \big\}$,

- $\mathcal{S} = \mathcal{S}_{\mathcal{K}} := \bigcup_{n<\omega} \mathcal{S}_n$; this is a tree with ω levels under $\prec_{\mathcal{K}}$ (equivalently under $\subseteq$, by our definition of each level).

We use this tree in our proof to test properties of the class $\mathcal{K}$. The key point about $\mathcal{S}_{\mathcal{K}}$ is that it contains information not just on models in the class of cardinality $\kappa = \mathrm{LST}(\mathcal{K})$ but more importantly on the way they embed into one another.

1.2 Formulas and sentences attached to $\mathcal{K}$

We now define by induction on $\gamma < \lambda^+$ formulas

$$\varphi_{M,\gamma,n}(\bar{x}_n),$$

for every n and $M \in \mathcal{S}_n$ (when $n = 0$ we may omit M).

Case 1 : $\gamma = 0$

If $n = 0$ then the formula $\varphi_{0,0}$ is $\top$ (the sentence denoting "truth").
Assume $n > 0$. Then

$$\varphi_{M,0,n}(\bar{x}_n) := \bigwedge \mathrm{Diag}_{\kappa}^n(M),$$

where $\mathrm{Diag}_{\kappa}^n(M)$ is the set $\left\{ \varphi(x_{\alpha_0}, \ldots, x_{\alpha_{k-1}}) \mid \alpha_0, \ldots, \alpha_{k-1} < \kappa \cdot n, \right.$ $\varphi(y_0, \ldots, y_{k-1})$ is an atomic or a negation of an atomic formula and $\left. M \models \varphi(a^*_{\alpha_0}, \ldots, a^*_{\alpha_{k-1}}) \right\}.$

Case 2 : γ a limit ordinal

Then

$$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta < \gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

Case 3 : $\gamma = \beta + 1$

Let $\varphi_{M,\gamma,n}(\bar{x}_n)$ be the formula

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{S}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha < \kappa} \bigvee_{\delta < \kappa \cdot (n+1)} z_\alpha = x_\delta \right]$$

Note: all the formulas constructed belong to $\mathbb{L}_{\lambda^+,\kappa^+}(\tau)$. When $n = 0$ our formulas are really *sentences* $\varphi_{\gamma,0}$, for $\gamma < \lambda^+$. These sentences may be understood as "external approximations" to the a.e.c. $\mathcal{K}$. Our first aim is to prove how these approximations end up characterizing the a.e.c. $\mathcal{K}$.

2   Characterizing $\mathcal{K}$ by its canonical sentence

In this section we prove the first main theorem:

Theorem 2.1. *There is a sentence $\psi_{\mathcal{K}}$ in the logic $\mathbb{L}_{\lambda^+,\kappa^+}(\tau)$ such that $\mathcal{K} = \mathrm{Mod}(\psi_{\mathcal{K}})$.*

Our first aim in this section is to prove that every model $M \in \mathcal{K}$ satisfies $\varphi_{\gamma,0}$, for all $\gamma < \lambda^+$.

In order to achieve this, we prove the following (more elaborate) statement, by induction on γ.

Claim 2.2. *Given $\gamma < \lambda^+$, $M \in \mathcal{K}$, $n < \omega$, $N \in \mathcal{S}_n$, $f : N \to M$ a $\prec_{\mathcal{K}}$-embedding (if $n = 0$, f is empty) then $M \models \varphi_{N,\gamma,n}[\langle f(a_\alpha^*) \mid \alpha < \kappa \cdot n \rangle]$.*

Before starting the proof, notice that in the statement of the Claim, when $n = 0$, we have that f is empty and $\varphi_{\gamma,0}$ is a sentence. Notice also as γ grows, the sentences $\varphi_{\gamma,0}$ capture ever more involved properties of the model M. Thus, when $\gamma = 0$, $\varphi_{0,0}$ holds trivially; for $\gamma = 1$, $M \models \varphi_{1,0}$ means M satisfies $\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M_{\mathrm{empt}} \\ N \in \mathcal{S}_1}} \exists \bar{x}_{=1} [\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha<\kappa} \bigvee_{\delta<\kappa\cdot 1} z_\alpha = x_\delta]$. This means that given any subset $Z \subseteq M$ of size at most κ, there is some $N \in \mathcal{S}_1$, the first level of the canonical tree, such that the image of N under some embedding $f : N \to M$, $f(X)$, covers X. In short, this amounts to saying that M is densely covered by images of models in $\mathcal{K}$ of size κ.

