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Strong Partition Realations Below the Power Set: Consistency Was Sierpinski Right? II.

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We continue here [Sh276] (see the introduction there) but we do not relay on it. The motivation was a conjecture of Galvin stating that $2^\omega \geq \omega_2$ + $\omega_2 \to [\omega_1]^n_{h(n)}$ is consistent for a suitable $h : \omega \to \omega$. In section 5 we disprove this and give similar negative results. In section 3 we prove the consistency of the conjecture replacing ω_2 by 2^ω, which is quite large, starting with an Erdős cardinal. In section 1 we present iteration lemmas which needs when we replace ω by a larger λ and in section 4 we generalize a theorem of Halpern and Lauchli replacing ω by a larger λ.

0. Preliminaries

Let $<^*_\chi$ be a well ordering of $\mathrm{H}(\chi)$, where $\mathrm{H}(\chi) = \{x :$ *the transitive closure of* x *has cardinality* $< \chi\}$, agreeing with the usual well-ordering of the

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ordinals. P (and Q, R) will denote forcing notions, i.e. partial orders with a minimal element $\emptyset = \emptyset_P$.

A forcing notion P is λ-closed if every increasing sequence of members of P, of length less than λ, has an upper bound.

If $P \in \mathrm{H}(\chi)$, then for a sequence $\bar{p} = \langle p_i : i < \gamma \rangle$ of members of P let $\underset{\sim}{\alpha} = \underset{\sim}{\alpha}_{\bar{p}} \stackrel{\text{def}}{=} \sup\{\underset{\sim}{j} : \{\beta_j : j < \underset{\sim}{j}\}$ has an upper bound in $P\}$ and define the *canonical upper bound of* $\bar{p}$, $\&\bar{p}$ as follows:

(a) the least upper bound of $\{p_i : i < \underset{\sim}{\alpha}\}$ in P if there exists such an element,

(b) the $<^*_\chi$-first upper bound of $\bar{p}$ if (a) can't be applied but there is such,

(c) p_0 if (a) and (b) fail, $\gamma > 0$,

(d) $\emptyset_P$ if $\gamma = 0$.

Let $p_0 \& p_1$ be the canonical upper bound of $\langle p_\ell : \ell < 2 \rangle$.

Take $[a]^\kappa = \{b \subseteq a : |b| = \kappa\}$ and $[a]^{<\kappa} = \bigcup_{\theta < \kappa} [a]^\theta$.

For sets of ordinals, A and B, define $H^{OP}_{A,B}$ as the maximal order preserving bijection between initial segments of A and B, i.e, it is the function with domain $\{\alpha \in A : \mathrm{otp}(\alpha \cap A) < \mathrm{otp}(B)\}$, and $H^{OP}_{A,B}(\alpha) = \beta$ if and only if $\alpha \in A$, $\beta \in B$ and $\mathrm{otp}(\alpha \cap A) = \mathrm{otp}(\beta \cap B)$.

Definition 0.1 $\lambda \rightarrow^+ (\alpha)^{<\omega}_\mu$ *holds provided whenever F is a function from $[\lambda]^{<\omega}$ to μ, $C \subseteq \lambda$ is a club then there is $A \subseteq C$ of order type α such that* $[w_1, w_2 \in [A]^{<\omega}, |w_1| = |w_2| \Rightarrow F(w_1) = F(w_2)]$.

Definition 0.2 $\lambda \rightarrow [\alpha]^n_{\kappa,\theta}$ if for every *function F from* $[\lambda]^n$ *to* κ there is $A \subseteq \lambda$ *of order type α such that $\{F(w) : w \in [A]^n\}$ has power $\leq \theta$.*

Definition 0.3 *A forcing notion P satisfies the Knaster condition (has property K) if for any $\{p_i : i < \omega_1\} \subset P$ there is an uncountable $A \subset \omega_1$ such that the conditions p_i and p_j are compatible whenever $i, j \in A$.*

1. Introduction

Concerning 1.1–1.3 see Shelah [Sh80], Shelah and Stanley [ShSt154, 154a].

Definition 1.1. *A forcing notion Q satisfies $*^{\varepsilon}_{\mu}$ where ε is a limit ordinal $< \mu$, if player I has a winning strategy in the following game:*

Playing: *the play finishes after ε moves.*
in the α^{th} the move:

Player I – if $\alpha \neq 0$ he chooses $\langle q^{\alpha}_{\zeta} : \zeta < \mu^+ \rangle$ such that $q^{\alpha}_{\zeta} \in Q$ and $(\forall \beta < \alpha)(\forall \zeta < \mu^+) p^{\beta}_{\zeta} \leq q^{\alpha}_{\zeta}$ and he chooses a regressive function $f_{\alpha} : \mu^+ \to \mu^+$ (i.e. $f_{\alpha}(i) < 1 + i$); if $\alpha = 0$ let $q^{\alpha}_{\zeta} = \emptyset_Q$, $f_{\alpha} = \emptyset$.

Player II – he chooses $\langle p^{\alpha}_{\zeta} : \zeta < \mu^+ \rangle$ such that $q^{\alpha}_{\zeta} \leq p^{\alpha}_{\zeta} \in Q$.

The outcome: *Player I wins provided whenever $\mu < \zeta < \xi < \mu^+$, $\mathrm{cf}(\zeta) = \mathrm{cf}(\xi) = \mu$ and $\wedge_{\beta < \varepsilon} f_{\beta}(\zeta) = f_{\beta}(\xi)$ the set $\{p^{\alpha}_{\zeta} : \alpha < \varepsilon\} \cup \{p^{\alpha}_{\xi} : \alpha < \varepsilon\}$ has an upper bound in Q.*

Definition 1.2. *We call $\langle P_i, Q_j : i \leq i(*),\ j < i(*) \rangle$ a $*^{\varepsilon}_{\mu}$-iteration provided that:*

(a) it is a $(< \mu)$-support iteration (μ is a regular cardinal)

(b) if $i_1 < i_2 \leq i(*)$, $\mathrm{cf}\, i_1 \neq \mu$ then P_{i_2}/P_{i_1} satisfies $*^{\varepsilon}_{\mu}$.

The Iteration Lemma 1.3. *If $\bar{Q} = \langle P_i, Q_j : i \leq i(*),\ j < i(*) \rangle$ is a $(< \mu)$-support iteration, (a) or (b) or (c) below hold, then it is a $*^{\varepsilon}_{\mu}$-iteration.*

(a) $i(*)$ is limit and $\bar{Q} \restriction j(*)$ is a $*^{\varepsilon}_{\mu}$-iteration for every $j(*) < i(*)$.

(b) $i(*) = j(*) + 1$, $\bar{Q} \restriction j(*)$ is a $*^{\varepsilon}_{\mu}$-iteration and $Q_{j(*)}$ satisfies $*^{\varepsilon}_{\mu}$ in $V^{P_{j(*)}}$.

(c) $i(*) = j(*) + 1$, $\mathrm{cf}\, j(*) = \mu^+$, $\bar{Q} \restriction j(*)$ is a $*^{\varepsilon}_{\mu}$-iteration and for every successor $i < j(*)$, $P_{i(*)}/P_i$ satisfies $*^{\varepsilon}_{\mu}$.

Proof. Left to the reader (after reading [Sh80] or [ShSt154a]).

Theorem 1.4. *Suppose $\mu = \mu^{<\mu} < \chi < \lambda$, and λ is a strongly inaccessible k^2_2-Mahlo cardinal, where k^2_2 is a suitable natural number (see 3.6(2) of [Sh289]), and assume $V = L$ for the simplicity. Then for some forcing notion P:*

(a) P is μ-complete, satisfies the μ^+-c.c., has cardinality λ, and $V^P \models$ "$2^{\mu} = \lambda$".

(b) $\Vdash_P \lambda \to [\mu^+]^2_3$ and even $\lambda \to [\mu^+]^2_{\kappa,2}$ for $\kappa < \mu$.

(c) if $\mu = \aleph_0$ then $\Vdash$ "MA_{χ}".

(d) if $\mu > \aleph_0$ then: $\Vdash_P$ "for every forcing notion Q of cardinality $\le \chi$, μ-complete satisfying $*^{\varepsilon}_{\mu}$, and for any dense sets $D_i \subseteq Q$ for $i < i_0 < \lambda$, there is a directed $G \subseteq Q$, $\wedge_i G \cap D_i \neq \emptyset$".

As the proof is very similar to [Sh276], (particularly after reading section 3) we do not give details. We shall define below just the systems needed to complete the proof. More general ones are implicit in [Sh289].

Convention 1.5. *We fix a one to one function $Cd = Cd_{\lambda,\mu}$ from ${}^{\mu>}\lambda$ onto λ.*

Remark. *Below we could have $\mathrm{otp}(B_x) = \mu^+ + 1$ with little change.*

Definition 1.6. *Let $\mu < \chi < \kappa \le \lambda$, $\lambda = \lambda^{<\mu}$, $\chi = \chi^{<\mu}$, $\mu = \mu^{<\mu}$.*

1) *We call x a $(\lambda, \kappa, \chi, \mu)$-precandidate if $x = \langle a^x_u : u \in I_x \rangle$ where for some set B_x (unique, in fact):*
   - (i) *$I_x = \{s : s \subseteq B_x, |x| \le 2\}$,*
   - (ii) *B_x is a subset of κ of order type μ^+,*
   - (iii) *a^x_u is a subset of λ of cardinality $\le \chi$ closed under Cd,*
   - (iv) *$a^x_u \cap B_x = u$,*
   - (v) *$a^x_u \cap a^x_v \subseteq a^x_{u \cap v}$,*
   - (vi) *if $u, v \in I_x$, $|u| = |v|$ then a^x_u and a^x_v have the same order type (and so $H^{OP}_{a^x_u, a^x_v}$ maps a^x_u onto a^x_v),*
   - (vii) *if $u_\ell, v_\ell \in I_x$ for $\ell = 1, 2$, $|u_1| = |v_1|$, $|u_2| = |v_2|$, $|u_1 \cup u_2| = |v_1 \cup v_2|$, $H^{OP}_{a^x_{u_1} \cup a^x_{u_2}, a^x_{v_1} \cup a^x_{v_2}}$ maps u_ℓ onto v_ℓ for $\ell = 1, 2$ then $H^{OP}_{a^x_{u_1}, a^x_{v_1}}$ and $H^{OP}_{a^x_{u_2}, a^x_{v_2}}$ are compatible.*

2) *We say x is a $(\lambda, \kappa, \chi, \mu)$-candidate if it has the form $\langle M^x_u : u \in I_x \rangle$ where*

(α)
   - (i) *$\langle |M^x_u| : u \in I_x \rangle$ is a $(\lambda, \kappa, \chi, \mu)$-precandidate (with $B_x \stackrel{\text{def}}{=} \cup I_x$)*
   - (ii) *L_x is a vocabulary with $\le \chi$-many $< \mu$-ary placespredicates and function symbols,*
   - (iii) *each M^x_u is an L_x-model,*
   - (iv) *for $u, v \in I_x$, $|u| = |v|$, $M^x_u \restriction (|M^x_u| \cap |M^x_v|)$ is a model, and in fact an elementary submodel of M^x_v, M^x_u and $M^x_{u \cap v}$.*

(β)
   - ($*$) *for $u, v \in I_x$, $|u| = |v|$, the function $H^{OP}_{|M^x_u|, |M^x_v|}$ is an isomorphism from M^x_u onto M^x_v.*

3) *The set $\mathfrak{A}$ is a $(\lambda, \kappa, \chi, \mu)$-system if*

(A) *each* $x \in \mathfrak{A}$ *is a* $(\lambda, \kappa, \chi, \mu)$*-candidate,*

(B) guessing: *if* L *is as in (2)*(α)*(ii),* M^* *is an* L*-model with universe* λ *then for some* $x \in \mathfrak{A}$, $s \in B_x \Rightarrow M^x_s \prec M^*$.

Definition 1.7. *1) We call the system* $\mathfrak{A}$ *disjoint when:*

$(*)$ *if* $x \neq y$ *are from* $\mathfrak{A}$ *and* $\mathrm{otp}(|M^x_\emptyset|) \leq \mathrm{otp}(|M^y_\emptyset|)$ then *for some* $B_1 \subseteq B_x$, $B_2 \subseteq B_y$ *we have*

a) $|B_1| + |B_2| < \mu^+$

b) *the sets*

$$\bigcup\{|M^x_s| : s \in [B_x \setminus B_1]^{\leq 2}\}$$

and

$$\bigcup\{|M^y_s| : s \in [B_y \setminus B_2]^{\leq 2}\}$$

have intersection $\subseteq M^y_\emptyset$.

2) *We call the system* $\mathfrak{A}$ *almost disjoint when:*

$(**)$ if $x, y \in \mathfrak{A}$, $\mathrm{otp}(|M^x_\emptyset|) \leq \mathrm{otp}(|M^y_\emptyset|)$ then *for some* $B_1 \subseteq B_x$, $B_2 \subseteq B_y$ *we have:*

(a) $|B_1| + |B_2| < \mu^+$,

(b) if $s \in [B_x \setminus B_1]^{\leq 2}$, $t \in [B_y \setminus B_2]^{\leq 2}$ *then* $|M^x_s| \cap |M^x_t| \subseteq |M^y_\emptyset|$.

2. Introducing the partition on trees

Definition 2.1. *Let*

1) $\mathrm{Per}({}^{\mu>}2) = \{T :$ *where*

(a) $T \subseteq {}^{\mu>}2$, $\langle\rangle \in T$,

(b) $(\forall \eta \in T)\,(\forall \alpha < \lg(\eta))\,\eta{\restriction}\alpha \in T$,

(c) *if* $\eta \in T \cap {}^{\alpha}2$, $\alpha < \beta < \mu$ *then for some* $\nu \in T \cap {}^{\beta}2$, $\eta \triangleleft \nu$,

(d) *if* $\eta \in T$ *then for some* ν, $\eta \triangleleft \nu$, $\nu\hat{\ }\langle 0\rangle \in T$, $\nu\hat{\ }\langle 1\rangle \in T$,

(e) *if* $\eta \in {}^{\delta}2$, $\delta < \mu$ *is a limit ordinal and* $\{\eta{\restriction}\alpha : \alpha < \delta\} \subseteq T$ then $\eta \in T$.

2) $\mathrm{Per}_f({}^{\mu>}2) = \Big\{T \in \mathrm{Per}({}^{\mu>}2)$: *if* $\alpha < \mu$ and $\nu_1, \nu_2 \in {}^{\alpha}2 \cap T$, *then*
$$[\bigwedge_{\ell=0}^{1} \nu_1{}^\frown\langle \ell \rangle \in T \Longleftrightarrow \bigwedge_{\ell=0}^{1} \nu_2{}^\frown\langle \ell \rangle \in T]\Big\}.$$

3) $\mathrm{Per}_u({}^{\mu>}2) = \{T \in \mathrm{Per}({}^{\mu>}2)$: *if* $\alpha < \mu$, $\nu_1 \neq \nu_2$ from ${}^{\alpha}2 \cap T$,
$$\text{then } \bigvee_{\ell=0}^{1} \bigvee_{m=1}^{2} \nu_m{}^\frown\langle \ell \rangle \notin T\}.$$

4) *For* $T \in \mathrm{Per}({}^{\mu>}2)$ *let* $\lim T = \{\eta \in {}^{\mu}2 : (\forall \alpha < \mu)\, \eta \restriction \alpha \in T\}$.

