CONTROLLING CLASSICAL CARDINAL CHARACTERISTICS WHILE COLLAPSING CARDINALS

MARTIN GOLDSTERN, JAKOB KELLNER, DIEGO A. MEJÍA, AND SAHARON SHELAH

Abstract. Given a forcing notion P that forces certain values to several classical cardinal characteristics of the reals, we show how we can compose P with a collapse (of a cardinal $\lambda > \kappa$ to κ) such that the composition still forces the previous values to these characteristics.

We also show how to force distinct values to $\mathfrak{m}$, $\mathfrak{p}$ and $\mathfrak{h}$ and also keeping all the values in Cichoń's diagram distint, using the Boolean Ultrapower method. (In our recent paper *Controlling cardinal characteristics without adding reals* the same was done for a newer Cichoń's Maximum construction which does not require large cardinals.)

Introduction

Cichoń's diagram (see Figure 1) lists ten cardinal characteristics of the continuum, which we will call *Cichoń-characteristics* (where we ignore the two "dependent" characteristics $\mathrm{add}(\mathcal{M}) = \min(\mathrm{cov}(\mathcal{M}), \mathfrak{b})$ and $\mathrm{cof}(\mathcal{M}) = \max(\mathrm{non}(\mathcal{M}), \mathfrak{d})$).

In many constructions that force given values to such characteristics we actually get something stronger, which we call "strong witnesses" (the objects $\bar{f}$ and $\bar{g}$ in Definition 1.10).

In this paper, we show how to collapse cardinals while preserving the strongly witnessed values for Cichoń-characteristics (and certain other types of characteristics).

With *Cichoń's Maximum* we denote the statement "all Cichoń-characteristics (including $\aleph_1$ and the continuum) are pairwise different". In [GKMSb] we show how to force Cichoń's Maximum (without using large cardinals).

In [GKMSa] we investigate how to preserve and how to change classical cardinal characteristics of the continuum in NNR extensions, i.e., extensions that do not add reals; and we show how this gives 13 pairwise different ones: ten from Cichoń's Maximum, plus $\mathfrak{m}$, $\mathfrak{p}$ and $\mathfrak{h}$ (see Definition 1.1). This construction is based on [GKMSb] (and accordingly does not use large cardinals).

The original Cichoń's Maximum construction [GKS19] uses Boolean ultrapowers (which makes large cardinals necessary). It turns out that it is possible to add $\mathfrak{m}$, $\mathfrak{p}$ and $\mathfrak{h}$ to this construction as well (see Figure 2); and as this construction seems interesting in its own right, we give the details in this paper. (But note that the result of [GKMSa] is stronger in the sense that we do not require large cardinals

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$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \boxed{} \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

FIGURE 1. Cichoń's diagram with the two "dependent" values removed, which are $\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$ and $\mathrm{cof}(\mathcal{M}) = \max(\mathrm{non}(\mathcal{M}), \mathfrak{d})$. An arrow $\mathfrak{x} \to \mathfrak{y}$ means that ZFC proves $\mathfrak{x} \leq \mathfrak{y}$.

there; the advantage of the result in this paper is that we can obtain singular values for $\mathrm{cov}(\mathcal{M})$ and $\mathfrak{d}$ in certain circumstances.)

Annotated Contents:

We will briefly review the Boolean ultrapower constructions in **Section 1**. We also describe how we can start with alternative initial forcings (for the left hand side of Cichoń's diagram), which for example allow us to get Cichoń's Maximum plus distinct values for $\mathfrak{m}$, $\mathfrak{p}$ and $\mathfrak{h}$, and allows two Cichoń-characteristics to be singular, namely, $\mathfrak{c}$ and either $\mathrm{cov}(\mathcal{M})$ or $\mathfrak{d}$ (the latter when the Cichoń's Maximum construction is based on [BCM21]). This contrasts the constructions in [GKMSa] without large cardinals, where only $\mathfrak{c}$ is allowed to be singular.

Part of the following Sections are parallel to [GKMSa], and we will regularly refer to that paper; this applies in particular to **Section 2** (and parts of Subsection 1.3), where we describe some classes of cardinal characteristics, and their behaviour under no-new-reals extension.

In **Section 3** we show how to add $\mathfrak{m}$, $\mathfrak{p}$ and $\mathfrak{h}$ to the Boolean ultrapower construction.

Also, the Boolean ultrapower method produces large gaps between the Cichoń values of the left hand side: the κ_i in Figure 3 are strongly compact (in the ground model; so as cofinalities are preserved they are still weakly inaccessible in the extension). We can get rid of these gaps using the main result of this paper:

In **Section 4** we show how we can collapse cardinals while keeping values for characteristics that are either strongly witnessed or small.

1. Preliminaries

1.1. **The characteristics.** In addition to the Cichoń-characteristics we will consider the following ones, whose definitions are well known.

Definition 1.1. Let $\mathcal{P}$ be a class of forcing notions.

(1) $\mathfrak{m}(\mathcal{P})$ denotes the minimal cardinal where Martin's axiom for the posets in $\mathcal{P}$ fails. More explicitly, it is the minimal κ such that, for some poset $Q \in \mathcal{P}$, there is a collection $\mathcal{D}$ of size κ of dense subsets of Q such that there is no filter in Q intersecting all the members of $\mathcal{D}$.
(2) $\mathfrak{m} := \mathfrak{m}(\mathrm{ccc})$.
(3) Write $a \subseteq^* b$ iff $a \setminus b$ is finite. Say that $a \in [\omega]^{\aleph_0}$ is a *pseudo-intersection* of $F \subseteq [\omega]^\omega$ if $a \subseteq^* b$ for all $b \in F$.

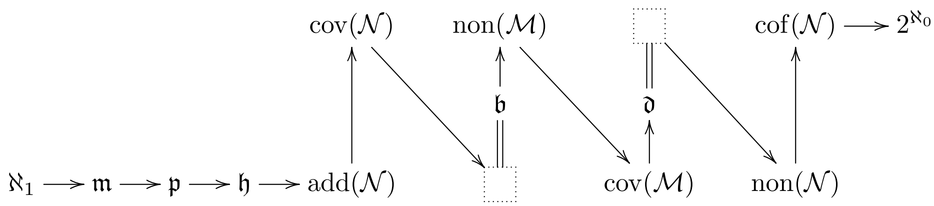

FIGURE 2. The model we construct in this paper; here $\mathfrak{x} \to \mathfrak{y}$ means that $\mathfrak{x} < \mathfrak{y}$ (when $\mathfrak{h}$ is omitted, any number of the $<$ signs can be replaced by $=$ as desired).
This model corresponds to "Version A" (vA*, Fig. 3). We also realise another ordering of the Cichoń values, called "Version B" (vB*, Fig. 4).

(4) The *pseudo-intersection number* $\mathfrak{p}$ is the smallest size of a filter base of a free filter on ω that has no pseudo-intersection in $[\omega]^{\aleph_0}$.
(5) The *tower number* $\mathfrak{t}$ is the smallest order type of a $\subseteq^*$-decreasing sequence in $[\omega]^{\aleph_0}$ without pseudo-intersection.
(6) The *distributivity number* $\mathfrak{h}$ is the smallest size of a collection of dense subsets of $([\omega]^{\aleph_0}, \subseteq^*)$ whose intersection is empty.
(7) A family $D \subseteq [\omega]^{\aleph_0}$ is *groupwise dense* if
  (i) $a \subseteq^* b$ and $b \in D$ implies $a \in D$, and
  (ii) whenever $(I_n : n < \omega)$ is an interval partition of ω, there is some $a \in [\omega]^{\aleph_0}$ such that $\bigcup_{n\in a} I_n \in D$.
  The *groupwise density number* $\mathfrak{g}$ is the smallest size of a collection of groupwise dense sets whose intersection is empty.

It is well known that ZFC proves the following (for references see [GKMSa]):

$$\begin{gathered}\mathfrak{m} \leq \mathfrak{p} = \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{g}, \quad \mathfrak{m} \leq \mathrm{add}(\mathcal{N}), \quad \mathfrak{t} \leq \mathrm{add}(\mathcal{M}), \quad \mathfrak{h} \leq \mathfrak{b}, \quad \mathfrak{g} \leq \mathrm{cof}(\mathfrak{d}), \\ 2^{<\mathfrak{t}} = \mathfrak{c} \text{ and } \mathrm{cof}(\mathfrak{c}) \geq \mathfrak{g},\end{gathered} \tag{1.2}$$

and all these cardinals are regular, with the possible exception of $\mathfrak{m}$, $\mathfrak{d}$ and $\mathfrak{c}$.

1.2. **The old constructions.** In this paper, we will build on two constructions from [GKS19, BCM21] and [KST19], which we call the "old constructions" and refer to as vA* and vB*, respectively. They force different values to several (or all) entries of Cichoń's diagram. We will not describe these constructions in detail, but refer to the respective papers instead.

The "basic versions" of the constructions do not require large cardinals and give us different values for the "left hand side":

Theorem 1.3. *Assume that $\aleph_1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ are regular cardinals and $\lambda_4 \leq \lambda_5 \leq \lambda_6$.*
[BCM21] *If λ_5 is a regular cardinal and $\lambda_6^{<\lambda_3} = \lambda_6$, then there is a f.s. iteration $\bar{P}^{\mathtt{vA}}$ of length of size λ_6 with cofinality λ_4, using iterands that are (σ, k)-linked for every $k \in \omega$, which forces*

$$\text{(vA)} \quad \mathrm{add}(\mathcal{N}) = \lambda_1,\ \mathrm{cov}(\mathcal{N}) = \lambda_2,\ \mathfrak{b} = \lambda_3,\ \mathrm{non}(\mathcal{M}) = \lambda_4, \quad \mathrm{cov}(\mathcal{M}) = \lambda_5,\ \textit{and}\ \mathfrak{d} = \mathfrak{c} = \lambda_6.$$

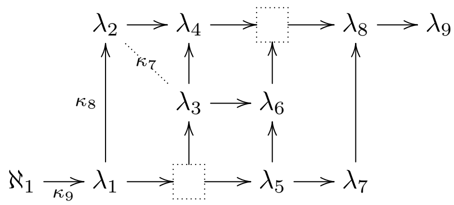

FIGURE 3. The vA* order.