When $\gamma = 2$, we know a bit more: parsing the sentence, $M \models \varphi_{2,0}$ means that in M,

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M_{\mathrm{empt}} \\ N \in \mathcal{S}_1}} \exists \bar{x}_{=1} \left[\varphi_{N,1,1}(\bar{x}_1) \wedge \bigwedge_{\alpha<\kappa} \bigvee_{\delta<\kappa\cdot 1} z_\alpha = x_\delta \right].$$

Parsing again, this means that

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M_{\mathrm{empt}} \\ N \in \mathcal{S}_1}} \exists \bar{x}_{=1} \left[\forall \bar{z'}_{[\kappa]} \bigvee_{\substack{N' \succ_{\mathcal{K}} N \\ N' \in \mathcal{S}_2}} \exists \bar{x}_{=2} \varphi_{N',0,2}(\bar{x}_2) \wedge \bigwedge_{\alpha<\kappa} \bigvee_{\delta<\kappa\cdot 2} z'_\alpha = x_\delta \right.$$

$$\left. \wedge \bigwedge_{\alpha<\kappa} \bigvee_{\delta<\kappa\cdot 2} z_\alpha = x_\delta \right].$$

What this long formula says is that given any subset $Z \subseteq M$ there is some N in level 1 of the tree $\mathcal{S}_{\mathcal{K}}$ and a map from N into M with image X_1

covering Z such that... for every subset $Z' \subseteq M$ some $\prec_{\mathcal{K}}$-extension of N in level 2 of the tree embeds into M, extending the original map, and covering also Z'.

PROOF   Let first $\gamma = 0$. Then we have either $n = 0$ in which case trivially $M \models \varphi_{0,0}(= \top)$ or $n > 0$. In the latter case $\varphi_{N,0,n} := \bigwedge \mathrm{Diag}_{\kappa}^{n}(N)$; if $f : N \to M$ is a $\prec_{\mathcal{K}}$-embedding, M satisfies this sentence as it satisfies each of the formulas $\varphi(y_0, \ldots y_{k-1})$ satisfied in N by the images of the $\prec_{\mathcal{K}}$-map f.

The case γ limit ordinal is an immediate consequence of the induction hypothesis.

Let now $\gamma = \beta + 1$ and assume that for every $M \in \mathcal{K}$, $n < \omega$, $N \in \mathcal{S}_n$, if $f : N \to M$ is a $\prec_{\mathcal{K}}$-embedding then $M \models \varphi_{N,\beta,n}[\langle f(a_{\alpha}^*) \mid \alpha < \kappa \cdot n \rangle]$.

Now, fix $M \in \mathcal{K}$, $n < \omega$, $N \in \mathcal{S}_n$ and $f : N \to M$ a $\mathcal{K}$-embedding. We want to check that $M \models \varphi_{N,\gamma,n}[\langle f(a_{\alpha}^*) \mid \alpha < \kappa \cdot n \rangle]$, i.e. we need to verify that

$$M \models \forall \bar{z}_{[\kappa]} \bigvee_{\substack{N' \succ_{\mathcal{K}} N \\ N' \in \mathcal{S}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N',\beta,n+1}(\bar{x}_n \,^\frown \bar{x}_{=n}) \wedge \bigwedge_{\alpha < \kappa} \bigvee_{\delta < \kappa \cdot (n+1)} z_{\alpha} = x_{\delta} \right]$$

when $\bar{x}_n$ is replaced in M by $\langle f(a_{\alpha}^*) \mid \alpha < \kappa \cdot n \rangle$.

