THERE ARE NOETHERIAN DOMAIN IN EVERY CARDINALITY WITH FREE ADDITIVE GROUP SH217

Saharon Shelah and Gershor Sageen

The Hebrew University of Jerusalem
Einstein Institute of Mathematics
Edmond J. Safra Campus, Givat Ram
Jerusalem 91904, Israel

Department of Mathematics
Hill Center-Busch Campus
Rutgers, The State University of New Jersey
110 Frelinghuysen Road
Piscataway, NJ 08854-8019 USA

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Theorem. *There are Noetherian rings (in fact domains) with a free additive group, in every infinite cardinality.*

Remark. 1) For $\aleph_1$ this was proved by O'Neill.
2) The work was done in Sept., '83.
3) We thank Fuchs for suggesting to us the problem.
4) This is an expanded version of [SgSh 217] which appears in the Notices of AMS.

Sketch of Proof. Let $\mathfrak{Z}$ be the ring of integers, X a set of distinct variables, $Z[X]$ the ring of polynomials over $\mathfrak{Z}, \mathfrak{Z}(X)$ its field of quotients and R_X the additive subgroup of $\mathfrak{Z}(X)$ generated by $\{p/q : p \in \mathfrak{Z}[X], q \in \mathfrak{Z}[X], p$ not divisible (nontrivially) by any integer$\} \subseteq \mathfrak{Z}(X)$. It is known that R_X is a Noetherian domain. Let for a ring R, R^+ be its additive group. For $Y \subseteq X$ we can define $Z[Y], Z(Y), R_Y$ similarly.

Lemma. *1) R^+_X is a free abelian group.*
2) If $n \geq 0, Y \subseteq X, x(1), \ldots, x(n) \in X \backslash Y$ pairwise distinct, $W = \{x(1), \ldots, x(n)\}$, $W(\ell) = W - \{x(\ell)\}$ then $R^+_{W \cup Y} / \sum\limits_{\ell=1}^{n} R^+_{W(\ell) \cup Y}$ is a free abelian group.

Proof. 1) Follows by 2) for $n = 0, Y = X$.
2) This is phrased because it is the natural way to prove 1) by induction on $|Y|$, for all n simultaneously (a degenerated case of [Sh 87a]). If $|Y| > \aleph_0$, let $Y = \{y(\alpha) : \alpha < \lambda\}$ with no repetitions, so $\lambda = |Y|, Y_\alpha = \{y(i) : i < \alpha\}$. It suffices for each $\alpha < \lambda$ to prove that $G_\alpha =: (R^+_{Y_{\alpha+1}} + \sum\limits_{\ell=1}^{n} R^+_{Y \cup W(\ell)}) / (\sum\limits_{\ell=1}^{n} R^+_{Y_{\alpha+1} \cup W(\ell)} + R^+_{Y_\alpha \cup W})$ is free.

We now show that G_α is isomorphic to $G'_\alpha = R^+_{Y_{\alpha+1}} / (\sum\limits_{\ell=1}^{n} R^+_{Y_{\alpha+1} \cup W(\ell)} + R^+_{Y_\alpha \cup W})$. For this it is enough to show $\big(\sum\limits_{\ell=1}^{n} R^+_{Y_\alpha \cup W(\ell)}\big) \cap R^+_{Y_{\alpha+1}} = \sum\limits_{\ell=1}^{n} R^+_{Y_{\alpha+1} \cup W(\ell)}$, as the right side is included in the left side trvially we have to show $\sum\limits_{\ell=1}^{n} \frac{p_\ell}{q_\ell} \in \sum\limits_{\ell=1}^{n} R^+_{Y_{\alpha+1} \cup W(\ell)}$ if $\frac{p_\ell}{q_\ell} \in R^+_{Y_1 \cup W(\ell)}$ and $\sum\limits_{\ell=1}^{n} \frac{p_\ell}{q_\ell} \in R^+_{Y_{\alpha+1} \cup W}$ which is easy by projections). But G'_α is free by induction hypothesis.