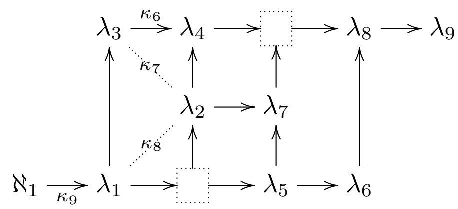

FIGURE 4. The vB* order.

[KST19] *If $\lambda_5 = \lambda_5^{<\lambda_4}$ and either $\lambda_2 = \lambda_3$,*[1] *or λ_3 is $\aleph_1$-inaccessible,*[2] *$\lambda_2 = \lambda_2^{<\lambda_2}$ and $\lambda_4^{\aleph_0} = \lambda_4$, then there is a f.s. iteration $\bar{P}^{\mathtt{vB}}$ of length of size λ_5 with cofinality λ_4, using iterands that are (σ, k)-linked for every $k \in \omega$, that forces*

$$(\mathtt{vB}) \quad \mathrm{add}(\mathcal{N}) = \lambda_1,\ \mathfrak{b} = \lambda_2,\ \mathrm{cov}(\mathcal{N}) = \lambda_3,\ \mathrm{non}(\mathcal{M}) = \lambda_4, \quad \textit{and } \mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda_5.$$

These consistency results correspond to λ_1–λ_6 of Figure 3, and to λ_1–λ_5 of Figure 4, respectively.

Remark 1.4. Note that the hypothesis for vB is weaker than the hypothesis in the original reference [KST19], even more, GCH is not assumed at all. This strengthening is a result of simple modifications, which are presented in [Mej19]. Moreover, note that $\mathfrak{d}$ can be singular in vA, while $\mathrm{cov}(\mathcal{M})$ can be singular in vB (also in the left-side models from [GMS16, GKS19]).

Both constructions can then be extended with Boolean ultrapowers (more precisely: compositions of finitely many successive Boolean ultrapowers), to make all values simultaneously different:

Theorem 1.5. *Assume $\aleph_1 < \lambda_1 < \lambda_2 < \lambda_3 \le \lambda_4 \le \lambda_5 \le \lambda_6 \le \lambda_7 \le \lambda_8 \le \lambda_9$.*

[BCM21] *If $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3$ such that*

- *(i) for $j = 7, 8, 9$, κ_j is strongly compact and $\lambda_j^{\kappa_j} = \lambda_j$,*
- *(ii) λ_i is regular for $i \neq 6$ and*
- *(iii) $\lambda_6^{<\lambda_3} = \lambda_6$,*

then there is a f.s. ccc iteration $P^{\mathtt{vA}*}$ (a Boolean ultrapower of $P^{\mathtt{vA}}$) that forces the constellation of Figure 3:

$$(\mathtt{vA}^*) \quad \mathrm{add}(\mathcal{N}) = \lambda_1,\ \mathrm{cov}(\mathcal{N}) = \lambda_2,\ \mathfrak{b} = \lambda_3,\ \mathrm{non}(\mathcal{M}) = \lambda_4, \quad \mathrm{cov}(\mathcal{M}) = \lambda_5,\ \mathfrak{d} = \lambda_6,\ \mathrm{non}(\mathcal{N}) = \lambda_7,\ \mathrm{cof}(\mathcal{N}) = \lambda_8,\ \textit{and } \mathfrak{c} = \lambda_9.$$

[KST19] *If $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4$ such that*

- *(i) for $j = 6, 7, 8, 9$, κ_j is strongly compact and $\lambda_j^{\kappa_j} = \lambda_j$,*
- *(ii) λ_i is regular for $i \neq 5$,*
- *(iii) $\lambda_2^{<\lambda_2} = \lambda_2$, $\lambda_4^{\aleph_0} = \lambda_4$, $\lambda_5^{<\lambda_4} = \lambda_5$, and*
- *(iv) λ_3 is $\aleph_1$-inaccessible,*

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[1] The result for the case $\lambda_2 = \lambda_3$ is easily obtained with techniques from Brendle [Bre91].

[2] A cardinal λ is *κ-inaccessible* if $\mu^\nu < \lambda$ for any $\mu < \lambda$ and $\nu < \kappa$.

then there is a f.s. ccc iteration $P^{\mathtt{vB}*}$ (a Boolean ultrapower of $P^{\mathtt{vB}}$) that forces the constellation of Figure 4:

$$(\mathtt{vB}^*)\quad \mathrm{add}(\mathcal{N})=\lambda_1,\ \mathfrak{b}=\lambda_2,\ \mathrm{cov}(\mathcal{N})=\lambda_3,\ \mathrm{non}(\mathcal{M})=\lambda_4,$$
$$\mathrm{cov}(\mathcal{M})=\lambda_5,\ \mathrm{non}(\mathcal{N})=\lambda_6,\ \mathfrak{d}=\lambda_7,\ \mathrm{cof}(\mathcal{N})=\lambda_8,\ \textit{and}\ \mathfrak{c}=\lambda_9.$$

More specifically: For $i=6,7,8,9$ let j_i be a complete embedding associated with some suitable Boolean ultrapower from the completion of $\mathrm{Coll}(\kappa_i,\lambda_i)$, which yields $\mathrm{cr}(j_i)=\kappa_i$ and $\mathrm{cof}(j_i(\kappa_i))=|j_i(\kappa_i)|=\lambda_i$ (see a bit more details at the end of Subsection 1.3). Then $P^{\mathtt{vA}*}=j_9(j_8(j_7(P^{\mathtt{vA}})))$ forces the constellation of Figure 3. Analogously, $P^{\mathtt{vB}*}=j_9(j_8(j_7(j_6(P^{\mathtt{vB}}))))$ forces the constellation of Figure 4.

Remark 1.6. In the original results of Theorem 1.5, all the inequalities are assumed to be strict (though in $\mathtt{vA}^*$ this is just from λ_6), but they can be equalities alternatively. Even more, whenever we change strict inequalities on the the right side to equalities, we may weaken the assumption by requiring fewer strongly compact cardinals. For example, if $\lambda_j=\lambda_{j+1}$ (for some $j=6,7,8,9$) then the compact cardinal κ_j is not required, furthermore, the weaker assumption $\lambda_{9-j}\le\lambda_{(9-j)+1}$ (for the dual cardinal characteristic, with $\lambda_0=\aleph_1$) is allowed in this case.

Moreover, $\mathfrak{d}$ is allowed to be singular in $\mathtt{vA}^*$, while $\mathrm{cov}(\mathcal{M})$ is allowed to be singular in $\mathtt{vB}^*$ (likewise in the construction from [GKS19]). See Remark 1.4.

Notation 1.7.
(1) Whenever we are investigating a characteristic $\mathfrak{x}$, we write $\lambda_{\mathfrak{x}}$ for the specific value we plan to force to it $\mathfrak{x}$. For example, for $\mathtt{vA}^*$ $\lambda_2=\lambda_{\mathrm{cov}(\mathcal{N})}$, whereas for $\mathtt{vB}^*$ $\lambda_2=\lambda_{\mathfrak{b}}$. We remark that we *do not* implicitly *assume* that $P\Vdash\mathfrak{x}=\lambda_{\mathfrak{x}}$ for the P under investigation; it is just an (implicit) declaration of intent.
(2) Whenever we base an argument on one of the old constructions above, and say "we can modify the construction to additionally force…", we implicitly assume that the desired values $\lambda_{\mathfrak{x}}$ for the "old" characteristics satisfy the assumptions we made in the "old" constructions (such as "$\lambda_{\mathfrak{x}}$ is regular").

See [GKMSa, Subsec. 2.3] for details on the history of the results of this section (and more).

1.3. Blass-uniform cardinal characteristics, LCU and COB.

A more detailed discussion on the concepts reviewed in this subsection can be found in [GKMSa, Subsec. 2.1].

Definition 1.8 ([GKMSa, Def. 2.1])**.** A *Blass-uniform cardinal characteristic* is a characteristic of the form

$$\mathfrak{d}_R:=\min\{|D|:D\subseteq\omega^\omega\text{ and }(\forall x\in\omega^\omega)\,(\exists y\in D)\ xRy\}$$

for some Borel[3] R.

Its dual cardinal

$$\mathfrak{b}_R:=\min\{|F|:F\subseteq\omega^\omega\text{ and }(\forall y\in\omega^\omega)\,(\exists x\in F)\ \neg xRy\}$$

is also Blass-uniform because $\mathfrak{b}_R=\mathfrak{d}_{R^\perp}$ where $xR^\perp y$ iff $\neg(yRx)$.

[3]More generally, it is just enough to assume that R is absolute between the extensions we consider.

In the practice, Blass-uniform cardinal characteristics are defined from a relation $R \subseteq X \times Y$ where X and Y are Polish spaces, but since we can translate such a relation to ω^ω using Borel isomorphisms, it is enough to discuss relations on ω^ω.

Systematic research on such cardinal characteristics started in the 1980s or possibly even earlier, see e.g. Fremlin [Fre84], Blass [Bla93, Bla10] and Vojtáš [Voj93].

Example 1.9. The following are pairs of dual Blass-uniform cardinals $(\mathfrak{b}_R, \mathfrak{d}_R)$ for natural Borel relations R:

(1) A cardinal on the left hand side of Cichoń's diagram and its dual on the right hand side: $(\mathrm{add}(\mathcal{N}), \mathrm{cof}(\mathcal{N}))$, $(\mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{N}))$, $(\mathrm{add}(\mathcal{M}), \mathrm{cof}(\mathcal{M}))$, $(\mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}))$, and $(\mathfrak{b}, \mathfrak{d})$.
(2) $(\mathfrak{s}, \mathfrak{r}) = (\mathfrak{b}_R, \mathfrak{d}_R)$ where $\mathfrak{s}$ is the splitting number, $\mathfrak{r}$ is the reaping number, and R is the relation on $[\omega]^{\aleph_0}$ defined by xRy iff "x does not split y".

