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The Hanf number in the strictly stable case

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We associate *Hanf numbers* $\mathrm{H}(\mathbf{t})$ to triples $\mathbf{t} = (T, T_1, p)$ where T and T_1 are theories and p is a type. We show that the Hanf number for the property: "there is a model M_1 of T_1 which omits p, but $M_1 \restriction \tau$ is saturated" is larger than the Hanf number of $\mathcal{L}_{\lambda^+,\kappa}$ but smaller than the Hanf number of $\mathcal{L}_{(2^\lambda)^+,\kappa}$ when T is stable with $\kappa = \kappa(T)$. In fact, surprisingly, we even characterise the Hanf number of $\mathbf{t}$ when we fix (T, λ) where T is a first order complete (and stable), $\lambda \geq |T|$ and demand $|T_1| \leq \lambda$.

1 Introduction

1.1 Background on results

This paper continues a series of papers by Baldwin and the present author [1, 2]. For each cardinal λ and theory T, we shall define a *Hanf number* $\mathrm{H}(\lambda, T)$ in § 2; if Φ is any property of theories, we can then define the λ-Hanf number of Φ as the supremum of $\mathrm{H}(\lambda, T)$ where T has property Φ.[1] Fixing a theory T, the Hanf number $\mathrm{H}(\lambda, T)$ can be seen as a measure of complexity of T. In [4], Newelski essentially asks what $\mathrm{H}(\lambda) := \sup\{\mathrm{H}(\lambda, T) : T$ is a theory$\}$ is. The mentioned predecessor papers [1, 2] gave partial answers to Newelski's question: in [1], Baldwin and the present author showed that the Hanf number for unstable theories is essentially equal to the Löwenheim number of second order logic; in [2], they showed that the Hanf number for superstable theories is bigger than the Hanf number of $\mathcal{L}_{(2^\lambda)^+,\aleph_0}$ but smaller than $\mathcal{L}_{\beth_2(\lambda)^+,\aleph_0}$. This fits with the description of the Hanf number as a measure of complexity: for unstable theories, it is very large, for superstable theories, it is quite small. Our original objective for this paper was to sort out the case of T *strictly stable*, i.e., stable but not superstable, which falls in the middle. We are trying to sort out when the Hanf numbers are manageable (e.g., for $\mathcal{L}_{\lambda^+,\aleph_0}$ where it is $\beth_\delta$ for $\delta < (2^\lambda)^+$) and when it is not (e.g., it is above a compact cardinal).

However, we ask a stronger question:

Question 1.1 Fix a complete first order theory T and a cardinal $\lambda \geq |T|$. Determine $\mathrm{H}(\lambda, T)$.

Clearly, this is a considerably more ambitious question. The paper [1] actually determines $\mathrm{H}(\lambda, T)$ when T is unstable, so we shall concentrate here on the case T is stable. For this case, we give a quite complete answer. For T strictly stable, our original case, we only need the following data to determine $\mathrm{H}(\lambda, T)$: the cardinals $|T|$ and $\kappa(T)$, a derived Boolean algebra $\mathbb{B}(T)$ of cardinality $|D(T)|$ where $D(T) = \bigcup\{D_n(T) : n < \omega\}$ and $D_n(T)$ is the set of complete n-types realised in models of T, and the truth value of "$2^{\aleph_0} > |D(T)| > |T|$ and T is unstable in $|D(T)|$ and T is superstable".

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[1] If T is unstable it is natural to assume that $\{\mu : \mu = \mu^{<\mu}\}$ is an unbounded class as otherwise for any T_1, λ we have $\mathrm{H}(T, T_1, \lambda) \leq \sup\{\mu^+ : \mu = \mu^{<\mu}\}$.

1.2 Description of the proof

In this argument, the infinitary logic $\mathcal{L}_{\lambda^+,\kappa}$ is central. A major point is to deal abstractly with what is essentially the Boolean algebra $\mathbb{B}_T$ of formulas over the empty set modulo T. In Definition 2.2, we introduce the logics $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}_T]$ and the union of these logics over the relevant $\mathbb{B}$s is called $\mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}$. (Moreover, $\mathcal{L}^{\mathrm{ba}}_{\lambda,\kappa}$ is equivalent to $\mathcal{L}_{\lambda,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda]$; cf. § 1.3 for definitions.) Then in Observation 2.4(4), we note that

$$\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}) \le \mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]) \le \mathrm{H}(\mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}) \le \mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\kappa}).$$

The main result shows that there is an exact equivalence between the classes $\mathbf{N}_{\lambda,T}$ of triples (T, T_1, p) and classes of the form Mod_ψ, $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ where $\mathbb{B}$ is the Boolean algebra of formulas over the empty set modulo T. Note that Grossberg and Vasey prove a generalization of the superstable case to a.e.c. by coding that does not use Boolean algebras [3, Theorem 4.8].

We now describe the proof of the main result, concentrating on the case of a stable theory T with $\kappa(T) = \kappa > \aleph_0$. First, recalling the characterization of saturated models (cf. § 1.3), it is natural to use $\mathcal{L}_{\lambda^+,\kappa}$ as the relevant logic for a triple $\mathbf{t} = (T, T_1, p)$. I.e., if $M_2 \in \mathrm{Mod}_\mathbf{t}$ and $\lambda > 2^{|T|}$, then there is a sentence $\psi_1 \in \mathcal{L}_{\lambda,\kappa}(\tau_T)$ saying M is κ-saturated. Thus, to be saturated it is enough to have that every countable infinite indiscernible set can be extended to one of cardinality $\|M\|$. Restricting ourselves to models of cardinality $\mu = \mu^{\aleph_0}$, there is ψ_2 in $\mathcal{L}_{|T|^+,\aleph_1}(\tau_T \cup \{F, F_n : n < \omega\})$ where F and F_n are unary function symbols.

This extension of τ_{T_1} is fine for our problem and describes how starting with a triple $\mathbf{t} = (T, T_1, p)$, we find $\psi \in \mathcal{L}_{\lambda^+,\kappa}(\tau_2)$ such that for $\mu = \mu^{\aleph_0} \ge \lambda$, there is $M \in \mathrm{Mod}_\mathbf{t}$ of cardinality μ if and only if there is $N \in \mathrm{Mod}_\psi$ of cardinality μ.

We need also translation in the other direction, i.e., given $\psi \in \mathcal{L}_{\lambda^+,\kappa}(\tau_\psi)$, we need to find a suitable theory T. Here $\mu = \mu^{<\kappa}$ is helpful: we can replace ψ by $\mathbf{t} = (T, T_1, p)$ such that T_1 codes enough set theory, and the main point is that in every model M_1 of $T_{1,\mathbf{t}}$, there are relation and function symbols pretending to code the set of sequences ${}^{<\kappa}M_1$. The main point is to understand why every such sequence is coded; this is done using the fact that $M_1 \restriction \tau_T$ is κ-saturated and $\kappa = \kappa(T)$.

This leaves the question: why is it enough to concentrate on cardinals $\mu = \mu^{<\kappa} > 2^\lambda$? We get the same Hanf numbers when we restrict ourselves to those cardinals, i.e., considering only the classes $\mathrm{spec}^2_\mathbf{t} \backslash 2^\lambda$. So when $\lambda \ge 2^{|T|}$, $\kappa > \aleph_0$ we get an accurate description of the family of classes $\mathrm{spec}^2_\mathbf{t}$ for $\mathbf{t} \in \mathbf{N}_{\lambda,T}$.

Note that allowing $\kappa = \aleph_0$ requires non-essential changes. However, the most natural case is $\lambda = |T| = |T_1|$, so allowing $\lambda < 2^{|T|}$; e.g., T says P_α (for $\alpha < \lambda$) are independent unary predicates. This requires the use of the Boolean algebra $\mathbb{B}_T$.

1.3 Preliminaries

In this section, we fix a complete first order theory T and define the relevant parameters (e.g., $\kappa(T)$ and $\mathbb{B}_T$) and recall the characterization of the existence of saturated models.

If τ is any vocabulary, then $\mathcal{L}(\tau)$ is the first order language in that vocabulary. If T is any first order theory, we write $\tau_T = \tau(T)$ for the vocabulary of T and $\mathcal{L}_{\tau(T)}$ for the language of T. Similarly, If M is any first order structure, we write $\tau_M = \tau(M)$ for the vocabulary of M. We write $|M|$ for the universe of M and $\|M\|$ for its cardinality. For a theory T let Mod_T be the class of models of T.

We introduce the following notation for sequences: $\bar{x}_{[u]} = \langle x_i : i \in u\rangle$; similarly, $\bar{y}_{[u]}$; e.g., $\bar{x}_{[\alpha]} = \langle x_i : i < \alpha\rangle$. As usual, if $\lambda \ge \kappa$, then $\mathcal{L}_{\lambda,\kappa}$ is the infinitary logic with transfinite connectives of size $< \lambda$ and transfinite quantifiers of size $< \kappa$; if τ is a vocabulary, we write $\mathcal{L}_{\lambda,\kappa}(\tau)$ for the corresponding language. This language is the closure of the set of atomic formulas under negation, conjunctions of the form $\bigwedge_{\alpha<\gamma} \varphi_\alpha$ (for $\gamma < \lambda$) and quantifications of the form $(\exists \bar{x}_{[u]})\varphi$ where $u \in [\kappa]^{<\kappa}$ (really just $(\exists \bar{x}_{[\varepsilon]})\varphi$ for $\varepsilon < \kappa$ suffices), but every formula has $< \kappa$ free variables.

In the following, T will always denote a complete and stable first order theory (if not said otherwise). Note that other notations (such as T_1) do not imply that the theory is necessarily complete or stable.

Let M be a model of T, $A \subseteq M$, and $\bar{a} \in {}^nM$. We write $\mathrm{tp}(\bar{a}, A, M) = \{\varphi(\bar{x}_{[n]}, \bar{b}) : \varphi(\bar{x}, \bar{y}) \in \mathcal{L}(\tau_M)$ and $\bar{b} \in {}^{\mathrm{lh}(\bar{y})}M$ and $M \models \varphi[\bar{a}, \bar{b}]\}$. Let $\mathrm{S}^n(A, M)$ be the set of complete n-types over A in M, i.e., $\{\mathrm{tp}(\bar{a}, A, N) : M \prec N$ and $\bar{a} \in {}^nN\}$; if $n = 1$ then we may omit n and we write $\mathrm{S}^n(M) = \mathrm{S}^n(|M|, M)$. We let $D_m(T) = \{\mathrm{tp}(\bar{a}, \varnothing, M) : \bar{a} \in {}^mM$ and $M \models T\}$ and $D(T) = \bigcup_m D_m(T)$. We recall that T is *stable in* λ or λ-*stable* when for every model M of

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T and $A \subseteq M$ of cardinality $\leq \lambda$ the set $\mathrm{S}(A, M)$ has cardinality $\leq \lambda$; T is *superstable* if and only if T is λ-stable for every sufficiently large λ.

We define $\kappa(T)$ to be the minimal κ such that if $A \subseteq M_* \in \mathrm{Mod}_T$ and $p \in \mathbf{S}(A, M)$ then there is $B \subseteq A$ of cardinality $< \kappa$ such that p does not fork over B (cf. [5, Chapter III]) and $\kappa_{\mathrm{reg}}(T) = \min\{\kappa : \kappa \text{ regular} \geq \kappa(T)\}$; thus, $\kappa_{\mathrm{reg}}(T)$ is the minimal regular κ such that T is stable in λ whenever $\lambda = \lambda^{<\kappa} + 2^{|T|}$; cf. [5, Chapter III].

We define $\lambda(T)$ to be the minimal λ such that T is stable in λ, i.e., if $M \models T$, then $\|M\| \leq |T| + \aleph_0$ implies that $|\mathbf{S}(M)| \leq \lambda$]; cf. [5, §§ III.5 & III.6].

By [5, § III.5 & § III.6], we have that $\kappa(T) \leq |T|^+$, $\lambda(T) = |D(T)|^{<\kappa(T)}$ except when $|D(T)| < 2^{\aleph_0}$, T is superstable and unstable in $|T|$. In this case $|D(T)| < 2^{\aleph_0} = \lambda(T)$ and $\lambda(T) = |D(T)|^{<\kappa(T)}$; cf. Fact 1.6. The point is that by [5, Chapter III]:

Fact 1.2 *Let T be a complete first order stable theory and let $\lambda \geq \aleph_1 + |T|$ be an infinite cardinal. Then T has a saturated model of cardinality λ if and only if T is λ-stable if and only if $\lambda = \lambda^{<\kappa(T)} + \lambda(T)$.*

If $\mathbb{B}$ is a Boolean algebra, we let $\mathrm{uf}(\mathbb{B})$ be the set of ultrafilters of $\mathbb{B}$.