The next claim completes the case "y countable".

Claim. *If* $Y \cup \{x(1), \ldots, x(n)\} \subseteq X, x(\ell) \in X \backslash Y$ *distinct,* $G = R^+_{W \cup Y}$, $I = \sum_i I_i, I_i = R^+_{W(i) \cup Y}$ *then* G/I *is free, when* Y *is countable.*

Proof. It suffices to prove:

(a) G/I is torsion free

(b) if $a_1, \ldots, a_k \in G/I$ are independent, then $\{m \in Z^+$:there are $\langle q_1, \ldots, q_k \rangle \in L$ such that $\sum_{i=1,\ldots,k} q_i a_i$ is divisible by m in $G/I\}$ is finite, where $L = \{\langle q_1, \ldots, q_k \rangle : q_i \in Z$, not all zero and they are with no common divisor$\}$.

Let $x_1(q) \in X$ for $q = 1, \ldots, n$ be new distinct variable and let $V = \{x_1(1), \ldots, x_1(n)\}$. For $u \subseteq \{1, \ldots, n\}$ let us define $h_u : R_{V \cup W \cup Y} \to R_{V \cup Y}$ an isomorphism $h_u(y) = y$ for $y \in Y, h_u(x(q)) = x_1(q)$ if $q \in u$, $h_u(x(q)) = x(q)$ if $q \notin u$. So let $a_1 + I, \ldots, a_k + I$ be independent.
Suppose $\langle q_1, \ldots, q_k \rangle \in L, m_0 m_1 \in Z \backslash \{0\}, m_0 m_1$ divides $\sum_i m_0 q_i a_i + I$. So for some $s \in R_{W \cup Y}$ and $p_\ell \in I_\ell$ for $\ell = 1, \ldots, n$ we have: $\sum_i m_0 q_i a_i = m_0 m_1 s + \sum_{\ell=1,\ldots,n} p_\ell$. Let u vary on subsets of $\{1, \ldots, n\}, b_u = \sum_u (-1)^{|u|} h_u(a_\ell) \in R_{V \cup W \cup Y}$, so $\sum_i m_0 q_i b_i = \sum_u (\sum_i m_0 q_i (h_u(a_i)) = m_0 m_1 \sum_u h_u(s) + \sum_{\ell=1,\ldots,n} \sum_u h_u(p_\ell))$. However for each $\ell = 1, \ldots, n$ we have $\sum_u h_u(p_\ell)$ is zero (as $x(\ell)$ does not appear in it). So $\sum_i m_0 q_i b_i$ is divisible by $m_0 m_0$ in $R^+_{V \cup W \cup Y}$. As $R^+_{V \cup W \cup Y}$ is free, it suffices to prove $\{b_i : i = 1, \ldots, k\}$ is independent, equivalently they are linearly independent (over the rationals) in $Z(Y \cup W \cup V)$. But, if not, we can substitute suitable numbers for $x_1(1), \ldots, x_1(n)$ and get contradiction to "$\{a_i + I : i = 1, \ldots, n\}$ is independent." That is let R' be a subring of $R_{V \cup W \cup Y}$ generated by $Z[X] \cup \{\frac{1}{q_1}, \ldots, \frac{1}{q_m}\}$ for some $m, q, \ldots, q_\ell \in Z[X]$ such that $h_u(a_i) \in R'$. Let g be a homomorphism from R' to $R_{W \cup Y}$ which is the identity on $R_{W \cup Y}$ and maps each $x_1(q)$ to an integer (so we require from $\langle g(x_i(q)) : q = 1, \ldots, n \rangle$ to make some finitely many polynomials over the integers nonzero which is possible). Now $\ell \in u \subseteq \{1, \ldots, n\} \Rightarrow h_u(a_i) \in I_\ell$. So it is enough to show that $\langle g(b_i) : i = 1, \ldots, k \rangle$ is linearly independent. But $g(b_i) = \sum_u g(h_u(a_i)) \in g h_\emptyset(a_i) + I = g(a_i) + I = a_i + I$.

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