Definition 1.10 ([GKMSa, Def. 2.3])**.** Fix a Borel relation R, λ a regular cardinal and μ an arbitrary cardinal. We define two properties:

Linearly cofinally unbounded: $\mathrm{LCU}_R(\lambda)$ means: There is a family $\bar f = (f_\alpha : \alpha < \lambda)$ of reals such that:

$$(\forall g \in \omega^\omega)\,(\exists \alpha \in \lambda)\,(\forall \beta \in \lambda \setminus \alpha)\;\neg f_\beta R g. \tag{1.11}$$

Cone of bounds: $\mathrm{COB}_R(\lambda, \mu)$ means: There is a $<\lambda$-directed partial order $\trianglelefteq$ on μ,[4] and a family $\bar g = (g_s : s \in \mu)$ of reals such that

$$(\forall f \in \omega^\omega)\,(\exists s \in \mu)\,(\forall t \trianglerighteq s)\; f R g_t. \tag{1.12}$$

Fact 1.13. $\mathrm{LCU}_R(\lambda)$ *implies* $\mathfrak{b}_R \le \lambda \le \mathfrak{d}_R$.
$\mathrm{COB}_R(\lambda, \mu)$ *implies* $\mathfrak{b}_R \ge \lambda$ *and* $\mathfrak{d}_R \le \mu$.

We often call the objects $\bar f$ in the definition of LCU and $(\trianglelefteq, \bar g)$ for COB "strong witnesses", and we say that the corresponding cardinal inequalities (or equalities) are "strongly witnessed". For example, "$(\mathfrak{b}, \mathfrak{d}) = (\lambda_\mathfrak{b}, \lambda_\mathfrak{d})$ is strongly witnessed" means: for the natural relation R (namely, the relation $\le^*$ of eventual dominance), we have $\mathrm{COB}_R(\lambda_\mathfrak{b}, \lambda_\mathfrak{d})$, $\mathrm{LCU}_R(\lambda_\mathfrak{b})$ and there is some regular $\lambda_0 \le \lambda_\mathfrak{d}$ such that $\mathrm{LCU}_R(\lambda)$ for all regular $\lambda \in [\lambda_0, \lambda_\mathfrak{d}]$ (this is to allow $\lambda_\mathfrak{d}$ to be singular as in `vA` and `vA`* of Theorems 1.3 and 1.5).

Remark 1.14. The old constructions ((`vA`), (`vB`) in Theorem 1.3) use that we can first force strong witnesses to the left hand side, and then preserve strong witnesses in Boolean ultrapowers, so that in the final model all Cichoń-characteristics are strongly witnessed. In more detail, for each dual pair $(\mathfrak{x}, \mathfrak{y})$ in Cichoń's diagram, there is a natural relation $R_\mathfrak{x}$ such that $(\mathfrak{x}, \mathfrak{y}) = (\mathfrak{b}_{R_\mathfrak{x}}, \mathfrak{d}_{R_\mathfrak{x}})$. We use these natural relations (with one exception[5]) as follows: The initial forcing (without Boolean

[4] I.e., every subset of μ of cardinality $<\lambda$ has a $\trianglelefteq$-upper bound

[5] The exception is the following: In `vA`, for the pair $(\mathfrak{x}, \mathfrak{y}) = (\mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}))$ it is forced $\mathrm{LCU}_{\neq^*}(\lambda_4)$, $\mathrm{LCU}_{\neq^*}(\lambda_5)$ and $\mathrm{COB}_{\neq^*}(\lambda_4, \lambda_5)$ (here $x \neq^* y$ iff $x(i) \neq y(i)$ for all but finitely many i); in `vB`, for $\mathfrak{x} = \mathrm{cov}(\mathcal{N})$, we use the natural relation $R_{\mathrm{cov}(\mathcal{N})}$ (defined as the set of all pairs (x, y) where the real y is in the F_σ set of full measure coded by x) only for COB. In this version, we do not know whether P forces $\mathrm{LCU}_{R_{\mathrm{cov}(\mathcal{N})}}(\lambda_{\mathrm{cov}(\mathcal{N})})$ (as we do not have sufficient preservation results for $R_{\mathrm{cov}(\mathcal{N})}$, more specifically, we do know whether (ρ, π)-linked posets are $R_{\mathrm{cov}(\mathcal{N})}$-good.) Instead, we use another relation R' (which defines different, anti-localization characteristics $(\mathfrak{b}_{R'}, \mathfrak{d}_{R'})$), for which ZFC proves $\mathrm{cov}(\mathcal{N}) \le \mathfrak{b}_{R'}$ and $\mathrm{non}(\mathcal{N}) \ge \mathfrak{d}_{R'}$. We can then show that P forces $\mathrm{LCU}_{R'}(\mu)$ for all regular $\lambda_{\mathrm{cov}(\mathcal{N})} \le \mu \le |\delta|$.

ultrapowers) is a f.s. iteration P of length δ and forces $\mathrm{LCU}_{R_{\mathfrak{x}}}(\mu)$ for all regular $\lambda_{\mathfrak{x}} \leq \mu \leq |\delta|$, and $\mathrm{COB}_{R_{\mathfrak{x}}}(\lambda_{\mathfrak{x}}, |\delta|)$.

Once we know that the initial forcing P gives strong witnesses for the desired values $\lambda_{\mathfrak{x}}$ for all "left-hand" values $\mathfrak{x}$ in Cichoń's diagram (and continuum for the cardinals $\geq \mathfrak{d}, \mathrm{non}(\mathcal{N})$ in `vA*` or $\geq \mathrm{cov}(\mathcal{M})$ in `vB*`), we use the following theorem to separate all the entries.

Theorem 1.15 ([KTT18, GKS19]). *Let $\nu < \kappa$ and $\lambda \neq \kappa$ be uncountable regular cardinals, R a Borel relation, and let P be a ν-cc poset forcing that λ is regular. Assume that $j : V \to M$ is an elementary embedding into a transitive class M satisfying:*

- *(i) The critical point of j is κ.*
- *(ii) M is $<\kappa$-closed.*[6]
- *(iii) For any cardinal $\theta > \kappa$ and any $<\theta$-directed partial order I, $j''I$ is cofinal in $j(I)$.*

Then:

- *(a) $j(P)$ is a ν-cc forcing.*
- *(b) If $P \Vdash \mathrm{LCU}_R(\lambda)$, then $j(P) \Vdash \mathrm{LCU}_R(\lambda)$.*
- *(c) If $\lambda < \kappa$ and $P \Vdash \mathrm{COB}_R(\lambda, \mu)$, then $j(P) \Vdash \mathrm{COB}_R(\lambda, |j(\mu)|)$.*
- *(d) If $\lambda > \kappa$ and $P \Vdash \mathrm{COB}_R(\lambda, \mu)$, then $j(P) \Vdash \mathrm{COB}_R(\lambda, \mu)$.*

Proof. We include the proof for completeness. Property (a) is immediate by (ii). First note that j satisfies the following additional properties.

- (iv) Whenever a is a set of size $<\kappa$, $j(a) = j''a$.
- (v) If $\mathrm{cof}(\alpha) \neq \kappa$ then $\mathrm{cof}(j(\alpha)) = \mathrm{cof}(\alpha)$.
- (vi) If $\theta > \kappa$, L is a set and $P \Vdash$"$(L, \dot{\trianglelefteq})$ is $<\theta$-directed" then $j(P) \Vdash$"$j''L$ is cofinal in $(j(L), j(\dot{\trianglelefteq}))$, and it is $<\theta$-directed".
- (vii) $j(P) \Vdash$"$\mathrm{cof}(j(\lambda)) = \lambda$".

Item (iv) follows from (i), and (v) follows from (iii). We show (vi). Let L^* be the set of nice P-names of members of L, and order it by $\dot{x} \leq \dot{y}$ iff $P \Vdash \dot{x} \dot{\trianglelefteq} \dot{y}$. It is clear that $\leq$ is $<\theta$-directed on L^*. On the other hand, since any nice $j(P)$-name of a member of $j(L)$ is already in M by (ii) and (a), $j(L^*)$ is equal to the set of nice $j(P)$-names of members of $j(L)$. Therefore, by (iii), $j''L^*$ is cofinal in $j(L^*)$. Note that $j''L^*$ is equal to the set of nice $j(P)$-names of members of $j''L$. Thus, (vi) follows.

For (vii), the case $\lambda < \kappa$ is immediate by (i) and (ii); when $\lambda > \kappa$, apply (vi) to $(L, \dot{\trianglelefteq}) = (\lambda, \leq)$ (the usual order) and $\theta = \lambda$.

To see (b), note that $M \models$"$j(P) \Vdash \mathrm{LCU}_R(j(\lambda))$" and, by (a) and (ii), the same holds inside V (because any nice name of an ordinal, represented by a maximal antichain on P, belongs to M, hence any nice name of a real, which in fact means that $j(P) \Vdash \mathrm{LCU}_R(\mathrm{cof}(j(\lambda)))$. By (vii) we are done.

Now assume $P \Vdash \mathrm{COB}_R(\lambda, \mu)$ witnessed by $(\dot{\trianglelefteq}, \dot{\bar{g}})$. This implies $M \models$"$j(P) \Vdash (j(\dot{\trianglelefteq}), j(\dot{\bar{g}}))$ witnesses $\mathrm{COB}_R(j(\lambda), j(\mu))$". If $\lambda < \kappa$ then $j(\lambda) = \lambda$ and it follows that $V \models$"$j(P) \Vdash \mathrm{COB}_R(\lambda, |j(\mu)|)$". In the case $\lambda > \kappa$ apply (vi) to conclude that $j(P)$ forces that $(j(\dot{\bar{g}}(\beta)) : \beta < \mu)$, with $j(\dot{\trianglelefteq})$ restricted to $j''\mu$, witnesses $\mathrm{COB}_R(\lambda, \mu)$. □

[6] I.e., $M^{<\kappa} \subseteq M$.

If κ is a strongly compact cardinal and $\theta^\kappa = \theta$, then there is an elementary embedding j associated with a Boolean ultrapower of the completion of $\mathrm{Coll}(\kappa, \theta)$ such that j satisfies (i)–(iii) of the preceding lemma and, in addition, for any cardinal $\lambda \geq \kappa$ such that either $\lambda \leq \theta$ or $\lambda^\kappa = \lambda$ holds, we have $\max\{\lambda, \theta\} \leq j(\lambda) < \max\{\lambda, \theta\}^+$ (see details in [KTT18, GKS19]). Therefore, using only this lemma, it is easy to see how to get from the old constructions (Theorem 1.3) to the Boolean ultrapowers (Theorem 1.5), as described in Remark 1.14 (see details in [BCM21, Thm. 5.7] for `vA`* and [KST19, Thm. 3.1] for `vB`*).

