Classifying classes of structures in model theory

Saharon Shelah

The Hebrew University of Jerusalem, Israel, and Rutgers University, NJ, USA

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Overview of the talk

We shall try to explain a new and surprising result that strongly indicates that there is more to be discovered about so-called dependent Theories; and we introduce some basic definitions, results and themes of model theory needed to explain it.
We shall not mention history nor any of the illustrious researchers who have contributed.
The talk will have two rounds:

- First round: A presentation of the result without details.
- Intermezzo: Some very basic notions of first order logic.
- Second round: Stability and Dependence again.

So let us start round one.

Classes of Structures

Groups theory investigates groups. Model theory investigates classes of structures, such as:

- K_{ring}, the class of rings,
- K_{field} the class of fields,
- K_{group}, the class of groups.

Central prototypical examples (which will appear later) are:

1. K_{lin}, the class of infinite linear orders. It turns out that it is more convenient to concentrate our attention to the (equally complicated) subclass K_{dlo} of dense linear orders without endpoints
2. K_{rg}, the class of random graphs. (I.e., the graphs such that any two disjoint finite sets A, B of nodes can be separated by a node x, i.e., x is connected to every member of A and not connected to any member of B.)

Note that in this talk we will only be interested in infinite structures.

Dividing Lines

Meta-Question

Can we find "useful/strong" dividing lines for the family of "reasonable" classes?

Our expectations:

- A high class has to contained many members (up to isomorphism) , or complicated ones, or members which are rigid in suitable sense;
- On the low side, we can prove strong negations of these properties (i.e., few members, not complicated)
  moreover we should understand the members of this class, they have a structure theory or classification (such as dimension)

A priori it is not clear that such dividing lines exist.
This very general setup covers a lot of ground, but it seems that we can say very little. Anyhow, we restrict ourselves to so called elementary classes, still a very comprehensive context, which we will explain now.

First order logic: Alphabet and Sentences

We concentrate exclusively on first order logic (we have much to say on other situations, but not here and now):
We first chose a suitable "alphabet" or "vocabulary", e.g.:

- For orders, we use the symbol $<$.
- For groups, we use symbols for multiplication, inverse and the unit.
- For fields, we use $+, \times, 0, 1$.
- For rings (with unit) the same as for fields.

We then define first order sentences: They can use the given alphabet, the symbol $=$, the connectives "and", "or", "implies", "not"; and also "for all x" and "there is an x", where x varies over the elements of the structure.
We are not allowed to use, e.g., infinite sentences, or "for all subsets A" (where A varies over the subsets of the structure).
For example, in the language of groups, the group axioms are first order sentences, but not the statements "every element has finite order" or "the group is simple".

Elementary classes

Definition

- The theory of a structure M is the set of first order sentences that are true in M.
- The elementary class K of a structure M consists of all structures N that have the same theory as M.

In this talk, we will only study elementary classes.

Examples:

- In the language of orders: The order $\mathbb{Q}$ defines an elementary class K_{dlo} called "dense linear orders". This class also contains the order $\mathbb{R}$.
- In the language of fields: $\mathbb{C}$ defines the elementary class "algebraically closed fields of characteristic 0". This class also contains the field of algebraic complex numbers.

Stability

A central dividing line is:

Definition

We say K is stable if it is

- neither as bad (i.e., as complicated) as K_{rg} (random graphs)
- nor as bad as K_{dlo} (dense linear orders).

We know that this is in fact an excellent dividing line:

- If an (elementary) class K is unstable, then it is complicated and has "non-structure" by various yardsticks.
- If K is stable, we have some simple structure (similar to dimension, and a kind of free amalgamation, called non-forking).

Examples:

- The "algebraically closed fields of characteristic 0" are stable (in a very strong way).
- "Dense linear orders" are unstable.
- The theory of the ring $\mathbb{N}$ (i.e., number theory) is unstable.