So let $\bar{c}_{[\kappa]} \in M$. By the LST axiom, there is some $M' \prec_{\mathcal{K}} M$ containing both $\bar{c}_{[\kappa]}$ and $\langle f(a_{\alpha}^*) \mid \alpha < \kappa \cdot n \rangle$, with $|M'| = \kappa$. By the isomorphism axioms there is $N' \succ_{\mathcal{K}} N$, $N' \in \mathcal{S}_{n+1}$, isomorphic to M' through an isomorphism f' extending f. We may now apply the induction hypothesis to N', f': since $f' : N' \to M$ is a $\prec_{\mathcal{K}}$-embedding, we have that $M \models \varphi_{N',\beta,n+1}[\langle a_{\alpha}^* \mid \alpha < \kappa \cdot (n+1) \rangle]$. But this enables us to conclude: N' is a witness for the disjunction on models $\prec_{\mathcal{K}}$-extending N, and the existential $\exists \bar{x}_{=n}$ is witnessed by $\langle a_{\alpha}^* \mid \alpha \in [\kappa \cdot n, \kappa \cdot (n+1)) \rangle$. As the original M' had been chosen to include the sequence $\bar{c}_{[\kappa]}$, the last part of the formula holds. $\square$Claim 2.2

Now we come to the main point:

Claim 2.3. *If M is a τ-model and $M \models \varphi_{\lambda+1,0}$ then $M \in \mathcal{K}$.*

PROOF   The plan of this proof is as follows: we build $\mathcal{G}$ a set of substructures of M of cardinality κ, each of them isomorphic to a model in $\mathcal{S}_1$ and such that $M \models \varphi_{N,\lambda,1}(\ldots)$ of the elements of the substructure; we prove that $\mathcal{G}$ is cofinal in M (using the fact that $M \models \varphi_{\lambda+1,0}$) and a directed set. We also prove that for elements of $\mathcal{G}$ being a submodel implies being a $\prec_{\mathcal{K}}$-submodel (this is the longest part of the proof, and requires a delicate combinatorial argument). We conclude that $M \in \mathcal{K}$, as it then ends up being the direct limit of the $\prec_{\mathcal{K}}$-directed system $\mathcal{G}$.

Let $\mathcal{G} := \{N^* \subseteq M \mid N^*$ has cardinality κ and for some $N \in \mathcal{S}_1$ there is a bijective $f : N \to N^*$ such that $M \models \varphi_{N,\lambda,1}[\langle f(a^*_\alpha) \mid \alpha < \kappa\rangle]\}$. In particular, such f's are isomorphisms from N to N^*.

We prove first

$$N^*_1 \subseteq N^*_2 \quad (N^*_\ell \in \mathcal{G}) \quad \text{then } N^*_1 \prec_{\mathcal{K}} N^*_2. \tag{1}$$

Fix $N^*_1 \subseteq N^*_2$, both in $\mathcal{G}$. Choose $(N^\ell_\eta, f^\ell_\eta)$ for $\ell = 1, 2$ and $\eta \in \mathrm{ds}(\lambda) := \{\nu \mid \nu$ a decreasing sequence of ordinals $< \lambda\}$ by induction on $\ell g(\eta)$ such that

1. $N^\ell_\eta \in \mathcal{S}_{\ell g(\eta)+1}$

2. f^ℓ_η embeds N^ℓ_η into M: $f^\ell_\eta(N^\ell_\eta) \subseteq M$

3. $M \models \varphi_{N^\ell_\eta,\mathrm{last}(\eta),\ell g(\eta)+1}[\langle f^\ell_\eta(a^*_\alpha \mid \alpha < \kappa \cdot (\ell g(\eta) + 1))\rangle]$ where $\mathrm{last}(\langle\rangle) = \lambda$, $\mathrm{last}(\nu^\frown\langle\alpha\rangle) = \alpha$

4. if $\nu \triangleleft \eta$ then $N^\ell_\nu \prec_{\mathcal{K}} N^\ell_\eta$ and $f^\ell_\nu \subseteq f^\ell_\eta$

5. $f^\ell_{\langle\rangle}(N^\ell_{\langle\rangle}) = N^*_\ell$

6. $f^1_\eta(N^1_\eta) \subseteq f^2_\eta(N^2_\eta)$ and $\nu \triangleleft \eta \Rightarrow f^2_\nu(N^2_\nu) \subseteq f^1_\eta(N^1_\eta)$.