2. Cardinal characteristics in extensions without new $<\kappa$-sequences

This section summarizes the technical results introduced in [GKMSa].

Lemma 2.1 ([GKMSa, Lemma 3.1]). *Assume that Q is θ-cc and $<\kappa$-distributive for κ regular uncountable, and let λ be a regular cardinal and R a Borel relation.*

(1) If $\mathrm{LCU}_R(\lambda)$, then $Q \Vdash \mathrm{LCU}_R(\mathrm{cof}(\lambda))$.
So if additionally $\lambda \leq \kappa$ or $\theta \leq \lambda$, then $Q \Vdash \mathrm{LCU}_R(\lambda)$.

(2) If $\mathrm{COB}_R(\lambda, \mu)$ and either $\lambda \leq \kappa$ or $\theta \leq \lambda$, then $Q \Vdash \mathrm{COB}_R(\lambda, |\mu|)$.
So for any λ, $\mathrm{COB}_R(\lambda, \mu)$ implies $Q \Vdash \mathrm{COB}_R(\min(|\lambda|, \kappa), |\mu|)$.

Lemma 2.2 ([GKMSa, Lemma 3.2]). *Assume that R is a Borel relation, P' is a complete subforcing of P, λ regular and μ is a cardinal, both preserved in the P-extension.*

(a) If $P \Vdash \mathrm{LCU}_R(\lambda)$ witnessed by some $\dot{\bar{f}}$, and $\dot{\bar{f}}$ is actually a P'-name, then $P' \Vdash \mathrm{LCU}_R(\lambda)$.

(b) If $P \Vdash \mathrm{COB}_R(\lambda, \mu)$ witnessed by some $(\dot{\trianglelefteq}, \dot{\bar{g}})$, and $(\dot{\trianglelefteq}, \dot{\bar{g}})$ is actually a P'-name, then $P' \Vdash \mathrm{COB}_R(\lambda, |\mu|)$.

We now review three properties of cardinal characteristics.

Definition 2.3 ([GKMSa, Def. 3.3]). Let $\mathfrak{x}$ be a cardinal characteristic.

(1) $\mathfrak{x}$ is $\mathfrak{t}$-*like*, if it has the following form: There is a formula $\psi(x)$ (possibly with, e.g., real parameters) absolute between universe extensions that do not add reals,[7] such that $\mathfrak{x}$ is the smallest cardinality λ of a set A of reals such that $\psi(A)$.

All Blass-uniform characteristics are $\mathfrak{t}$-like; other examples are $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{u}$, $\mathfrak{a}$ and $\mathfrak{i}$.

(2) $\mathfrak{x}$ is called $\mathfrak{h}$-*like*, if it satisfies the same, but with A being a family of sets of reals (instead of just a set of reals).

Note that $\mathfrak{t}$-like implies $\mathfrak{h}$-like, as we can include "the family of sets of reals is a family of singletons" in ψ. Examples are $\mathfrak{h}$ and $\mathfrak{g}$.

(3) $\mathfrak{x}$ is called $\mathfrak{m}$-*like*, if it has the following form: There is a formula φ (possibly with, e.g., real parameters) such that $\mathfrak{x}$ is the smallest cardinality λ such that $H(\leq\lambda) \vDash \varphi$.

Any infinite $\mathfrak{t}$-like characteristic is $\mathfrak{m}$-like: If ψ witnesses $\mathfrak{t}$-like, then we can use $\varphi = (\exists A)\,[\psi(A) \& (\forall a \in A)\ a \text{ is a real}]$ to get $\mathfrak{m}$-like (since $H(\leq\lambda)$ contains all reals). Examples are[8] $\mathfrak{m}$, $\mathfrak{m}(\text{Knaster})$, etc.

[7] Concretely, if $M_1 \subseteq M_2$ are transitive (possibly class) models of a fixed, large fragment of ZFC, with the same reals, then ψ is absolute between M_1 and M_2.

[8] $\mathfrak{m}$ can be characterized as the smallest λ such that there is in $H(\leq\lambda)$ a ccc forcing Q and a family $\bar{D}$ of dense subsets of Q such that "there is no filter $F \subseteq Q$ meeting all D_i" holds.

Lemma 2.4 ([GKMSa, Lemma 3.4]). *Let $V_1 \subseteq V_2$ be models (possibly classes) of set theory (or a sufficient fragment), V_2 transitive and V_1 is either transitive or an elementary submodel of $H^{V_2}(\chi)$ for some large enough regular χ, such that $V_1 \cap \omega^\omega = V_2 \cap \omega^\omega$.*

(a) *If $\mathfrak{x}$ is $\mathfrak{h}$-like, then $V_1 \vDash \mathfrak{x} = \lambda$ implies $V_2 \vDash \mathfrak{x} \leq |\lambda|$.*

In addition, whenever κ is uncountable regular in V_1 and $V_1^{<\kappa} \cap V_2 \subseteq V_1$:

(b) *If $\mathfrak{x}$ is $\mathfrak{m}$-like, then $V_1 \vDash \mathfrak{x} \geq \kappa$ iff $V_2 \vDash \mathfrak{x} \geq \kappa$.*
(c) *If $\mathfrak{x}$ is $\mathfrak{m}$-like and $\lambda < \kappa$, then $V_1 \vDash \mathfrak{x} = \lambda$ iff $V_2 \vDash \mathfrak{x} = \lambda$.*
(d) *If $\mathfrak{x}$ is $\mathfrak{t}$-like and $\lambda = \kappa$, then $V_1 \vDash \mathfrak{x} = \lambda$ implies $V_2 \vDash \mathfrak{x} = \lambda$.*

We apply this to three situations: Boolean ultrapowers, extensions by distributive forcings, and complete subforcings:

Corollary 2.5 ([GKMSa, Cor. 3.5]). *Assume that κ is uncountable regular, $P \Vdash \mathfrak{x} = \lambda$, and*

(i) *<u>either</u> Q is a P-name for a $<\kappa$-distributive forcing, and we set $P^+ := P * Q$ and $j(\lambda) := \lambda$;*
(ii) *<u>or</u> P is ν-cc for some $\nu < \kappa$, $j : V \to M$ is a complete embedding into a transitive $<\kappa$-closed model M, $\mathrm{cr}(j) \geq \kappa$, and we set $P^+ := j(P)$,*
(iii) *<u>or</u> P is κ-cc, $M \preceq H(\chi)$ is $<\kappa$-closed, and we set $P^+ := P \cap M$ and $j(\lambda) := |\lambda \cap M|$. (So P^+ is a complete subposet of P; and if $\lambda \leq \kappa$ then $j(\lambda) = \lambda$.)*

Then we get:

(a) *If $\mathfrak{x}$ is $\mathfrak{m}$-like and $\lambda \geq \kappa$, then $P^+ \Vdash \mathfrak{x} \geq \kappa$.*
(b) *If $\mathfrak{x}$ is $\mathfrak{m}$-like and $\lambda < \kappa$, then $P^+ \Vdash \mathfrak{x} = \lambda$.*
(c) *If $\mathfrak{x}$ is $\mathfrak{h}$-like then $P^+ \Vdash \mathfrak{x} \leq |j(\lambda)|$. Concretely,*
for (i): $P^+ \Vdash \mathfrak{x} \leq |\lambda|$;
for (ii): $P^+ \Vdash \mathfrak{x} \leq |j(\lambda)|$;
for (iii): $P^+ \Vdash \mathfrak{x} \leq |\lambda \cap M|$.
(d) *So if $\mathfrak{x}$ is $\mathfrak{t}$-like and $\lambda = \kappa$, then for (i) and (iii) we get $P^+ \vDash \mathfrak{x} = \kappa$.*

2.1. **On the role of large cardinals in our construction.** It is known that NNR extensions will preserve Blass-uniform characteristics in the absence of at least some large cardinals. More specifically:

Lemma 2.6. *Assume that $0^\#$ does not exist. Let $V_1 \subseteq V_2$ be transitive class models with the same reals, and assume $V_1 \models \mathfrak{x} = \lambda$ for some Blass-uniform $\mathfrak{x}$. Then $V_2 \models \mathfrak{x} = |\lambda|$.*

(This is inspired by the deeper observation of Mildenberger [Mil98, Prop. 2.1], who uses the covering lemma [DJ82] for the Dodd-Jensen core model to show that in *cardinality preserving* NNR extensions, a measurable in an inner model is required to change the value of a Blass-uniform characteristic.)

Proof. Fix a bijection in V_1 between the reals and some ordinal α. Assume that in V_2, $X \subseteq \omega^\omega$ witnesses that $\aleph_1 \leq \mathfrak{x} \leq \mu < |\lambda|$. Using the bijection, we can interpret X as a subset of α. According to Jensen's covering lemma in V_2, there is in L (and thus in V_1) some $X' \supseteq X$ such that $|X'| = |X|$ in V_2, in particular $|X'|^{V_2} < \lambda$. Therefore, $|X'|^{V_1} < \lambda$ as well; and, by absoluteness, V_1 thinks that X' witnesses $\mathfrak{x} < \lambda$, a contradiction. $\square$

Recall the "old" Boolean ultrapower constructions vA*: Assume that we start with a forcing notion P forcing $\mathfrak{d} = 2^{\aleph_0} = \lambda_6$. We now use the elementary embedding $j = j_7 : V \to M$ with critical point κ_7, and set $P' := j(P)$. As we have seen, P' still forces $\mathfrak{d} = \lambda_6$, but $2^{\aleph_0} = \lambda_7 = |j(\kappa_7)|$.

So let G be a P'-generic filter over V (which is also M-generic). Set $V_1 := M[G]$ and $V_2 := V[G]$. Then V_1 is a $<\kappa$-complete submodel of V_2. By elementaricity, $M \models j(P) \Vdash \mathfrak{d} = j(\lambda_6)$. So $V_1 \models \mathfrak{d} = j(\lambda_6)$, whereas $V_2 \models \mathfrak{d} = \lambda_6 < |j(\lambda_6)|$.