Counting types

One reason why stability is such a good dividing line, is that it is connected with counting so-called complete types. More on types later, for now just an example:

Every real $r \in \mathbb{R} \setminus \mathbb{Q}$ defines a type over the dense linear order $\mathbb{Q}$: Basically, the type consists of the statements "$x < q$" for $q > r$ and "$x > q$" for $q < r$. There are other types over $\mathbb{Q}$, such as "$x > q$" for all q (i.e., $+\infty$).

If we define types appropriately, we get:

Theorem

K is stable iff for every $M \in K$ there are "few" complete types over M.

(Few means: at most $||M||^{\aleph_0}$ many.)

Dependent theories

As much as the stable/unstable dividing line is great, we would like the positive (or: "low") side to cover more ground. This motivates

Definition

K is dependent, if it is not as bad/complicated as K_{rg} (random graphs).

So dependent meets "half the requirement for being stable".
On this family we know much less, still

Thesis

The dividing line dependent/independent is important.

The theorem promised in the beginning says:

The Recounting Theorem

If we count the complete types suitably (i.e., count them modulo some equivalence), then the dependent classes K are exactly the ones with few types over nice enough $M \in K$.

An example: fields

Question

For which fields F is their elementary class stable? dependent?

Stable fields include:

- For any p, the class of algebraically closed fields of characteristic p.
- For $p > 0$, the class of separably closed fields of characteristic p.
- Any finite field (but this is dull, since the elementary class only has one element modulo isomorphism).

Are there any more stable fields? We do not know.
Is the family of dependent fields significantly wider family of classes?
Dependent fields include:

- The reals,
- Many formal power series fields,
- the p-adics.

Applications

Disclaimer: For me, applications are not the aim, or "the test" for the merits of a theory, but naturally applications are expected; so here is a

Theorem

There are substantial applications.

E.g., see the work on the Mordell-Lang conjecture.

Thesis

Looking at the behaviour of the structures M
in a class K which have some uncountable cardinal κ
will help finding such dividing lines,
which might turn out helpful
even for those who (unlike me) have no interest in such question per se.

Other examples of dividing lines that are easily explained:

Categoricity: For every (elementary) class K, one of the following occurs:

1. For every uncountable cardinal λ there is (modulo isomorphism) exactly one structure M in K which has cardinality λ.
2. For no uncountable cardinal λ does the above hold.

Main gap For every (elementary) class K, one of the following occurs:

1. for every cardinal λ, the number of structures in K which have cardinality λ is maximal, i.e., there are 2^λ many.
2. For every cardinal $\lambda = \aleph_\alpha$, the number above is bounded by a fixed function of λ (which is much smaller than 2^λ for "typical" λ).

(So structures are in some appropriate sense not more complicated than trees.)

Such uniform dichotomic behaviour indicate it is a real dividing lines.

Usually, proving there are few structure indicate that we can understand them.

Intermezzo

The first order language exemplified on fields. Some basic notions of first order logic: elementary submodels and types.

First order language for field: definable sets

Given a field M we consider the naturally defined subsets of M, and more generally of M^n, where the definitions can use parameters from a subset $A \subseteq M$.

- Most widely used: The set of those $\bar{x}$ solving an equation $\sigma(\bar{x}) = 0$ where σ is a polynomial with coefficients from A.
- But we may also look at the set of solutions of k many equations, the set of non-solutions and, e.g., the set of $\bar{y}$ for which the following equation is solvable: $\sigma_1(\bar{x}, \bar{y}) = 0$.
- Generally, the family of first order definitions $\varphi = \varphi(\bar{x})$ is the closure of the family of "roots of polynomials" by intersection (of two), complement and projections (ie the set on n-tuples which can be lengthen to an $n + m$-tuples satisfying a formula φ).
- Again, note that conditions speaking about infinite sequences and about "for every subset of M" are not allowed.

This family has better closure properties than, say, the roots of polynomials; hence sometimes you better investigate it, even if you are interested just in polynomials.

The elementary class of a field

Definition

Let M be a field.