The induction: if $\ell g(\eta) = 0$ let $f^\ell_\eta = f^\ell_{\langle\rangle}$ be a one-to-one function from $\langle a^*_\alpha \mid \alpha < \kappa\rangle$ onto N^*_ℓ; as $\|N^*_\ell\| = \kappa$ there is a model N^ℓ_η with universe $\langle a^*_\alpha \mid \alpha < \kappa\rangle$ such that f^ℓ_η is an isomorphism from N^ℓ_η onto N^*_ℓ. Since $\mathrm{last}(\langle\rangle) = \lambda$ and by definition of $\mathcal{G}$ we have $M \models \varphi_{N^\ell_{\langle\rangle},\lambda,1}[f^\ell_{\langle\rangle}(a^*_\alpha) \mid \alpha < \kappa]$, this choice satisfies the relevant clauses (1, 2, 3, 5 and the first part of 6).

If $\ell g(\eta) = n = m + 1$ we first choose (f^1_η, N^1_η). From the inductive definition of $\varphi_{N^1_{\eta\restriction m,\mathrm{last}(\eta\restriction m),m}}$ with $\bar{z}_{[\kappa]}$ an enumeration of $\langle f^2_{\eta\restriction m}(a^*_\alpha) \mid \alpha < \kappa \cdot m\rangle$, the sequence $\bar{x}_{=n}$ gives us the map f^1_η, with domain N^1_η (a witness of the disjunction in the formula), and $N^1_\eta \supseteq N^2_{\eta\restriction m}$. (While doing this, we make sure the new function $f^1_\eta \supseteq f^1_{\eta\restriction m}$.)

Now to choose (f^2_η, N^2_η) we use a symmetric argument and the inductive definition of $\varphi_{N^2_{\eta\restriction m,\mathrm{last}(\eta\restriction m),m}}$ with $\bar{z}_{[\kappa]}$ enumerating $\langle f^1_{\eta\restriction n}(a^*_\alpha) \mid \alpha < \kappa \cdot n\rangle$; as before, the sequence $\bar{x}_{=n}$ gives us the map f^2_η, with domain N^2_η. Again we make sure $f^2_\eta \supseteq f^2_{\eta\restriction m}$.

In both construction steps the model obtained is a $\prec_{\mathcal{K}}$-extension, since it is given by the disjunction inside the formula $\varphi_{N^\ell_{\eta\restriction m,\mathrm{last}(\eta\restriction m),m}}$.

This finishes the inductive construction of the well-founded tree of models and functions $(N^\ell_\eta, f^\ell_\eta)_{\eta\in\mathrm{ds}(\lambda)}$.

Let us now check why having carried the induction suffices.

We apply a partition theorem on well founded trees due to Komjath and the first author [3]. In [2], Gruenhut and the first author provide the following useful form.

Theorem 2.4 (Komjath-Shelah, [3])**.** *Let α be an ordinal and μ a cardinal. Set $\nu = \left(|\alpha|^{\mu^{\aleph_0}}\right)^+$ and let $F(ds(\nu^+)) \to \mu$ be a colouring of the tree of finite descending sequences of ordinals $< \lambda$. Then there is an embedding $\varphi : ds(\alpha) \to ds(\nu)$ and a function $c : \omega \to \mu$ such that for every $\eta \in ds(\alpha)$ of length $n + 1$*

$$F(\varphi(\eta)) = c(n).$$

In our case, the number of colors μ is $\kappa^{|\tau|+\kappa} = 2^\kappa$. So, the corresponding ν is $\left(|\alpha|^{\mu^{\aleph_0}}\right)^+ = \left(|\alpha|^{(2^\kappa)^{\aleph_0}}\right)^+ = \left(|\alpha|^{2^\kappa}\right)^+ = \beth_2(\kappa)^+$ and $\nu^+ = \lambda$. Our coloring (given by the choice of the models N_η^ℓ and maps f_η^ℓ for $\eta \in ds(\lambda)$) is therefore a mapping

$$F : ds(\lambda) \to \mu$$

and the partition theorem provides a sequence $(\eta_n)_{n<\omega}$, $\eta_n \in ds(\alpha)$ such that:

$$k \leqslant m \leqslant n, \ell \in \{1, 2\} \Rightarrow N_{\eta_m \restriction k}^\ell = N_{\eta_n \restriction k}^\ell.$$