Observation 1.3 *For every Boolean algebra $\mathbb{B}_1$ of cardinality $\leq \lambda$ and $\kappa \leq \lambda^+$ there is a Boolean algebra $\mathbb{B}_2$ of cardinality λ such that $|\mathrm{uf}(\mathbb{B}_2)| = \sum\{|\mathrm{uf}(\mathbb{B}_1)|^{\vartheta} : \vartheta < \kappa\} + \lambda$.*

Proof. Without loss of generality $|\mathbb{B}_1| = \lambda$, as otherwise we replace $\mathbb{B}$, by $\mathbb{B}_1 \oplus \mathbb{B}^0_\lambda$ where $\mathbb{B}^0_\lambda$ is the Boolean algebra of finite and co-finite subsets of λ. If $|\mathbb{B}_1| = \lambda$, $\kappa = \vartheta^+$, and $\vartheta \leq \lambda$, we define the Boolean algebra $\mathbb{B}_2$ as the free product of ϑ copies of $\mathbb{B}_1$.

If κ is a limit cardinal $\leq \lambda$ and $|\mathbb{B}_1| = \lambda$ let $\mathbb{B}_{2,\vartheta}$ be as above for $\vartheta < \kappa$ and $\mathbb{B}_2$ the disjoint sum of $\langle \mathbb{B}_{2,\vartheta} : \vartheta < \kappa \rangle$ so essentially except for one ultrafilter, all ultrafilters on $\mathbb{B}_2$ are ultrafilters on some $\mathbb{B}_{2,\vartheta}$ so $|\mathrm{uf}(\mathbb{B}_2)| = 1 + \sum_{\vartheta<\kappa} |\mathrm{uf}(\mathbb{B}_{2,\vartheta})|$. □

We are now going to define the relevant Boolean algebras: Fix a model M. For a formula $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}(\tau_M)$ and $\bar{a} \in {}^{\mathrm{lh}(\bar{y})}M$ let $\varphi(M, \bar{a}) = \{\bar{b} \in {}^{\mathrm{lh}(\bar{x})}M : M \models \varphi[\bar{b}, \bar{a}]\}$. (This includes the special case of $\mathrm{lh}(\bar{y}) = 0$; in this case, we write $\varphi(M) = \{\bar{b} \in {}^{\mathrm{lh}(\bar{x})}M : M \models \varphi[\bar{b}]\}$.) Now we let $\mathbb{B}_{M,m}$ be the Boolean algebra of subsets of ${}^mM$ consisting of the sets $\{\varphi(M) : \varphi = \varphi(\bar{x}_{[m]})\}$. Let $\bar{\mathbb{B}}_M = \langle \mathbb{B}_{M,m} : m < \omega \rangle$; *par abus de langage*, let $\mathrm{uf}(\bar{\mathbb{B}}_M) = \bigcup_m \mathrm{uf}(\mathbb{B}_{M,m})$. Let $\mathbb{B}_M$ be the direct sum of $\bar{\mathbb{B}}_M$ so $\langle 1_{\mathbb{B}_{M,m}} : m < \omega \rangle$ be a maximal antichain of $\mathbb{B}_M$, $\mathbb{B}_M{\restriction}\{x \in \mathbb{B}_M : x \leq 1_{\mathbb{B}_{M,m}}\} = \mathbb{B}_{M,m}$, and $\bigcup\{\mathbb{B}_{M,m} : m < \omega\}$ generates $\mathbb{B}_M$. There is a unique ultrafilter disjoint from $\{1_{\mathbb{B}_{M,n}} : n < \omega\}$; we write $\text{tr-ufil}(\mathbb{B}_M)$ for it (for "trivial ultrafilter") and $\mathrm{uf}^-(\mathbb{B}_M) = \mathrm{uf}(\mathbb{B}_M) \backslash \{\text{tr-ufil}(\mathbb{B}_M)\}$.[2] We write $\lambda'(M)$ for the cardinality of $\mathrm{uf}(\mathbb{B}_M)$.

If T is a theory, then $\mathbb{B}_{T,m}$ is the Boolean algebra of the formulas $\varphi(\bar{x}_{[m]}) \in \mathcal{L}(\tau_T)$ modulo equivalence over T, so $\varphi_1(\bar{x}_{[m]}) \leq \varphi_2(\bar{x}_{[m]})$ if and only if $T \vdash \varphi_1(\bar{x}_{[m]}) \to \varphi_2(\bar{x}_{[m]})$; i.e., the elements are actually $\equiv_T$-equivalence classes of formulas. We introduce the notation for $\bar{\mathbb{B}}_T$ and $\mathrm{uf}(\bar{\mathbb{B}}_T)$ analogous to that of $\bar{\mathbb{B}}_M$ and $\mathrm{uf}(\bar{\mathbb{B}}_M)$. We let $\lambda'(T) := \lambda'(M)$ when $M \models T$.

We let $\mathbb{B}^{\mathrm{fr}}_\lambda$ be the Boolean algebra generated freely by $\{\mathbf{a}_\alpha : \alpha < \lambda\}$ so $\mathrm{uf}(\mathbb{B}^{\mathrm{fr}}_\lambda)$ has cardinality 2^λ.

Let $\mathrm{EQ}_T = \{\varphi(\bar{x}_{[n]}, \bar{y}_{[n]}) : n < \omega, \varphi \in \mathcal{L}(\tau_T)$ and for every model M of T, $\{(\bar{a}, \bar{b}) : \bar{a}, \bar{b} \in M$ and $M \models \varphi[\bar{a}, \bar{b}]\}$ is an equivalence relation on ${}^nM$ with finitely many equivalence classes$\}$. A model M is $\aleph_\varepsilon$-saturated when for every triple (b, A, N) satisfying $A \subseteq M \prec N$, $b \in N$, A finite, there is $b' \in M$ realizing the type $\{\varphi(x, b; \bar{a}) : \bar{a} \subseteq A, \varphi(x, y, \bar{a})$ is an equivalence relation with finitely many equivalence classes in $M\}$, this type is called $\mathrm{stp}(b, A, N)$; cf. [5, Chapter III].

We might be interested in the Boolean algebra of formulas which are *almost* over $\varnothing$, i.e., $\varphi(\bar{x}_m, \bar{a})$, $\bar{a} \in {}^{\mathrm{lh}(\bar{y})}M$ where $\varphi(\bar{x}_m, \bar{y}) \in \mathcal{L}(\tau_T)$ satisfies $\varphi(\bar{x}_m, \bar{y})$ such that for some $\vartheta(\bar{x}_m, \bar{y}_m) \in \mathrm{EQ}^m_M$, we have

$$M \models (\forall \bar{z})(\forall \bar{x}_m, \bar{y}_m)[\vartheta(\bar{x}_m, \bar{y}_m) \to (\varphi(\bar{x}_m, \bar{z}) \equiv \varphi_n(\bar{y}_m, \bar{z})].$$

However, this is not necessary here.

Observation 1.4 *The Boolean algebra $\mathbb{B}_{M,m}$ essentially depends just on* $\mathrm{Th}(M)$*, i.e., if $T = \mathrm{Th}(M)$ then $\mathbb{B}_{M,m}$ is isomorphic to $\mathbb{B}_{T,m}$ where an isomorphism j is defined as follows: $\varphi(\bar{x}_{[m]}) \in \mathcal{L}(\tau_T)$ implies that $j(\varphi(M)) = \varphi(\bar{x}_{[m]})/{\equiv_T}$, so $\lambda'(T)$ is well defined. (Similarly for other notions defined above.) Furthermore, the sets* $\mathrm{uf}^-(\mathbb{B}_M)$

[2] The point is that we should like to say that the set of ultrafilters of $\mathbb{B}_M$ is the union of the set of ultrafilters of $\mathbb{B}_{M,m}$ for $m < \omega$. However, one ultrafilter, viz. the trivial ultrafilter, does not fit; cf. Observation 1.4.

and $\mathrm{uf}(\mathbb{B}_M)$ *have the same cardinality, in fact, there is a natural one-to-one mapping* π *from* $\mathrm{uf}(\bar{\mathbb{B}}_M)$ *onto* $\mathrm{uf}^-(\mathbb{B}_M)$ *such that if* $D \in \mathrm{uf}(\mathbb{B}_{M,m})$*, then* $\pi(D) = \{a \in \mathbb{B}_{M,m} : a \cap 1_{\mathbb{B}_{M,m}} \in D\}$.

Fact 1.5 (Shelah; [5, Lemma III.3.10]) *Let* T *be a complete and stable first order theory with* $\kappa = \kappa(T)$ *and let* M *be an uncountable model of* T.

Case 1: $\kappa > \aleph_0$. The model M is saturated if and only if

(1a) if $I \subseteq M$ is an infinite indiscernible set then there is an indiscernible set $J \subseteq M$ extending I of cardinality $\|M\|$ (without loss of generality, T is countable) and

(1b) M is κ-saturated.

Case 2: $\kappa = \aleph_0$. The model M is saturated if and only if

(2a) if $A \subseteq M$ is finite and $a \in M\backslash\mathrm{acl}(A)$ then there is an indiscernible set J over A in M based on A such that $a \in J$ and J is of cardinality $\|M\|$ and

(2b) M is $\aleph_\varepsilon$-saturated.

Fact 1.6 (Shelah; [5, II.5.9, II.5.10, & II.5.11]) *Let* T *be a complete and stable first order theory. If* $\kappa(T) > \aleph_0$ *then* $\lambda(T) = |D(T)|^{<\kappa_{\mathrm{reg}}(T)}$. *If* $\kappa(T) = \aleph_0$ *then* $\lambda(T) = |D(T)|$ *or* $\lambda(T) = 2^{\aleph_0} + |D(T)|$. *In the latter case, there must be some finite* $A \subseteq M$, $M \in \mathrm{Mod}_T$, *such that the set* $\{\mathrm{stp}(a, A) : a \in M\}$ *has cardinality continuum.*

Definition 1.7 Let ϑ be a cardinal and $\tau^{\mathrm{eq}}_\vartheta = \{E_i : i < \vartheta\}$ where each E_i a two-place predicate. We define $T^{\mathrm{eq},0}_\vartheta$ be the universal theory included in $\mathcal{L}(\tau^{\mathrm{eq}}_\vartheta)$ such that for a $\tau^{\mathrm{eq}}_\vartheta$-model M, $M \models T^{\mathrm{eq},0}_\vartheta$ if and only if E^M_i is an equivalence relation and E^M_j refines E^M_i for $i < j < \vartheta$. Finally, let $T^{\mathrm{eq}}_\vartheta$ be the model completion of $T^{\mathrm{eq},0}_\vartheta$.

Claim 1.8 (Basic properties of non-forking; [5, Chapter III]) *A model* $M_\delta = \bigcup_{i<\delta} M_i$ *is* λ*-saturated if* $\langle M_i : i < \delta\rangle$ *is a* $\prec$*-increasing sequence of models of* T, T *is stable,* $\kappa(T) \leq \mathrm{cf}(\delta)$, *and each* M_i *is* λ*-saturated.*

2 The frame

In this section, we define all of the key notions for this paper, including the logics $\mathcal{L}_{\lambda,\kappa}[\mathbb{B}]$ via which we shall characterise Hanf numbers and look at the relations among such logics (cf. Definition 2.2 & Claims 2.6 & 2.7). We then deal with representations.

Let T be a complete first order theory such that $\lambda \geq |T|$. We let $\mathbf{N}_{\lambda,T}$ be the class of triples $\mathbf{t} = (T, T_1, p) = (T_{\mathbf{t}}, T_{1,\mathbf{t}}, p_{\mathbf{t}})$ such that $T_{\mathbf{t}} = T$, $T_1 \supseteq T$ is a first order theory, $|\tau(T_1)| \leq \lambda$, and $p(x)$ is an $\mathcal{L}(\tau_{T_1})$-type (not necessarily complete).

For $\mathbf{t} \in \mathbf{N}_{\lambda,T}$, we say $M_1 \models \mathbf{t}$ or $M_1 \in \mathrm{Mod}_{\mathbf{t}}$ or M_1 *is a model of* $\mathbf{t}$ if

(a) $M_1 \models T_{1,\mathbf{t}}$ and M_1 a τ_{T_1}-model;

(b) M_1 omits the type $p_{\mathbf{t}}(x)$; and

(c) $M_1{\restriction}\tau_T$ is saturated.

Let $\mathrm{Mod}^0_{\mathbf{t}} = \{M \in \mathrm{Mod}_{\mathbf{t}} : \|M\| \geq |T_{1,\mathbf{t}}| + \aleph_1\}$. Let $\mathbf{N} = \{\mathbf{t} : \mathbf{t} \in \mathbf{N}_{\lambda,T}$ for some λ and $T\}$. For $\lambda \geq \mu$ let $\mathbf{N}_{\lambda,\mu} = \{\mathbf{t} \in \mathbf{N} : |T_{\mathbf{t}}| \leq \mu$ and $|T_{1,\mathbf{t}}| \leq \lambda\}$ and $\mathbf{N}_\lambda = \mathbf{N}_{\lambda,\lambda}$. Let $\mathrm{spec}_{\mathbf{t}} = \{\|M\| : M \models \mathbf{t}\}$ for $\mathbf{t} \in \mathbf{N}_{\lambda,T}$.