1. $\varphi[M]$ is the set of tuples (of appropriate length) satisfying φ in M.
2. The elementary class K_M is the class of the fields N such that for every φ we have $\varphi[N] = \emptyset$ iff $\varphi[M] = \emptyset$.
3. $M \prec N$, i.e., M is elementary submodel of N, iff $\varphi[M] = \varphi[N] \restriction M$ for every relevant φ.

It is easy to see that $M \prec N$ implies $N \in K_M$.

Complete Types

Back to general first order classes. To understand the notion "stable", we need other fundamental notions: elementary submodel, and complete types over a structure.

A first approximation to the definition is:

Definition

- Let N be a structure, $a \in N$, $M \subseteq N$. The type of a over M, $tp(a, M, N)$, is the set of formulas $\varphi(x)$ with parameters in M such that $\varphi(a)$ holds in N.
- If M is a substructure of N (e.g., a subgroup), then M is elementary submodel of N if all (first order) sentences with parameters in M hold in M iff they hold in N.
- For a structure M, let $\mathbf{S}(M)$ be the family of types $tp(a, M, N)$ for any $M \prec N$ and $a \in N$.

Examples of types

Let $M \subseteq N$ be two models, and let $a \in N$.
Recall that the type of a over M, $tp(a, M, N)$, is the set of sentences $\varphi(x)$ with parameters in M such that $\varphi(a)$ holds in N.
Let us ignore the types of elements $a \in M$, as they are easy to understand; so assume $a \in N \setminus M$.

- Assume that $M \subseteq N$ are algebraically closed fields.
  Then all elements $b \in N \setminus M$ have the same type, so there is only one nontrivial type over M.
- If M is a dense linear order, there are always many nontrivial types: for example, every real number determines a type over $\mathbb{Q}$.
- Similarly for random graphs; every partition of a random graph M into two disjoint sets determines a type.

Another definition of types

There is an alternative, indirect definition for $\mathbf{S}(M)$, the family of complete types of M, which might be more accessible:
First, we define "f is an elementary embedding of M into N" by : f is an isomorphism for M onto some M' such that $M' \prec N$.

Definition

$\mathbf{S}(M)$ consists of all (a, M, N) with $M \prec N$ and $a \in N$, where we identify $\mathrm{tp}(a_1, M, N_1)$ and $\mathrm{tp}(a_2, M, N_2)$ iff there is a mapping fixing M which takes $a_1 \mapsto a_2$.
In more detail: if there are M^+, f_1, f_2 such that

- $M \prec M^+$,
- f_1 is an elementary embedding of N_1 into M^+ over M
- f_2 is an elementary embedding of N_2 into M^+ over M
- and $f_1(a_1) = f_2(a_2)$.

Round 2

We again look at stability and dependency.

The stable/unstable division

This is a major, well researched dividing line and, as mentioned, very useful.

Recall that $\mathbf{S}(M) = \{\mathrm{tp}(a, M, N) : M \preceq N, a \in N\}$

Thesis

If in K there are Ms with large $\mathbf{S}(M)$, say $|\mathbf{S}(M)| > |M|$, it is a sign of complexity. If there are few then we can expect to understand them.

Definition/Theorem

- K is stable in an infinite cardinal λ iff:
  ($M \in K$, M has λ elements) implies ($\mathbf{S}(M)$ has λ elements).
- K is stable iff it is stable in some λ.
- Equivalently, K is stable iff ($M \in K$, M has λ elements) implies ($\mathbf{S}(M)$ has at most $\lambda^{\aleph_0}$ elements) (for all λ).
- Equivalently, K is stable iff it is neither as complicated as K_{lin} nor as K_{rg}.

Dependence

Definition

K is dependent iff for some formula $\varphi = \varphi(\bar{x}, \bar{y})$ and $M \in K$ considering $\varphi[M]$ as a graph, it has an induced sub-graph which is random.

Question

But is dependent/independent a significant dividing line? E.g., can we understand dependent classes? Are non-dependent ones complicated?