5) *For* $T \in \mathrm{Per}_f({}^{\mu>}2)$ *let* $\mathrm{clp}_T : T \to {}^{\mu>}2$ *be the unique one-to-one function from* $\mathrm{sp}(T) \overset{\mathrm{def}}{=} \{\eta \in T : \eta{}^\frown\langle 0 \rangle \in T, \eta{}^\frown\langle 1 \rangle \in T\}$ *onto* ${}^{\mu>}2$, *which preserves* $\triangleleft$ *and lexicographic order.*

6) *Let* $SP(T) = \{\mathrm{lg}(\eta) : \eta \in \mathrm{sp}(T)\}$, $\mathrm{sp}(\eta, \nu) = \min\{i : \eta(i) \neq \nu(i)$ *or* $i = \mathrm{lg}(\eta)$ *or* $i = \mathrm{lg}(\nu)\}$.

Definition 2.2. *1) For cardinals* μ, σ *and* $n < \omega$ *and* $T \in \mathrm{Per}({}^{\mu>}2)$ *let* $\mathrm{Col}^n_\sigma(T) = \{d : d$ *is a function from* $\cup_{\alpha<\mu}[{}^{\alpha}2]^n \cap T$ *to* $\sigma\}$. *We will write* $d(\nu_0, \dots, \nu_{n-1})$ *for* $d(\{\nu_0, \dots, \nu_{n-1}\})$.

2) *Let* $<^*_\alpha$ *denote a well ordering of* ${}^{\alpha}2$ *(in this section it is arbitrary). We call* $d \in \mathrm{Col}^n_\sigma(T)$ *end-homogeneous for* $\langle <^*_\alpha : \alpha < \mu \rangle$ *provided that: if* $\alpha < \beta$ *are from* $\mathrm{SP}(T)$, $\{\nu_0, \dots, \nu_{n-1}\} \subseteq {}^{\beta}2 \cap T$, $\langle \nu_\ell \restriction \alpha : \ell < n \rangle$ *are pairwise distinct and* $\bigwedge_{\ell,m} [\nu_\ell <^*_\beta \nu_m \Longleftrightarrow \nu_\ell \restriction \alpha <^*_\alpha \nu_m \restriction \alpha]$ *then*
$$d(\nu_0, \dots, \nu_{n-1}) = d(\nu_0 \restriction \alpha, \dots, \nu_{n-1} \restriction \alpha).$$

3) *Let* $\mathrm{Eh\,Col}^n_\sigma(T) = \{d \in \mathrm{Col}^n_\sigma(T) : d$ *is end-homogeneous* $\}$ *(for some* $\langle <^*_\alpha : \alpha < \mu \rangle$*).*

4) *For* $\nu_0, \dots, \nu_{n-1}, \eta_0, \dots, \eta_{n-1}$ *from* ${}^{\mu>}2$, *we say* $\bar\nu = \langle \nu_0, \dots, \nu_{n-1} \rangle$ *and* $\bar\eta = \langle \eta_o, \dots, \eta_{n-1} \rangle$ *are strongly similar for* $\langle <^*_\alpha : \alpha < \mu \rangle$ *if:*
   (i) $\mathrm{lg}(\nu_\ell) = \mathrm{lg}(\eta_\ell)$
   (ii) $\mathrm{sp}(\nu_\ell, \nu_m) = \mathrm{sp}(\eta_\ell, \eta_m)$
   (iii) if $\ell_1, \ell_2, \ell_3, \ell_4 < n$ *and* $\alpha = \mathrm{sp}(\nu_{\ell_1}, \nu_{\ell_2})$ *then*
$$\nu_{\ell_3} \restriction \alpha <^*_\alpha \nu_{\ell_4} \restriction \alpha \Longleftrightarrow \eta_{\ell_3} \restriction \alpha <^*_\alpha \eta_{\ell_4} \restriction \alpha \quad \text{and} \quad \nu_{\ell_3}(\alpha) = \eta_{\ell_3}(\alpha)$$

5) *For* $\nu^a_0, \dots, \nu^a_{n-1}, \nu^b_0, \dots, \nu^b_{n-1}$ *from* ${}^{\mu>}2$ we say $\bar\nu^a = \langle \nu^a_0, \dots, \nu^a_{n-1} \rangle$ and $\bar\nu^b = \langle \nu^b_0, \dots, \nu^b_{n-1} \rangle$ *are similar if the truth values of (i)–(iii) below doe not depend on* $t \in \{a, b\}$ *for any* $\ell(1), \ell(2), \ell(3), \ell(4) < n$*:*

(i) $\lg(\nu^t_{\ell(1)}) < \lg(\nu^t_{\ell(2)})$

(ii) $\mathrm{sp}(\nu^t_{\ell(1)}, \nu^t_{\ell(2)}) < \mathrm{sp}(\nu^t_{\ell(3)}, \nu^t_{\ell(4)})$

(iii) *for* $\alpha = \mathrm{sp}(\nu^t_{\ell(1)}, \nu^t_{\ell(2)})$,

$$\nu^t_{\ell(3)}\restriction\alpha <^*_\alpha \nu^t_{\ell(4)}\restriction\alpha$$

and

$$\nu^t_{\ell(3)}(\alpha) = 0.$$

6) We say $d \in \mathrm{Col}^n_\sigma(T)$ *is almost homogeneous [homogeneous] on* $T_1 \subseteq T$ *(for* $\langle <^*_\alpha : \alpha < \mu\rangle$*)* if *for every* $\alpha \in \mathrm{SP}(T_1)$, $\bar\nu$, $\bar\eta \in [{}^\alpha 2]^n \cap T_1$ *which are strongly similar [similar] we have* $d(\bar\nu) = d(\bar\eta)$.

7) We say $\langle <^*_\alpha : \alpha < \mu\rangle$ *is nice to* $T \in \mathrm{Per}({}^{\mu>}2)$*, provided that: if* $\alpha < \beta$ *are from* $\mathrm{SP}(T)$, $(\alpha, \beta) \cap \mathrm{SP}(T) = \emptyset$, $\eta_1 \neq \eta_2 \in {}^\beta 2 \cap T$, $[\eta_1\restriction\alpha <^*_\alpha \eta_2\restriction\alpha$ *or* $\eta_1\restriction\alpha = \eta_2\restriction\alpha$, $\eta_1(\alpha) < \eta_2(\alpha)]$ *then* $\eta_1 <^*_\beta \eta_2$.

Definition 2.3. *1)* $\mathrm{Pr}_{eht}(\mu, n, \sigma)$ *means: for every* $d \in \mathrm{Col}^n_\sigma({}^{\mu>}2)$ *for some* $T \in \mathrm{Per}({}^{\mu>}2)$*,* d *is end homogeneous on* T*.*

2) $\mathrm{Pr}_{aht}(\mu, n, \sigma)$ *means for every* $d \in \mathrm{Col}^n_\sigma({}^{\mu>}2)$ *for some* $T \in \mathrm{Per}({}^{\mu>}2)$*,* d *is almost homogeneous on* T*.*

3) $\mathrm{Pr}_{ht}(\mu, n, \sigma)$ *means for every* $d \in \mathrm{Col}^n_\sigma({}^{\mu>}2)$ *for some* $T \in \mathrm{Per}({}^{\mu>}2)$*,* d *is homogeneous on* T*.*

4) For $x \in \{eht, aht, ht\}$*,* $\mathrm{Pr}^f_x(\mu, n, \sigma)$ *is defined like* $\mathrm{Pr}_x(\mu, n, \sigma)$ *but we demand* $T \in \mathrm{Per}_f({}^{\mu>}2)$*.*

5) If above we replace eht*,* aht*,* ht *by* $ehtn$*,* $ahtn$*,* htn*, respectively, this means* $\langle <^*_\alpha : \alpha < \mu\rangle$ *is fixed apriori.*

6) Replacing n *by "*$< \kappa$*",* σ *by* $\bar\sigma = \langle \sigma_\ell : \ell < \kappa\rangle$ *for* $\kappa \leq \aleph_0$*, means that* $\langle d_n : n < \kappa\rangle$ *are given,* $d_n \in \mathrm{Col}^n_\sigma({}^{\mu>}2)$ *and the conclusion holds for all* d_n $(n < \kappa)$ *simultaneously. Replacing "*σ*" by "*$< \sigma$*" means that the assertion holds for every* $\sigma_1 < \sigma$*.*

Definition 2.4. *1)* $\mathrm{Pr}_{aht}(\mu, n, \sigma(1), \sigma(2))$ *means: for every* $d \in \mathrm{Col}^n_{\sigma(1)}$ $({}^{\mu>}2)$ *for some* $T \in \mathrm{Per}({}^{\mu>}2)$ *and* $\langle <^*_\alpha : \alpha < \mu\rangle$ *for every* $\bar\eta \in \bigcup\{[{}^\alpha 2]^n \cap T : \alpha \in \mathrm{SP}(T)\}$*,*

$$\Big\{d(\bar\nu) : \bar\nu \in \bigcup\{[{}^\alpha 2]^n \cap T_1 : \alpha \in \mathrm{SP}((T_1)\},\ \bar\eta \text{ and } \bar\nu \text{ are strongly similar for } \langle <^*_\alpha : \alpha < \mu\rangle\Big\}$$

has cardinality $< \sigma(2)$.

2) $\Pr_{ht}(\mu, n, \sigma(1), \sigma(2))$ *is defined similarly with "similar" instead of "strongly similar".*

3) $\Pr_x\left(\mu, < \kappa, \langle \sigma^1_\ell : \ell < \kappa \rangle \, \langle \sigma^2_\ell : \ell < \kappa \rangle\right)$, $\Pr^f_x(\mu, n, \sigma(1), \sigma(2))$, $\Pr^f_x(\mu, < \aleph_0, \bar\sigma^1, \bar\sigma^2)$ *are defined in the same way.*

There are many obvious implications.

Fact 2.5. *1) For every* $T \in \mathrm{Per}({}^{\mu >}2)$ *there is a* $T_1 \subseteq T$, $T_1 \in \mathrm{Per}_u({}^{\mu>}2)$.

2) *In defining* $\Pr^f_x(\mu, n, \sigma)$ *we can demand* $T \subseteq T_0$ *for any* $T_0 \in \mathrm{Per}_f({}^{\mu>}2)$, *similarly for* $\Pr^f_x(\mu, < \kappa, \sigma)$.

3) *The obvious monotonicity holds.*

Claim 2.6. *1) Suppose* μ *is regular,* $\sigma \geq \aleph_0$ *and* $\Pr^f_{eht}(\mu, n, < \sigma)$. *Then* $\Pr^f_{aht}(\mu, n, < \sigma)$ *holds.*

2) *If* μ *is weakly compact and* $\Pr^f_{aht}(\mu, n, < \sigma)$, $\sigma < \mu$, *then* $\Pr^f_{ht}(\mu, n, < \sigma)$ *holds.*

3) *If* μ *is Ramsey and* $\Pr^f_{aht}(\mu, < \aleph_0, < \sigma)$, $\sigma < \mu$, *then* $\Pr^f_{ht}(\mu, < \aleph_0, < \sigma)$.

4) *If* $\mu = \omega$, *in the "nice" version, the orders* $\langle <^*_\alpha : \alpha < \mu \rangle$ *disappear.*

Proof. : Check it.

The following theorem is a quite strong positive result for $\mu = \omega$. Halpern Lauchli proved 2.7(1), Laver proved 2.7(2) (and hence (3)), Pincus pointed out that Halpern Lauchli's proof can be modified to get 2.7(2), and then $\Pr^f_{eht}(\omega, n, < \sigma)$ and (by it) $\Pr^f_{ht}(\omega, n, < \sigma)$ are easy.

Theorem 2.7. *1) If* $d \in \mathrm{Col}^n_\sigma({}^{\omega>}2)$, $\sigma < \aleph_0$, then *there are* $T_0, \ldots, T_{n-1} \in \mathrm{Per}_f({}^{\omega>}2)$ *and* $k_0 < k_1 < \ldots < k_\ell < \ldots$ *and* $s < \sigma$ *such that for every* $\ell < \omega$: if $\mu_0 \in T_0$, $\mu_1 \in T_1, \ldots, \nu_{n-1} \in T_{n-1}$, $\bigwedge_{m<n} \lg(\nu_m) = k_\ell$, *then* $d(\nu_0, \ldots, \nu_{n-1}) = s$.

2) *We can demand in (1) that*

$$\mathrm{SP}(T_\ell) = \{k_0, k_1, \ldots\}$$

3) $\Pr^f_{htn}(\omega, n, \sigma)$ *for* $\sigma < \aleph_0$.

4) $\Pr^f_{htn}\left(\omega, < \aleph_0, \langle \sigma^1_n : n < \omega \rangle, \langle \sigma^2_n : n < \omega \rangle\right)$ *if* $\sigma^1_n < \aleph_0$ *and* $\langle \sigma^2_n : n < \omega \rangle$ *diverge to infinity.*

Definition 2.8. *Let d be a function with domain $\supseteq [A]^n$, A be a set of ordinals, F be a one-to-one function from A to ${}^{\alpha(*)}2$, $<^*_\alpha$ be a well ordering of ${}^\alpha 2$ for $\alpha \le \alpha(*)$ such that $F(\alpha) <^*_\alpha F(\beta) \iff \alpha < \beta$, and σ be a cardinal.*

1) *We say d is (F,σ)-canonical on A if for any $\alpha_1 < \cdots < \alpha_n \in A$,*

$$\Big|\Big\{d(\beta_1,\ldots,\beta_n) : \langle F(\beta_1),\ldots,F(\beta_n)\rangle \text{ similar to } \langle F(\alpha_1),\ldots,F(\alpha_n)\rangle\Big\}\Big| \le \sigma.$$

2) *We define "almost (F,σ)-canonical" similarly using strongly similar instead of "similar".*

3. Consistency of a strong partition below the continuum

This section is dedicated to the proof of

Theorem 3.1. *Suppose λ is the first Erdős cardinal, i.e. the first such that $\lambda \to (\omega_1)^{<\omega}_2$. Then, if A is a Cohen subset of λ, in $V[A]$ for some $\aleph_1$–c.c. forcing notion P of cardinality λ, $\Vdash_P$ "$MA_{\aleph_1}(Knaster) + 2^{\aleph_0} = \lambda$" and:*

1.) $\Vdash_P$ "$\lambda \to [\aleph_1]^n_{h(n)}$" for suitable $h : \omega \mapsto \omega$ (explicitly defined below).

2.) In V^P for any colorings d_n of λ, where d_n is n-place, and for any divergent $\langle \sigma_n : n < \omega\rangle$ (see below), there is a $W \subseteq \lambda$, $|W| = \aleph_1$ and a function $F : W \mapsto {}^\omega 2$ such that: d_n is $(F,\sigma_n)-$ canonical on W for each n. (See definition 2.8 above.)

Remark 3.2. *$h(n)$ is $n!$ times the number of $u \in [{}^\omega 2]^n$ satisfying (if $\eta_1,\eta_2,\eta_3,\eta_4 \in u$ are distinct then $\mathrm{sp}(\eta_1,\eta_2)$, $\mathrm{sp}(\eta_3,\eta_4)$ are distinct) up to strong similarity for any nice $\langle <^*_\alpha : \alpha < \omega\rangle$.*

2) A sequence $\langle \sigma_n : n < \omega\rangle$ is divergent if $\forall m\ \exists k\ \forall n \ge k\ \sigma_n \ge m$.