The *Hanf number* $\mathrm{H}(\lambda, T)$ is the minimal μ such that if $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ and $\mathbf{t}$ has a model of cardinality $\geq \mu$ then $\mathbf{t}$ has models of arbitrarily large cardinality; cf. Definition 2.2 (3). Equivalently, $\mathrm{H}(\lambda, T) = \sup\{\mathrm{H}(\mathbf{t}) : \mathrm{H}(\mathbf{t}) < \infty, \mathbf{t} \in \mathbf{N}_{\lambda,T}\}$ where $\mathrm{H}(\mathbf{t}) = \sup\{\|M\|^+ : M \in \mathrm{Mod}_{\mathbf{t}}\}$. Let $\lambda(\mathbf{t}) := \lambda(T_{\mathbf{t}}) + |T_{1,\mathbf{t}}|$, recalling the definitions in § 1.3.

From now on, we shall assume that T is stable and that $\mathbf{t} = (T, T_1, p) \in \mathbf{N}_{\lambda,T}$ for $\lambda \geq |T|$ unless said otherwise. Furthermore, we assume that $\kappa = \kappa_{\mathrm{reg}}(T)$ (cf. § 1.3).

Claim 2.1 *If* $M \in \mathrm{Mod}_{\mathbf{t}}$ *has cardinality* $\mu > \aleph_0$ *then* $\mu = \mu^{<\kappa(T)} + |\lambda(T)|$. *If* $M \in \mathrm{Mod}_{\mathbf{t}}$ *and* $\lambda(\mathbf{t}) \leq \mu = \mu^{<\kappa(T)} < \|M\|$ *and* $A \subseteq M$ *is of cardinality* μ, *then there is some* $N \in \mathrm{Mod}_{\mathbf{t}}$ *of cardinality* μ *such that* $A \subseteq N \prec M$.

Recall that there is a countable, superstable, $\aleph_0$-categorical theory T that is not $\aleph_0$-stable. So to simplify Claim 2.1, we ignore $\aleph_0$.

Proof of Claim 2.1. The first claim follows from Fact 1.2. For the second claim, note that also $\mu = \mu^{<\kappa_{\rm reg}(T)}$ by cardinal arithmetic and hence $\kappa_{\rm reg}(T) \le \mu$; we choose M_i by induction on $i < \kappa_{\rm reg}(T)$ such that

(a) if i is even, then $M_i \prec M$ and $\|M_i\| = \mu$;

(b) if i is odd, then $M_i \restriction \tau(T_{\mathbf{t}}) \prec M \restriction \tau(T_{\mathbf{t}})$, $\|M_i\| = \mu$ and M_i is saturated;

(c) if $j < i$, then $A \cup |M_j| \subseteq |M_i|$.

There is no problem to carry the induction and then $M' = \bigcup\{M_{2i} : i < \kappa_{\rm reg}(T)\} = \bigcup\{M_{2i+1} : i < \kappa_i(T)\}$ is as required: $M' \prec M$ by (a), (c), and Tarski-Vaught; $\|M'\| = \mu$ since $\mu^{<\kappa(T)} = \mu$ and $M' \restriction \tau(T)$ is saturated by (b), (c), and Claim 1.8. □

We conclude that for understanding the Hanf number of $\mathbf{t}$, it is enough to consider cardinals $\mu = \mu^{<\kappa(T)} \ge \lambda(\mathbf{t})$. Now we turn to the logics of the form $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$; first we define them.

Definition 2.2 Assume that $\lambda \ge \kappa = {\rm cf}(\kappa)$ and let $\mathbb{B}$ be a Boolean algebra of cardinality $\le \lambda$ with ${\rm uf}(\mathbb{B})$, the set of ultrafilters on $\mathbb{B}$.

(1) Then let ${\rm voc}_\lambda[\mathbb{B}]$ be the class of vocabularies τ of cardinality $\le \lambda$ such that there are individual constants c_b for every $b \in \mathbb{B}$, unary predicate symbols P and Q, and a binary predicate R, as well as possibly additional symbols in τ. For $\tau \in {\rm voc}_\lambda[\mathbb{B}]$, let $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau)$ be the set of sentences $\psi \in \mathcal{L}_{\lambda^+,\kappa}(\tau)$ but we stipulate that from ψ we can reconstruct the triple $(\lambda^+, \kappa, \mathbb{B})$ hence $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$, (e.g., demand $<_\bullet \in \tau$ is a two-place predicate and $<_\bullet^M = \{(a,b) : \mathbb{B} \models c_a < c_b\}$. (Note that ψ has $\le \lambda$ sub-formulas.) Omitting τ means $\tau = \tau_\psi$ is the minimal $\tau \in {\rm voc}_\lambda[\mathbb{B}]$ such that $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau)$.

(2) For $\tau \in {\rm voc}_\lambda[\mathbb{B}]$ and $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau)$, let ${\rm Mod}^1_\psi[\mathbb{B}]$ be the class of models M of ψ (which are τ_ψ-models if not said otherwise) such that

(a) $P^M = \{c^M_b : b \in \mathbb{B}\}$;

(b) $\langle c^M_b : b \in \mathbb{B}\rangle$ are pairwise distinct;

(c) $R \subseteq P^M \times Q^M$;

(d) for every $a \in Q^M$ the set ${\rm uf}^M(a) := \{b \in \mathbb{B} : M \models c_b \mathrel{R} a\}$ belongs to ${\rm uf}(\mathbb{B})$;

(e) if $a_1 \ne a_2$ are from Q^M then ${\rm uf}^M(a_1) \ne {\rm uf}^M(a_2)$;

(f) for every $u \in {\rm uf}(\mathbb{B})$ there is $a \in Q^M$ such that $M \models \bigwedge_{i<\lambda, b \in u} c_b \mathrel{R} a$. (By clause (e) the element a is unique.)

Note that clauses (a) to (e) can be expressed in $\mathcal{L}_{\lambda^+,\aleph_0}$, but when $|{\rm uf}(\mathbb{B})| > \lambda$, then clause (f) cannot.

(3) Let ${\rm Mod}^2_\psi[\mathbb{B}]$ be the class of $M \in {\rm Mod}^1_\psi[\mathbb{B}]$ such that $\|M\| = \|M\|^{<\kappa}$. In that case, it follows that $\|M\| \ge |{\rm uf}(\mathbb{B})|$.

(4) For $\iota = 1, 2$ and $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ let ${\rm spec}^\iota_\psi[\mathbb{B}] = \{\|M\| : M \in {\rm Mod}^\iota_\psi[\mathbb{B}]\}$. Writing ${\rm Mod}^\iota_\psi$, ${\rm spec}^\iota_\psi$ we mean $\iota \in \{1,2\}$ and may omit ι when $\iota = 2$ (because this is the main case for us); cf. Observation 2.4(1) below and $\mathbb{B}$ can be reconstructed from ψ.

(5) Let ${\rm H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}])$ be the first μ such that if $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ and there is $M \in {\rm Mod}_\psi[\mathbb{B}]$ of cardinality $\ge \mu$ then $\{\|M\| : M \in {\rm Mod}_\psi[\mathbb{B}]\}$ is an unbounded class of cardinals.

(6) Let $\mathcal{L}^{\rm ba}_{\lambda^+,\kappa} := \bigcup\{\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}] : \mathbb{B}$ a Boolean algebra of cardinality $\le \lambda\}$; so every sentence of $\mathcal{L}^{\rm ba}_{\lambda^+,\kappa}(\tau)$ is a sentence in $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau)$ for some $\mathbb{B}$ as above.[3] We may stipulate that the set of elements of $\mathbb{B}$ is a cardinal $\le \lambda$ and $c_i \in \tau$ for $i < \lambda$. We define ${\rm H}(\mathcal{L}^{\rm ba}_{\lambda^+,\kappa})$ similarly; yes, this is just $\sup\{{\rm H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]) : \mathbb{B}$ as above$\}$.

Having defined the sets $(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}])(\tau)$ of sentences, the relevant classes of models ${\rm Mod}^\iota_\psi[\mathbb{B}]$, the spectrum ${\rm spec}^\iota_\psi[\mathbb{B}]$, and the Hanf numbers we should now try to understand how Boolean algebras are ordered according to their complexity by their Hanf numbers; here, the free Boolean algebra $\mathbb{B}^{\rm fr}_\lambda$ with λ generators is the most complicated among them.

Claim 2.3 *(1) For every Boolean algebra $\mathbb{B}_1$ of cardinality λ or just $\le \lambda$ and $\psi_1 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}_1]$ there is $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}^{\rm fr}_\lambda]$ such that ${\rm spec}^\iota_{\psi_1} \backslash 2^\lambda = {\rm spec}^\iota_\psi \backslash 2^\lambda$ for $\iota = 1, 2$ and ${\rm H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}_1]) \le {\rm H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}^{\rm fr}_\lambda])$.*

(2) If $\mathbb{B}_1, \mathbb{B}_2$ are Boolean algebras of cardinality $\le \lambda$ and $\mathbb{B}_1$ is a homomorphic image of $\mathbb{B}_2$, by h_2, then for every $\psi_1 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}_1]$ there is $\psi_2 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}_2]$ such that ${\rm spec}^\iota_{\psi_1}[\mathbb{B}_1] \backslash \|\mathbb{B}_2\| = {\rm spec}^\iota_{\psi_2}[\mathbb{B}_1] \backslash \mathbb{B}_2$ for $\iota = 1, 2$ and ${\rm H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}_1]) \le {\rm H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}_2])$.

[3] So every sentence $\psi \in \mathcal{L}^{\rm ba}_{\lambda^+,\kappa}$ fixes a Boolean algebra $\mathbb{B}$ as above and a vocabulary of cardinality $\le \lambda$ from ${\rm voc}_\lambda[\mathbb{B}]$ as described.

(3) For every $\psi_1 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ *there are* $\psi_2, \psi_2', \psi_2'' \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ *such that (a)* $\mathrm{spec}^1_{\psi_2}[\mathbb{B}] = \{\mu : \mu = \mu^{<\kappa} \in \mathrm{spec}^1_{\psi_1}[\mathbb{B}]\} = \mathrm{spec}^2_{\psi_1}[\mathbb{B}]$*, (b)* $\mathrm{spec}^1_{\psi_2'}[\mathbb{B}] = \{\mu^{<\kappa} : \mu \in \mathrm{spec}^1_{\psi_1}[\mathbb{B}]\}$*, and (c)* $\mathrm{spec}^1_{\psi_2''}[\mathbb{B}] = \{\mu : \mu \geq \lambda + \|\mathbb{B}\|$ *and* $\mu \in \mathrm{spec}^1_{\psi_1}[\mathbb{B}]\}$.

Concerning Claim 2.3 (3a) & (3b), recall that if $\mu > 2^{<\kappa}$ then $(\mu^{<\kappa})^{<\kappa} = \mu$; cf. [6].

P r o o f o f C l a i m 2.3. (1) Let h be a homomorphism from $\mathbb{B}^{\mathrm{fr}}_\lambda$ onto $\mathbb{B}_1$, exists as $\mathbb{B}_1$ is a Boolean algebra of cardinality $\leq \lambda$. Now apply part (2).

(2) Let $I := \mathrm{Ker}(h) := \{a \in \mathbb{B}^{\mathrm{fr}}_\lambda : h(a) = 0\}$ and let $h_1 : \mathbb{B}_1 \to \mathbb{B}_2$ be such that $a \in \mathbb{B}_1$ implies that $h(h_2(a)) = a$. Let $\mathbb{B}_1'$ be the Boolean algebra with set of elements $\mathrm{Rang}(h_1)$ such that h_1 is an isomorphism from $\mathbb{B}_1$ onto $\mathbb{B}_1'$. Let ψ_1' be like ψ_1 replacing $\mathbb{B}_1$ by $\mathbb{B}_1'$ and the predicate P by a predicate P'. The rest should be clear.

(3) Should be clear but we elaborate. For clause (a), let $\tau_2 = \tau(\psi_1) \cup \{F_{i,j} : i < j < \kappa\}$ with $F_{i,j} \notin \tau(\psi)$ be pairwise distinct unary function symbols. Let $\psi_2 = \psi_1 \wedge \varphi_2$ where

$$\varphi_2 = \bigwedge_{0<j<\kappa} (\forall \ldots, x_i, \ldots)_{i<j} (\exists y)[\bigwedge_{i<j} F_{i,j}(y) = x_i].$$

Now think!