We therefore obtain $(N_k^\ell, g_{k,n}^\ell)_{k \leqslant n}$ such that

- $N_k^1 \subseteq N_k^2 \subseteq N_{k+1}^1$ and
- $g_{k,n}^\ell$ is an isomorphism from N_k^ℓ onto $N_{\eta_n \restriction k}^\ell$.

Hence $N_n^\ell \prec_{\mathcal{K}} N_{n+1}^\ell$ and so $\langle N_n^\ell \mid n < \omega \rangle$ is $\prec_{\mathcal{K}}$-increasing. Let $N_\ell := \bigcup_n N_n^\ell$. Then clearly $N_1 = N_2$; call this model N. Since we then have $N_n^1 \prec_{\mathcal{K}} N$, $N_n^2 \prec_{\mathcal{K}} N$ and $N_n^1 \subseteq N_n^2$ by the coherence axiom for a.e.c.'s we have that $N_n^1 \prec_{\mathcal{K}} N_n^2$. In particular, when $n = 0$ we get that $N_1^* \prec_{\mathcal{K}} N_2^*$.

Finally, we also have that

$$\mathcal{G} \text{ is cofinal in } [M]^{\leqslant \kappa}, \tag{2}$$

as $M \models \varphi_{\lambda+1,0}$ and the definition of the sentence $\varphi_{\lambda,0}$ says that every $Z \subseteq M$ can be covered by some N^* of cardinality κ isomorphic to *some* $N \in \mathcal{S}_1$ such that $M \models \varphi_{N,\lambda,1}(\langle f(a_\alpha^*) \mid \alpha < \kappa \rangle)\ldots$ but this means $N^* \in \mathcal{G}$. Also, $\mathcal{G}$ is a directed system.

Finally, putting together (1) and (2), we conclude that every τ-model M such that $M \models \varphi_{\lambda+1,0}$ must be in the class: $M = \bigcup \mathcal{G}$, and $\mathcal{G}$ is a $\prec_{\mathcal{K}}$-directed system. Since $\mathcal{K}$ is an a.e.c, the limit of this $\prec_{\mathcal{K}}$-directed system must be an element of $\mathcal{K}$, therefore $M \in \mathcal{K}$. $\hspace{3cm}$ $\square_{\text{Lemma 2.3}}$

Lastly, we complete the proof of Theorem 2.1: Claims 2.2 and 2.3 provide the definability in the class, as clearly $\varphi_{\gamma,0} \in \mathbb{L}_{\lambda^+,\kappa^+}(\tau_{\mathcal{K}})$.

$\hspace{11cm}$ $\square_{\text{Theorem 2.1}}$

3 Strong embeddings and definability

We now focus on the relation $\prec_{\mathcal{K}}$ of our a.e.c. $\mathcal{K}$: we characterize it in $\mathbb{L}_{\lambda^+,\kappa^+}$. We prove a syntactic criterion for being a $\prec_{\mathcal{K}}$-substructure (given that we already have that $M_1 \subseteq M_2$) *in terms of satisfiability in* M_2 of certain formulas on tuples from M_1. This may be regarded as a very strong analog of a "Tarski-Vaught" criterion for a.e.c.'s.

It is worth mentioning we will continue using in a crucial way both the canonical tree $\mathcal{S}_{\mathcal{K}}$ of our a.e.c., and the partition theorem on well-founded trees.