Hence, for this specific constellation of models, some large cardinals (at least $0^{\#}$) are required (for our construction we actually use strongly compact cardinals).

3. Applications

For notation simplicity, we declare that "1-Knaster" means "ccc", and "ω-Knaster" means "precaliber $\aleph_1$". Corollary 2.5 gives us 11 characteristics:

Lemma 3.1. *Given $\aleph_1 \leq \lambda_{\mathfrak{m}} < \kappa_9$ regular and $1 \leq k_0 \leq \omega$, we can modify $P^{\mathtt{vA}*}$ (and also $P^{\mathtt{vB}*}$) so that we additionally force:*

(1) $\mathfrak{m}(k\text{-Knaster}) = \aleph_1$ *for* $1 \leq k < k_0$,
(2) $\mathfrak{m}(k\text{-Knaster}) = \lambda_{\mathfrak{m}}$ *for* $k \geq k_0$,
(3) $\mathfrak{p} \geq \kappa_9$.

Proof. As in [GKMSa, Sect. 4 & 5], we can modify $P^{\mathtt{vA}}$ to construct a ccc poset P' forcing the same as $P^{\mathtt{vA}}$ and, in addition, $\mathfrak{p} = \mathfrak{b}$, and both (1) and (2). Apply Boolean ultrapowers to P' as in the "old" construction, resulting in P^*. We can apply Corollary 2.5(ii), more specifically the consequences (a) and (b): (b) implies that P^* forces (1) and (2), while (a) implies that P^* forces $\mathfrak{p} \geq \kappa_9$. And just as in the "old" construction, we can use Theorem 1.15 to show that P^* forces the desired values to the Cichoń-characteristics. □

The following lemma is useful to modify $\mathfrak{g}$ and $\mathfrak{c}$ via complete subposets, while preserving $\mathfrak{m}$-like and Blass-uniform values from the original poset.

Lemma 3.2 ([GKMSa, Lemma 6.3])**.** *Assume the following:*

(1) $\aleph_1 \leq \kappa \leq \nu \leq \mu$, *where* κ *and* ν *are regular and* $\mu = \mu^{<\kappa} \geq \nu$,
(2) P *is a* κ*-cc poset forcing* $\mathfrak{c} > \mu$.
(3) For some Borel relations R^1_i $(i \in I_1)$ *on* ω^ω *and some regular* $\lambda^1_i \leq \mu$: P *forces* $\mathrm{LCU}_{R^1_i}(\lambda^1_i)$
(4) For some Borel relations R^2_i $(i \in I_2)$ *on* ω^ω, $\lambda^2_i \leq \mu$ *regular and a cardinal* $\vartheta^2_i \leq \mu$: P *forces* $\mathrm{COB}_{R^2_i}(\lambda^2_i, \vartheta^2_i)$.
(5) For some $\mathfrak{m}$*-like characteristics* $\mathfrak{y}_j$ $(j \in J)$ *and* $\lambda_j < \kappa$: $P \Vdash \mathfrak{y}_j = \lambda_j$.
(6) For some $\mathfrak{m}$*-like characteristics* $\mathfrak{y}'_k$ $(k \in K)$: $P \Vdash \mathfrak{y}'_k \geq \kappa$.
(7) $|I_1 \cup I_2 \cup J \cup K| \leq \mu$.

Then there is a complete subforcing P' of P of size μ forcing

(a) $\mathfrak{y}_j = \lambda_j$, $\mathfrak{y}'_k \geq \kappa$, $\mathrm{LCU}_{R^1_i}(\lambda^1_i)$ *and* $\mathrm{COB}_{R^2_{i'}}(\lambda^2_{i'}, \vartheta^2_{i'})$ *for all* $i \in I_1$, $i' \in I_2$, $j \in J$ *and* $k \in K$;
(b) $\mathfrak{c} = \mu$ *and* $\mathfrak{g} \leq \nu$.

Remark 3.3. So we can preserve $\mathrm{COB}_R(\lambda, \theta)$ provided both λ and θ are $\leq \mu$.

For larger λ or θ this is generally not possible. E.g., if $\lambda > \mu$, then $\mathrm{COB}_R(\lambda, \theta)$ will fail in the P'-extension as it implies $\mathfrak{b}_R \geq \lambda > \mu = \mathfrak{c}$; in fact, according to [GKMSb, Lemma 1.6], we actually get $P' \Vdash \mathrm{COB}_R(\nu,\nu)$. However we do get the following (the proof is straightforward):

If we assume, in addition to the conditions of Lemma 3.2, that $\mu^{<\nu} = \mu$ and $P \Vdash \mathrm{COB}_R(\lambda,\theta)$ for some Borel relation R and $\lambda \le \nu$ (now allowing also $\theta > \mu$), then we can construct P' such that $P' \Vdash \mathrm{COB}_R(\lambda,\mu)$. (But this only gives us $\mathfrak{d}_R \le \mu = \mathfrak{c}$.)

In the case of $\nu < \lambda$, we have $P \Vdash \mathrm{COB}_R(\nu,\theta)$, so as we have just seen we can get $P' \Vdash \mathrm{COB}_R(\nu,\min(\mu,\theta))$ (which implies $\mathfrak{b}_R \ge \nu$, which is a bit better than the $\mathfrak{b}_R \ge \kappa$ we get from (6)).

The following two results deal with $\mathfrak{p}$.

Lemma 3.4 ([GKMSa, Lemma 7.2]). *Assume $\xi^{<\xi} = \xi$, P is ξ-cc, and set $Q = \xi^{<\xi}$ (ordered by extension). Then P forces that Q^V preserves all cardinals and cofinalities. Assume $P \Vdash \mathfrak{x} = \lambda$ (in particular that λ is a cardinal), and let R be a Borel relation.*

- *(a) If $\mathfrak{x}$ is $\mathfrak{m}$-like: $\lambda < \xi$ implies $P \times Q \Vdash \mathfrak{x} = \lambda$; $\lambda \ge \xi$ implies $P \times Q \Vdash \mathfrak{x} \ge \xi$.*
- *(b) If $\mathfrak{x}$ is $\mathfrak{h}$-like: $P \times Q \Vdash \mathfrak{x} \le \lambda$.*
- *(c) $P \Vdash \mathrm{LCU}_R(\lambda)$ implies $P \times Q \Vdash \mathrm{LCU}_R(\lambda)$.*
- *(d) $P \Vdash \mathrm{COB}_R(\lambda,\mu)$ implies $P \times Q \Vdash \mathrm{COB}_R(\lambda,\mu)$.*

Lemma 3.5 ([DS], [GKMSa, Lemma 7.3]). *Assume that $\xi = \xi^{<\xi}$ and P is a ξ-cc poset that forces $\xi \le \mathfrak{p}$. In the P-extension V', let $Q = (\xi^{<\xi})^V$. Then,*

- *(a) $P \times Q = P * Q$ forces $\mathfrak{p} = \xi$*
- *(b) If in addition P forces $\xi \le \mathfrak{p} = \mathfrak{h} = \kappa$ then $P \times Q$ forces $\mathfrak{h} = \kappa$.*

We are now ready to prove the consistency of 13 pairwise different classical characteristics. Note that the following result allows $\mathfrak{d}$ and $\mathfrak{c}$ singular. A similar result with $\mathrm{cov}(\mathcal{M})$ and $\mathfrak{c}$ singular can be obtained if we base the initial construction in [GKS19] instead of [BCM21].

Theorem 3.6. *Assume $\aleph_1 \le \lambda_{\mathfrak{m}} \le \lambda_{\mathfrak{p}} \le \lambda_{\mathfrak{h}} \le \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 \le \lambda_4 \le \lambda_5 \le \lambda_6 \le \lambda_7 \le \lambda_8 \le \mu$ such that*

- *(i) For $j = 7, 8, 9$, κ_i is strongly compact,*
- *(ii) $\lambda_j^{\kappa_j} = \lambda_j$ for $j = 7, 8$,*
- *(iii) λ_i is regular for $i \ne 6$*
- *(iv) $\lambda_{\mathfrak{p}}^{<\lambda_{\mathfrak{p}}} = \lambda_{\mathfrak{p}}$*
- *(v) $\lambda_6^{<\lambda_3} = \lambda_6$,*
- *(vi) $\mu^{<\lambda_{\mathfrak{h}}} = \mu$.*

Then there is a $\lambda_{\mathfrak{p}}^+$-cc poset P which preserves cofinalities and forces (1) and (2) of Lemma 3.1, and

$$\mathfrak{p} = \lambda_{\mathfrak{p}},\ \mathfrak{h} = \mathfrak{g} = \lambda_{\mathfrak{h}},\ \mathrm{add}(\mathcal{N}) = \lambda_1,\ \mathrm{cov}(\mathcal{N}) = \lambda_2,\ \mathfrak{b} = \lambda_3,\ \mathrm{non}(\mathcal{M}) = \lambda_4,$$
$$\mathrm{cov}(\mathcal{M}) = \lambda_5,\ \mathfrak{d} = \lambda_6,\ \mathrm{non}(\mathcal{N}) = \lambda_7,\ \mathrm{cof}(\mathcal{N}) = \lambda_8,\ \textit{and}\ \mathfrak{c} = \mu.$$

Proof. Let P^* be the ccc poset obtained in the proof of Lemma 3.1 for $\lambda_9 := (\mu^{\kappa_9})^+$ (the modification of $P^{\mathtt{vA}*}$). This is a ccc poset of size λ_9 that forces the values of the Cichoń-characteristics as in Theorem 1.5 ($\mathtt{vA}$*) with strong witnesses, and forces (1) and (2) of Lemma 3.1 and $\mathfrak{p} \ge \kappa_9$ whenever $\lambda_{\mathfrak{m}} < \kappa_9$, but in the case $\lambda_{\mathfrak{m}} = \kappa_9$ it forces $\mathfrak{m}(k_0\text{-Knaster}) \ge \kappa_9$ instead of (2).