Dense linear orders: Few types modulo conjugacy

Lately have tried to recount $\mathbf{S}(\mathbb{Q},<)$; recall there were continuum many members (one for each irrational, at least). But this time I succeed to count only up to 6!
How come? This time we count only up to conjugacy. Now for any two irrational numbers b, c there is an automorphism of the rational order taking the cut induced by b to the cut induced by c. So all the irrationals contribute just one type up to conjugacy.
What about others? there are

1. the trivial types ($x_0 = a, a \in \mathbb{Q}$),
2. $+\infty$,
3. $-\infty$,
4. $a + \epsilon$, ϵ "infinitesimal"
5. $a - \epsilon$.

Altogether six families, giving six conjugacy classes

Random graphs: Many types modulo conjugacy

Generally we can consider only models with lots of automorphisms, so-called saturated models.
So maybe for all elementary classes we get few types up to conjugacy?
But consider the class K_{rg} of random graphs:
For any $M \in K_{\mathrm{rg}}$ and $A \subseteq M$ recall that there is a type coding A, so we should count the number of isomorphism types of the pairs (M, A) , and it is not hard to see that it is large.

For transparency assume GCH, the generalized continuum Hypothesis, i.e., assume that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all α. Then every elementary class K has in cardinality $\lambda = \aleph_{\alpha+1}$ a (unique) so-called saturated model $M_{K,\lambda}$.
So our question is:

Question

Given K, when is $\mathbf{S}(M_{K,\aleph_{\alpha+1}})$/conj small?

There are examples showing that possibly the number may be:

- small = constant,
- medium $\sim |\alpha|$,
- large $= 2^{\aleph_{\alpha+1}}$.

Why?
What are the possibilities?
Let us consider some examples:

Stable classes

Let K be an algebraically closed field (of characteristic 0, say). We have the following types modulo conjugacy:

- the algebraic elements, (countably many types)
- transcendental elements inside $M_{K,\lambda}$, ($||M||$ many types, but only one conjugacy class)
- transcendental elements outside of $M_{K,\lambda}$. (Only one type)

So:

Example

For the class K of algebraically closed fields, we get $|\mathbf{S}(M_{K,\aleph_{\alpha+1}})/\text{ conj}| = \aleph_0$.

In fact:

Theorem

If K is stable, then the number of types/conj is $\leq 2^{\aleph_0}$ and is constant.

Unstable classes

As mentioned, K_{rg} has many types, more generally:

Theorem

If T is independent (= as bad as T_{rg}) then

$$|\mathbf{S}(M_{K_T,\aleph_{\alpha+1}})/\ conj| \geq 2^{\aleph_{\alpha+1}}$$

We are left with the main question: What about the (unstable but) dependent classes?
The obvious example is K_{dlo}: A cut has two cofinalities. So we have two cardinals, one is λ by saturation, the other is any cardinal $\aleph_\beta \leq \aleph_{\alpha+1}$.
Hence $|\mathbf{S}(K_{\mathrm{dlo}}, \aleph_{\alpha+1})/\ \mathrm{conj}| \geq |\alpha|$. A more careful analysis shows that this lower bound is (almost) also an upper bound:

Example

$$|\mathbf{S}(K_{\mathrm{dlo}}, \aleph_{\alpha+1})/\,\mathrm{conj}| \sim |\alpha|$$

It turns out that there is a general theorem:

Main Theorem: Recounting Theorem

Let K be dependent, and $\lambda = \aleph_{\alpha+1}$ be large enough ($> \beth_\omega$). Then

$$|\mathbf{S}(M_{K,\aleph_{\alpha+1}})/\operatorname{conj}| \leq |\alpha|^{\aleph_0}$$

and $|\mathbf{S}(M_{K,\aleph_{\alpha+1}})/\operatorname{conj}| \geq |\alpha|$ if K is unstable.

Thesis

The theorem above is a strong indication that being dependent is a major dividing line, that there is much to be understood on dependent classes and more non-structure about independent classes

Proving this we are forced to understand structures in such classes.