Theorem 3.1. *Let $\mathcal{K}$ be an a.e.c., $\tau = \tau(\mathcal{K}) \leqslant \kappa = \mathrm{LST}(\mathcal{K})$, $\lambda = \beth_2(\kappa)^{++}$. Then, given τ-models $M_1 \subseteq M_2$, the following are equivalent:*

(A) $M_1 \prec_{\mathcal{K}} M_2$

(B) *if $\bar{a}_\ell \in {}^{\kappa \geqslant}(M_\ell)$ for $\ell = 1, 2$ and $\gamma < \lambda$ then there are $\bar{b}_\ell$, N_ℓ and f_ℓ for $\ell = 1, 2$ such that:*
for $\ell = 1, 2$,

    (a) *$\bar{b}_\ell \in {}^{\kappa \geqslant}(M_\ell)$ and $N_\ell \in \mathcal{S}_\ell$*

    (b) *$\mathrm{Rang}(\bar{a}_\ell) \subseteq \mathrm{Rang}(\bar{b}_\ell)$*

    (c) *f_ℓ is an isomorphism from N_ℓ onto $M_\ell \upharpoonright \mathrm{Rang}(\bar{b}_\ell)$*

    (d) *$\mathrm{Rang}(\bar{b}_1) \subseteq \mathrm{Rang}(\bar{b}_2)$*

    (e) *$N_1 \subseteq N_2$*

    (f) *$M_\ell \models \varphi_{N_\ell,\gamma,\ell}[\langle f_\ell(a^*_\alpha) \mid \alpha < \kappa \cdot \ell \rangle]$.*

(C) *if $\bar{a} \in {}^{\kappa \geqslant}(M_1)$ then there are $\bar{b}$, N and f such that*

    (a) *$\bar{b} \in {}^{\kappa \geqslant}(M_1)$ and $N \in \mathcal{S}_1$*

    (b) *$\mathrm{Rang}(\bar{a}) \subseteq \mathrm{Rang}(\bar{b})$*

    (c) *f is an isomorphism from N onto $M_1 \upharpoonright \mathrm{Rang}(\bar{b})$*

    (d) *$M_2 \models \varphi_{N,\lambda+1,1}[\langle f(a^*_\alpha) \mid \alpha < \kappa \rangle]$.*

PROOF   *(A)⇒ (B)*: Let $\bar{a}_\ell \in {}^{\kappa\geqslant}(M_\ell)$ for $\ell = 1, 2$ and let $\gamma < \lambda$. Choose first $N_1^* \prec_{\mathcal{K}} M_1$ of cardinality $\leqslant \kappa$ including $\text{Rang}(\bar{a}_1)$ and next, choose $N_2^* \prec_{\mathcal{K}} M_2$ including $N_1^* \cup \bar{a}_2$, of cardinality κ. Let $\bar{b}_\ell$ enumerate N_ℓ^* and let (N_1, f_1, N_2, f_2) be such that

1. $N_1 \in \mathcal{S}_1, N_2 \in \mathcal{S}_2, N_1 \subseteq N_2$ and

2. f_ℓ is an isomorphism from N_ℓ onto N_ℓ^* for $\ell = 1, 2$.

This is possible: since $M_1 \prec_{\mathcal{K}} M_2$ and $N_\ell^* \prec_{\mathcal{K}} M_\ell$ for $\ell = 1, 2$, we also have that $N_1^* \prec_{\mathcal{K}} N_2^*$. Therefore there are corresponding models $N_1 \subseteq N_2$ in the canonical tree, at levels 1 and 2 (as these must satisfy $N_1 \prec_{\mathcal{K}} N_2$).

We then have that $f_\ell : N_\ell \to M_\ell$ is a $\mathcal{K}$-embedding from elements N_1 and N_2 in the canonical tree $\mathcal{S}$. By Claim 2.2, we may conclude that

$$M_1 \models \varphi_{N_1,\gamma,1}[\langle f(a_\alpha^*) \mid \alpha < \kappa \rangle]$$

and

$$M_2 \models \varphi_{N_2,\gamma,2}[\langle f(a_\alpha^*) \mid \alpha < \kappa \cdot 2 \rangle],$$

for each $\gamma < \lambda$.