Notation 3.3. *For a sequence $a = \langle \alpha_i,\, e^*_i : i < \alpha\rangle$, we call $b \subseteq \alpha$ closed if*

(i) $i \in b \Rightarrow a_i \subseteq b$

(ii) if $i < \alpha$, $e^*_i = 1$ and $\sup(b \cap i) = i$ then $i \in b$.

Definition 3.4. *Let $\mathfrak{K}$ be the family of $\bar{Q} = \langle P_i, \underset{\sim}{Q}_j, a_j, e^*_j : j < \alpha,\ i \le \alpha\rangle$ such that*

(a) $a_i \subseteq i$, $|a_i| \leq \aleph_1$,

(b) a_i is closed for $\langle a_j, e^*_j : j < i\rangle$, $e^*_i \in \{0,1\}$, and $[e^*_i = 1 \Rightarrow \mathrm{cf}\, i = \aleph_1]$

(c) P_i is a forcing notion, $\underset{\sim}{Q}_j$ is a P_j-name of a forcing notion of power $\aleph_1$ with minimal element $\emptyset$ or $\emptyset_j$ and for simplicity the underlying set of $\underset{\sim}{Q}_j$ is $\subseteq [\omega_1]^{<\aleph_0}$ (we do not lose by this).

(d) $P_\beta = \{p : p$ is a function whose domain is a finite subset of β and for $i \in \mathrm{dom}(p)$, $\Vdash_{P_i}$ "$f(i) \in \underset{\sim}{Q}_i$"$\}$ with the order $p \leq q$ if and only if for $i \in \mathrm{dom}(p)$, $q{\restriction}i \Vdash_{P_i}$ "$p(i) \leq q(i)$".

(e) for $j < i$, $\underset{\sim}{Q}_j$ is a P_j-name involving only antichains contained in $\{p \in P_j : \mathrm{dom}(p) \subseteq a_j\}$.

For $p \in P_i$, $j < i$, $j \notin \mathrm{dom}\, p$ we let $p(j) = \emptyset$. Note for $p \in P_i$, $j \leq i$, $p{\restriction}j \in P_j$

Definition 3.5. *For $\bar{Q} \in \mathfrak{K}$ as above (so $\alpha = \lg(\bar{Q})$):*
1) for any $b \subseteq \beta \leq \alpha$ closed for $\langle a_i, e^*_i : i < \beta\rangle$ we define P^{cn}_b [by simultaneous induction on β]:

$$P^{\mathrm{cn}}_b = \{p \in P_\beta : \mathrm{dom}\, p \subseteq b, \text{ and for } i \in \mathrm{dom}\, p,\ p(i) \text{ is a canonical name}\}$$

i.e., for any x, $\{p \in P^{\mathrm{cn}}_{a_i} : p \Vdash_{P_i}$ "$p(i) = x$" or $p \Vdash_{P_i}$ "$p(i) \neq x$" $\}$ is a predense subset of P_i.
2) For $\bar{Q}$ as above, $\alpha = \lg(\bar{Q})$, take $\bar{Q}{\restriction}\beta = \langle P_i, \underset{\sim}{Q}_j, a_j : i \leq \beta,\ j < \beta\rangle$ for $\beta \leq \alpha$ and the order is the order in P_α (if $\beta \geq \alpha$, $\bar{Q}{\restriction}\beta = \bar{Q}$).

3) "b *closed for* $\bar{Q}$ means "b closed for $\langle a_i, e^*_i : i < \lg \bar{Q}\rangle$".

Fact 3.6. *1) if $\bar{Q} \in \mathfrak{K}$ then $\bar{Q}{\restriction}\beta \in \mathfrak{K}$.*
2) Suppose $b \subseteq c \subseteq \beta \leq \lg(\bar{\theta})$, b and c are closed for $\bar{Q} \in \mathfrak{K}$.

(i) If $p \in P^{\mathrm{cn}}_c$ then $p{\restriction}b \in P^{\mathrm{cn}}_b$.

(ii) If $p, q \in P^{\mathrm{cn}}_c$ and $p \leq q$ then $p{\restriction}b \leq q{\restriction}c$.

(iii) $P^{\mathrm{cn}}_c \lessdot P_\beta$. 3) $\lg \bar{Q}$ is closed for $\bar{Q}$.

4) if $\bar{Q} \in \mathfrak{K}$, $\alpha = \lg \bar{Q}$ then P^{cn}_α is a dense subset of P_α.
5) If b is closed for $\bar{Q}$, $p, q \in P^{\mathrm{cn}}_{\lg \bar{Q}}$, $p \leq q$ in $P_{\lg \bar{Q}}$ and $i \in \mathrm{dom}\, p$ then $q{\restriction}a_i \Vdash_{P_i}$ "$p(i) \leq q(i)$" hence $\Vdash_{P^{\mathrm{cn}}_{a_i}}$ "$p(i) \leq_{Q_i} q(i)$".

Definition 3.7. *Suppose $W = (W, \leq)$ is a finite partial order and $\bar{Q} \in \mathfrak{K}$.*
1) $IN_W(\bar{Q})$ is the set of $\bar{b}$-s satisfying (α)–(γ) below:

(α) $\bar{b} = \langle b_w : w \in W \rangle$ *is an indexed set of* $\bar{Q}$*-closed subsets of* $\lg(\bar{Q})$,

(β) $W \models$ "$w_1 \le w_2$" $\Rightarrow b_{w_1} \subseteq b_{w_2}$,

(γ) $\zeta \in b_{w_1} \cap b_{w_2}$, $w_1 \le w$, $w_2 \le w$ *then* $(\exists u \in W)\zeta \in b_u \wedge u \le w_1 \wedge u \le w_2$.
We assume $\bar{b}$ *codes* $(W, \le)$.

2) For $\bar{b} \in IN_W(\bar{Q})$, *let*

$$\bar{Q}[\bar{b}] \stackrel{\text{def}}{=} \{\langle p_w : w \in W \rangle : p_w \in P^{\text{cn}}_{b_w}, [W \models w_1 \le w_2 \Rightarrow p_{w_2} \restriction b_{w_1} = p_{w_1}]\}$$

with ordering $\bar{Q}[\bar{b}] \models \bar{p}^1 \le \bar{p}^2$ *iff* $\bigwedge_{w \in W} p^1_w \le p^2_w$.
3) Let $\mathfrak{K}^1$ *be the family of* $\bar{Q} \in \mathfrak{K}$ *such that for every* $\beta \le \lg(\bar{Q})$ *and* $(\bar{Q} \restriction \beta)$*-closed* b, P_β *and* $P_\beta / P^{\text{cn}}_b$ *satisfy the Knaster condition.*

Fact 3.8. *Suppose* $\bar{Q} \in \mathfrak{K}^1$, $(W, \le)$ *is a finite partial order,* $\bar{b} \in IN_W(\bar{Q})$ *and* $\bar{p} \in \bar{Q}[\bar{b}]$.
1) If $w \in W$, $p_w \le q \in P^{\text{cn}}_{b_w}$ then *there is* $\bar{r} \in \bar{Q}[\bar{b}]$, $q \le r_w$, $\bar{p} \le \bar{r}$, *in fact*

$$r_u(\gamma) = \begin{cases} p_u(\gamma) & \text{if } \gamma \in \operatorname{Dom} p_u \setminus \operatorname{Dom} q \\ p_u(\gamma) \ \& \ q(\gamma) & \text{if } \gamma \in b_u \cap \operatorname{Dom} q \text{ and for some } v \in W, \\ & v \le u,\ v \le w \text{ and } \gamma \in b_v \\ p_u(\gamma) & \text{if } \gamma \in b_u \cap \operatorname{dom} q \text{ but the previous case fails} \end{cases}$$

2) Suppose $(W_1, \le)$ *is a submodel of* $(W_2, \le)$, *both finite partial orders,* $\bar{b}^l \in IN_{W_l}(\bar{Q})$, $\bar{b}^1_w = \bar{b}^2_w$ *for* $w \in W_1$.
(α) *If* $\bar{q} \in \bar{Q}[\bar{b}^2]$ *then* $\langle q_w : w \in W_1 \rangle \in \bar{Q}[\bar{b}^1]$.
(β) *If* $\bar{p} \in \bar{Q}[\bar{b}^1]$ then *there is* $\bar{q} \in \bar{Q}[\bar{b}^2]$, $\bar{q} \restriction W_1 = \bar{p}$, *in fact* $q_w(\gamma)$ *is* $p_u(\gamma)$ *if* $u \in W_1$, $\gamma \in b_u$, $u \le w$, *provided that*
(**) *if* $w_1, w_2 \in W_1$, $w \in W_2$, $w_1 \le w$, $w_2 \le w$ *and* $\zeta \in b_{w_1} \cap b_{w_2}$ *then for some* $v \in W_1$, $\zeta \in b_v$, $v \le w_1$, $v \le w_2$.
(this guarantees that if there are several u*'s as above we shall get the same value).*
3) If $\bar{Q} \in \mathfrak{K}^1$ *then* $\bar{Q}[\bar{b}]$ *satisfies the Knaster condition. If* $\emptyset$ *is the minimal element of* W *(i.e.* $u \in W \Rightarrow W \models \emptyset \le u$*) then* $\bar{Q}[\bar{b}]/P^{\text{cn}}_{b_\emptyset}$ *also satisfies the Knaster condition and so* $\lessdot \bar{Q}[\bar{b}]$, *when we identify* $p \in P^{\text{cn}}_b$ *with* $\langle p : w \in W \rangle$.

Proof. 1) It is easy to check that each $r_u(\gamma)$ is in $P^{\text{cn}}_{b_u}$. So, in order to prove $\bar{r} \in \bar{Q}[\bar{b}]$, we assume $W \models u_1 \le u_2$ and has to prove that $r_{u_2} \restriction b_{u_1} = r_{u_1}$. Let $\zeta \in b_{u_1}$.

<u>First case:</u> $\zeta \notin \operatorname{Dom}(p_{u_1}) \cup \operatorname{Dom} q$.

So $\zeta \notin \mathrm{Dom}(r_{u_1})$ (by the definition of r_{u_1}) and $\zeta \notin \mathrm{Dom}\, p_{u_2}$ (as $\bar{p} \in \bar{Q}[\bar{b}]$) hence $\zeta \notin (\mathrm{Dom}\, p_{u_2}) \cup (\mathrm{Dom}\, q)$ hence $\zeta \notin \mathrm{Dom}(r_{u_2})$ by the choice of r_{u_2}, so we have finished.

<u>Second case:</u> $\zeta \in \mathrm{Dom}\, p_{u_1} \setminus \mathrm{Dom}\, q$.

As $\bar{p} \in \bar{Q}[\bar{b}]$ we have $p_{u_1}(\zeta) = p_{u_2}(\zeta)$, and by their definition, $r_{u_1}(\zeta) = p_{u_1}(\zeta)$, $r_{u_2}(\zeta) = p_{u_2}(\zeta)$.

<u>Third case:</u> $\zeta \in \mathrm{Dom}\, q$ and $(\exists v \in W)$ $(\zeta \in b_v \wedge v \leq u_1 \wedge v \leq w)$. By the definition of $r_{u_1}(\zeta)$, we have $r_{u_1}(\zeta) = p_{u_1}(\zeta) \& q(\zeta)$, also the same v witnesses $r_{u_2}(\zeta) = p_{u_2}(\zeta) \& q(\zeta)$, (as $\zeta \in b_v \wedge v \leq u_1 \wedge v \leq w \Rightarrow \zeta \in b_v \wedge v \leq u_2 \wedge v \leq w$) and of course $p_{u_1}(\zeta) = p_{u_2}(\zeta)$ (as $\bar{p} \in \bar{Q}[\bar{b}]$).

<u>Fourth case:</u> $\zeta \in \mathrm{Dom}\, q$ and $\neg(\exists v \in W)$ $(\zeta \in b_v \wedge v \leq u_1 \wedge v \leq w)$.

By the definition of $r_{u_1}(\zeta)$ we have $r_{u_1}(\zeta) = p_{u_1}(\zeta)$. It is enough to prove that $r_{u_2}(\zeta) = p_{u_2}(\zeta)$ as we know that $p_{u_1}(\zeta) = p_{u_2}(\zeta)$ (because $\bar{p} \in \bar{Q}[\bar{b}]$, $u_1 \leq u_2$). If not, then for some $v_0 \in W$, $\zeta \in b_{v_0} \wedge v_0 \leq u_2 \wedge v_0 \leq w$. But $\bar{b} \in \mathrm{IN}_W(\bar{Q})$, hence (see Def. 3.7(1) condition (γ) applied with ζ, w_1, w_2, w there standing for ζ, v_0, u_1, u_2 here) we know that for some $v \in W$, $\zeta \in v \wedge v \leq v_0 \wedge v \leq u_1$. As $(W, \leq)$ is a partial order, $v \leq v_0$ and $v_0 \leq w$, we can conclude $v \leq w$. So v contradicts our being in the fourth case. So we have finished the fourth case.

Hence we have finished proving $\bar{r} \in \bar{Q}[\bar{b}]$. We also have to prove $q \leq r_w$, but for $\zeta \in \mathrm{Dom}\, q$ we have $\zeta \in b_w$ (as $q \in P^{\mathrm{cn}}_w$ is on assumption) and $r_w(\zeta) = q(\zeta)$ because $r_w(\zeta)$ is defined by the second case of the definition as $(\exists v \in W)$ $(\zeta \in b_w \wedge v \leq w \wedge v \leq w)$, i.e. $v = w$.

Lastly we have to prove that $\bar{p} \leq \bar{r}$ (in $\bar{Q}[\bar{b}]$). So let $u \in W$, $\zeta \in \mathrm{Dom}\, p_u$ and we have to prove $r_u \restriction \zeta \Vdash_{P_\zeta}$ "$p_u(\zeta) \leq_{P_\zeta} r_u(\zeta)$". As $r_u(\zeta)$ is $p_u(\zeta)$ or $p_u(\zeta) \& q(\zeta)$ this is obvoius.

2) Immediate.
3) We prove this by induction on $|W|$.
For $|W| = 0$ this is totally trivial.
For $|W| = 1, 2$ this is assumed.
For $|W| > 2$ fix $\bar{p}^i \in \bar{Q}[\bar{b}]$ for $i < \omega_1$. Choose a maximal element $v \in W$ and let $c = \bigcup\{b_w : W \models w < v\}$. Clearly c is closed for $\bar{Q}$.

We know that P^{cn}_c, $P^{\mathrm{cn}}_{b_v}/P^{\mathrm{cn}}_c$ are Knaster by the induction hypothesis. We also know that $p^i_v \restriction c \in P^{\mathrm{cn}}_c$ for $i < \omega_1$, hence for some $r \in P^{\mathrm{cn}}_c$,

$$r \Vdash \text{"} \underset{\sim}{A} \overset{\mathrm{def}}{=} \left\{ i < \omega_1 : p^i_v \restriction c \in \underset{\sim}{G}_{P^{\mathrm{cn}}_c} \right\} \text{ is uncountable"}$$

hence

$\Vdash$ "there is an uncountable $A^1 \subseteq \underset{\sim}{A}$ such that

$$\left[i, j \in A^1 \Rightarrow p^i_v,\, p^j_v \text{ are compatible in } P^{\rm cn}_{b_v} / \underset{\sim}{G}_{P^{\rm cn}_c}\right].$$

Fix a $P^{\rm cn}_c$-name $\underset{\sim}{A}^1$ for such an A^1.