By application of Lemma 3.2 to $\kappa = \nu = \lambda_{\mathfrak{h}}$ and to μ, we find a complete subposet P' of P^* forcing (1) and (2) of Lemma 3.1, $\lambda_{\mathfrak{h}} \leq \mathfrak{p} \leq \mathfrak{g} \leq \lambda_{\mathfrak{h}}$ (so they are equalities), $\mathfrak{c} = \mu$ and that the values of the other cardinals in Cichoń's diagram are the same values forced by P^*, even with strong witnesses. This is clear in the case $\lambda_{\mathfrak{m}} < \lambda_{\mathfrak{h}}$, but the case $\lambda_{\mathfrak{m}} = \lambda_{\mathfrak{h}}$ (even $\lambda_{\mathfrak{m}} = \kappa_9$) is also fine because P' would force $\lambda_{\mathfrak{m}} \leq \mathfrak{m}(k_0\text{-Knaster}) \leq \mathfrak{m}(\text{precaliber}) \leq \mathfrak{p} \leq \mathfrak{g} \leq \lambda_{\mathfrak{m}}$.

If $\lambda_{\mathfrak{p}} = \lambda_{\mathfrak{h}}$ then we would be done, so assume that $\lambda_{\mathfrak{p}} < \lambda_{\mathfrak{h}}$. Hence, by Lemmas 3.4 and 3.5, $P := P' \times (\lambda_{\mathfrak{p}}^{<\lambda_{\mathfrak{p}}})$ is as required. It is clear that P forces $\mathfrak{m}(k_0\text{-Knaster}) = \mathfrak{m}(\text{precaliber}) = \lambda_{\mathfrak{m}}$ when $\lambda_{\mathfrak{m}} < \lambda_{\mathfrak{p}}$, but the same happens when $\lambda_{\mathfrak{m}} = \lambda_{\mathfrak{p}}$ because P would force $\lambda_{\mathfrak{m}} \leq \mathfrak{m}(k_0\text{-Knaster}) \leq \mathfrak{m}(\text{precaliber}) \leq \mathfrak{p} \leq \lambda_{\mathfrak{m}}$. □

The same argument works to get a similar version of the previous result for `vB`* where $\mathrm{cov}(\mathcal{M})$ and $\mathfrak{c}$ are allowed to be singular.

Theorem 3.7. *Assume* $\aleph_1 \leq \lambda_{\mathfrak{m}} \leq \lambda_{\mathfrak{p}} \leq \lambda_{\mathfrak{h}} < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 \leq \lambda_5 \leq \lambda_6 \leq \lambda_7 \leq \lambda_8 \leq \mu$ *such that*

(i) for $j = 6, 7, 8, 9$, κ_j *is strongly compact,*
(ii) $\lambda_j^{\kappa_j} = \lambda_j$ *for* $j = 6, 7, 8$,
(iii) λ_i *is regular for* $i \neq 5$,
(iv) $\lambda_{\mathfrak{p}}^{<\lambda_{\mathfrak{p}}} = \lambda_{\mathfrak{p}}$
(v) $\lambda_2^{<\lambda_2} = \lambda_2$, $\lambda_4^{\aleph_0} = \lambda_4$, $\lambda_5^{<\lambda_4} = \lambda_5$,
(vi) λ_3 *is* $\aleph_1$*-inaccessible, and*
(vii) $\mu^{<\lambda_{\mathfrak{h}}} = \mu$.

Then there is a $\lambda_{\mathfrak{p}}^+$*-cc poset* P*, preserving cofinalities, that forces (1) and (2) of Lemma 3.1, and*

$$\mathfrak{p} = \lambda_{\mathfrak{p}},\ \mathfrak{h} = \mathfrak{g} = \lambda_{\mathfrak{h}},\ \mathrm{add}(\mathcal{N}) = \lambda_1,\ \mathfrak{b} = \lambda_2,\ \mathrm{cov}(\mathcal{N}) = \lambda_3,\ \mathrm{non}(\mathcal{M}) = \lambda_4,$$
$$\mathrm{cov}(\mathcal{M}) = \lambda_5,\ \mathrm{non}(\mathcal{N}) = \lambda_6,\ \mathfrak{d} = \lambda_7,\ \mathrm{cof}(\mathcal{N}) = \lambda_8,\ \textit{and } \mathfrak{c} = \mu.$$

In the previous proof we can preserve all characteristics only because, before applying Lemma 3.2, $\mathrm{cof}(\mathcal{N})$ (equal to λ_8) is smaller than the continuum (equal to λ_9). In particular, if we use version `vA` without large cardinals, and we cannot further increase the continuum above $\mathrm{cof}(\mathcal{N})$, then the methods of this section only ensure a model of (1) and (2) of Lemma 3.1 plus

$$\mathfrak{p} = \lambda_{\mathfrak{p}},\ \mathfrak{g} = \mathfrak{h} = \lambda_{\mathfrak{h}},$$
$$\min\{\lambda_1, \lambda_{\mathfrak{h}}\} \leq \mathrm{add}(\mathcal{N}) \leq \lambda_1,\ \min\{\lambda_2, \lambda_{\mathfrak{h}}\} \leq \mathrm{cov}(\mathcal{N}) \leq \lambda_2,\ \min\{\lambda_3, \lambda_{\mathfrak{h}}\} \leq \mathfrak{b} \leq \lambda_3,$$
$$\mathrm{non}(\mathcal{M}) = \lambda_4,\ \mathrm{cov}(\mathcal{M}) = \lambda_5,\ \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c} = \mu,$$

whenever $\aleph_1 \leq \lambda_{\mathfrak{m}} \leq \lambda_{\mathfrak{p}} = \lambda_{\mathfrak{p}}^{<\lambda_{\mathfrak{p}}} \leq \lambda_{\mathfrak{h}} \leq \lambda_3$ are regular, $\lambda_{\mathfrak{m}} \leq \lambda_1$ and $\lambda_5 \leq \mu = \mu^{<\lambda_3} < \lambda_6$, where λ_i (i=1,. . . ,6) are as in `vA` (see Remark 3.3). That is, some left side Cichoń-characteristics do not get decided unless $\lambda_{\mathfrak{x}} \leq \lambda_{\mathfrak{h}}$ Hence, it is unclear whether $\mathfrak{h}$ gets separated from all the left side characteristics. A similar situation occurs with version `vB`: we may lose $\mathfrak{x} \geq \lambda_{\mathfrak{x}}$ for any left side characteristic $\mathfrak{x}$ when $\lambda_{\mathfrak{h}} < \lambda_{\mathfrak{x}}$.

4. Reducing gaps (or getting rid of them)

We start with the following well-known result.

Lemma 4.1 (Easton's lemma)**.** *Let κ be an uncountable cardinal, P a κ-cc poset and let Q be a $<\kappa$-closed poset. Then P forces that Q is $<\kappa$-distributive.*

Proof. For successor cardinals, this is proved in [Jec03, Lemma 15.19], but the same argument is valid for any regular cardinal. Singular cardinals are also fine because, for κ singular, $<\kappa$-closed implies $<\kappa^+$-closed. □

As mentioned in Remark 1.6, we can choose right side Cichoń-characteristics rather arbitrarily or even choose them to be equal (equality allows a construction from fewer compact cardinals). However, large gaps were required between some left side cardinals. We deal with this problem now, and show that we can reasonably assign arbitrary values to all characteristics, and in particular set any "reasonable selection" of them equal.

Let us introduce notation to describe this effect:

Definition 4.2. Let $\bar{\mathfrak{x}} = (\mathfrak{x}_i : i < n)$ be a finite sequence of cardinal characteristics (i.e., of definitions). Say that $\bar{\mathfrak{x}}$ is a $<$-*consistent sequence* if the statement $\mathfrak{x}_0 < \dots < \mathfrak{x}_{n-1}$ is consistent with ZFC (perhaps modulo large cardinals).

A consistent sequence $\bar{\mathfrak{x}}$ is $\leq$-*consistent* if, in the previous chain of inequalities, it is consistent to replace any desired instance of $<$ with $=$. More formally, for any interval partition $(I_k : k < m)$ of $\{0, \dots, n-1\}$, it is consistent that $\mathfrak{x}_i = \mathfrak{x}_j$ for any $i, j \in I_k$, and $\mathfrak{x}_i < \mathfrak{x}_j$ whenever $i \in I_k$, $j \in I_{k'}$ and $k < k' < m$.

For example, the sequence

$$(\aleph_1, \mathrm{add}(\mathcal{N}), \mathrm{cov}(\mathcal{M}), \mathfrak{b}, \mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}), \mathfrak{d})$$

is $\leq$-consistent, as well as

$$(\aleph_1, \mathrm{add}(\mathcal{N}), \mathfrak{b}, \mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M})),$$

see Theorem 1.3. Previously, it had not been known whether the sequences of ten Cichoń-characteristics from [GKS19, BCM21, KST19] are $\leq$-consistent: It is not immediate that cardinals on the left side can be equal while separating everything on the right side. The reason is that, to separate cardinals on the right side, it is necessary to have a strongly compact cardinal between the dual pair of cardinals on the left, thus the left side gets separated as well. But thanks to the collapsing method of this section, we can equalize cardinals on the left as well. As a result, we obtain the following:[9]

Lemma 4.3. *The sequences*

$$(\aleph_1, \mathfrak{m}, \mathfrak{p}, \mathrm{add}(\mathcal{N}), \mathrm{cov}(\mathcal{M}), \mathfrak{b}, \mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}), \mathfrak{d}, \mathrm{non}(\mathcal{N}), \mathrm{cof}(\mathcal{N}), \mathfrak{c})\,\textit{and}$$
$$(\aleph_1, \mathfrak{m}, \mathfrak{p}, \mathrm{add}(\mathcal{N}), \mathfrak{b}, \mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}), \mathrm{non}(\mathcal{N}), \mathfrak{d}, \mathrm{cof}(\mathcal{N}), \mathfrak{c})$$

are $\leq$-consistent (modulo large cardinals).

(Note that we lose $\mathfrak{h}$ in the process.)

To prove this claim, we use the following:

Assumption 4.4. (1) κ is regular uncountable.
(2) $\theta \geq \kappa$, $\theta = \theta^{<\kappa}$.
(3) P is κ-cc and forces that $\mathfrak{x} = \lambda$ for some characteristic $\mathfrak{x}$ (so in particular λ is a cardinal in the P-extension).