For clause (b), let $\tau_2' = \tau(\psi_1) \cup \{F_{i,j} : i < j < \kappa\} \cup \{P_j : j \leq \kappa\}$ with $F_{i,j}$ as above $F_{i,j}, P_j \notin \tau(\psi_1)$ for $j \leq \kappa$ be pairwise distinct unary predicates and let $P = P_\kappa$. Let $\psi_1^P \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ be such that for a $(\tau(\psi_1) \cup \{P\})$-model M, $M \models \psi_1^P$ if and only if $(M \restriction P^M) \restriction \tau(\psi_1)$ is a $\tau(\psi_1)$-model and is a model of T. Lastly, let $\psi_2 = \psi_1^P \wedge \varphi_2'$ where φ_2' is the conjunction of

1. $\varphi_2^0 = (\forall x)(P(x) \vee \bigvee_{i<\kappa} P_i(x)) \wedge \bigwedge_{i<j\leq\kappa} \neg(\exists x)(P_i(x) \wedge P_j(x))$, so φ_2^0 says $\langle P^M \rangle ^\frown \langle P_j^M : j < \kappa \rangle$ is a partition of M, the universe of the model;
2. $\varphi^1_{2,i,j} = (\forall x)(P(F_{i,j}(x))$ for $i < j < \kappa$;
3. $\varphi^2_{2,j} = (\forall x, y)[x \neq y \wedge P_j(x) \to \bigvee_{i<j} F_{i,j}(x) \neq F_{i,j}(y)]$;
4. $\varphi^3_{2,j} = (\forall \ldots, x_i, \ldots)_{i<j}(\bigwedge_{i<j} P(x_i) \to (\exists y)(P_j(y) \wedge \bigwedge_{i<j} F_{i,j}(y) = x_i)$.

Now check! Finally, clause (c) is even easier noting that "$\geq \|\mathbb{B}\|$" holds by the definitions. □

Observation 2.4 *Let* $\mathbb{B}$ *be a Boolean algebra of cardinality* $\leq \lambda$ *and* $\kappa \leq \lambda^+$.

(1) If $\mathrm{uf}(\mathbb{B})$ *has cardinality* $\leq \lambda$ *(hence* $\mathbb{B}$ *has cardinality* $\leq \lambda$*), then* $\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]) = \mathrm{H}(\mathcal{L}_{\lambda^+,\kappa})$.

(2) In the definition of $\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}])$ *it does not matter if we use* $\mathrm{Mod}^1_\psi[\mathbb{B}]$ *or* $\mathrm{Mod}^2_\psi[\mathbb{B}]$.

(3) For every $\mu < \mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}])$*, we have* $2^\mu < \mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}])$*; hence* $\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}])$ *is a strong limit cardinal of cofinality* $> \lambda$.

(4) We have that $\mathrm{H}(\mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}) < \mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\kappa})$.

(5) We have $\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}) \leq \mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]) \leq \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda] = \mathrm{H}(\mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}) < \mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\kappa})$.

(6) If $\mathbb{B}^{\mathrm{fr}}_\lambda$ *is the free Boolean algebra of cardinality* λ *and* $\kappa = \aleph_0$*, then* $\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}) < \beth_{(2^\lambda)^+} < \mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda])$*. Also for any* $\kappa \geq \aleph_0$*, we have* $\mathrm{H}(\mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}) < \mathrm{H}(\mathcal{L}_{\lambda^+,\lambda^+})$.

(7) If $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ *and* $\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]) \leq \sup\{\|M\| : M \in \mathrm{Mod}_\psi[\mathbb{B}]\}$ *then* $\infty = \sup\{\|M\| : M \in \mathrm{Mod}_\psi[\mathbb{B}]\}$ *hence* $\mathrm{cf}(\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}])) \leq 2^\lambda$*).*

(8) Like part (7) for $\psi \in \mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}$ *and* $\mathrm{Mod}^{\mathrm{ba}}_\psi$.

P r o o f . (1) is easy. For (2), we first observe that as the Hanf number is easily $> 2^\lambda \geq |\mathrm{uf}(\mathbb{B})|$, we can ignore models of cardinality $< 2^\lambda$. Now, if $\psi_1 \in \mathcal{L}_{\lambda,\kappa}[\mathbb{B}](\tau)$ and $\sup(\mathrm{spec}^1_{\psi_1}) < \infty$, then

$$\sup(\mathrm{spec}^2_{\psi_1}) \leq \sup(\mathrm{spec}^1_{\psi_1}) \leq (\sup(\mathrm{spec}^2_{\psi_1}))^{<\kappa} < \infty \qquad (*)$$

[Why? The first inequality holds because $\mathrm{spec}^1_\psi \supseteq \mathrm{spec}^2_\psi$; the second inequality by Claim 2.1.]

We can conclude that spec^1_ψ is bounded if and only if spec^2_ψ is bounded and the Hanf number of the logic $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ using Mod^1_ψ is smaller or equal to the Hanf number of the logic $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ using Mod^2_ψ. The other inequality holds by Claim 2.3 (3b) and (*).

Alternatively, if $\psi_1 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ then by Claim 2.3 (3b), there is $\psi'_2 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ such that $\sup(\mathrm{spec}^1_{\psi_1}) < \infty$ implies $\sup(\mathrm{spec}^1_{\psi_1}) \leq \sup(\mathrm{spec}^2_{\psi'_2}) < \infty$, hence the Hanf number using spec^1_ψ's is $\leq$ the Hanf number using spec^2_ψ's. Moreover, above we get $\sup(\mathrm{spec}^1_{\psi_1}) \leq \sup(\mathrm{spec}^2_{\psi'_2}) = \sup(\mathrm{spec}^1_{\psi'_2})$ as $\mathrm{spec}^2_{\psi'_2} = \mathrm{spec}^1_{\psi'_2}$. On the other hand, by Claim 2.3 (3a), if $\psi_1 \in \mathcal{L}_{\lambda,\kappa}[\mathbb{B}]$ then there is $\psi_2 \in \mathcal{L}_{\lambda,\kappa}[\mathbb{B}]$ such that $\mathrm{spec}^1_{\psi_2} = \mathrm{spec}^2_{\psi_1}$ so $\sup(\mathrm{spec}^2_{\psi_1}) < \infty$ implies $\sup \mathrm{spec}^2_{\psi_1} = \sup \mathrm{spec}^1_{\psi_2} < \infty$ so also the other inequality holds.

(3) For any $\psi_1 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$, we can find $\psi_2 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ such that $\tau_{\psi_1} \subseteq \tau_{\psi_2}$ and $P_*, R_* \in \tau_{\psi_2} \backslash \tau_{\psi_1}$ are unary and binary predicates, respectively, and

$$M_2 \in \mathrm{Mod}^\iota_{\psi_2}[\mathbb{B}] \text{ if and only if (a) } (M_2 \restriction P_*^{M_2} \restriction \tau_{\psi_1}) \in \mathrm{Mod}_{\psi_1}[\mathbb{B}] \text{ and}$$
$$\text{(b) } M_2 \models (\forall y, z)(\exists x)[P_*(x) \wedge (R(x, y) \equiv \neg R(x, z))],$$
$$\text{hence } |P_*^{M_2}| \leq \|M_2\| \leq 2^{|P_*(M_2)|}.$$

Clearly for every $M_1 \in \mathrm{Mod}^1_{\psi_1}[\mathbb{B}]$ and $\mu = \mu^{<\kappa} \in [\|M_1\|, 2^{\|M_1\|}]$ there is $M_2 \in \mathrm{Mod}^1_{\psi_2}[\mathbb{B}]$ of cardinality μ. This clearly suffices for the first statement. The second is easy, too.

For (4), let $\mathbf{K}_{\lambda^+,\kappa}$ be the class of pairs $(\psi, \mathbb{B})$ such that $\mathbb{B}$ is a Boolean algebra of cardinality $\leq \lambda$, $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$. For $(\psi, \mathbb{B}) \in \mathbf{K}_{\lambda^+,\kappa}$ let $\mathrm{H}(\psi, \mathbb{B}) = \bigcup\{\mu^+ : \mu \in \mathrm{spec}^2_\psi(\mathbb{B})\}$. Clearly up to isomorphism (of vocabularies) $\mathbf{K}_{\lambda^+,\kappa}$ has cardinality $\leq 2^\lambda$ and hence $\mathbf{C}_{\lambda^+,\kappa} := \{\mathrm{H}(\psi, \mathbb{B}) : (\psi, \mathbb{B}) \in \mathbf{K}_{\lambda^+,\kappa}\}$ has cardinality $\leq 2^\lambda$. So let $\langle(\psi_i, \mathbb{B}_i) : i < 2^\lambda\rangle$ be such that $(\psi_i, \mathbb{B}_i)$ is as above and $\mathbf{C}_{\lambda^+,\kappa} \backslash \{\infty\} = \{\mu_i : i < 2^\lambda\}$ where $\mu_i = \mathrm{H}(\psi_i, \mathbb{B}_i) = \bigcup\{\mu^+ : \mu \in \mathrm{spec}^1_{\psi_i}[\mathbb{B}_i]\}$ for $i < 2^\lambda$. Now we can find $\psi \in \mathcal{L}_{(2^\lambda)^+,\kappa}$ such that $M \models \psi$ if and only if $<^M$ is a linear order of $|M|$ and for arbitrarily large $a \in M$ there are $i < 2^\lambda$ and $N \in \mathrm{Mod}^2_{\psi_i}[\mathbb{B}_i]$ with universe $\{b : b <^M a\}$. Together with part (3), clearly $\infty > \sup(\mathrm{spec}_\psi) = \max(\mathrm{spec}_\psi) = \bigcup\{\mu_i : i < 2^\lambda\}$ so we are done.

For the first inequality of (5), "$\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}) \leq \mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}])$", we refer to the definition of $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$. For the second inequality of (5), "$\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]) \leq \mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda])$", use Claim 2.3(1). For the third inequality of (5), "$\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda]) = \mathrm{H}(\mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa})$", use the definition of the latter and the second inequality. Finally, for the fourth inequality of (5), "$\mathrm{H}(\mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}) < \mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\kappa})$", recall item (4).

The first inequality of (6), "$\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}) < \beth_{(2^\lambda)^+}$", is well known; cf., e.g. [5, Theorems VII.5.4 & VII.5.5] recalling $\kappa = \aleph_0$. The second inequality of (6), "$\beth_{(2^\lambda)^+} < \mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda])$", holds by part (4) and by the equality in part (5). For the third inequality of (6), note that there is $\psi \in \mathcal{L}_{\lambda^+,\lambda^+}$ such that: $M \models \psi$ if and only if P^M, Q^M, R^M are as in the definition of $\mathrm{Mod}^1_\psi[\mathbb{B}]$, $F_i^M (i < \lambda)$ are as in Claim 2.3 (3a) for Q^M, i.e.,

$$M \models (\forall \ldots, x_i, \ldots)_{i<\lambda}(\exists y)[\bigwedge_{i<\lambda} Q(x_i) \rightarrow (\exists y)(Q(y) \wedge \bigwedge_{i<\lambda} F_i(y) = x_i)],$$

and $<^M$ is a well ordering of Q^M.

The proof of (7) is similar to the end of the proof of part (4), replacing ψ_i by ψ_1; i.e., we can find $\psi_1 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ such that $M_1 \models \psi_1$ if and only if for some $< \in \tau(\psi_1)$, $<^{M_1}$ is a linear order of $|M_1|$ such that for arbitrarily large $b \in M_1$, $M_1 \restriction \{a : a <^{M_1}\} \restriction \tau_\psi$ is a model of ψ. Clearly this suffices.

For (8), assume $\mu < \mathrm{H}(\mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa})$; hence, by the definition there is $\psi \in \mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}$ such that $\{\|M\| : M \models \psi\}$ is bounded but has a member $\geq \mu$. By the definition of $\mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}$ for some Boolean algebras $\mathbb{B}$ of cardinality $\leq \lambda$, we have $\psi \in \mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}[\mathbb{B}]$; now apply part (2). □

The following Claims 2.6 & 2.7, using Definition 2.5, show another way to represent the logic $\mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}$, or, equivalently, the logic $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda]$. This will help us later to characterise the Hanf numbers.

Definition 2.5 We are modifying some of the concepts defined in Definition 2.2. Recall the definition of $\mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}$ (Definition 2.2 (6)), $\mathrm{Mod}^1_\psi(\mathbb{B})$ (Definition 2.2 (3)), and $\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}])$ (Definition 2.2 (5)).

Let $\mathcal{L}^*_{\lambda^+,\kappa}$ be defined like $\mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}$, except that you replace $\langle c_b : b \in \mathbb{B}\rangle$ by $\langle c_i : i < \lambda\rangle$ and $\mathrm{uf}(\mathbb{B})$ by $\wp(\{c_i : i < \lambda\})$. For $\psi \in \mathcal{L}^*_{\lambda^+,\kappa}$ let Mod^*_ψ be defined like $\mathrm{Mod}^1_\psi(\mathbb{B})$, except that you replace $\mathrm{uf}(\mathbb{B})$ by $\wp(\lambda)$. Let $\mathrm{H}(\mathcal{L}^*_{\lambda^+,\kappa})$ be defined like $\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}])$. For $\psi \in \mathcal{L}^*_{\lambda^+,\kappa}$ let $\mathrm{spec}^*_\psi = \mathrm{spec}^{1,*}_\psi = \{\|M\| : M \in \mathrm{Mod}^*_\psi\}$; and $\mathrm{spec}^{2,*}_\psi = \{\|M\| : M \in \mathrm{Mod}^*_\psi$ and $\|M\| = \|M\|^{<\kappa}\}$; for transparency, we shall stipulate that from ψ we can reconstruct $\mathcal{L}^*_{\lambda^+,\kappa}$.