Let $A^2 = \left\{i < \omega_1 : \exists q \in P^{\rm cn}_c,\ q \Vdash i \in \underset{\sim}{A}^1\right\}$. Necessarily $|A_2| = \aleph_1$, and for $i \in A^2$ there is $q^i \in P^{\rm cn}_c$, $q^i \Vdash i \in A^1$, and w.l.o.g. $p^i_v \restriction c \le q^i$. Note that $p^i_v \& q^i \in P^{\rm cn}_c$.

For $i \in A^2$ let, $\bar r^i$ be defined using 3.8(1) (with $\bar p^i$, $p^i_v \& q^i$). Let $W_1 = W \setminus \{v\}$, $\bar b' = \langle b_w : w \in W_1\rangle$.

By the induction hypothesis applied to W_1, $\bar b'$, $\bar r^i \restriction W_1$, for $i \in A^2$ there is an uncountable $A^3 \subseteq A^2$ and for $i < j$ in A^3, there is $\bar r^{i,j} \in \bar Q[\bar b']$, $\bar r^i \restriction W_1 \le \bar r^{i,j}$, and $\bar r^j \restriction W_1 \le \bar r^{i,j}$. Now define $r^{i,j}_c \in P^{\rm cn}_c$ as follows: its domain is $\bigcup\left\{\operatorname{dom} r^{i,j}_w : W \models w < v\right\}$, $r^{i,j}_c \restriction (\operatorname{dom} r^{i,j}_w) = r^{i,j}_w$ whenever $W \models w < v$. Why is this a definition? As if $W \models w_1 \le v \wedge w_2 \le v$, $\zeta \in b_{w_1} \wedge \zeta \in b_{w_2}$ then for some $u \in W$, $u \le w_1 \wedge u \le w_2$ and $\zeta \in u$. It is easy to check that $r^{i,j}_c \in P^{\rm cn}_c$. Now $r^{i,j}_c \Vdash_{P^{\rm cn}_c}$ "$p^i_{b_v}$, $p^j_{b_v}$ are compatible in $P^{\rm cn}_{b_v}/P^{\rm cn}_c$".

So there is $r \in P^{\rm cn}_{b_v}$ such that $r^{i,j}_c \le r$, $p^i_{b_v} \le r$, $p^j_{b_v} \le r$. As in part (1) of 3.8 we can combine r and $\bar r^{i,j}$ to a common upper bound of $\bar p^i$, $\bar p^j$ in $\bar Q[\bar b]$. ■

Claim 3.9. *If $e = 0, 1$ and δ is a limit ordinal, and $P_i, \underset{\sim}{Q}_i$, $\alpha_i, e^*_i (i < \delta)$ are such that for each $\alpha < \delta$, $\bar Q^\alpha = \langle P_i, \underset{\sim}{Q}_j, \alpha_j, e^*_j : i \le \alpha,\ j < \alpha\rangle$ belongs to $\mathfrak{K}^\ell$, then for a unique P_δ, $\bar Q = \langle P_i, \underset{\sim}{Q}_j, \alpha_j, e^*_j : i \le \delta,\ j < \delta\rangle$ belongs to $\mathfrak{K}^\ell$.*

Proof. We define P_δ by (d) of Definition 3.4. The least easy problem is to verify the Knaster conditions (for $\bar Q \in \mathfrak{K}^1$). The proof is like the preservation of the c.c.c. under iteration for limit stages. ■

Convention 3.9A. *By 3.9 we shall not distinguish strictly between $\langle P_i, \underset{\sim}{Q}_j$, $\alpha_j, e^*_j : i \le \delta,\ j < \delta\rangle$ and $\langle P_i, \underset{\sim}{Q}_i, \alpha_i, e^*_i : i < \delta\rangle$.*

Claim 3.10. *If $\bar Q \in \mathfrak{K}^\ell$, $\alpha = \lg(\bar Q)$, $a \subset \alpha$ is closed for $\bar Q$, $|a| \le \aleph_1$, $\underset{\sim}{Q}_1$ is a $P^{\rm cn}_a$-name of a forcing notion satisfying (in V^{P_α}) the Knaster condition, its underlying set is a subset of $[\omega_1]^{<\aleph_0}$ then there is a unique $\bar Q^1 \in \mathfrak{K}^\ell$, $\lg(\bar Q_1) = \alpha + 1$, $Q^1_\alpha = \underset{\sim}{Q}$, $\bar Q \restriction \alpha = \bar Q$.*

Proof. Left to the reader. ■

Proof of Theorem 3.1.

A Stage: We force by $\mathfrak{K}^1_{<\lambda} = \{\bar{Q} \in \mathfrak{K}^1 : \lg(\bar{Q}) < \lambda, \bar{Q} \in H(\lambda)\}$ ordered by being an initial segment (which is equivalent to forcing a Cohen subset of λ). The generic object is essentially $\bar{Q}^* \in \mathfrak{K}^1_\lambda$, $\lg(\bar{Q}^*) = \lambda$, and then we force by $P_\lambda = \lim \bar{Q}^*$. Clearly $\mathfrak{K}^\ell_{<\lambda}$ is a λ-complete forcing notion of cardinality λ, and P_λ satisfies the c.c.c. Clearly it suffices to prove part (2) of 3.1.

Suppose $\underset{\sim}{d}_n$ is a name of a function from $[\lambda]^n$ to $\underset{\sim}{k}_n$ for $n < \omega$, $\underset{\sim}{\sigma}_n < \omega$, $\langle \sigma_n : n < \omega \rangle$ diverges (i.e. $\forall m\ \exists k\ \forall n \geq k\ \sigma_n \geq m$) and for some $\bar{Q}^0 \in \mathfrak{K}^1_{<\lambda}$.

$$\bar{Q}^0 \Vdash_{\mathfrak{K}^1_{<\lambda}} \text{"there is } p \in \underset{\sim}{P}_\lambda\ [p \Vdash_{P_\lambda} \langle \underset{\sim}{d}_n : n < \omega \rangle \text{ is a counterexample to (2) of 3.1"}].$$

In V we can define $\langle \bar{Q}^\zeta : \zeta < \lambda \rangle$, $\bar{Q}^\zeta \in \mathfrak{K}^1_{<\lambda}$, $\zeta < \xi \Rightarrow \bar{Q}^\zeta = \bar{Q}^\xi \restriction \lg(\bar{Q}^\zeta)$, in $\bar{Q}^{\zeta+1}$, $e^*_{\lg(\bar{Q}_\zeta)} = 1$, $\bar{Q}^{\zeta+1}$ forces (in $\mathfrak{K}^1_{<\lambda}$) a value to p and the $\underset{\sim}{P}_\lambda$-names $\underset{\sim}{d}_n \restriction \zeta$, $\underset{\sim}{\sigma}_n$, $\underset{\sim}{k}_n$ for $n < \omega$, i.e. the values here are still P_λ-names. Let $\bar{Q}^*$ be the limit of the $\bar{Q}^\xi$-s. So $\bar{Q}^* \in \mathfrak{K}^1$, $\lg(\bar{Q}^*) = \lambda$, $\bar{Q}^* = \langle P^*_i, \underset{\sim}{Q}^*_j, \alpha^*_j, e^*_j : i \leq \lambda, j < \lambda \rangle$, and the P^*_λ-names $\underset{\sim}{d}_n$, $\underset{\sim}{\sigma}_n$, $\underset{\sim}{k}_n$ are defined such that in $V^{P^*_\lambda}$, $\underset{\sim}{d}_n$, $\underset{\sim}{\sigma}_n$, $\underset{\sim}{k}_n$ contradict (2) (as any P^*_λ-name of a bounded subset of λ is a $P^*_{\lg(\bar{Q}^\xi)}$-name for some $\xi < \lambda$).

B Stage: Let $\chi = \kappa^+$and $<^*_\chi$ be a well-ordering of $\mathrm{H}(\chi)$. Now we can apply $\lambda \to (\omega_1)^{<\omega}_2$ to get δ, B, N_s (for $s \in [B]^{<\aleph_0}$) and $\mathbf{h}_{s,t}$ (for $s, t \in [B]^{<\aleph_0}$, $|s| = |t|$) such that:

- (a) $B \subseteq \lambda$, $\mathrm{otp}(B) = \omega_1$, $\sup B = \delta$,
- (b) $N_s \prec (H(\chi), \in, <^*_\chi)$, $\bar{Q}^* \in N_s$, $\langle \underset{\sim}{d}, \underset{\sim}{\sigma}_n, \underset{\sim}{k}_n : n < \omega \rangle \in N_s$,
- (c) $N_s \cap N_t = N_{s \cap t}$,
- (d) $N_s \cap B = s$,
- (e) if $s = t \cap \alpha$, $t \in [B]^{<\aleph_0}$ then $N_s \cap \lambda$ is an initial segment of N_t,
- (f) $\mathbf{h}_{s,t}$ is an isomorphism from N_t onto N_s (when defined)
- (g) $\mathbf{h}_{t,s} = \mathbf{h}^{-1}_{s,t}$
- (h) $p_0 \in N_s$, $p_0 \Vdash_{P_\lambda}$ "$\langle \underset{\sim}{d}_n, \underset{\sim}{\sigma}_n, \underset{\sim}{k}_n : n < \rangle$ is a counterexample",
- (i) $\omega_1 \subseteq N_s$, $|N_s| = \aleph_1$ and if $\gamma \in N_s$, $\mathrm{cf}\,\gamma > \aleph_1$ then $\mathrm{cf}(\sup(\gamma \cap N_s)) = \omega_1$.

Let $\bar{Q} = \bar{Q}^* \restriction \delta$, $P = P^*_\delta$ and $P_a = P^{\rm cn}_a$ (for $\bar{Q}$), where a is closed for $\bar{Q}$. Note: $P^*_\lambda \cap N_s = P^*_\delta \cap N_s = P_{\sup \lambda \cap N_s} \cap N_s = P_s \cap N_s$. Note also $\gamma \in \lambda \cap N_s \Rightarrow a^*_\gamma \subseteq \lambda \cap N_s$.

C Stage: It suffices to show that we can define $\underset{\sim}{Q}_\delta$ in V^{P_δ} which forces a subset W of B of cardinality $\aleph_1$ and $\underset{\sim}{F} : W \to {}^\omega 2$ which exemplify the desired conclusion in (2), and prove that $\underset{\sim}{Q}_\delta$ satisfies the $\aleph_1$-c.c.c. (in V^{P_δ} (and has cardinality $\aleph_1$)) and moreover (see Definitions 3.4 and 3.7(3)) we also define $a_\delta = \bigcup_{s \in [B]^{<\aleph_0}} N_s$, $e_\delta = 1$, $\bar{Q}' = \bar{Q}\hat{\ }\langle P^*_\delta, \underset{\sim}{Q}_\delta, a_\delta, e_\delta \rangle$ and prove $\bar{Q}' \in \mathfrak{K}^1$.
We let $\underset{\sim}{d}(u) = \underset{\sim}{d}_{|u|}(u)$.

Let $F : \omega_1 \to {}^\omega 2$ be one-to-one such that $\forall \eta \in {}^{\omega >} 2\ \exists^{\aleph_1} \alpha < \omega_1\ [\eta \triangleleft F(\alpha)]$. (This will not be the needed $\underset{\sim}{F}$, just notation).

For $s, t \in [B]^{<\aleph_0}$, we say $s \equiv^n_F t$ if $|s| = |t|$ and $\forall \xi \in s$, $\forall \zeta \in t [\xi = \mathbf{h}_{s,t}(\zeta) \Rightarrow F(\xi) \restriction n = F(\zeta) \restriction n]$. Let $I_n = I_n(F) = \{ s \in [B]^{<\aleph_0} : (\forall \zeta \neq \xi \in s), [F(\zeta) \restriction n \neq F(\xi) \restriction n] \}$.

We define R_n as follows: a sequence $\langle p_s : s \in I_n \rangle \in R_n$ if and only if

(i) for $s \in I_n$, $p_s \in P^*_\lambda \cap N_s$,

(ii) for some c_s we have $p_s \Vdash$ "$\underset{\sim}{d}(s) = c_s$",

(iii) for $s, t \in I_n$, $s \equiv^n_F t \Rightarrow \mathbf{h}_{s,t}(p_t) = p_s$,

(iv) for $s, t \in I_n$, $p_s \restriction N_{s \cap t} = p_t \restriction N_{s \cap t}$.

R^-_n is defined similarly omitting (ii).

For $x = \langle p_s : s \in I_n \rangle$ let $n(x) = n$, $p^x_s = p_s$, and (if defined) $c^x_s = c_s$. Note that we could replace $x \in R_n$ by a finite subsequence. Let $R = \bigcup_{n<\omega} R_n$, $R^- = \bigcup_{n<\omega} R^-_n$. We define an order on $R^- : x \leq y$ if and only if $n(x) \leq n(y)$, and $[s \in I_{n(x)} \wedge t \in I_{n(y)} \wedge s \subseteq t \Rightarrow p^x_s \leq p^y_t]$.

D Stage: Note the following facts::

D(α) **Subfact:** If $x \in R^-_n$, $t \in I_n$ and $p^x_t \leq p^1 \in P^*_\delta \cap N_t$, then there is y such that $x \leq y \in R^-_n$, $p^y_t = p^1$.

Proof. We let for $s \in I_n$

$$p^y_s \overset{\rm def}{=} \& \left\{ \mathbf{h}_{s_1,t_1}(p^1 \restriction N_{t_1}) : s_1 \subseteq s, t_1 \subseteq t, s_1 \equiv^n_F t_1 \right\} \& p^x_s.$$

(This notation means that p^y_s is a function whose domain is the union of the domains of the conditions mentioned, and for each coordinate we take the canonical upper bound, see preliminaries.) Why is p^y_s well defined?

Suppose $\beta \in N_s \cap \lambda$ (for $\beta \in \lambda \setminus N_s$, clearly $p^y_s(\beta) = \emptyset_\beta$), $s_\ell \subseteq s$, $t_\ell \subseteq t$, $s_\ell \equiv^n_F t_\ell$ for $\ell = 1, 2$ and $\beta \in \mathrm{Dom}\left[\mathbf{h}_{s_\ell,t_\ell}(p^1 \restriction N_{t_\ell})\right]$, and it suffices to show that $p^x_s(\beta)$, $\mathbf{h}_{s_1,t_1}(p^1 \restriction N_{t_1})(\beta)$, $\mathbf{h}_{s_2,t_2}(p^1 \restriction N_{t_2})(\beta)$ are pairwise comparable. Let $u = \bigcap\{v \in [B]^{<\aleph_0} : \beta \in N_v\}$, necessarily $u \subseteq s_1 \cap s_2$, and let $u_\ell = \mathbf{h}^{-1}_{s_\ell,t_\ell}(u)$. As $s_\ell, t_\ell, t \in I_n$, $s_\ell \equiv^n_F t_\ell$ and $u_\ell \subseteq t_\ell \subseteq t$, necessarily $u_1 = u_2$. Thus $\gamma \overset{\text{def}}{=} \mathbf{h}^{-1}_{u,v}(\beta) = \mathbf{h}^{-1}_{s_\ell,t_\ell}(\beta)$ and so the last two conditions are equal.