[9]Each sequence yields 2^{11} many consistency results (not all of them new, obviously; CH is one of them).

(4) Q is $<\kappa$-closed.
(5) $P \Vdash Q$ is θ^+-cc.[10]
(6) We set $P^+ := P \times Q = P * Q$. We call the P^+-extension V'' and the intermediate P-extension V'.

(We will actually have $|Q| = \theta$, which implies (5)).
Let us list a few simple facts:

(P1) In V', all V-cardinals $\geq\kappa$ are still cardinals, and Q is a $<\kappa$-distributive forcing (due to Easton's lemma). So we can apply Lemma 2.1 and Corollary 2.5.
(P2) Let μ be the successor (in V or equivalently in V') of θ. So in V', Q is μ-cc and preserves all cardinals $\leq \kappa$ as well as all cardinals $\geq \mu$.
(P3) So if $V \models$"$\kappa \leq \nu \leq \theta$", then in V'', $\kappa \leq |\nu| < \mu$. The V'' successor of κ is $\leq \mu$.

We now apply it to a collapse:

Lemma 4.5. *Let R be a Borel relation, κ be regular, $\theta > \kappa$, $\theta^{<\kappa} = \theta$, P κ-cc, and set $Q :=$ Coll(κ, θ), i.e., the set of partial functions $f : \kappa \to \theta$ of size $< \kappa$. Then:*

(a) $P \times Q$ forces $|\theta| = \kappa$.
(b) If P forces that λ is a cardinal then

$$P \times Q \Vdash |\lambda| = \begin{cases} \kappa & \text{if (in } V\text{) } \kappa \leq \lambda \leq \theta \\ \lambda & \text{otherwise.} \end{cases}$$

(c) If $\mathfrak{x}$ is $\mathfrak{m}$-like, $\lambda < \kappa$ and $P \Vdash \mathfrak{x} = \lambda$, then $P \times Q \Vdash \mathfrak{x} = \lambda$.
(d) If $\mathfrak{x}$ is $\mathfrak{m}$-like and $P \Vdash \mathfrak{x} \geq \kappa$, then $P \times Q \Vdash \mathfrak{x} \geq \kappa$.
(e) If R is a Borel relation then
  (i) $P \Vdash$ "λ regular and $\mathrm{LCU}_R(\lambda)$" implies $P \times Q \Vdash \mathrm{LCU}_R(|\lambda|)$.
  (ii) $P \Vdash$ "λ is regular and $\mathrm{COB}_R(\lambda, \mu)$" implies $P \times Q \Vdash \mathrm{COB}_R(|\lambda|, |\mu|)$.

Proof. As mentioned, Assumption 4.4 is met; in particular, P forces that $\check{Q}$ is $<\kappa$-distributive (by 4.4(P2)), so we can use Lemma 2.1 and Corollary 2.5. Also note that, whenever $\kappa < \lambda \leq \theta$ and $P \Vdash$"λ is regular", $P \times Q$ forces $\mathrm{cof}(\lambda) = \kappa = |\lambda|$. □

So we can start, e.g., with a forcing P_0 as in Theorem 3.6: P_0 is $\lambda_{\mathfrak{p}}^+$-cc, and forces strictly increasing values to the characteristics in the first, say, sequence of Lemma 4.3.

We now pick some $\kappa_0 < \theta_0$, satisfying $\lambda_{\mathfrak{p}} < \kappa_0$ and the assumptions of the previous Lemma, i.e., κ_0 is regular and $\theta_0^{<\kappa_0} = \theta_0$. Let Q_0 be the collapse of θ_0 to κ_0, a forcing of size θ_0. So $P_1 := P_0 \times Q_0$ is θ_0^+-cc and, according to the previous Lemma, still forces the "same" values (and in fact strong witnesses) to the Cichoń-characteristics (including the case that any value λ_i with $\kappa_0 < \lambda_i \leq \theta_0$ is collapsed to $|\lambda_i| = \kappa_0$). The $\mathfrak{m}$-like invariants below κ_0, i.e., $\mathfrak{m}$ and $\mathfrak{p}$, are also unchanged.

We now pick another pair $\theta_0 < \kappa_1 < \theta_1$ (with the same requirements) and take the product of P_1 with the collapse Q_1 of θ_1 to κ_1, etc.

In the end, we get $P_0 \times Q_0 \times \cdots \times Q_n$. Each characteristic which by P was forced to have value λ now is forced to have value $|\lambda|$, which is κ_m if $\kappa_m \leq \lambda \leq \theta_m$ for some m, and λ otherwise. This immediately gives the

[10] I.e., P forces that all antichains of Q have size $\leq\theta$.

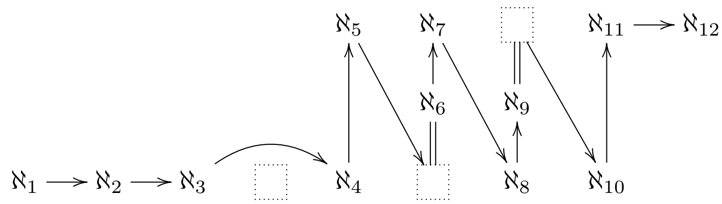

FIGURE 5. A possible assignment for Figure 2 (note that we lose control of $\mathfrak{h}$): $\mathfrak{m} = \aleph_2$, $\mathfrak{p} = \aleph_3$, $\lambda_i = \aleph_{3+i}$ for $i = 1, \ldots, 9$.

Proof of Lemma 4.3. We start with GCH, and construct the inital forcing to already result in the desired (in)equalities between $\aleph_1, \mathfrak{m}, \mathfrak{p}$ and to result in pairwise different regular Cichoń values λ_i and $\mathfrak{p} < \mathrm{add}(\mathcal{N})$.

Let $(I_m)_{m \in M}$ be the interval partition of the sequence $(\mathfrak{p}, \mathrm{add}(\mathcal{N}), \ldots, \mathfrak{c})$ indicating which characteristics we want to identify. For each non-singleton I_m, let κ_m be the value of the smallest characteristic in I_m, and θ_m the largest. Note that $\theta_m < \kappa_{m+1} < \theta_{m+1}$. Then $P_0 \times Q_0 \times \cdots \times Q_{M-1}$ forces that all characteristics in I_m have value κ_m, as desired. □

Similarly and easily we get the following:

Lemma 4.6. *We can assign the values $\aleph_1, \aleph_2, \ldots, \aleph_{12}$ to the first sequence of Lemma 4.3 (as in Figure 5).*

We can do the same for the second sequence.

Proof. Again, start with GCH and P_0 forcing the desired values for $\mathfrak{m}$ and $\mathfrak{p}$ (now $\aleph_2$ and $\aleph_3$) and pairwise distinct regular Cichoń values λ_i. Then pick $\kappa_0 = \lambda_{\mathfrak{p}}^+ = \aleph_4$ and $\theta_0 = \lambda_1$ (which then becomes $\aleph_4$ after the collapse). Then set $\kappa_1 = \lambda_1^+$ (which would be $\aleph_5$ after the first collapse), and $\theta_1 = \lambda_2$, etc. □

We can of course just as well assign the values $(\aleph_{\omega \cdot m+1})_{1 \le m \le 12}$ instead of $(\aleph_m)_{1 \le m \le 12}$. It is a bit awkward to make precise the (not entirely correct) claim "we can assign whatever we want"; nevertheless we will do just that in the rest of this section.

Theorem 4.7. *Assume GCH. Let $1 \le k_0 \le \omega$, let $1 \le \alpha_{\mathfrak{m}} \le \alpha_{\mathfrak{p}} \le \alpha_1 \le \cdots \le \alpha_9$ be a sequence of successor ordinals, and $\kappa_9 < \kappa_8 < \kappa_7$ compact cardinals with $\kappa_9 > \alpha_9$. Then there is a poset P which forces (1) and (2) of Lemma 3.1 for $\lambda_{\mathfrak{m}} = \aleph_{\alpha_{\mathfrak{m}}}$ and, in addition,*

$$\mathfrak{p} = \aleph_{\alpha_{\mathfrak{p}}},\ \mathrm{add}(\mathcal{N}) = \aleph_{\alpha_1},\ \mathrm{cov}(\mathcal{N}) = \aleph_{\alpha_2},\ \mathfrak{b} = \aleph_{\alpha_3},\ \mathrm{non}(\mathcal{M}) = \aleph_{\alpha_4},$$
$$\mathrm{cov}(\mathcal{M}) = \aleph_{\alpha_5},\ \mathfrak{d} = \aleph_{\alpha_6},\ \mathrm{non}(\mathcal{N}) = \aleph_{\alpha_7},\ \mathrm{cof}(\mathcal{N}) = \aleph_{\alpha_8},\ \textit{and } \mathfrak{c} = \aleph_{\alpha_9}.$$

Actually, we will prove something more general: We first formulate this more general result for the case $\alpha_1 < \alpha_2 < \alpha_3$; as explained in Remark 4.9, there are variants of the theorem which allow $\alpha_1 = \alpha_2$ and/or $\alpha_2 = \alpha_3$.