The following claim essentially tells us that for determining the Hanf number of $\mathcal{L}^{\mathrm{ba}}_{\lambda^+,\kappa}$, we may use the "worst possible" Boolean algebra, $\mathbb{B}^{\mathrm{fr}}_\lambda$ and $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda]$ is essentially equal to $\mathcal{L}^*_{\lambda^+,\kappa}$.

Claim 2.6 *In the natural definition of* $\mathrm{H}(\mathcal{L}^*_{\lambda^+,\kappa})$ *it does not matter if we use* $\mathrm{spec}^{1,*}_\psi$ *or* $\mathrm{spec}^{2,*}_\psi$ *for* $\psi \in \mathcal{L}^+_{\lambda^+,\kappa}$. *The Hanf number* $\mathrm{H}(\mathcal{L}^*_{\lambda^+,\kappa})$ *is a strong limit cardinal of cofinality* $> \lambda$ *with* $\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa}) < \mathrm{H}(\mathcal{L}^*_{\lambda^+,\kappa}) < \mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\kappa})$. *If* $\psi \in \mathcal{L}^*_{\lambda^+,\kappa}$ *and* $\mathrm{H}(\mathcal{L}^*_{\lambda^+,\kappa}) \leq \sup\{\|M\| : M \in \mathrm{Mod}_\psi\}$, *then* $\infty = \sup\{\|M\| : M \in \mathrm{Mod}_\psi\}$. *Furthermore, for every* $\psi_1 \in \mathcal{L}^*_{\lambda^+,\kappa}$ *there are* $\psi_2, \psi_2', \psi_2'' \in \mathcal{L}^*_{\lambda^+,\kappa}$ *such that*

$$\mathrm{spec}^*_{\psi_2} = \{\mu : \mu = \mu^{<\kappa} \in \mathrm{spec}^*_{\psi_1}[\mathbb{B}]\} = \mathrm{spec}^{2,*}_{\psi_1},$$

$$\mathrm{spec}^*_{\psi_2'} = \{\mu^{<\kappa} : \mu \in \mathrm{spec}^{1,*}_{\psi_1}\}, \text{ and}$$

$$\mathrm{spec}^*_{\psi_2''} = \{\mu : \mu \geq \lambda \text{ and } \mu \in \mathrm{spec}^{1,*}_{\psi_1}[\mathbb{B}]\}.$$

P r o o f . Similar to Observation 2.4 & Claim 2.3(3). □

Claim 2.7 *For every* $\psi_1 \in \mathcal{L}^*_{\lambda^+,\kappa}$ *there is* $\psi_2 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda]$ *such that* $\{\|M\| : M \in \mathrm{Mod}^{\mathrm{ba}}_{\psi_1}\} = \{\|M\| : M \in \mathrm{Mod}^*_{\psi_2}[\mathbb{B}]\}$, *i.e.,* $\mathrm{spec}^*_{\psi_1} = \mathrm{spec}_{\psi_2}[\mathbb{B}]$. *For every* $\psi_2 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda]$ *there is* $\psi_1 \in \mathcal{L}^*_{\lambda^+,\kappa}$ *such that* $\mathrm{spec}^*_{\psi_2} = \{\mu : \mu \geq \lambda$ *and* $\mu \in \mathrm{spec}^{1,*}_{\psi_1}[\mathbb{B}]\}$.

P r o o f . This follows from the fact that (A) implies (B) where the statements (A) and (B) are defined as follows:

> Assume $\mathbb{B}$ is the Boolean algebra generated freely by $\langle b_i : i < \lambda\rangle$, M is a model, $P_1^M = \{b_i : i < \lambda\}$, $P_2^M = \mathbb{B}$, $Q_1^M = \wp(\lambda)$, $Q_2^M = \mathrm{uf}(\mathbb{B})$, $R_1^M = \{(c_i, u) : u \subseteq \lambda, i \in u\}$ and $R_2^M = \{((c, D) : c \in \mathbb{B}, D \in \mathrm{uf}(\mathbb{B}))$ and $c \in D\}$, $c_{\bar{b}} \in \bar{c}(M)$ and $c_b^M = b$ for $b \in \mathbb{B}$. (A)

> If N is a model of $\mathrm{Th}(M)$ omitting the type $p(x) = \{P(x) \wedge x \neq c_b : b \in \mathbb{B}\}$ then if N satisfies the demands in Definition 2.2(2) of $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}^{\mathrm{fr}}_\lambda]$ (with P_2, Q_2, R_2 here standing for P, Q, R there), then N satisfies the demands in Definition 2.5(1) of $\mathcal{L}^*_{\lambda^+,\kappa}$ (with P_1, Q_1, R_1 here standing for P, Q, R there). (B)

□

We shall now connect the logics defined to first order theories. The easy part is to start with a Boolean algebra $\mathbb{B}$ and construct a related first order theory T.

Claim 2.8 *Let* $\mathbb{B}$ *be a Boolean algebra of of cardinality* $\leq \lambda$ *and let* $\kappa \leq \lambda^+$. *Then there are theories* $T^1 := T^1_{\mathbb{B},\kappa}$ *and* $T^2 := T^2_{\mathbb{B},\kappa}$ *such that*

1. T^1 *is complete, stable, and has elimination of quantifiers,* $|T^1| = \lambda$, $\kappa(T^1) = \kappa$, *and* $\lambda'(T^1)$ *is the cardinality of* $\mathrm{uf}(\mathbb{B})$ *(cf. definitions in § 1.3); in fact,* $\mathbb{B}_T$ *is not much more complicated than* $\mathbb{B}$, *but we shall not elaborate, see below;*
2. T^2 *is complete and stable,* $|T^2| = \lambda$, $\kappa(T^2) = \kappa$, *and* $\lambda'(T^2) = \lambda + 2^{\aleph_0}$.

P r o o f . Easy, but we elaborate.

For claim 1., we let $\tau^1_* := \tau^1_{\mathbb{B},\kappa} := \{P_b : b \in \mathbb{B}\} \cup \{Q_\vartheta : \vartheta < \kappa \text{ is infinite}\} \cup \{E_{\vartheta,i} : \vartheta < \kappa \text{ is infinite}, i < \vartheta\}$ where P_b, Q_ϑ are unary predicates and $E_{\vartheta,i}$ is a binary predicate. Then, we define a universal theory $T^1_0 \subseteq \mathcal{L}(\tau_*)$ such that a τ_*-model M satisfies T^1_0 if and only if

(a) $b \mapsto P^M_b$ embeds $\mathbb{B}$ into the Boolean algebra $\wp(P^M_{1_\mathbb{B}})$ so $P^M_{0_\mathbb{B}} = \varnothing$;

(b) $\langle P^M_{1_\mathbb{B}}\rangle^\frown\langle Q^M_\vartheta : \vartheta < \kappa\rangle$ are pairwise disjoint;

(c) $E^M_{\vartheta,i}$ is an equivalence relation on Q^M_ϑ; in particular, if $aE^M_{\vartheta,i}b$, then $a, b \in Q^M_\vartheta$;

(d) if $i < j < \vartheta$, then $E^M_{\vartheta,j}$ refines $E^M_{\vartheta,i}$.

We observe that $T^1_0 \subseteq \mathcal{L}(\tau_*)$ is a well defined universal theory and Mod_{T_0} has the amalgamation and the joint embedding properties.

Let $\mathbb{T}$ be the set of signatures $\tau \subseteq \tau_*$ such that $P, P_{1_\mathbb{B}}, P_{0_B} \in \tau$, if $E_{\vartheta,i} \in \tau$, then $Q_\vartheta \in \tau$, and if $\mathbb{B} \models$ "$b \cap c = a \wedge -b = d$" and $\{P_b, P_c\} \subseteq \tau$, then $\{P_a, P_d\} \subseteq \tau$. For any $\tau \in \mathbb{T}$, let $T^1_{0,\tau}$ be defined like T^1_0 but restricting ourselves to predicates from τ. Now for any $\tau \in \mathbb{T}$,

(i) if M is a τ-model of $T^1_{0,\tau}$, then M can be expanded to a τ_*-model of T^1_0;

(ii) $T^1_{0,\tau}$ has the amalgamation and the joint embedding properties,

(iii) if $M_1 \subseteq M_2$ are models of $T^1_{0,\tau}$ and $\tau \subseteq \tau_1 \in \mathbb{T}$ and N_1 is a τ_1-model expanding M_2 then there is a τ_1-model N_2 expanding M_1 and extending N_1.

[Why? Easy, e.g., clause (b) by disjoint union.]

For finite $\tau \in \mathbb{T}$, $T^1_{0,\tau}$ has a model completion called $T^1_{1,\tau}$ which has elimination of quantifiers (†). [Why? Because τ is a relational finite vocabulary and $T^1_{0,\tau}$ is universal with amalgamation and joint embedding properties.]

If $\tau_1 \subseteq \tau_2$ are from $\mathbb{T}$, then $T^1_{1,\tau_1} \subseteq T^1_{1,\tau_2}$. [Why? By (iii) and (†).] Now, let $T^1 := T^1_{\mathbb{B},\kappa} := \bigcup\{T^1_{1,\tau} : \tau \in \mathbb{T} \text{ finite}\}$ be the model completion of T^1_0. This has elimination of quantifiers. [Why? Follows from the above.] We observe that if $\tau \in \mathbb{T}$ is finite, then $T^1_{1,\tau}$ is $\aleph_0$-categorical and $\aleph_0$-stable, T^1 is stable, $\kappa(T^1) = \kappa$, $|\lambda'(T^1)| = |\mathbb{B}| + \aleph_0$, and $\lambda(T^1) = \min\{\mu : \mu \geq \lambda \text{ and } \mu^{<\kappa} = \mu\}$. [Why? Consider the monster $\mathfrak{C} = \mathfrak{C}_{T_{1,\tau}}$ and use automorphisms.] So $T^1 := T^1_{\mathbb{B},\kappa}$ is as promised.

For claim 2., we modify the above construction and obtain T^2_0: we add $Q_0, E_{0,n} (n < \omega)$ with Q_0 unary and $E_{0,n}$ binary, also Q^M_0 is disjoint to $Q^M_\vartheta (\vartheta \in [\aleph_0, \kappa))$ and to $P^M_{1_\mathbb{B}}$, $E^M_{0,n}$ is an equivalence relation on P^M_0; $E^M_{0,0}$ has one equivalence class; $E^M_{0,n+1}$ refines $E^M_{0,n}$ and divides each $E^M_{0,n}$ equivalence class to at most 2 equivalence classes. □

We would like to translate "$M \models \psi$ for $\psi \in \mathcal{L}_{\lambda^+,\kappa}$" to "$M \in \mathrm{Mod}_\mathbf{t}$", i.e., when $\kappa(T) \geq \kappa$ and, in particular, when $\kappa > \aleph_0$. However, the following is the "translation of $\psi \in \mathcal{L}_{\lambda^+,\kappa}(\tau_0)$"; i.e., it deals strictly with the logic $\mathcal{L}_{\lambda^+,\kappa}$; in particular, without the presence of a Boolean algebra. Our aim is to do some of the work of Theorem 2.10 in which we are really interested. So, Theorem 2.9 is not directly related to any of the **t**'s as there is no saturation requirement; moreover stability appears neither in Theorem 2.9 nor in Theorem 2.10.

Note, furthermore, that in Theorem 2.9 we can let κ_1 be such that $\kappa = \kappa_1^+$ or $\kappa_1 = \kappa$ if κ is a limit cardinal and let $\Upsilon = \kappa_1 + 1$ and omit F_{κ_1} and P_{κ_1}.

Theorem 2.9 (Representation Theorem for $\mathcal{L}_{\lambda^+,\kappa}$) *Assume $\psi \in \mathcal{L}_{\lambda^+,\kappa}(\tau_0)$, so, of course, $|\tau_0| \leq \lambda$. Let Υ be κ if $\kappa \leq \lambda$ and $\lambda + 1$ if $\kappa = \lambda^+$. We shall define a vocabulary $\tau_1 \supseteq \tau_0$ of cardinality λ with additional unary function symbols $\bar{F} = \langle F_i : i < \Upsilon\rangle$ with no repetitions, additional unary predicate symbols $\bar{P} = \langle P_i : i < \Upsilon\rangle$ with no repetitions, additional constant symbols $\langle c_i : i < \lambda\rangle$, and only additional unary predicate symbol P_*. We define a quantifier-free $\mathcal{L}(\tau_1)$-type $p(x) := \{P_*(x) \wedge x \neq c_i : i < \lambda\}$. Then we can find a first order theory T_1 in the vocabulary τ_1 such that the following conditions on a τ_0-model M_0 are equivalent:*

(i) $M_0 \models \psi$ and $\|M_0\| = \|M_0\|^{<\kappa} + \lambda^{<\kappa}$;

(ii) there is a τ_1-expansion M_1 of M_0 to a model of T_1 omitting $p(x)$ such that

(α) $\langle P^{M_1}_i : i < \Upsilon\rangle$ is a partition of $|M_1|$ and

(β) if $i < \Upsilon$ and $\langle a_j : j < i\rangle$ is a sequence of elements of M_1 (of length i) then for some $b \in P^{M_1}_i$ we have that $j < i$ implies $F^{M_1}_j(b) = a_j$.