Now $p^x_s(\beta) = p^x_u(\beta) = \mathbf{h}_{u,v}(p^x_s(\gamma)) \le \mathbf{h}_{s_\ell,t_\ell}((p^x_t \restriction N_{t_\ell})(\gamma)) = \Big(\mathbf{h}_{s_\ell,t_\ell}(p^x_t \restriction N_{t_\ell})\Big)(\beta)$.

We leave to the reader checking the other requirements. ■

D(β) Subfact: If $x \in R^-_n$, $t \in I$ then $\bigcup\{p^x_s : s \in I_n,\ s \subseteq t\}$ (as union of functions) exists and belongs to $P^*_\lambda \cap N_t$.

Proof. See (iv) in the definition of R^-_n. ■

D(γ) Subfact: If $x \le y$, $x \in R_n$, $y \in R^-_n$, *then* $y \in R_n$.

Proof. Check it. ■

D(δ) Subfact: If $x \in R^-_n$, $n < m$, *then* there is $y \in R_m$, $x \le y$.

Proof. By subfact D(β) we can find $x^1 = \langle p^1_t : t \in I_m\rangle \in inR^-_m$ with $x \le x^1$. Using repeatedly subfact D(α) we can increase x^1 (finitely many times) to get $y \in R_m$. ■

D(ε) Subfact: If $x \in R^-_n$, $s, t \in I_n$, $s \equiv^n_F t$, $p^x_s \le r_1 \in P^*_\lambda \cap N_s$, $p^x_t \le r_2 \in P^*_\lambda \cap N_t$, $(\forall \zeta \in t)\,[F(\zeta)(n) \ne (F(\mathbf{h}_{s,t}(\zeta)))(n)]$ (or just $p^x_{s_1} \restriction s_1 = \mathbf{h}_{s,t}(p^x_{t_1} \restriction t_1)$ where $t_1 \overset{\text{def}}{=} \{\xi \in t : F(\xi)(n) = (F(\mathbf{h}_{s,t}(\xi)))(n)\}$, $s_1 \overset{\text{def}}{=} \{\mathbf{h}_{s,t}(\xi) : \xi \in t_1\}$), *then* there is $y \in R_{n+1}$, $x \le y$ such that $r_1 = p^y_s$ and $r_2 = p^y_t$.

Proof. Left to the reader. ■

E Stage † **:**

† We will have $T \subset {}^{\omega>}2$ gotten by 2.7(2) and then want to get a subtree with as few as possible colors, we can find one isomorphic to ${}^{\omega>}2$, and there restrict ourselves to $\cup_n T^*_n$.

We define: $T^*_k \subseteq {}^{2^k \geq}2$ by induction on k as follows:

$$\begin{aligned} T^*_0 =& \{\langle\rangle, \langle 1\rangle\} \\ T^*_{k+1} =& \{\nu : \nu \in T^*_k \text{ or } 2^k < \lg(\nu) \leq 2^{k+1} \ , \nu\restriction 2^k \in T^*_k \text{ and} \\ & [2^k \leq i < 2^{k+1} \wedge \nu(i) = 1] \Rightarrow i = 2^k + (\sum_{m<2^k} \nu(i)2^m)]\}. \end{aligned}$$

We define

$$\text{Tr Emb}(k,n) = \Big\{ h : h \text{ a is function from } T^*_k \text{ into } {}^{n\geq}2 \text{ such that}$$
$$\begin{aligned} & \text{for } \nu, \rho \in T^*_k : \\ & [\eta = \nu \Leftrightarrow h(\eta) = h(\nu)] \\ & [\eta \triangleleft \nu \Leftrightarrow h(\eta) \triangleleft h(\nu)] \\ & [\lg(\eta) = \lg(\nu) \Rightarrow \lg(h(\eta) = \lg(h(\nu)] \\ & [\nu = \eta\char`^\langle i\rangle \Rightarrow (h(\nu))[\lg(h(\eta))] = i] \\ & [\,\lg(\eta) = {}^k2 \Rightarrow \lg(h(\eta)) = n] \Big\}. \end{aligned}$$

$$\begin{aligned} \mathbf{T}(k,n) =& \{\text{Rang } h : h \in \text{Tr Emb}(k,n)\}, \\ \mathbf{T}(*,n) =& \bigcup_k \mathbf{T}(k,n), \\ \mathbf{T}(k,*) =& \bigcup_k \mathbf{T}(k,n). \end{aligned}$$

For $T \in \mathbf{T}(k,*)$ let $n(T)$ be the unique n such that $T \in \mathbf{T}(k,n)$ and let

$$\begin{aligned} B_T =& \{\alpha \in B : F(\alpha)\restriction n(T) \text{ is a maximal member of } T\}, \\ fs_T =& \Big\{ t \subseteq B_T : \eta \in t \wedge \nu \in t \wedge \eta \neq \nu \Rightarrow \eta\restriction n(T) \neq \nu\restriction n(T)\}, \\ \Theta_T =& \Big\{ \langle p_s : s \in fs_T\rangle : p_s \in P \cap N_s, [s \subseteq t \wedge \{s,t\} \subseteq fs_T \Rightarrow p_s = p_t\restriction N_s] \Big\}. \end{aligned}$$

Let further

$$\begin{aligned} \Theta_k =& \bigcup \{\Theta_T : T \in \mathbf{T}(k,*)\} \\ \Theta =& \bigcup_k \Theta_k. \end{aligned}$$

For $\bar p \in \Theta$, $\mathbf{n}_{\bar p} = \mathbf{n}(\bar p)$, $T_{\bar p}$ are defined naturally.

For $\bar{p}, \bar{q} \in \Theta$, $\bar{p} \leq \bar{q}$ iff $\mathbf{n}_{\bar{p}} \leq \mathbf{n}_{\bar{q}}$ and for every $s \in fs_{T_{\bar{p}}}$ we have $p_s \leq q_s$.

F Stage: Let $\underset{\sim}{g} : \omega \to \omega$, $\underset{\sim}{g} \in N_s$, $\underset{\sim}{g}$ grows fast enough relative $\langle \sigma_n : n < \omega \rangle$. We define a game Gm. A play of the game lasts after ω moves, in the n^{th} move player I chooses $\bar{p}^n \in \Theta_n$ and a function h_n satisfying the restrictions below and then player II chooses $\bar{q}_n \in \Theta_n$, such that $\bar{p}_n \leq \bar{q}_n$ (so $T_{\bar{p}_n} = T_{\bar{q}_n}$). Player I loses the play if sometimes he has no legal move; if he never loses, he wins. The restrictions player I has to satisfy are:

(a) for $m < n$, $\bar{q}_m \leq \bar{p}_n$, p^n_s forces a value to $\underset{\sim}{g} \restriction (n+1)$,

(b) h_n is a function from $[B_{T_{\bar{p}_n}}]^{\leq g(n)}$ to ω,

(c) if $m < n \Rightarrow h_n, h_m$ are compatible,

(d) If $m < n$, $\ell < g(m)$, $s \in [B_{T_{\bar{p}_n}}]^{\ell}$, *then* $p^n_s \Vdash \underset{\sim}{d}(s) = h_n(s)$,

(e) Let $s_1, s_2 \in \operatorname{Dom} h_n$. Then $h_n(s_1) = h_n(s_2)$ whenever s_1, s_2 are similar over n which means:

(i) $\Big(F\big(H^{OP}_{s_2,s_1}(\zeta)\big)\Big) \restriction \mathbf{n}[\bar{p}^n] = \big(F(\zeta)\big) \restriction \mathbf{n}[\bar{p}^n]$ for $\zeta \in s_1$,

(ii) $H^{OP}_{s_2,s_1}$ preserves the relations $\operatorname{sp}\big(F(\zeta_1), F(\zeta_2)\big) < \operatorname{sp}\big(F(\zeta_3), F(\zeta_4)\big)$ and $F(\zeta_3)\Big(\operatorname{sp}(F(\zeta_1), F(\zeta_2))\Big) = i$ (in the interesting case $\zeta_3 \neq \zeta_1, \zeta_2$ implies i $= 0$).

G Stage/Claim: Player I has a winning strategy in this game.

Proof. As the game is closed, it is determined, so we assume player II has a winning strategy , and eventually we shall get a contradiction. We define by induction on n, $\bar{r}^n$ and Φ^n such that

(a) $\bar{r}^n \in R_n$, $\bar{r}^n \leq \bar{r}^{n+1}$,

(b) Φ^n is a finite set of initial segments of plays of the game,

(c) in each member of Φ^n player II uses his winning strategy,

(d) if y belongs to Φ^n then it has the form $\langle \bar{p}^{y,\ell}, h^{y,\ell}, \bar{q}^{y,\ell} : \ell \leq m(y) \rangle$; let $h_y = h^{y,n_y}$ and $T_y = T_{\bar{q}^y, m(y)}$; also $T_y \subseteq^{n \geq} 2$, $q^{y,\ell}_s \leq r^n_s$ for $s \in fs_{T_y}$.

(e) $\Phi_n \subseteq \Phi_{n+1}$, Φ_n is closed under taking the initial segments and the empty sequence (which too is an initial segment of a play) belongs to Φ_0.

(f) For any $y \in \Phi_n$ and T, h *either* for some $z \in \Phi_{n+1}$, $n_z = n_y + 1$, $y = z \restriction (n_y + 1)$, $T_z = T$ and $h_z = h$ *or* player I has no legal $(n_y+1)^{\text{th}}$ move $\bar{p}^n, h^n$ (after y was played) such that $T_{\bar{p}^n} = T$, $h^n = h$, and $p^n_s = r^n_s$ for $s \in fs_T$ (or always $\leq$ or always $\geq$).

There is no problem to carry the definition. Now $\langle \bar{r}^n_s : n < \omega \rangle$ define a function d^*: if $\eta_1, \dots, \eta_k \in^m 2$ are distinct then $d^*(\langle \eta_1, \dots, \eta_k \rangle) = c$ *iff* for every (equivalently some) $\zeta_1 < \cdots < \zeta_k$ from B, $\eta_\ell \triangleleft F(\zeta_\ell)$ and $r^k_{\{\zeta_1,\dots,\zeta_k\}} \Vdash$ "$\underset{\sim}{d}_k(\{\zeta_1, \dots, \zeta_k\}) = c$".

Now apply 2.7(2) to this coloring, get $T^* \subseteq^{\omega>} 2$ as there. Now player I could have chosen initial segments of this T^* (in the n^{th} move in Φ_n) and we get easily a contradiction. ■

H Stage: We fix a winning strategy for player I (whose existence is guaranteed by stage G).

We define a forcing notion Q^*. We have $(r, y, f) \in Q^*$ *iff*

(i) $r \in P^{\text{cn}}_{a_\delta}$

(ii) $y = \langle \bar{p}^\ell, h^\ell, \bar{q}^\ell : \ell \le m(y) \rangle$ is an initial segment of a play of Gm in which player I uses his winning strategy

(iii) f is a finite function from B to $\{0, 1\}$ such that $f^{-1}(\{1\}) \in fs_{T_y}$ (where $T_y = T_{\bar{q}^{m(y)}}$).

(iv) $r = q^{y,m(y)}_{f^{-1}(\{1\})}$.

The *Order* is the natural one.

I Stage: If $\underset{\sim}{J} \subseteq P^{\text{cn}}_{a_\delta}$ is dense open then $\{(r, y, f) \in Q^* : r \in \underset{\sim}{J}\}$ is dense in Q^*.

Proof. By 3.8(1) (by the appropriate renaming). ■

J Stage: We define Q_δ in V^{P_δ} as $\{(r, y, f) \in Q^* : r \in \underset{\sim}{G}_{P_\delta}\}$, the order is as in Q^*.

The main point left is to prove the Knaster condition for the partial ordered set $\bar{Q}^* = \bar{Q}\hat{\ }\langle P_\delta, \underset{\sim}{Q}_\delta, a_\delta, e_\delta \rangle$ demanded in the definition of $\mathfrak{K}^1$. This will follow by 3.8(3) (after you choose meaning and renamings) as done in stages K,L below.

K Stage: So let $i < \delta$, $\text{cf}(i) \ne \aleph_1$, and we shall prove that $P^+_{\delta+1}/P_i$ satisfies the Knaster condition. Let $p_\alpha \in P^*_{\delta+1}$ for $\alpha < \omega_1$, and we should find $p \in P_i$, $p \Vdash_{P_i}$"*there is an unbounded* $A \subseteq \{\alpha : p_\alpha \restriction i \in \underset{\sim}{G}_{P_i}\}$ *such that for any* $\alpha, \beta \in A$, p_α, p_β *are compatible in* $P^*_{\delta+1}/\underset{\sim}{G}_{P_i}$".

Without loss of generality:

(a) $p_\alpha \in P^{cn}_{\delta+1}$.

(b) for some $\langle i_\alpha : \alpha < \omega_1 \rangle$ increasing continuous with limit δ we have: $i_0 > i$, $\mathrm{cf}\, i_\alpha \neq \aleph_1$, $p_\alpha \restriction \delta \in P_{i_{\alpha+1}}$, $p_\alpha \restriction i_\alpha \in P_{i_0}$.

Let $p^0_\alpha = p^\alpha \restriction i_0$, $p^1_\alpha = p_\alpha \restriction \delta = p_\alpha \restriction i_{\alpha+1}$, $p_\alpha(\delta) = (r_\alpha, y_\alpha, f_\alpha)$, so without loss of generality

(c) $r_\alpha \in P_{i_{\alpha+1}}$, $r_\alpha \restriction i_\alpha \in P_{i_0}$, $m(y_\alpha) = m^*$,

(d) $\mathrm{Dom}\, f_\alpha \subseteq i_0 \cup [i_\alpha, i_{\alpha+1})$,

(e) $f_\alpha \restriction i_0$ is constant (remember $\mathrm{otp}(B) = \omega_1$,

(f) if $\mathrm{Dom}\, f_\alpha = \{j^\alpha_0, \dots j^\alpha_{k_\alpha - 1}\}$ then $k_\alpha = k$, $[j^\alpha_\ell < i_\alpha \Leftrightarrow \ell < k^*]$, $\bigwedge_{\ell < k^*} j^\alpha_\ell = j^\ell$, $f(j^\alpha_\ell) = f(j^\beta_\ell)$, $F(j^\alpha_\ell)) \restriction m(y_\alpha) = F(j^\beta_\ell) \restriction m(y_\beta)$.

The main problem is the compatibility of the $q^{y_\alpha, m(y_\alpha)}$. Now by the definition Θ_α (in stage E) and 3.8(3) this holds. $\blacksquare$

L Stage: If $c \subset \delta + 1$ is closed for $\bar{Q}^*$, then $P^*_{\delta+1}/P^{cn}_c$ satisfies the Knaster condition.

If c is bounded in δ, choose a successor $i \in (\sup c, \delta)$ for $\bar{Q} \restriction i \in \mathfrak{K}_1$. We know that P_i/P^{cn}_c satisfies the Knaster condition and by stage K, $P^*_{\delta+1}/P_i$ also satisfies the Knaster condition; as it is preserved by composition we have finished the stage.