(B)⇒ (C): let $\bar{a} \in {}^{\kappa\geqslant}(M_1)$. We need $\bar{b}$, $N \in \mathcal{S}_1$ and $f : N \to M_1 \upharpoonright \text{Rang}(\bar{b})$ such that

$$M_2 \models \varphi_{N,\lambda+1,1}[\langle f(a_\alpha^* \mid \alpha < \kappa)\rangle]. \tag{3}$$

(B) provides a model $N = N_1 \in \mathcal{S}_1$ and elements $\bar{b} = \bar{b}_1$, as well as an isomorphism $f : N \to \text{Rang}(\bar{b})$. We now check that (B) also implies 3.

Recall the definition of $\varphi_{N,\lambda+1,1}$ (as applied to $[\langle f(a_\alpha^* \mid \alpha < \kappa)\rangle]$). This formula holds in M_2 if for every $\bar{c}_{[\kappa]}$ (of size κ) in M_2, **for some** $\prec_{\mathcal{K}}$-extension N' of N in $\mathcal{S}_2$ we have that

$$M_2 \models \exists \bar{x}_{=2} \varphi_{N',\lambda,2}[\langle f(a_\alpha^* \mid \alpha < \kappa)\rangle^\frown \bar{x}_{=2}] \tag{4}$$

and the elements $\bar{c}_{[\kappa]}$ are "covered" by the list of elements (of length $\kappa \cdot 2$) $\langle f(a_\alpha^* \mid \alpha < \kappa)\rangle^\frown \bar{x}_{=2}$. But the remaining part of clause (B) provides just this: there is *some* $N' = N_2 \in \mathcal{S}_2$, extending $N = N_1$ such that for each $\gamma < \lambda$, and an isomorphism f' from N' into some $\prec_{\mathcal{K}}$-submodel N^* of M_2 containing $\text{Rang}(\bar{c}_{[\kappa]})$ such that $M_2 \models \varphi_{N',\gamma,2}[\langle f'(a_\alpha^* \mid \alpha < \kappa \times 2)\rangle]$. The submodel N' witnesses the disjunction on models and $\langle f'(a_\alpha^*) \mid \alpha \in [\kappa, \kappa \cdot 2)\rangle$ witnesses the existential $\bar{x}_{=2}$.

(C)⇒ (A): assuming (C) means that for every κ-tuple $\bar{a}$ from M_1 there are a model $N \in \mathcal{S}_1$, a κ-tuple $\bar{b}$ from M_1 containing $\bar{a}$ and an isomorphism from N onto $M_1 \upharpoonright \text{Rang}(\bar{b})$ such that

$$M_2 \models \varphi_{N,\lambda+1,1}[\langle f(a_\alpha^*) \mid \alpha < \kappa \rangle].$$

This means that for each $\bar{c}$ included in M_2 (of length κ) there are some extension N' of N with $N' \in \mathcal{S}_2$ and some $\bar{d}$ included in M_2, of length κ, such that

$$M_2 \models \varphi_{N',\lambda,2}[\langle f(a_\alpha^*) \mid \alpha < \kappa \rangle^\frown \bar{d}]$$

and such that $\mathrm{Rang}(\bar{c}) \subseteq \mathrm{Rang}([\langle f(a_\alpha^*)\rangle]^\frown \bar{d})$.

Consider first the family

$$\mathcal{G}_1 := \left\{ N_1^* \subseteq M_1 \mid \exists N_1 \in \mathcal{S}_1 \exists f : N_1 \overset{\approx}{\to} N_1^* [M_2 \models \varphi_{N_1,\lambda+1,1}(f(a_\alpha^*)_{\alpha<\kappa})] \right\};$$

by part (d) of the hypothesis $\mathcal{G}_1$ is a directed family, cofinal in M_1.