P r o o f . What we shall do is essentially add Skolem functions, and coding sequences of length $< \kappa$. In particular, using the function symbols F_ε (for $\varepsilon < \kappa$) we can replace quantifying over ε-tuple $\langle x_\zeta : \zeta < \varepsilon\rangle$ by quantifying by one x.

Note that as ψ has no free variables, without loss of generality, every subformula φ of ψ has a set of free variables equal to $\{x_i : i < \varepsilon\}$ for some $\varepsilon = \varepsilon_\varphi < \kappa$ such that if φ is a sub-formula of ψ and $\varphi = \bigwedge_{i<j} \varphi_i$ then $\varepsilon_{\varphi_i} = \varepsilon_\varphi$. Let Δ be the set of sub-formulas of ψ so without loss of generality (a syntactical rewriting), there is a list $\langle \varphi_i(\bar{x}_{[\varepsilon(i)]}) : i < i(*)\rangle$ for some $i(*) \leq \lambda$ of Δ such that $\varepsilon(0) = 0$, $\varphi_0 = \psi$ and $\bar{x}_{[\varepsilon(i)]}$ is a sequence of length $< \kappa$ of variables, in fact, $\bar{x}_{[\varepsilon(i)]} = \langle x_\varepsilon : \varepsilon < \varepsilon(i)\rangle$ and $\varepsilon(i) < \kappa$.

For any τ_0-model M such that $\|M\| = \|M\|^{<\kappa} + \lambda^{<\kappa}$, we say N *codes* M when

(a) N expands M,

(b) the sequences $\langle F_i^N : i < \Upsilon\rangle$ and $\langle P_i^N : i < \Upsilon\rangle$ satisfy conditions (α) and (β) of (ii) in the theorem (with N instead of M_1),

(c) $Q_i^N = \{b \in P_{\varepsilon(i)}^N : M \models \varphi_i[\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i)\rangle]\}$ for $i < i(*)$,

(d) $\langle c_i^N : i < \lambda\rangle$ are pairwise distinct and $P_*^N = \{c_i^N : i < \lambda\}$,

(e) if $\varphi_i(\bar{x}_{\varepsilon(i)}) = \bigwedge_{j<j(i)} \varphi_{i,j}(\bar{x}_{\varepsilon(i)})$, so, for some $k(i, j) < i(*)$ we have $\varphi_{i,j}(\bar{x}_{\varepsilon(i)}) = \varphi_{k(i,j)}(\bar{x}_{\varepsilon(k(i,j))})$ and $\varepsilon(k(i, j)) = \varepsilon(i)$, then $F_{1,i} \in \tau(N)$ is unary and for $b \in P_{\varepsilon(i)}^N$, we have that

(α) $N \models F_{1,i}(b) = c_j \wedge \neg\varphi_i(\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i)\rangle)$ implies $M \models \neg\varphi_{i,j}(\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i)\rangle)$ (which means if $\varphi_{i,j} = \varphi_{k(i,j)}$ and $N \models \neg Q_i(b) \wedge c_j = F_{1,i}(b)$, then $M \models \neg Q_{k(i,j)}[b]$) and, of course,

(β) if $M \models \varphi_i(\langle f_\varepsilon(b) : \varepsilon < \varepsilon(i)\rangle)$ and $j < \varepsilon(i)$ then $M \models \varphi_{i,j}(\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i)\rangle)$, and

(f) if $\varphi_i(\bar{x}_{\varepsilon(i)}) = (\exists \bar{x}_{[\varepsilon(i),\zeta(i))})\varphi_{j_1(i)}(\bar{x}_{\varepsilon(i)}, \bar{x}_{[\varepsilon(i),\zeta(i))})$ and $F_\varepsilon(b) = a_\varepsilon$ for $\varepsilon < \varepsilon(i)$ then the following are equivalent:

(α) $M_1 \models \varphi_i[\langle a_\varepsilon : \varepsilon < \varepsilon(i)\rangle]$, or, equivalently, $M_1 \models \varphi_1[\langle F_\varepsilon(b) : \varepsilon < \varepsilon(i)\rangle]$ and

(β) $M_1 \models (\exists y)\varphi_{j_1(i)}(\langle a_\varepsilon : \varepsilon < \varepsilon(i)\rangle, \langle F_\zeta(y) : \zeta \in [\varepsilon(i), \zeta(i)]\rangle$.

We now let τ_1 be $\tau_\psi \cup \{F_\varepsilon, P_\varepsilon : \varepsilon < \Upsilon\} \cup \{Q_i : i < i(*)\} \cup \{F_{1,i} : i < i(*)$ and φ_i is a conjunction$\}$ and $T_1 := \bigcap\{\mathrm{Th}(N)$: there is a τ_0-model $M \models \psi$ such that $\|M\| = \|M\|^{<\kappa} + \lambda$ and N code $M\}$ and check that the tuple $(\tau_1, T_1, p(x), \bar{F}, \bar{P})$ is as required. □

So, how does Theorem 2.9 help for our main aim? It starts to translate $\psi \in \mathcal{L}_{\lambda^+,\kappa}(\tau_0)$ to $\mathbf{t} = (\tau_1, T_1, p(x))$, so instead having blocks of quantifiers $(\exists \bar{x}_{[\varepsilon]})$, $\varepsilon < \kappa$ we have $(\exists x)$, i.e., by the sequence of functions $\langle F_i : i < \varepsilon\rangle$ we code any ε-tuple by one element. This will help later to make "the $\tau(T_\mathbf{t})$-reduct is saturated" equivalent to the existence of suitable coding.

Recalling the definition of $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ (Definition 2.2 (6)), we get this section's main result: translating from $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ to a representation. This representation is naturally more complicated than the one for $\psi \in \mathcal{L}_{\lambda^+,\aleph_0}$.

Theorem 2.10 (Representation Theorem for $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$) *Assume $\mathbb{B}$ is a Boolean algebra of cardinality $\leq \lambda$ and for notational transparency no $b \in \mathbb{B}$ is an ordinal and $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau_0)$. We shall define a vocabulary $\tau_1 \supseteq \tau_0$ of cardinality λ with additional unary function symbols $\bar{F} = \langle F_i : i < \Upsilon\rangle$ with no repetitions, additional unary predicate symbols $\bar{P} = \langle P_i : i < \Upsilon\rangle$ with no repetitions, additional constant symbols $\langle c_i : i < \lambda\rangle$, and an additional unary predicate symbol P_*. We define a quantifier-free $\mathcal{L}(\tau_1)$-type $p(x) := \{P_*(x) \wedge x \neq c_i : i < \lambda\}$. Then we can find a first order theory T_1 in the vocabulary τ_1 such that the following conditions on a τ_0-model M_0 are equivalent:*

(i) $M_0 \in \mathrm{Mod}^2_\psi[\mathbb{B}]$, so $M_0 \models \psi$ and $\|M_0\| = \|M_0\|^{<\kappa} + \lambda^{<\kappa}$ and

(ii) there is a τ_1-expansion M_1 of M_0 to a model of T_1 omitting $p(x)$ such that

(α) $\langle P_i^{M_1} : i < \kappa\rangle$ is a partition of $|M_1|$,

(β) if $i < \kappa$ and for each $j < i$, we have $a_j \in M_1$, then for some $b \in P_i^{M_1}$ we have that $j < i$ implies $F_j^{M_1}(b) = a_j$,

(γ) for each $b \in \mathbb{B}$, c_b is an additional constant symbol in $\tau_1 \backslash \tau_0$ with no repetition, P and Q are unary predicate symbols in τ_1 unary and R is a binary predicate symbol in τ_1 such that $P^{M_1} = \{c_b^{M_1} : b \in \mathbb{B}\}$, $R^{M_1} \subseteq P^{M_1} \times Q^{M_1}$, and for every $b \in Q^{M_1}$ the set $u(b, M_1) := \{c_b \in P^{M_1} : (c_b, b) \in R^{M_1}\}$ is an ultrafilter of $\mathbb{B}$ and for every ultrafilter D of the Boolean algebra $\mathbb{B}$ there is one and only one $b \in Q^{M_1}$ such that $u(b, M_1) = D$.

P r o o f . First, note that the symbols P, Q, and c_b (for $b \in \mathbb{B}$) are in τ_ψ as in Definition 2.2. Second, we repeat the proof of Theorem 2.9 or just quote it: there is $\tau_* \supset \tau_\psi$, $|\tau_*| = \lambda$ with $F_\varepsilon, P_\varepsilon, F_{1,\varepsilon}, c_\varepsilon, Q \in \tau_*$ as in Theorem 2.9. Third, we prove the claimed equivalence of Theorem 2.10. The direction "(ii)⇒(i)" holds as in Theorem 2.9. For the other direction, assume $M_0 \in \mathrm{Mod}^2_\psi[\mathbb{B}]$ and we choose M_1 as in condition (ii) of Theorem 2.9. Then, lastly, clause (γ) of condition (ii) holds because $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ and M_1 expands M_0. □

Claim 2.11 *Let $\kappa \leq \lambda^+$ be singular (hence $\kappa^+ \leq \lambda^+$). Then for every $\psi_1 \in \mathcal{L}_{\lambda^+,\kappa^+}[\mathbb{B}]$ there is $\psi_2 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ such that* $\mathrm{spec}^2_{\psi_2} = \mathrm{spec}^2_{\psi_1}$.

P r o o f . Easy. □

The only requirement in the equivalence of Theorem 2.10 that is not expressible by a sentence of $\mathcal{L}_{\lambda^+,\kappa}$ (even with extra predicates) is "for every ultrafilter D of the Boolean algebra $\mathbb{B}$ there is one and only one $b \in Q^{M_1}$ such that $u(b, M_1) = D$"; cf. (ii)(γ).

As indicated above, $\mathbb{B}^{\mathrm{fr}}_\lambda$ is the most complicated Boolean algebra for our purpose. So it is natural to wonder about the order among the relevant Boolean algebras which we intend to comment on elsewhere.

3 Real equality for every theory

3.1 Answering the original and the new question

The original question for this work was about the strictly stable case, i.e., fixing $\kappa > \aleph_0$, dealing with $\{\mathbf{t} \in \mathbf{N}_\lambda : \kappa(T_\mathbf{t}) = \kappa\}$, so we deal with this case first.

In this case, Theorem 3.1 tells us that for strictly stable T and $\lambda \geq |T|$, the family of classes $\mathrm{Mod}_\mathbf{t}$ for $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ and the family of classes $\mathrm{Mod}^2_\psi[\mathbb{B}]$ for $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ where $\kappa = \kappa_{\mathrm{reg}}(T)$ and $\mathbb{B}$ is the Boolean algebra $\mathbb{B}_T$ from the definitions in § 1.3 are very similar. How this is proved? For one direction, we start with $\mathbf{t} \in N_{\lambda,T}$; so the (essential) non-first order part of the demand $M \in \mathrm{Mod}_\mathbf{t}$ is "$M{\restriction}\tau(T_\mathbf{t})$ is saturated". At first glance, we need (in addition to the first order theory and the omission of a type) to say some things on eliminating $u \in [M]^{<\|M\|}$ and relation on it, but because T is stable, this can be expressed by (cf. Fact 1.5):

(a) $M{\restriction}\tau(T_\mathbf{t})$ is $\kappa_{\mathrm{reg}}(T)$-saturated and
(b) if $I \subseteq M$ is an infinite indiscernible set in $M{\restriction}\tau(T_\mathbf{t})$, $|I| = \aleph_0$, then we can find an indiscernible set $J \supseteq I$ in $M{\restriction}\tau(T_\mathbf{t})$ of cardinality $\|M\|$.

So, the use of $\mathcal{L}_{\lambda^+,\kappa}$, where $\kappa = \kappa_{\mathrm{reg}}(T)$, is natural. If $2^{|T|} \leq \lambda$, this is obvious; otherwise, we have to be more careful. We use the Boolean algebra $\mathbb{B} = \mathbb{B}_T$ and $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ rather than $\mathcal{L}_{\lambda^+,\kappa}$ to express $M{\restriction}\tau(T_\mathbf{t})$ is $\aleph_0$-saturated, so by $\kappa_{\mathrm{reg}}(T)$-sequence homogeneity, this is enough.