So assume c is unbounded in δ and it is easy too. So as seen in stage J, we have finished the proof of 3.1. $\blacksquare$

Theorem 3.11. *If $\lambda \geq \beth_\omega$, P is the forcing notion of adding λ Cohen reals then*

$(*)_1$ *in V^P, if $n < \omega$ $d : [\lambda]^{\leq n} \to \sigma$, $\sigma < \aleph_0$, then for some c.c.c. forcing notion Q we have $\Vdash_Q$* "there are an uncountable $A \subseteq \lambda$ and an one-to-one $F : A \to^\omega 2$ such that d is F-canonical on A" *(see notation in §2).*

$(*)_2$ *if in V, $\lambda \geq \mu \to_{\mathrm{wsp}} (\kappa)_{\aleph_0}$ (see [Sh289]) and in V^P, $d : [\mu]^{\leq n} \to \sigma$, $\sigma < \aleph_0$ then in V^P for some c.c.c. forcing notion Q we have $\Vdash_Q$* "there are $A \in [\mu]^\kappa$ and one-to-one $F : A \to^\omega 2$ such that d is F-canonical on A" *(see §2,).*

$(*)_3$ *if in V, $\lambda \geq \mu \to_{\mathrm{wsp}} (\aleph_1)^n_{\aleph_2}$ and in V^P $d : [\mu]^{\leq n} \to \sigma$, $\sigma < \aleph_0$ then in V^P for every $\alpha < \omega_1$ and $F : \alpha \to^\omega 2$ for some $A \subseteq \mu$ of order type α and $F' : A \to^\omega 2$, $F'(\beta) \overset{\mathrm{def}}{=} F(\mathrm{otp}(A \cap \beta))$, d is F'-canonical on A.*

$(*)_4$ *in V^P, $2^{\aleph_0} \to (\alpha, n)^3$ for every $\alpha < \omega_1$, $n < \omega$. Really, assuming $V \models$ GCH, we have $\aleph_{n^1_3} \to (\alpha, n)$ see [Sh289].*

Proof. Similar to the proof of 3.1. Superficially we need more indiscernibility then we get, but getting $\langle M_u : u \in [B]^{\leq n}\rangle$ we ignore $d(\{\alpha,\beta\})$ when there is no u with $\{\alpha,\beta\} \in M_u$.

Theorem 3.12. *If λ is strongly inaccessible ω-Mahlo, $\mu < \lambda$, then for some c.c.c. forcing notion P of cardinality λ, V^P satisfies*

(a) MA_μ

(b) $2^{\aleph_0} = \lambda = 2^\kappa$ for $\kappa < \lambda$

(c) $\lambda \to [\aleph_1]^n_{\sigma,h(n)}$ for $n < \omega$, $\sigma < \aleph_0$, $h(n)$ is as in 3.1.

Proof. Again, like 3.1.

4. Partition theorem for trees on large cardinals

Lemma 4.1 *Suppose $\mu > \sigma + \aleph_0$ and*

$(*)_\mu$ for every μ-complete forcing notion P, in V^P, μ is measurable.

Then

(1) for $n < \omega$, $Pr^f_{eht}(\mu, n, \sigma)$.

(2) $Pr^f_{eht}(\mu, <\aleph_0, \sigma)$, if there is $\lambda > \mu$, $\lambda \to \left(\mu^+\right)^{<\omega}_2$.

(3) In both cases we can have the Pr^f_{ehtn} version, and even choose the $\langle <^*_\alpha : \alpha < \mu\rangle$ in any of the following ways.

(a) We are given $\langle <^0_\alpha : \alpha < \mu\rangle$, and we let for $\eta, \nu \in {}^\alpha 2 \cap T$, $\alpha \in SP(T)$ (T is the subtree we consider):

$\eta <^*_\alpha \nu$ if and only if $\mathrm{clp}_T(\eta) <^0_\beta \mathrm{clp}_T(\nu)$ where $\beta = \mathrm{otp}(\alpha \cap SP(T))$ and $\mathrm{clp}_T(\eta) = \langle \eta(j) : j \in \mathrm{lg}(\eta), j \in \mathrm{SP}(T)\rangle$.

(b) We are given $\langle <^0_\alpha : \alpha < \mu\rangle$, we let that for $\nu, \eta \in {}^\alpha 2 \cap T$, $\alpha \in SP(T)$: $\eta <^*_\alpha \nu$ if and only if $n{\restriction}(\beta+1) <^0_{\beta+1} \nu{\restriction}(\beta+1)$ where $\beta = \sup(\alpha \cap SP(T))$.

Remark. *1) $(*)_\mu$ holds for a supercompact after Laver treatment. On hypermeasurable see Gitik Shelah [GiSh344].*
2) We can in $(*)_\mu$ restrict ourselves to the forcing notion P actually used. For it by Gitik [Gi] much smaller large cardinals suffice.
3) The proof of 4.1 is a generalization of a proof of Harrington to Halpern Lauchli theorem from 1978.

Conclusion 4.2. *In 4.1 we can get $Pr^f_{ht}(\mu, n, \sigma)$ (even with (3)).*

Proof of 4.2. We do the parallel to 4.1(1). By $(*)_\mu$, μ is weakly compact hence by 2.6(2) it is enough to prove $Pr^f_{aht}(\mu, n, \sigma)$. This follows from 4.1(1) by 2.6(1). ■

Proof of Lemma 4.1. 1), 2). Let $\kappa \le \omega$, $\sigma(n) < \mu$, $d_n \in \mathrm{Col}^n_{\sigma(n)}({}^{\mu >}2)$ for $n < \kappa$.

Choose λ such that $\lambda \to (\mu^+)^{<2\kappa}_{2^\mu}$ (there is such a λ by assumption for (2) and by $\kappa < \omega$ for (1)). Let Q be the forcing notion $({}^{\mu>}2, \triangleleft)$, and $P = P_\lambda$ be $\{f : \mathrm{dom}(f)$ *is a subset of* λ *of cardinality* $< \mu$, $f(i) \in Q\}$ ordered naturally. For $i \notin \mathrm{dom}(f)$, take $f(i) = <>$; Let $\underset{\sim}{\eta}_i$ be the P-name for $\{f(i) : f \in \underset{\sim}{G}_P\}$. Let $\underset{\sim}{D}$ be a P-name of a normal ultrafilter over μ (in V^P). For each $n < \omega$, $d \in \mathrm{Col}^n_{\sigma(n)}({}^{\mu>}2)$, $j < \sigma(n)$ and $u = \{\alpha_0, \dots, \alpha_{n-1}\}$, where $\alpha_0 < \dots < \alpha_{n-1} < \lambda$, let $\underset{\sim}{A}^j_d(u)$ be the P_λ-name of the set

$$A^j_d(u) = \Big\{ i < \mu : \langle \underset{\sim}{\eta}_{\alpha_\ell} \restriction i : \ell < n \rangle \text{ are pairwise distinct and } j = d(\eta_{\alpha_0} \restriction i, \dots, \eta_{\alpha_{n-1}} \restriction i) \Big\}.$$

So $\underset{\sim}{A}^j_d(u)$ is a P_λ-name of a subset of μ, and for $j(1) < j(2) < \sigma(n)$ we have $\Vdash_{P_\lambda}$ "$\underset{\sim}{A}^{j(1)}_d(u) \cap \underset{\sim}{A}^{j(2)}_d(u) = \emptyset$, *and* $\bigcup_{j<\sigma(n)} \underset{\sim}{A}^j_d(u)$ *is a co-bounded subset of* μ". As $\Vdash_P$ "$\mathfrak{D}$ is μ-complete uniform ultrafilter on μ", in V^P there is exactly one $j < \sigma(n)$ with $A^j_d(u) \in \mathfrak{D}$. Let $\underset{\sim}{j}_d(u)$ be the P-name of this j.

Let $I_d(u) \subseteq P$ be a maximal antichain of P, each member of $I_d(u)$ forces a value to $\underset{\sim}{j}_d(u)$. Let $W_d(u) = \bigcup\{\mathrm{dom}(p) : p \in I_d(u)\}$ and $W(u) = \bigcup\{W_{d_n}(u) : n < \kappa\}$. So $W_d(u)$ is a subset of λ of cardinaltiy $\le \mu$ as well as $W(u)$ (as P satisfies the μ^+-c.c. and $p \in P \Rightarrow |\mathrm{dom}(p)| < \mu$).

As $\lambda \to (\mu^{++})^{<2\kappa}_{2^\mu}$, $d_n \in \mathrm{Col}^n_{\sigma_n}({}^{\mu>}2)$ there is a subset Z of λ of cardinality μ^{++} and set $W^+(u)$ for each $u \in [Z]^{<\kappa}$ such that:

(i) $W^+(u_1) \cap W^+(u_2) = W^+(u_1 \cap u_2)$,

(ii) $W(u) \subseteq W^+(u)$ if $u \in [Z]^{<\kappa}$,

(iii) if $|u_1| = |u_2| < \kappa$ and $u_1, u_2 \subseteq Z$ then $W^+(u_1)$ and $W^+(u_2)$ have the same order type and note that $H[u_1, u_2] \overset{\text{def}}{=} H^{OP}_{W^+(u_1), W^+(u_2)}$, induces naturally a map from $P \restriction u_1 \overset{\text{def}}{=} \{p \in P : \mathrm{dom}(p) \subseteq W^+(u_1)\}$ to $P \restriction u_2 \overset{\text{def}}{=} \{p \in P : \mathrm{dom}(p) \subseteq W^+(u_2)\}$.

(iv) if $u_1, u_2 \in [Z]^{<\kappa}$, $|u_1| = |u_2|$ then $H[u_1, u_2]$ maps $I_{d_n}(u_1)$ onto $I_{d_n}(u_2)$ and: $q \Vdash$ "$\underset{\sim}{j}_d(u_1) = j$" $\Leftrightarrow H[u_1, u_2](q) \Vdash$ "$\underset{\sim}{j}_d(u_2) = j$" ,

(v) if $u_1 \subseteq u_2 \in [Z]^{<\kappa}$, $u_3 \subseteq u_4 \in [Z]^{<\kappa}$, $|u_4| = |u_2|$, $H^{OP}_{u_2,u_4}$ maps u_1 onto u_3 *then* $H[u_1, u_3] \subseteq H[u_2, u_4]$.

Let $\gamma(i)$ be the i^{th} member of Z.

Let $s(m)$ be the set of the first m members of Z and $R_n = \{p \in P : \text{dom}(p) \subseteq W^+(s(n)) - \bigcup_{t \subset s(n)} W^+(t)\}$.

We define by induction on $\alpha < \mu$ a function F_α and $p_u \in R_{|u|}$ for $u \in \bigcup_{\beta<\alpha} [{}^\beta 2]^{<\kappa}$ where we let $\emptyset_\beta$ be the empty subset of $[{}^\beta 2]$ and we behave as if $[\beta \neq \gamma \Rightarrow \emptyset_\beta \neq \emptyset_\gamma]$ and we also define $\zeta(\beta) < \mu$, such that:

(i) F_α is a function from ${}^{\alpha>}2$ into ${}^{\mu>}2$, extending F_β for $\beta < \alpha$,

(ii) F_α maps ${}^\beta 2$ to ${}^{\zeta(\beta)}2$ for some $\zeta(\beta) < \mu$ and $\beta_1 < \beta_2 < \alpha \Rightarrow \zeta(\beta_1) < \zeta(\beta_2)$,

(iii) $\eta \triangleleft \nu \in^{\alpha>} 2$ implies $F_\alpha(\eta) \triangleleft F_\alpha(\nu)$,

(iv) for $\eta \in^\beta 2$, $\beta + 1 < \alpha$ and $\ell < 2 we have F_\alpha(\eta)\hat{\ }\langle \ell \rangle \triangleleft F_\alpha(\eta\hat{\ }\langle \ell \rangle)$,

(v) $p_u \in R_m$ whenever $u \in [{}^\beta 2]^m$, $m < \kappa$, $\beta < \alpha$ and for $u(1) \in [Z]^m$ let $p_{u,u(1)} = H[s(|u|), u(1)](p_u)$.

(vi) $\eta \in^\beta 2$, $\beta < \alpha$, then $p_{\{\eta\}}(\min Z) = F_\alpha(\eta)$.

(vii) if $\beta < \alpha$, $u \in [{}^\beta 2]^n$, $n < \kappa$, $h : u \to s(n)$ one-to-one onto (not necessarily order preserving) *then* for some $c(u, h) < \sigma(n)$:

$$\bigcup_{t \subseteq u} p_{t,h''(t)} \Vdash_{P_\lambda} \text{“}\underset{\sim}{d}_n(\underset{\sim}{\eta}_{\gamma(0)}, \ldots, \underset{\sim}{\eta}_{\gamma(n-1)}) = c(u, h)\text{”},$$

(Note: as $p_u \in R_{|u|}$ the domains of the conditions in this union are pairwise disjoint.)

(viii) If n, u, β, h are as in (vii), $u = \{\nu_0, \ldots, \nu_{n-1}\}$, $\nu_\ell \triangleleft \rho_\ell \in^\gamma 2$, $\beta \leq \gamma < \alpha$ then $d_n(F_\alpha(\rho_0), \ldots, F_\alpha(\rho_{n-1})) = c(u, h)$ where h is the unique function from u onto $s(n)$ such that $[h(\nu_\ell) \leq h(\nu_m) \Rightarrow \rho_\ell <^*_\gamma \rho_m]$.

(ix) if $\beta < \gamma < \alpha$, $\nu_1, \ldots, \nu_{n-1} \in^\gamma 2$, $n < \kappa$, and $\nu_0 \restriction \beta, \ldots, \nu_{n-1} \restriction \beta$ are pairwise distinct then:

$$p_{\{\nu_0 \restriction \beta, \ldots, \nu_n \restriction \beta\}} \subseteq p_{\{\nu_0, \ldots, \nu_{n-1}\}}.$$

<u>For α limit</u>: no problem.

<u>For $\alpha + 1, \alpha$ limit</u>: we try to define $F_\alpha(\eta)$ for $\eta \in^\alpha 2$ such that $\bigcup_{\beta<\alpha} F_{\beta+1}(\eta \restriction \beta) \triangleleft F_\alpha(\eta)$ and (viii) holds. Let $\zeta = \bigcup_{\beta<\alpha} \zeta(\beta)$, and for $\eta \in^\alpha 2$, $F^0_\alpha(\eta) =$

$\bigcup_{\beta<\alpha} F_\alpha(\eta\restriction\beta)$ and for $u \in [{}^\alpha 2]^{<\kappa}$, $p^0_u \stackrel{\rm def}{=} \bigcup\{p^0_{\{\nu\restriction\beta:\nu\in u\}} : \beta<\alpha,\ |u| = |\{\nu\restriction\beta : \nu \in u\}|\}$. Clearly $p^0_u \in R_{|u|}$.

Then let $h : {}^\alpha 2 \to Z$ be one-to-one, such that $\eta <^*_\alpha \nu \Leftrightarrow h(\eta) < h(\nu)$ and let $p \stackrel{\rm def}{=} \bigcup\{p^0_{u,u(1)} : u(1) \in [Z]^{<\kappa},\ u \in [{}^\alpha 2]^{<\kappa},\ |u(1)| = |u|,\ h''(u) = u(1)\}$.