Now fix $N_1^* \in \mathcal{G}_1$ and let

$$\mathcal{G}_{N_1^*,2} := \left\{ N_2^* \subseteq M_2 \mid N_1^* \prec_{\mathcal{K}} N_2^* \text{ and} \right.$$

$$\left. \exists N_2 \in \mathcal{S}_2 \exists f_2 : N_2 \overset{\approx}{\to} N_2^* [M_2 \models \varphi_{N_2,\lambda,2}(f_2(a_\alpha^*)_{\alpha<\kappa\cdot2})] \right\}.$$

Now build a tree of models as in the proof of Claim 2.3 inside $\mathcal{S}_{\mathcal{K}}$, indexed by $\mathrm{ds}(\lambda)$, and use the partition theorem on well-founded trees to conclude that

$$N_{2,1}^* \subseteq N_{2,2}^*, N_{2,\ell}^* \in \mathcal{G}_{N_1^*,2}(\ell = 1,2) \Rightarrow N_{2,1}^* \prec_{\mathcal{K}} N_{2,2}^*. \qquad (5)$$

Now, one of the consequences of $M_2 \models \varphi_{N_1,\lambda+1,1}(f(a_\alpha^*)_{\alpha<\kappa})$ (for the model N_1 in $\mathcal{S}_1$ corresponding to N_1^* and for the map f) is precisely that $\mathcal{G}_{N_1^*,2}$ is cofinal in M_2 and a directed family, $\prec_{\mathcal{K}}$-directed also, by (5). Therefore, by the union axiom of a.e.c.'s we may conclude that $N_2^* \prec_{\mathcal{K}} M_2$; since we also had $N_1^* \prec_{\mathcal{K}} N_2^*$, we have that $N_1^* \prec_{\mathcal{K}} M_2$.

Since N_1^* was an arbitrary member of $\mathcal{G}_1$, we may conclude that all members of $\mathcal{G}_1$ are $\prec_{\mathcal{K}}$-elementary in M_2. By another application of the partition relation, the family $\mathcal{G}_1$ also has the property that $M_{1,1}^* \subseteq M_{1,2}^*$ in the family implies $M_{1,1}^* \prec_{\mathcal{K}} M_{1,2}^*$. So, applying again the union axiom, we may conclude that $M_1 = \bigcup \mathcal{G}_1 \prec_{\mathcal{K}} M_2$.

$$\square_{\text{Theorem 3.1}}$$

The previous criterion for $M_1 \prec_{\mathcal{K}} M_2$, given $M_1 \subseteq M_2$, is admittedly quite sophisticated compared with the classical Tarski-Vaught criterion for elementarity in first order logic. There are, however, some interesting parallels.

- In part (C) of our criterion, we only evaluate the formula at the "large model" M_2. This is one of the crucial aspects of the Tarski-Vaught criterion, as it allows construction "from below" of elementary submodels.

- The aspect of our criterion that is definitely less within reach is a version of "capturing existential formulas". We are in a sense exactly doing *that* but in the more complex world of a.e.c.'s. Satisfying a formula of the form $\varphi_{N,\lambda+1,1}(\dots)$ at a subset of elements of the *small* model M_1, when parsing the formula, in a way reflects the possibility of being able to realize, according to M_2, all "possible extensions" of small models, reflecting them correctly to M_1. The partition relation on well-founded trees of course ends up being the key in our case.

4 Around the logic of an a.e.c.

The logic usually called L^1_κ from Shelah's paper [6] satisfies Interpolation and a weak form of compactness: strong undefinability of well-order. Furthermore, it satisfies a Lindström-like maximality theorem for these properties (as well as union of ω-chains of models). The logic L^1_κ, however, lacks a well-defined syntax. Väänänen and Villaveces [7] have produced a logic with a clearly defined (and relatively symple) syntax, whose Δ-closure (a notion appearing first in [4]) is L^1_κ, and which satisfies several of the good properties of that logic (of course, strong undefinability of well-order but also closure under unions of chains). Also, Dzamonja and Väänänen have linked chain logic [1] to L^1_κ.

All of these logics are close to our constructions in this paper: the sentence $\varphi_{\lambda+1,0}$ belongs to $\mathbb{L}_{\lambda^+,\kappa^+}$ and L^1_μ lies in between two logics of the form $L_{\mu,\aleph_0}$ and $L_{\mu,\mu}$. Our sentence $\varphi_{\lambda+1,0}$ belongs to L^1_μ. However, it is not clear if this is the minimal logic for which this is the case.

The question of which is the minimal logic capturing an a.e.c. remains still partially open. Our theorems in this article provide a substantial advance in this direction.

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