Note, that on the one hand, $M \in \mathrm{Mod}_\mathbf{t}$ implies that $\|M\| \in \mathbf{C}_T = \{\mu : \mu = \mu^{<\kappa(T)} + \lambda(T)\}$ (cf. Claim 2.1); on the other hand, for $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$, $M \models \psi$ does not imply that. Still, we know that $\mathrm{spec}^1_\psi = \{\|M\| : M \models \psi\}$ and $\mathrm{spec}^2_\psi = \mathrm{spec}^1_\psi \cap \mathbf{C}_T$ are closed enough (cf. Observation 2.4, in particular (1)). Recall that $\mathbb{B} = \mathbb{B}^{\mathrm{fr}}_\lambda$ is the worst case.

For superstable T, we fix (λ, T); here, e.g., the case of $\mathrm{Th}({}^\omega 2, E_n)_n$ with $E_n = \{(\eta, \nu) : \eta, \nu \in {}^\omega 2, \eta{\restriction}n = \nu{\restriction}n\}$ makes us work somewhat more.

Theorem 3.1 *Assume T is a stable first order complete of cardinality $\leq \lambda$ and $\kappa = \kappa_{\mathrm{reg}}(T) = \min\{\vartheta : \vartheta$ regular and $\vartheta \geq \kappa(T)\}$ and $\lambda(T) = \min\{\lambda : T$ stable in $\lambda\}$ and let $\mathbb{B} = \mathbb{B}_T$ (cf. the definitions in § 1.3). Assume furthermore that $\kappa(T) > \aleph_0$ (i.e., T is not superstable).*

1. *We have* $\{\mathrm{spec}_\mathbf{t} : \mathbf{t} \in \mathbf{N}_{\lambda,T}\} = \{\mathrm{spec}^2_\psi[\mathbb{B}] : \psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]\}$.
2. *If $\tau_0 = \tau_T$ and $\psi_0 = \wedge\{\varphi : \varphi \in T\}$ or just $\tau_T \subseteq \tau_0$, $|\tau_0| \leq \lambda$, $\psi_0 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau_0)$ and $M \in \mathrm{Mod}_{\psi_0}[\mathbb{B}]$ implies that $M \models T$ then there is $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ such that* $\mathrm{spec}^2_{\psi_0}[\mathbb{B}] = \mathrm{spec}_\mathbf{t}$.
3. *If $\mathbf{t} \in \mathbf{N}_{\lambda,T}$ then for some $\psi_1 \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau_1)$, $\tau_1 \supseteq \tau(T_2)$ and* $\mathrm{spec}^1_{\psi_1}[\mathbb{B}] = \mathrm{spec}_\mathbf{t} = \mathrm{spec}^2_{\psi_1}[\mathbb{B}]$.

The proof gives more, viz. that the two contexts have the same PC classes. This proof is divided to two subsections each to one direction.

Proof. Claim 1. follows from 2. and 3. Claim 2. is proved in § 3.3 and claim 3. is proved in § 3.2 (i.e., by Claim 3.4 noting Hypothesis 3.3). □

Corollary 3.2 *If T is a complete and stable first order theory, $\kappa = \kappa(T) > \aleph_0$, and $|T| \leq \lambda$, then $\mathrm{H}(\lambda, T)$ is bigger than $\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa})$, but smaller than $\mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\kappa})$.*

Proof. First, assume T is strictly stable, i.e., $\kappa(T) > \aleph_0$. The "bigger than $\mathrm{H}(\mathcal{L}_{\lambda^+,\kappa})$" follows from claim 2. in Theorem 3.1 recalling the first inequality in Observation 2.4 (5).

The "smaller than $\mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\kappa})$" follows from claim 3. in Theorem 3.1 recalling the second and third inequality of Observation 2.4 (5). We are left with the case that T is superstable: then, we quote [2, Theorem 1.2], or Claims 3.5 & 3.6. □

3.2 Proving claim 3. of Theorem 3.1

Hypothesis 3.3 *In this entire section, we fix* $\mathbf{t} = (T, T_1, p) \in \mathbf{N}_{\lambda,T}$ *such that* T *is a complete and stable first order theory (so,* $\lambda \geq |T_1| \geq |T|$*) and let* $\mathbb{B} = \mathbb{B}_T$, $\kappa = \kappa_\tau(T)$. *This mean, without loss of generality,* P *and* Q *are unary predicate symbols,* R *is a binary predicate symbol,* c_b *(for* $b \in \mathbb{B}$*) are individual constant symbols without repetitions and all of them are not in* $\tau(T_1)$ *and* $\tau_2 = \tau(T_1) \cup \{P, Q, R, c_b : b \in \mathbb{B}\}$.

Claim 3.4 *Assume* $\kappa > \aleph_0$. *There is* $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau_1)$ *such that* $\mathrm{Mod}_{\mathbf{t}} = \{N{\restriction}\tau(T_1) : N \models \psi$ *so* $\tau(N) = \tau(\psi) \supseteq \tau_1\}$.

P r o o f . Note that in the proofs of Claims 3.5 & 3.6, we use this proof stating the changes; there $\kappa(T) = \aleph_0$, i.e., T is superstable.

Stage A: Without loss of generality we can replace T by T^{eq} (no need for new elements: we can extend T_1 to have a copy of M^{eq} with new predicates and an isomorphism). The use of T^{eq} is anyhow just for transparency. For $\vartheta = \mathrm{cf}(\vartheta) < \kappa_{\mathrm{reg}}(T)$ choose a sequence $\bar\varphi_\vartheta = \langle \varphi_{\vartheta,i}(x, \bar y_{\vartheta,i}) : i < \vartheta\rangle$ witnessing $\vartheta < \kappa_{\mathrm{reg}}(T)$ equivalently $\vartheta < \kappa(T)$.

Stage B: We remind the reader of the definition of EQ_T (§ 1.3). Let

$$\tau = \tau(T_1) \cup \{P, Q, R, S_{\varphi(\bar x_{[n]},\bar y_{[n]})}, G_n, c_b, Q_\vartheta, <_\vartheta, F_i, P_i, F_{1,i} : b \in \mathbb{B}, i < \kappa, \varphi(\bar x_n, \bar y_n) \in \mathrm{EQ}_T, n < \omega\},$$

where the union is disjoint second set is without repetitions. Here, P_i and Q_ϑ are unary predicates, c_b are individual constants, R is a binary predicate, $S_{\varphi(\bar x_{[n]})}$ is an n-place function for $\varphi(\bar x_{[n]}) \in \mathcal{L}(\tau_T)$, F_i is unary function symbol (for $i < \kappa$), $F_{1,n}$ is an n-place function symbol, and G_n is an n-place function symbol. For a while, fix $M_1 \in \mathrm{Mod}_{\mathbf{t}}$. Note that $\|M_1\| = \|M_1\|^{<\kappa} \geq \lambda(T)$ (by Fact 1.2). We are now going to define a set $\mathscr{M}[M_1]$; in the definition, we do not use "$\kappa > \aleph_0$". Let $M = M_1{\restriction}\tau(T)$; a model N belongs to $\mathscr{M}[M_1]$ if it satisfies the following conditions (a) to (j):

- (a) The model N is a τ-expansion of M_1.
- (b) The sets P^N, Q^N, R, and $\langle c_b^N : b \in \mathbb{B}\rangle$ code $\mathbb{B}_T$ and $\mathrm{uf}(\mathbb{B}_T)$ (cf. the definitions in § 1.3 and condition (ii) in Theorem 2.10).
- (c1) If π is the canonical isomorphism from $\mathbb{B}_T$ onto $\mathbb{B}_M$, $\bar a \in {}^nM$, $\varphi = \varphi(\bar x_{[n]}, \bar y_{[n]}) \in \mathrm{EQ}_n(T)$, and $\pi(c) = \varphi(M, \bar a)$, then $\mathrm{S}^N_\varphi(\bar a) = b^N_c$.
- (c2) We have that $Q^N = \{d_D$: for some $m, D \in \mathrm{uf}(\mathbb{B}_{T,m})\}$ and $R^N = \{(c^N_b, d_D) : b \in \mathbb{B}_{T,m}$ and $D \in \mathrm{uf}(\mathbb{B}_{T,m}), b \in D\}$ where d_D belongs to $\bigcap\{\pi(c) : c \in D\}$.
- (d) The model $N{\restriction}\tau_T$ is κ-saturated.
- (e1) If $\kappa > \aleph_0$ and $\langle a_n : n < \omega\rangle$ is an indiscernible set in M then for some $b, a \mapsto G^N_2(a, b)$ is a one-to-one function from M onto an indiscernible set which includes $\{a_n : n < \omega\}$.[4]
- (e2) If $\kappa = \aleph_0$, $\bar c \in {}^nM$, and $b \in M$ is not algebraic over $\bar c$, then $a \mapsto G^N_{n+2}(a, b, \bar c)$ is a one-to-one function, $G^N_{n+2}(b, b, \bar c) = b$, and $\{G^N_{n+2}(a, b, \bar c) : a \in M\}$ is an indiscernible set over $\bar c$ based on $\bar c$, all in M.
- (f1) The function $F^N_{1,m}$ is a function from ${}^mM$ to Q^N such that if $\bar a \in {}^mM$ then $d = F^N_{1,m}(\bar a)$ is the member of Q^N coding $\mathrm{stp}(\bar a, \varnothing, M)$, i.e., if $D \in \mathrm{uf}(\mathbb{B}_T)$, then we have that $F_{1,m}(\bar a) = d_D$ if and only if $\mathrm{stp}(\bar a, \varnothing, M) = D$.
- (f2) If $D \in \mathrm{uf}(\mathbb{B}_{T,m})$ and $1_{\mathbb{B}_{T,m}} \in D$, then for some $\bar a \in {}^mM$, we have that $F^N_{1,m}(\bar a) = d_D$. (Recall that $\mathbb{B}_{T,m} = \mathbb{B}_T{\restriction}1_{\mathbb{B}_{T,m}}$.)
- (g) For every $i < \kappa$ and $\bar a = \langle a_j : j < i\rangle \in {}^iM$ there is some $b \in P^N_i$ such that for all $j < i$, we have $F^N_j(b) = a_j$.
- (h) The sequence $\langle P^N_i : i < \kappa\rangle$ is a partition of N.
- (i) Fix any regular $\vartheta < \kappa_{\mathrm{reg}}(T)$. If $\eta \in {}^{\vartheta \geq}\|M_1\|$, let $\bar a^\vartheta_\eta := \langle F^N_i(\pi_\vartheta(\eta)) : \ell < \mathrm{lh}(\bar y_{\vartheta,i})\rangle$. Then we have that

[4] Note that when $\kappa > \aleph_0$, we can use a two-place function symbol G.

(i1) $Q^N_\vartheta = \bigcup\{P^N_i : i \leq \vartheta\}$ and $(Q^N_\vartheta, <^N_\vartheta)$ is a partial order which is a tree with ϑ levels isomorphic to $({}^{\vartheta \geq}\|M_1\|, \triangleleft)$ (say, $\pi_\vartheta : {}^{\vartheta >}\|M_1\| \to Q^N_\vartheta$ is such an isomorphism),

(i2) $b_1 <^N_\vartheta b_2$ if and only if for some $i_1 < i_2 < \vartheta$ we have that $b_1 \in P^N_{i_1}$, $b_2 \in P^N_{i_2}$, and $j < i_1$ implies that $F^N_j(b_1) = F^N_j(b_2)$,

(i3) if $i < \vartheta$, $\eta \in {}^i\|M_1\|$, and $\alpha < \beta < \|M_1\|$, then $N \models \neg(\exists x)((\varphi_{\vartheta,i}(x, \bar{a}^\vartheta_{\eta\frown\langle\alpha\rangle}) \wedge \varphi_i(x, \bar{a}^\vartheta_{\eta\frown\langle\beta\rangle}))$,

(i4) if $n < \omega$, $i_0 < \ldots < i_{n-1} < \vartheta$, $\eta_k \in {}^{(i_k)}\|M_\ell\|$ (for $k < n$), and $\eta_0 \triangleleft \eta_1 \triangleleft \ldots \eta_{n-1}$, then $N \models (\exists x)(\bigwedge_{k<n} \varphi_{i_k}(x, \bar{a}^\vartheta_{\eta_k}))$,

(i5) if $i < j \leq \vartheta$ and $\eta \in {}^j\|M_1\|$, then $F_{\vartheta,j,i}(\pi(\eta)) = \pi(\eta\restriction i)$,

(i6) if $c \in Q^N_\vartheta$, then there is some $\eta \in {}^{\vartheta\geq}\|M_1\|$ such that $F^N_\vartheta(c) = \pi_\vartheta(\eta)$ and, writing $j_\eta := \mathrm{lh}(\eta)$, we have
(i6–1) if $i < j_\eta$, then $N \models \varphi_{\vartheta,i}[c, \bar{a}^0_{\eta\restriction(i_1)}]$ and
(i6–2) if $j_\eta < \vartheta$, then $\alpha < \|M_1\|$ implies that $N \models \neg\varphi_{J_\eta}[c, \bar{a}_{\eta\frown\langle\alpha\rangle}]$,

(i7) $F^N_{\vartheta,2}$ is a binary function such that if $\eta \in {}^{\vartheta>}\|M_1\|$, then $\langle F^N_{\vartheta,i}(c, \pi_\vartheta(\eta)) : c \in \|M_1\|\rangle$ lists with no repetitions $\langle \pi_\vartheta(\eta\frown\langle\alpha\rangle) : \alpha < \|M_1\|\rangle$,

(i8) $F^N_{i,1,\vartheta}$ or $F^N_{\vartheta,1}$ is a unary function such that for every $c \in M$, $F_{1,\vartheta}(c) = \pi(\eta)$ for some $\eta \in {}^{\vartheta\geq}\|M_1\|$ and for any $i \leq \vartheta$ and $\nu \in {}^i\|M_1\|$, we have that c realises $\{\varphi_j(x, \bar{a}^\vartheta_{\nu\restriction j}) : j < i$ if and only if $\nu \trianglelefteq \eta\}$.