For any generic $G \subseteq P_\lambda$ to which p belongs, $\beta < \alpha$ and ordinals $i_0 < \dots < i_{n-1}$ from Z such that $\langle h^{-1}(i_\ell)\restriction\beta : \ell < n\rangle$ are pairwise distinct we have that

$$B_{\{i_\ell:\ell<n\},\beta} = \Big\{\xi < \mu : d_n(\eta_{i_0}\restriction\xi, \dots, \eta_{i_{n-1}}\restriction\xi) = c(u,h^*)\Big\},$$

belongs to $\mathfrak{D}[G]$, where $u = \{h^{-1}(i_\ell)\restriction\beta : \ell < n\}$ and $h^* : u \to s(|u|)$ is defined by $h^*(h^{-1}(i_\ell)\restriction\beta) = H^{OP}_{\{i_\ell:\ell<n\},s(n)}(i_\ell)$. Really every large enough $\beta < \mu$ can serve so we omit it. As $\mathfrak{D}[G]$ is μ-complete uniform ultrafilter on μ, we can find $\xi \in (\zeta, \kappa)$ such that $\xi \in B_u$ for every $u \in [{}^\alpha 2]^n$, $n < \kappa$. We let for $\nu \in {}^\alpha 2$, $F_\alpha(\nu) = \underset{\sim}{\eta}_{h(i)}[G]\restriction\xi$, and we let $p_u = p^0_u$ except when $u = \{\nu\}$, then:

$$p_u(i) = \begin{cases} p^0_u(i) & i \neq \gamma(0) \\ F_{\alpha+1}(\nu) & i = \gamma(0) \end{cases}.$$

<u>For $\alpha+1$, α is a successor</u>: First for $\eta \in {}^{\alpha-1} 2$ define $F(\eta\char`^\langle\ell\rangle) = F_\alpha(\eta)\char`^\langle\ell\rangle$. Next we let $\{(u_i,h_i) : i < i^*\}$, list all pairs (u,h), $u \in [{}^\alpha 2]^{\le n}$, $h : u \to s(|u|)$, one-to-one onto. Now, we define by induction on $i \le i^*$, $p^i_u (u \in [{}^\alpha 2]^{<\kappa})$ such that :

(a) $p^i_u \in R_{|u|}$,

(b) p^i_u increases with i,

(c) for $i+1$, (vii) holds for (u_i,h_i),

(d) if $\nu_m \in {}^\alpha 2$ for $m < n$, $n < \kappa$, $\langle \nu_m\restriction(\alpha-1) : m < n\rangle$ are pairwise distinct, *then* $p_{\{\nu_m\restriction(\alpha-1)\ :m<n\}} \le p^0_{\{\nu_m:m<n\}}$,

(e) if $\nu \in {}^\alpha 2$, $\nu(\alpha-1) = \ell$ *then* $p^0_{\{\nu\}}(0) = F_\alpha(\nu\restriction(\alpha-1))\char`^\langle\ell\rangle$.

There is no problem to carry the induction.

Now $F_{\alpha+1}\restriction{}^\alpha 2$ is to be defined as in the second case, starting with $\eta \to p^{i^*}_{\{\eta\}}(\eta)$.

<u>For $\alpha = 0, 1$</u>: Left to the reader.

So we have finished the induction hence the proof of 4.1(1), (2).

3) Left to the reader (the only influence is the choice of h in stage of the induction). ■

5. Somewhat complimentary negative partition relation in ZFC

The negative results here suffice to show that the value we have for $2^{\aleph_0}$ in §3 is reasonable. In particular the Galvin conjecture is wrong and that for every $n < \omega$ for some $m < \omega$, $\aleph_n \not\to [\aleph_1]^m_{\aleph_0}$.

See Erdos Hajnal Máté Rado [EHMR] for

Fact 5.1. *If $2^{<\mu} < \lambda \le 2^\mu$, $\mu \not\to [\mu]^n_\sigma$ then $\lambda \not\to [(2^{<\mu})^+]^{n+1}_\sigma$.*

This shows that if e.g. in 1.4 we want to increase the exponents, to 3 (and still $\mu = \mu^{<\mu}$) e.g. μ cannot be successor (when $\sigma \le \aleph_0$) (by [Sh276], 3.5(2)).

Definition 5.2. *$Pr_{np}(\lambda, \mu, \bar\sigma)$, where $\bar\sigma = \langle \sigma_n : n < \omega \rangle$, means that there are functions $F_n : [\lambda]^n \to \sigma_n$ such that for every $W \in [\lambda]^\mu$ for some n, $F''_n([W]^n) = \sigma(n)$. The negation of this property is denoted by $NPr_{np}(\lambda, \mu, \bar\sigma)$.*

If $\sigma_n = \sigma$ we write σ instead of $\langle \sigma_n : n < \omega \rangle$.

Remark 5.2A. *1) Note that $\lambda \to [\mu]^{<\omega}_\sigma$ means: if $F : [\lambda]^{<\omega} \to \sigma$ then for some $A \in [\lambda]^\mu$, $F''([A]^{<\omega}) \ne \sigma$. So for $\lambda \ge \mu \ge \sigma = \aleph_0$, $\lambda \not\to [\mu]^{<\omega}_\sigma$, (use $F : F(\alpha) = |\alpha|$) and $Pr_{np}(\lambda, \mu, \sigma)$ is stronger than $\lambda \not\to [\mu]^{<\omega}_\sigma$.*
2) We do not write down the monotonicity properties of Pr_{np} — they are obvious.

Claim 5.3 *1) We can (in 5.2) w.l.o.g. use $F_{n,m} : [\lambda]^n \to \sigma_n$ for $n, m < \omega$ and obvious monotonicity properties holds, and $\lambda \ge \mu \ge n$.*
2) Suppose $NPr_{np}(\lambda, \mu, \kappa)$ and $\kappa \not\to [\kappa]^n_\sigma$ or even $\kappa \not\to [\kappa]^{<\omega}_\sigma$. Then the following case of Chang conjecture holds:

(*) for every model M with universe λ and countable vocabulary, there is an elementary submodel N of M of cardinality μ,

$$|N \cap \kappa| < \kappa$$

3) If $NPr_{np}(\lambda, \aleph_1, \aleph_0)$ then $(\lambda, \aleph_1) \to (\aleph_1, \aleph_0)$.

Proof. Easy.

Theorem 5.4. *Suppose $Pr_{np}(\lambda_0, \mu, \aleph_0)$, μ regular $> \aleph_0$ and $\lambda_1 \geq \lambda_0$, and no $\mu' \in (\lambda_0, \lambda_1)$ is μ'-Mahlo. Then $Pr_{np}(\lambda_1, \mu, \aleph_0)$.*

Proof. Let $\chi = \beth_8(\lambda_1)^+$, let $\{F^0_{n,m} : m < \omega\}$ list the definable n-place functions in the model $(H(\chi), \in, <^*_\chi)$, with $\lambda_0, \mu, \lambda_1$ as parameters, let $F^1_{n,m}(\alpha_0, \dots, \alpha_{n-1})$ (for $\alpha_0, \dots, \alpha_{n-1} < \lambda_1$) be $F^0_{n,m}(\alpha_0, \dots, \alpha_{n-1})$ if it is an ordinal $< \lambda_1$ and zero otherwise. Let $F_{n,m}(\alpha_0, \dots, \alpha_{n-1})$ (for $\alpha_0, \dots, \alpha_{n-1} < \lambda_1$) be $F^0_{n,m}(\alpha_0, \dots, \alpha_{n-1})$ if it is an ordinal $< \omega$ and zero otherwise. We shall show that $F_{n,m}(n, m < \omega)$ exemplify $Pr_{np}(\lambda_1, \mu, \aleph_0)$ (see 5.3(1)).

So suppose $W \in [\lambda_1]^\mu$ is a counterexample to $Pr(\lambda_1, \mu, \aleph_0)$ i.e. for no $n, m, F''_{n,m}([W]^n) = \omega$. Let W^* be the closure of W under $F^1_{n,m}(n, m < \omega)$. Let N be the Skolem Hull of W in $(H(\chi), \in, <^*_\chi)$, so clearly $N \cap \lambda_1 = W^*$. Note $W^* \subseteq \lambda_1$, $|W^*| = \mu$. Also as $\mathrm{cf}(\mu) > \aleph_0$ if $A \subseteq W^*$, $|A| = \mu$ then for some $n, m < \omega$ and $u_i \in [W]^n$ (for $i < \mu$), $F^1_{n,m}(u_i) \in A$ and $[i < j < \mu \Rightarrow F^1_{n,m}(u_i) \neq F^1_{n,m}(u_i)]$. It is easy to check that also $W^1 = \{F^1_{n,m}(u_i) : i < \mu\}$ is a counterexample to $Pr(\lambda_1, \mu, \sigma)$. In particular, for $n, m < \omega$, $W_{n,m} = \{F^1_{n,m}(u) : u \in [W]^n\}$ is a counterexample if it has power μ. W.l.o.g. W is a counterexample with minimal $\delta \overset{\text{def}}{=} \sup(W) = \cup\{\alpha + 1 : \alpha \in W\}$. The above discussion shows that $|W^* \cap \alpha| < \mu$ for $\alpha < \delta$. Obviously $\mathrm{cf}\,\delta = \mu^+$. Let $\langle \alpha_i : i < \mu \rangle$ be a strictly increasing sequence of members of W^*, converging to δ, such that for limit i we have $\alpha_i = \min(W^* - \bigcup_{j<i}(\alpha_j + 1)$. Let $N = \bigcup_{i<\mu} N_i$, $N_i \prec N$, $|N_i| < \mu$, N_i increasing continuous and w.l.o.g. $N_i \cap \delta = N \cap \alpha_i$.

<u>α Fact</u>: δ is $> \lambda_0$.
Proof. Otherwise we then get an easy contradiction to $Pr(\lambda_0, \mu, \sigma))$ as choosing the $F^0_{n,m}$ we allowed λ_0 as a parameter.

<u>β Fact</u>: If F is a unary function definable in N, $F(\alpha)$ is a club of α for every limit ordinal $\alpha(< \lambda_1)$ *then* for some club C of μ we have

$$(\forall j \in C \setminus \{\min C\})(\exists i_1 < j)(\forall i \in (i_1, j))[i \in C \Rightarrow \alpha_i \in F(\alpha_j)].$$

Proof. For some club C_0 of μ we have $j \in C_0 \Rightarrow (N_j, \{\alpha_i : i < j\}, W) \prec (N, \{\alpha_i : i < \mu\}, W)$.

We let $C = C'_0 = \mathrm{acc}(C)$ (= set of accumulation points of C_0).

We check C is as required; suppose j is a counterexample. So $j = \sup(j \cap C)$ (otherwise choose $i_1 = \max(j \cap C)$). So we can define, by induction on n, i_n, such that:

(a) $i_n < i_{n+1} < j$

(b) $\alpha_{i_n} \notin F(\alpha_j)$

(c) $(\alpha_{i_n}, \alpha_{i_{n+1}}) \cap F(\alpha_j) \neq \emptyset$.

Why (C_0')? $\models$ "$F(\alpha_j)$ is unbounded below α_j" hence $N \models$ "$F(\alpha_j)$ is unbounded below α_j", but in N, $\{\alpha_i : i \in C_0,\ i < j\}$ is unbounded below α_j.

Clearly for some $n, m, \alpha_j \in W_{n,m}$ (see above). Now we can repeat the proof of [Sh276,3.3(2)] (see mainly the end) using only members of $W_{n,m}$. Note: here we use the number of colors being $\aleph_0$.

β^+ Fact: Wolog the C in Fact β is μ.
Proof: Renaming.

γ Fact: δ is a limit cardinal.
Proof: Suppose not. Now δ cannot be a successor cardinal (as $\mathrm{cf}\,\delta = \mu \leq \lambda_0 < \delta$) hence for every large enough i, $|\alpha_i| = |\delta|$, so $|\delta| \in W^* \subseteq N$ and $|\delta|^+ \in W^*$.

So $W^* \cap |\delta|$ has cardinality $< \mu$ hence order-type some $\gamma^* < \mu$. Choose $i^* < \mu$ limit such that $[j < i^* \Rightarrow j + \gamma^* < i^*]$. There is a definable function F of $(H(\chi), \in, <^*_\chi)$ such that for every limit ordinal α, $F(\alpha)$ is a club of α, $0 \in F(\alpha)$, if $|\alpha| < \alpha$, $F(\alpha) \cap |\alpha| = \emptyset$, $\mathrm{otp}(F(\alpha)) = \mathrm{cf}\,\alpha$.

So in N there is a closed unbounded subset $C_{\alpha_j} = F(\alpha_j)$ of α_j of order type $\leq \mathrm{cf}\,\alpha_j \leq |\delta|$, hence $C_{\alpha_j} \cap N$ has order type $\leq \gamma^*$, hence for i^* chosen above unboundedly many $i < i^*$, $\alpha_i \notin C_{\alpha_{i^*}}$. We can finish by fact β^+.

δ Fact: For each $i < \mu$, α_i is a cardinal.
Proof: If $|\alpha_i| < i$ then $|\alpha_i| \in N_i$, but then $|\alpha_i|^+ \in N_i$ contradicting to Fact γ, by which $|\alpha_i|^+ < \delta$, as we have assumed $N_i \cap \delta = N \cap \alpha_i$.

ε Fact: For a club of $i < \mu$, α_i is a regular cardinal.
(Proof: if $S = \{i : \alpha_i \text{ singular}\}$ is stationary, then the function $\alpha_i \to \mathrm{cf}(\alpha_i)$ is regressive on S. By Fodor lemma, for some $\alpha^* < \delta$, $\{i < \mu : \mathrm{cf}\,\alpha_i < \alpha^*\}$ is stationary. As $|N \cap \alpha^*| < \mu$ for some β^*, $\{i < \mu : \mathrm{cf}\,\alpha_i = \beta^*\}$ is stationary. Let $F_{1,m}(\alpha)$ be a club of α of order type $\mathrm{cf}(\alpha)$, and by fact β we get a contradiction as in fact γ.

ζ Fact: For a club of $i < \mu$, α_i is Mahlo.
Proof: Use $F_{1,m}(\alpha) =$ a club of α which, if α is a successor cardinal or inaccessible not Mahlo, then it contains no inaccessible, and continue as in fact γ.