Theorem 4.8. *Assume GCH and $1 \le k_0 \le \omega$. Let $1 \le \alpha_{\mathfrak{m}} \le \alpha_{\mathfrak{p}} \le \alpha_1 < \alpha_2 < \alpha_3 \le \alpha_4 \le \ldots \le \alpha_9$ be ordinals and assume that there are strongly compact cardinals $\kappa_9 < \kappa_8 < \kappa_7$ such that*

(i) $\alpha_{\mathfrak{p}} \le \kappa_9$, $\alpha_1 < \kappa_8$ and $\alpha_2 < \kappa_7$;

(ii) *for* $i = 1, 2, 3$, $\aleph_{\beta_{i-1}+(\alpha_i-\alpha_{i-1})}$ *is regular,*[11] *where* $\beta_i := \max\{\alpha_i, \kappa_{10-i}+1\}$ *and* $\alpha_0 = \beta_0 = 0$;

(iii) *for* $i \geq 4$, $i \neq 6$, $\aleph_{\beta_3+(\alpha_i-\alpha_3)}$ *is regular;*

(iv) $\mathrm{cof}(\aleph_{\beta_3+(\alpha_6-\alpha_3)}) \geq \aleph_{\beta_3}$*; and*

(v) $\aleph_{\alpha_\mathfrak{m}}$ *and* $\aleph_{\alpha_\mathfrak{p}}$ *are regular.*

Then we get a poset P *as in the previous Theorem.*

Proof. For $4 \leq i \leq 9$ put $\beta_i := \beta_3 + (\alpha_i - \alpha_3)$. Also set $\lambda_\mathfrak{m} := \aleph_{\alpha_\mathfrak{m}}$, $\lambda_\mathfrak{p} := \aleph_{\alpha_\mathfrak{p}}$ and $\lambda_i := \aleph_{\beta_i}$ for $1 \leq i \leq 9$. Note that λ_i is regular for $i \neq 6$, $\mathrm{cof}(\lambda_6) \geq \lambda_3$ and $\lambda_\mathfrak{m} \leq \lambda_\mathfrak{p} \leq \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 \leq \lambda_4 \leq \ldots \leq \lambda_9$. In the case $\alpha_\mathfrak{p} < \alpha_1$ let P be the $\lambda_\mathfrak{p}^+$-cc poset corresponding to Theorem 3.6 (the modification of $P^{\mathtt{vA}*}$), otherwise let P be the ccc poset corresponding to Lemma 3.1 and forcing $\mathfrak{p} \geq \kappa_9$ and $\mathfrak{m}(k_0\text{-Knaster}) = \mathfrak{m}(\text{precaliber}) = \lambda_\mathfrak{m}$ (or just $\mathfrak{m}(k_0\text{-Knaster}) \geq \kappa_9$ when $\alpha_\mathfrak{m} = \kappa_9$).[12]

Step 1. We first assume $\alpha_\mathfrak{p} < \alpha_1$. In the case $\kappa_9 < \alpha_1$ we have $\beta_1 = \alpha_1$, so let $P_1 := P$; in the case $\alpha_1 \leq \kappa_9$, we have $\beta_1 = \kappa_9 + 1$ and $\lambda_1 = \kappa_9^+$. Put $\kappa_1 := \aleph_{\alpha_1}$ and $P_1 := P \times \mathrm{Coll}(\kappa_1, \lambda_1)$. It is clear that κ_1 is regular and $\kappa_1 \leq \lambda_1$ so, by Lemma 4.5, P_1 forces $\mathrm{add}(\mathcal{N}) = \aleph_{\alpha_1}$ and that the values of the other cardinals are the same as in the P-extension. Even more, for any $\xi \geq \kappa_8$, P_1 forces $\aleph_\xi = \aleph_\xi^V$ because, in the ground model, κ_8 is an $\aleph$-fixed point between $\aleph_{\beta_1}$ and $\aleph_{\beta_2}$ (and thus between β_1 and β_2).

Now assume $\alpha_\mathfrak{p} = \alpha_1$ (so P is ccc) and let $\kappa_1 := \lambda_\mathfrak{p} = \aleph_{\alpha_1}$. Since $\alpha_1 = \alpha_\mathfrak{p} \leq \kappa_9$, we have $\lambda_1 = \kappa_9^+$, so we set $P_1 := P \times \mathrm{Coll}(\kappa_1, \lambda_1 \times \kappa_1)$. This poset forces the same as the above, but for $\mathfrak{p}$ we just now $\mathfrak{p} \geq \kappa_1$ (or just $\mathfrak{m}(k_0\text{-Knaster}) \geq \kappa_9$ when $\alpha_p = \kappa_9$), but $\mathfrak{p} \leq \kappa_1$ also holds because $\mathrm{Coll}(\kappa_1, \lambda_1 \times \kappa_1)$ adds a $\kappa_1^{<\kappa_1}$-generic function (see the proof of Lemma 3.5).

Step 2. In the case $\kappa_8 < \alpha_2$ put $P_2 := P_1$; otherwise, we have $\beta_2 = \kappa_8 + 1$ and $\lambda_2 = \kappa_8^+$. Set $\kappa_2 := \aleph_{\beta_1+(\alpha_2-\alpha_1)}$ and $P_2 := P_1 \times \mathrm{Coll}(\kappa_2, \lambda_2)$. It is clear that $\kappa_2 < \lambda_2$, so Lemma 4.5 applies, i.e., P_2 forces $\mathrm{cov}(\mathcal{N}) = \kappa_2$ and that the values of the other cardinals are the same as in the P_1-extension. Also note that P_1 forces $\kappa_2 = \aleph_{\alpha_2}$, and this value remains unaltered in the P_2-extension. Furthermore P_2 forces $\aleph_\xi = \aleph_\xi^V$ for any $\xi \geq \kappa_7$.

Step 3. In the case $\kappa_7 < \alpha_3$ put $P_3 := P_2$; otherwise, set $\kappa_3 := \aleph_{\beta_2+(\alpha_3-\alpha_2)}$ and $P_3 := P_2 \times \mathrm{Coll}(\kappa_3, \lambda_3)$. Note that P_3 forces $\mathfrak{b} = \kappa_3 = \aleph_{\alpha_3}$ and that the other values are the same as forced by P_2. Hence, P_3 is as desired, e.g., $\mathrm{non}(\mathcal{M}) = \lambda_4 = \aleph_{\beta_4}^V = \aleph_{\alpha_4}$. □

Remark 4.9. Theorem 4.8 also holds when $\alpha_1 \leq \alpha_2 \leq \alpha_3$, but depending on the equalities the hypothesis may change. For example, in the case $\alpha_1 = \alpha_2 < \alpha_3$, hypothesis (ii) is modified by: $\beta_1 = \kappa_9 + 1$, $\beta_2 = \kappa_8 + 1$, $\beta_3 = \max\{\alpha_3, \kappa_7 + 1\}$ and both $\aleph_{\alpha_1}$ and $\kappa_3 := \aleph_{\beta_2+(\alpha_3-\alpha_2)}$ are regular. For the proof, the idea is first collapse $\lambda_2 := \aleph_{\beta_2}$ to $\kappa_1 := \aleph_{\alpha_1}$ (as in step 1 of the proof, considering similar cases for $\alpha_\mathfrak{p}$), and then (possibly) collapse $\lambda_3 := \aleph_{\beta_3}$ to κ_3 (as in step 3). This guarantees that the sequence of cardinals in the previous theorem is $\leq$-consistent.

A similar result (and remark about $\leq$-consistency) applies to $\mathtt{vB}*$.

[11] This is equivalent to say that α_i is either a successor ordinal or a weakly inaccessible larger than β_{i-1}.

[12] This distinction is necessary: the forcing P from Theorem 3.6 is not $\lambda_\mathfrak{p}$-cc, so we would not be able to apply Lemma 4.5 to $P \times \mathrm{Coll}(\lambda_\mathfrak{p}, \kappa_9^+)$.

Theorem 4.10. *Assume GCH and* $1 \leq k_0 \leq \omega$. *Let* $1 \leq \alpha_{\mathfrak{m}} \leq \alpha_{\mathfrak{p}} \leq \alpha_1 < \alpha_2 < \alpha_3 \leq \alpha_4 \leq \ldots \leq \alpha_9$ *be ordinals and assume that there are strongly compact cardinals* $\kappa_9 < \kappa_8 < \kappa_7 < \kappa_6$ *such that*

(i) $\alpha_{\mathfrak{p}} \leq \kappa_9$, $\alpha_1 < \kappa_8$, $\alpha_2 < \kappa_7$, *and* $\alpha_3 < \kappa_6$;
(ii) *for* $i = 1, 2, 3, 4$, $\aleph_{\beta_{i-1}+(\alpha_i-\alpha_{i-1})}$ *is regular, where* $\beta_i := \max\{\alpha_i, \kappa_{10-i}+1\}$ *and* $\alpha_0 = \beta_0 = 0$;
(iii) *for* $i \geq 6$, $\aleph_{\beta_4+(\alpha_i-\alpha_4)}$ *is regular;*
(iv) $\mathrm{cof}(\aleph_{\beta_4+(\alpha_5-\alpha_4)}) \geq \aleph_{\beta_4}$;
(v) β_3 *is* <u>*not*</u> *the successor of a cardinal with countable cofinality; and*
(vi) $\aleph_{\alpha_{\mathfrak{m}}}$ *and* $\aleph_{\alpha_{\mathfrak{p}}}$ *are regular.*

Then there is a poset that forces (1) and (2) of Lemma 3.1 for $\lambda_{\mathfrak{m}} = \aleph_{\alpha_{\mathfrak{m}}}$ *and*

$$\mathfrak{p} = \aleph_{\alpha_{\mathfrak{p}}},\ \mathrm{add}(\mathcal{N}) = \aleph_{\alpha_1},\ \mathfrak{b} = \aleph_{\alpha_2},\ \mathrm{cov}(\mathcal{N}) = \aleph_{\alpha_3},\ \mathrm{non}(\mathcal{M}) = \aleph_{\alpha_4},$$
$$\mathrm{cov}(\mathcal{M}) = \aleph_{\alpha_5},\ \mathrm{non}(\mathcal{N}) = \aleph_{\alpha_6},\ \mathfrak{d} = \aleph_{\alpha_7},\ \mathrm{cof}(\mathcal{N}) = \aleph_{\alpha_8},\ \textit{and}\ \mathfrak{c} = \aleph_{\alpha_9}.$$

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Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstrasse 8-10/104, 1040 Vienna, Austria.
Email address: goldstern@tuwien.ac.at
URL: http://www.tuwien.ac.at/goldstern/

Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstrasse 8-10/104, 1040 Vienna, Austria.
Email address: kellner@fsmat.at
URL: http://dmg.tuwien.ac.at/kellner/

Creative Science Course (Mathematics), Faculty of Science, Shizuoka University, Ohya 836, Suruga-ku, Shizuoka-shi, Japan 422-8529.
Email address: diego.mejia@shizuoka.ac.jp
URL: http://www.researchgate.com/profile/Diego_Mejia2

Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, and Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA.
Email address: shlhetal@math.huji.ac.il
URL: http://shelah.logic.at