(j) If $j < \kappa$ has cofinality ϑ and $\langle i_j(\iota) : \iota < \vartheta\rangle$ is an increasing sequence of ordinals with limit j, $b_i \in M_2$ (for $i < j$), $d \in N$, $F^N_{\vartheta,2}(d) \in P^N_\vartheta$, and $\iota < \vartheta \wedge i_* < i_j(\iota)$ implies that $F^N_{i_*}(F^N_\iota(d)) = b_{i_*}$, then there is $d' \in P_j$ such that $i_* < j$ implies that $F_{i_*}(d') = b_{i_*}$.

Finally, we define $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau)$ to be such that a τ-model N satisfies ψ if and only if for a relevant large enough subset Λ of $\mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}](\tau)$ of cardinality $\leq \lambda$, $\psi = \bigwedge\{\varphi \in \Lambda$: if $M_1 \in \mathrm{Mod}_{\mathbf{t}}$, and $N \in \mathscr{M}[M_1]$ then $N \models \varphi\}$. Here, by "Λ is large enough", we mean that for each of the clauses (ψ1) to (ψ4) below, the sentence expressing "the τ-model satisfies clause (ψi)" belongs to Λ. (Note that clause (ψ4) means all of the clauses (a) to (j) of the definition of $\mathscr{M}[M_1]$ above.)

(ψ1) $N\restriction\tau_T$ is a model of T,
(ψ2) $N\restriction\tau_{T_1}$ is a model of T_1,
(ψ3) $N\restriction\tau_{T_1}$ omits p, and
(ψ4) $N \in \mathscr{M}[N\restriction\tau_{T_1}]$.

Now we observe that

$(*)_1$ $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$,
$(*)_2$ every $M_1 \in \mathrm{Mod}_{\mathbf{t}}$ can be expanded to a model from Mod^*_ψ (cf. Definition 2.5; this is more than being a model of ψ!), and
$(*)_3$ if $N \in \mathrm{Mod}_\psi$, then $N\restriction\tau(T_1) \in \mathrm{Mod}_{\mathbf{t}}$.

[Why? For $(*)_1$, read the definition of ψ; for $(*)_2$, read the definitions of $\mathscr{M}[M_1]$ and ψ. For $(*)_3$, ask first: why is $M_1 = N\restriction\tau_{T_1}$ a model of T_1? Since $M_1 \in \mathrm{Mod}_t$ and $N \in \mathscr{M}[M_1]$, we have that $N\restriction\tau(T_1)$ is M_1 by (a) in the definition of $\mathscr{M}[M_1]$. Secondly, why does M_1 omit $p_{\mathbf{t}}$? Recalling clause (ψ3) and the choice of ψ, this should be clear. Thirdly, why is $M = N\restriction\tau_T$ saturated? It realises every $p \in D_m(T) = \mathbf{S}^m(\varnothing, M)$, by condition (e) in the definition of $\mathscr{M}[M_1]$ and it is κ-saturated by condition (d) in the definition of $\mathscr{M}[M_1]$. By condition (e1) of the definition of $\mathscr{M}[M_1]$, every indiscernible subset of cardinality $\aleph_0$ can be extended to one of cardinality $\|M\|$. By the last two sentences, M is saturated by *Case 1* of Fact 1.5.]

So we are done. □

Claim 3.5 *The same statement as in Claim 3.4, but T is superstable and $\lambda(T) \leq \lambda$.*

Proof. We follow the proof of Claim 3.4 with some changes. The proof of "$M = N\restriction\tau_T$ is saturated" inside the proof of $(*)_3$ is different: there is a saturated $M_* \in \mathrm{Mod}_T$ of cardinality $\leq \lambda$ and we can demand on ψ that $N \models \psi$ implies M_* is elementarily embeddable into $N\restriction\tau_T$ and $N\restriction\tau_T$ is $\aleph_0$-sequence homogeneous. Note that if $M_* \prec M \in \mathrm{Mod}_T$ and M is $\aleph_0$-sequence homogeneous implies M is $\aleph_\varepsilon$-saturated (cf. the definitions in § 1.3).

Another difference is that in this case, *Case 2* in Fact 1.5 means that M is saturated if M is $\aleph_\varepsilon$-saturated and for every finite $A \subseteq M$ and $a \in M \backslash \mathrm{acl}(A)$ there is an indiscernible set $I \subseteq M$ over A of cardinality $\|M\|$ based on A (i.e., $\mathrm{Av}(M, I)$ does not fork over A) to which a belongs. Thus, (e2) in the definition of $\mathscr{M}[M_1]$ in the proof of Claim 3.4 implies that M is saturated. □

Claim 3.6 *(1) The same statement as in Claim 3.4, but T is superstable and $2^{\aleph_0} \leq \lambda$. (2) The same statement as in Claim 3.4, but T is superstable and $|D(T)| > |T|$.*

Proof. This proof is similar to the proof of Claim 3.5. However, the problem is how ψ guarantees that $N{\restriction}\tau_T$ is $\aleph_\varepsilon$-saturated. As the model is $\aleph_0$-saturated it suffices to prove that for every m and $D \in \mathrm{uf}(\mathbb{B}_{T,m+1})$ (equivalently, $p \in D_{m+1}(T)$ for some $\bar{a}^\frown\langle c\rangle \in {}^{m+1}N$ realizing p), we have if $N{\restriction}\tau_T \prec M'$ and $c' \in M'$ realises $\mathrm{tp}(c, \bar{a}, N{\restriction}\tau_T)$ then some $c'' \in N{\restriction}\tau_T$ realises $\mathrm{stp}(c', \bar{a}, M')$ in M'. We write $(*)$ for this statement.

Let $p = \mathrm{tp}(c, \bar{a}, M)$; we let $\lambda_* = \lambda(p)$, $\langle E_\alpha(x_0, x_1; \bar{y}_{[m]}) : \alpha < \lambda_*\rangle$ be as in [5, III.5.1, p. 123] and let 2^{λ_*} be the cardinality of the set $\{\mathrm{stp}(c', a, M') : M \prec M', c' \in M, c \text{ realises } \mathrm{tp}(c; \bar{a}, M')\}$ from $(*)$. Hence it suffices to prove $2^\lambda \leq |D(T)|$.

Case 1: $\lambda_* = \aleph_0$. If $2^{\aleph_0} \leq \lambda$, this is easy. If $|D(T)| > |T|$, then for some m, there is an independent sequence $\langle \varphi_n(\bar{x}_{[m]}) : n < \omega\rangle$ of formulas of $\mathcal{L}(\tau_T)$ over T (i.e., if $M \in \mathrm{Mod}_T$ then any non-trivial finite Boolean combination of them is realised in M) and we continue as in the second case.

Case 2: $\lambda_* > \aleph_0$. In this case, by [5, III.5.9 & III.5.10, p. 126] there is an sequence $\langle \varphi_i(x, \bar{y}_{[m]}) : i < \lambda_*\rangle$ of formulas from $\mathcal{L}(\tau_T)$ which is independent over T, so $\mathbb{B}^{\mathrm{fr}}_{\lambda_*}$ is embeddable into $\mathbb{B}_{T,m+1}$. Hence ψ says that the Boolean algebra $\wp(\lambda_*)$ is interpreted in N for every relevant λ_*, but $\lambda_* \leq |T|$.

From this, it is easy to see that ψ ensures $(*)$. □

3.3 Proving claim 2. of Theorem 3.1

Hypothesis 3.7 *In this section, we assume that T is a complete first order theory with $\lambda \geq |T|$, $\lambda^+ \geq \kappa$, and $\mathbb{B} = \mathbb{B}_T$.*

Claim 3.8 *Assume $\psi \in \mathcal{L}_{\lambda^+,\kappa}[\mathbb{B}]$ and $\kappa = \kappa_{\mathrm{reg}}(T) < \infty$, so T is stable. Then there is $\mathbf{t} = (T, T_1, p) \in \mathbf{N}_{\lambda,T}$ such that $\tau(T_1) \supseteq \tau(\psi)$ and $\mathrm{Mod}_{\mathbf{t}} = \{N{\restriction}\tau(\psi) : N \in \mathrm{Mod}_\psi[\mathbb{B}]\}$.*

Proof. We apply Theorem 2.10 to $\mathbb{B}$ and ψ and get a tuple $(\tau_1, T_1, p(*), \bar{F}, \bar{P})$ as in Theorems 2.9 & 2.10; without loss of generality, $\tau_1 \cap \tau(T) = \varnothing$. Now we imitate the proof of Claim 3.4. □

3.4 Elaborating on the missing case

In § 3.2, we covered most theories T, but not all.

Hypothesis 3.9 *In this section, we assume that T is superstable of cardinality λ, $\lambda(T) > \lambda$, $2^{\aleph_0} > \lambda$, and $\lambda \geq |D(T)|$.*

Claim 3.10 *There is an m, a model $M \in \mathrm{Mod}_T$, and $\bar{a} \in {}^mM$ such that $\{\mathrm{stp}(c, \bar{a}, M) : c \in M\}$ is of cardinality $2^{\aleph_0}$.*

Proof. This follows from [5]; for completeness, we elaborate. As $\lambda \geq |D(T)|$ there is an $\aleph_0$-saturated model M of T of cardinality λ. Moreover, without loss of generalityif $A \subseteq M$ is finite and $a \in M$ is not algebraic over A, then there is $I \subseteq M$ of cardinality λ which is indiscernible over A, based on A and $a \in I$.

Also without loss of generality, if $A \subseteq M$ is finite and $\mathscr{P}_{M,A} = \{\mathrm{stp}(c, A, M') : M \prec M' \text{ and } c \in M'\}$ has cardinality $\leq \lambda$ then all of them are realised in M. Since $\lambda(T) > \lambda$, M is not saturated, hence for some $\bar{a} \in {}^{\omega>}M$, we have that $|\mathscr{P}_{M,\bar{a}}| \geq \lambda$, which easily implies $|\mathscr{P}_{M,\bar{a}}| \geq 2^{\aleph_0}$. If $|\mathscr{P}_{M,\bar{a}}| > 2^{\aleph_0}$, then by [5, § III.5], we get a contradiction to $\lambda \geq |D(T)|$. □

Definition 3.11 For any model M and a sequence $\bar{a}$ from M (or a set), let $\mathbb{B}_{M,\bar{a},m}$ be the Boolean algebra of subsets of ${}^mM$ of the form $\varphi(M, \bar{c})$, where $\varphi(\bar{x}_{[m]}, \bar{z}) \in \mathcal{L}(\tau_M)$, $\bar{b} \in {}^{\mathrm{lh}(\bar{z})}M$ and $\varphi(\bar{x}, \bar{c})$ is almost over $\bar{a}$ which means: for some $\vartheta(\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{z}) \in \mathcal{L}(\tau_M)$, we have that

1. in M, $\vartheta(\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{a}) \vdash \varphi(\bar{x}_{[m]}, \bar{c}) \equiv \varphi(\bar{y}_{[m]}, \bar{c})$ and
2. $\vartheta(\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{a})$ defines in M an equivalence relation with finitely many equivalence classes.

Claim 3.12 *Let T be a theory as in Hypothesis 3.9, let M, $\bar{a}$, and m be as in Claim 3.10, and $\mathbb{B} = \mathbb{B}_{M,\bar{a},m}$. Then the results of Claim 3.4 & Theorem 3.1 hold if we use $\mathbb{B}$ instead of $\mathbb{B}_T$.*

Proof. As above, really $m = 1$ suffices; in particular, if $p \in \mathbf{S}(\bar{a}, M)$, $\bar{a} \in {}^{m}M$, and $M \in \mathrm{Mod}_T$, then $\lambda_*(p) \leq \aleph_0$ (otherwise by [5, Lemmas III.5.9, III.5.10, & III.5.11], we have $|\mathbf{S}^{2m}(\bar{a}, m)| \geq 2^{\lambda_*(p)} > \lambda$, contradiction). □

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