ξ Fact: For a club of $i < \mu$, α_i is α_i-Mahlo.
Proof: Let $F_{1,m(0)}(\alpha) = \sup\{\zeta : \alpha \text{ is } \zeta\text{-Mahlo}\}$. If the set $\{i < \mu : \alpha_i$ is not α_i-Mahlo$\}$ is stationary then as before for some $\gamma \in N$, $\{i : F_{1,m(0)}(\alpha_i) = \gamma\}$ is stationary and let $F_{1,m(1)}(\alpha)$ — a club of α such that if α is not $(\gamma+1)$-Mahlo then the club has no γ-Mahlo member. Finish as in the proof of fact δ. ■

Remark 5.4.A. *We can continue and say more.*

Lemma 5.5 *1) Suppose $\lambda > \mu > \theta$ are regular cardinals, $n \geq 2$ and*
(i) for every regular cardinal κ, if $\lambda > \kappa \geq \theta$ then $\kappa \not\to [\theta]^{<\omega}_{\sigma(1)}$.
(ii) for some $\alpha(*) < \mu$ for every regular $\kappa \in (\alpha(*), \lambda)$, $\kappa \not\to [\alpha(*)]^n_{\sigma(2)}$.
Then
(a) $\lambda \not\to [\mu]^{n+1}_\sigma$ where $\sigma = \min\{\sigma(1), \sigma(2)\}$,
(b) there are functions $d_2 : [\lambda]^{n+1} \to \sigma(2)$, $d_1 : [\lambda]^3 \to \sigma(1)$ such that for every $W \in [\lambda]^\mu$, $d_1''([W]^3) = \sigma(1)$ or $d_2''([W]^{n+1}) = \sigma(2)$.
2) Suppose $\lambda > \mu > \theta$ are regular cardinals, and
(i) for every regular $\kappa \in [\theta, \lambda)$, $\kappa \not\to [\theta]^{<\omega}_{\sigma(1)}$,
(ii) $\sup\{\kappa < \lambda : \kappa$ regular$\} \not\to [\mu]^n_{\sigma(2)}$.
Then
(a) $\lambda \not\to [\mu]^{2n}_\sigma$ where $\sigma = \min\{\sigma(1), \sigma(2)\}$
(b) there are functions $d_1 : [\lambda]^3 \to \sigma(1)$, $d_2 : [\lambda]^{2n} \to \sigma(2)$ such that for every $W \in [\lambda]^\mu$, $d_1''([W]^3) = \sigma(1)$ or $d_2''([W]^{2n} = \sigma(2)$.

Remark. *The proof is similar to that of [Sh276] 3.3,3.2.*

Proof. 1) We choose for each i, $0 < i < \lambda_i$, C_i such that: if i is a successor ordinal, $C_i = \{i-1, 0\}$; if i is a limit ordinal, C_i is a club of i of order type $\mathrm{cf}\, i$, $0 \in C_i$, $[\mathrm{cf}\, i < i \Rightarrow \mathrm{cf}\, i < \min(C_i - \{0\})]$ and $C_i \setminus \mathrm{acc}(C_i)$ contains only successor ordinals.

Now for $\alpha < \beta$, $\alpha > 0$ we define by induction on ℓ, $\gamma^+_\ell(\beta,\alpha)$, $\gamma^-_\ell(\beta,\alpha)$, and then $\kappa(\beta,\alpha)$, $\varepsilon(\beta,\alpha)$.

(A) $\gamma^+_0(\beta,\alpha) = \beta$, $\gamma^-_0(\beta,\alpha) = 0$.

(B) if $\gamma^+_\ell(\beta,\alpha)$ is defined and $> \alpha$ and α is not an accumulation point of $C_{\gamma^+_\ell(\beta,\alpha)}$ then we let $\gamma^-_{\ell+1}(\beta,\alpha)$ be the maximal member of $C_{\gamma^+_\ell(\beta,\alpha)}$ which is $< \alpha$ and $\gamma^+_{\ell+1}(\beta,\alpha)$ is the minimal member of $C_{\gamma^+_\ell(\beta,\alpha)}$ which is $\geq \alpha$ (by the choice of $C_{\gamma^+_\ell(\beta,\alpha)}$ and the demands on $\gamma^+_\ell(\beta,\alpha)$ they are well defined).

So

(B1) (a) $\gamma^-_\ell(\beta,\alpha) < \alpha \leq \gamma^+_\ell(\beta,\alpha)$, and if the equality holds then $\gamma^+_{\ell+1}(\beta,\alpha)$ is not defined.

(b) $\gamma^+_{\ell+1}(\beta,\alpha) < \gamma^+_\ell(\beta,\alpha)$ when both are defined.

(C) Let $k = k(\beta,\alpha)$ be the maximal number k such that $\gamma^+_k(\beta,\alpha)$ is defined (it is well defined as $\langle \gamma^+_\ell(\beta,\alpha) : \ell < \omega \rangle$ is strictly decreasing). So

(C1) $\gamma^+_{k(\beta,\alpha)}(\beta,\alpha) = \alpha$ or $\gamma^+_{k(\beta,\alpha)} > \alpha$, $\gamma^+_{k(\beta,\alpha)}$ is a limit ordinal and α is an accumulation point of $C_{\gamma^+_{k(\beta,\alpha)}}(\beta,\alpha)$.

(D) For $m \leq k(\beta,\alpha)$ let us define

$$\varepsilon_m(\beta,\alpha) = \max\{\gamma^-_\ell(\beta,\alpha) + 1 : \ell \leq m\}.$$

Note

(D1) (a) $\varepsilon_m(\beta,\alpha) \leq \alpha$ (if defined),

(b) if α is limit then $\varepsilon_m(\beta,\alpha) < \alpha$ (if defined),

(c) if $\varepsilon_m(\beta,\alpha) \leq \xi \leq \alpha$ *then* for every $\ell \leq m$ we have

$$\gamma^+_\ell(\beta,\alpha) = \gamma^+_\ell(\beta,\xi), \quad \gamma^-_\ell(\beta,\alpha) = \gamma^-_\ell(\beta,\xi), \quad \varepsilon_\ell(\beta,\alpha) = \varepsilon_\ell(\beta,\xi).$$

(explanation for (c): if $\varepsilon_m(\beta,\alpha) < \alpha$ this is easy (check the definition) and if $\varepsilon_m(\beta,\alpha) = \alpha$, necessarily $\xi = \alpha$ and it is trivial).

(d) if $\ell \leq m$ then $\varepsilon_\ell(\beta,\alpha) \leq \varepsilon_m(\beta,\alpha)$

For a regular $\kappa \in (\alpha(*),\lambda)$ let $g^1_\kappa : [\kappa]^{<\omega} \to \sigma(2)$ exemplify $\kappa \not\to [\theta]^{<\omega}_{\sigma(1)}$ and for every regular cardinal $\kappa \in [\theta,\lambda)$ let $g^2_\kappa : [\kappa]^n \to \sigma(2)$ exemplify $\kappa \not\to [\alpha(*)]^n_{\sigma(2)}$. Let us define the colourings:

Let $\alpha_0 > \alpha_1 > \ldots > \alpha_n$. Remember $n \geq 2$.

Let $n = n(\alpha_0,\alpha_1,\alpha_2)$ be the maximal natural number such that:

(i) $\varepsilon_n(\alpha_0,\alpha_1) < \alpha_0$ is well defined,

(ii) for $\ell \leq n$, $\gamma^-_\ell(\alpha_0,\alpha_1) = \gamma^-_\ell(\alpha_0,\alpha_2)$.

We define $d_2(\alpha_0,\alpha_1,\ldots,\alpha_n)$ as $g^2_\kappa(\beta_1,\ldots,\beta_n)$ where

$$\kappa = \mathrm{cf}\,(\gamma^+_{n(\alpha_0,\alpha_1,\alpha_2)}(\alpha_0,\alpha_1)),$$
$$\beta_\ell = \mathrm{otp}\left[\alpha_\ell \cap C_{\gamma^+_{n(\alpha_0,\alpha_1,\alpha_2)}(\alpha_0,\alpha_1)}\right].$$

Next we define $d_1(\alpha_0,\alpha_1,\alpha_2)$.

Let $i(*)=\sup\left[C_{\gamma^+_n(\alpha_0,\alpha_2)}\cap C_{\gamma^+_n(\alpha_1,\alpha_2)}\right]$ where $n=n(\alpha_0,\alpha_1,\alpha_2)$, E be the equivalence relation on $C_{\gamma^+_n(\alpha_0,\alpha_1)}\setminus i(*)$ defined by

$$\gamma_1 E\gamma_2 \Leftrightarrow \forall\gamma\in C_{\gamma^+_n(\alpha_0,\alpha_2)}[\gamma_1<\gamma\leftrightarrow\gamma_2<\gamma].$$

If the set $w=\left\{\gamma\in C_{\gamma^+_n(\alpha_0,\alpha_1)}:\gamma>i(*),\ \gamma=\min\gamma/E\right\}$ is finite, we let $d_1(\alpha_0,\alpha_1,\alpha_2)$ be $g^1_\kappa(\{\beta_\gamma:\gamma\in w\})$ where $\kappa=\left|C_{\gamma^+_n(\alpha_0,\alpha_1)}\right|$, $\beta_\gamma=\text{otp}\left(\gamma\cap C_{\gamma^+_n(\alpha_0,\alpha_1)}\right)$.

We have defined d_1, d_2 required in condition (b) (though have not yet proved that they work) We still have to define d (exemplifying $\lambda\not\rightarrow[\mu]^{n+1}_\ell$). Let $n\geq 3$, for $\alpha_0>\alpha_1>\ldots>\alpha_n$, we let $d(\alpha_0,\ldots,\alpha_n)$ be $d_1(\alpha_0,\alpha_1,\alpha_2)$ if w defined during the definition has odd number of members and $d_2(\alpha_0,\ldots,\alpha_n)$ otherwise.

Now suppose Y is a subset of λ of order type μ, and let $\delta=\sup Y$. Let M be a model with universe λ and with relations Y and $\{(i,j):i\in C_j\}$. Let $\langle N_i:i<\mu\rangle$ be an increasing continuous sequence of elementary submodels of M of cardinality $<\mu$ such that $\alpha(i)=\alpha_i=\min(Y\setminus N_i)$ belongs to N_{i+1}, $\sup(N\cap\alpha_i)=\sup(N\cap\delta)$. Let $N=\bigcup\limits_{i<\mu}N_i$. Let $\delta(i)=\delta_i\stackrel{\text{def}}{=}\sup(N_i\cap\alpha_i)$, so $0<\delta_i\leq\alpha_i$, and let $n=n_i$ be the first natural number such that δ_i an accumulation point of $C^i\stackrel{\text{def}}{=}C_{\gamma^+_n(\alpha_i,\delta(i))}$, let $\varepsilon_i=\varepsilon_{n(i)}(\alpha_i,\delta_i)$. Note that $\gamma^+_n(\alpha_i,\delta_i)=\gamma^+_n(\alpha_i,\varepsilon_i)$ hence it belongs to N.

<u>Case I</u>: *For some (limit) $i<\mu$, $\text{cf}(i)\geq\theta$ and $(\forall\gamma<i)[\gamma+\alpha(*)<i]$ such that for arbitrarily large $j<i$, $C^i\cap N_j$ is bounded in $N_j\cap\delta=N_j\cap\delta_j$.*
This is just like the last part in the proof of [Sh276],3.3 using g^1_κ and d_1 for $\kappa=\text{cf}(\gamma^+_{n_i}(\alpha_i,\delta_i)$.

<u>Case II</u>: *Not case I.*

Let $S_0=\{i<\mu:(\forall\alpha<i)[\gamma+\alpha(*)<i],\ \text{cf}(i)=\theta\}$. So for every $i\in S_0$ for some $j(i)<i$, $(\forall j)\Big[j\in(j(i),i)\Rightarrow C^i\cap N_j$ is unbounded in $\delta_j\Big]$. But as $C^i\cap\delta_i$ is a club of δ_i, clearly $(\forall j)\Big[j\in(j(i),i)\Rightarrow\delta_j\in C^i\Big]$.

We can also demand $j(i)>\varepsilon_{n(\alpha(i),\delta(i))}(\alpha(i),\delta(i))$.

As S_0 is stationary, (by not case I) for some stationary $S_1\subseteq S_0$ and $n(*)$, $j(*)$ we have $(\forall i\in S_1)\Big[j(i)=j(*)\wedge n(\alpha(i),\delta_i)=n(*)\Big]$.

Choose $i(*) \in S_1$, $i(*) = \sup(i(*) \cap S_1)$, such that the order type of $S_1 \cap i(*)$ is $i(*) > \alpha(*)$. Now if $i_2 < i_1 \in S_1 \cap i(*)$ then $n(\alpha_{i(*)}, \alpha_{i_1}, \alpha_{i_2}) = n(*)$. Now $L_{i(*)} \stackrel{\text{def}}{=} \left\{ \text{otp}(\alpha_i \cap C^{i(*)}) : i \in S_1 \cap i(*) \right\}$ are pairwise distinct and are ordinals $< \kappa \stackrel{\text{def}}{=} |C^{i(*)}|$, and the set has order type $\alpha(*)$. Now apply the definitions of d_2 and g^2_κ on $L_{i(*)}$.

2) The proof is like the proof of part (1) but for $\alpha_0 > \alpha_1 > \cdots$ we let $d_2(\alpha_0, \ldots, \alpha_{2n-1}) = g^2_\kappa(\beta_0, \ldots, \beta_n)$ where

$$\beta_\ell \stackrel{\text{def}}{=} \text{otp}\,(C_{\gamma_n^+(\beta_{2\ell},\beta_{2\ell+1})}(\beta_{2\ell}, \beta_{2\ell+1}) \cap \beta_{2\ell+1})$$

and in case II note that the analysis gives μ possible β_ℓ's so that we can apply the definition of g^2_κ.

Definition 5.7. *Let $\lambda \not\to_{\text{stg}} [\mu]^n_\theta$ mean: if $d : [\lambda]^n \to \theta$, and $\langle \alpha_i : i < \mu \rangle$ is strictly increasingly continuous and for $i < j < \mu$, $\gamma_{i,j} \in [\alpha_i, \alpha_{i+1})$ then*

$$\theta = \left\{ d(w) : \text{ for some } j < \mu,\ w \in [\{\gamma_{i,j} : i < j\}]^n \right\}.$$

Lemma 5.8. *1) $\aleph_t \not\to [\aleph_1]^{n+1}_{\aleph_0}$ for $n \geq 1$.*
2) $\aleph_n \not\to_{\text{stg}} [\aleph_1]^{n+1}_{\aleph_0}$ for $n \geq 1$.

Proof. 1) For $n = 2$ this is a theorem of Torodčevič, and if it holds for $n \geq 2$ by 5.5(1) we get that it holds for n+1 (with n, λ, μ, θ, $\alpha(*)$, $\sigma(1)$, $\sigma(2)$ there corresponding to $n+1$, $\aleph_{n+1}$, $\aleph_1$, $\aleph_0$,$\aleph_0$, $\aleph_0$,$\aleph_0$ here).
2) Similar.

References

[EHMR] P. Erdős, A. Hajnal, A. Máté, R. Rado, Combinatorial set theory: partition relations for cardinals, Disq. Math. Hung., no 13, (1984)

[Gi] M. Gitik, Measurability preserved by κ-complete forcing notion

[GiSh344] M. Gitik, S. Shelah, On certain indiscernibility of strong cardinals and a question of Hajnal, Archive für Math. Logic, 28(1989) 35–42

[HL] Halpern, Lauchli,

[Sh80] S. Shelah, A Weak Generalization of MA to Higher Cardinals, Israel J. of Math, [BSF], 30(1978), 297-306.

[Sh289] S. Shelah, Consisting of Partition Theorem for Graphs and Models, Proc. of the Torontp 8/87 Conference in General Topology.

[Sh276] S. Shelah, Was Sierpinsky right I ? Israel J. Math. 62 (1988) 355–380

[ShSt154] S. Shelah and L. Stanley, Generalized Martin Axiom and the Souslin Hypothesis for Higher Cardinals, Israel J. of Math, [BSF], 43(1982), 225-236.

[ShSt154a] S. Shelah and L. Stanley, Corrigendum to "Generalized Martin's Axiom and Souslin's Hypothesis for Higher Cardinal, Israel J. of Math, 53(1986), 309-314.

[To] S. Todorčevič, Partitions of pairs of uncountable ordinals, Acta MAth, 159 (l987), 261–294

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