A MORE GENERAL ITERABLE CONDITION ENSURING $\aleph_1$ IS NOT COLLAPSED

SH311

Saharon Shelah

Institute of Mathematics
The Hebrew University
91 904 Jerusalem, Israel

Rutgers University
Mathematics Department
New Brunswick, NJ USA

Abstract. In a self-contained way, we deal with revised countable support iterated forcing for the reals; improve theorems on preservation of a property UP, weaker than semi proper, and hopefully improve the presentation. We continue [Sh:b, Ch.X,XI] (or see [Sh:f, Ch.X,XI]), and Gitik Shelah [GiSh 191] and [Sh:f, Ch.XIII,XIV] and particularly Ch.XV; concerning "no new reals" see lately Larson Shelah [\LarSh:746]. In particular, we fulfill some promises from [Sh:f] and give a more streamlined version.

F144, based on Chapter XV of PIF
I would like to thank Alice Leonhardt for the beautiful typing.
Previous Version - 01/Feb/27
Latest Revision - 06/July/25

ANNOTATED CONTENT

any stationary subset of it. They are all relatives of "semi-properness" for strictly $(\mathbb{I}, \mathbf{W})$-suitable models, they speak on $\mathbb{I}$-tagged trees of countable models.]

[We prove that satisfying $UP^1(\mathbf{W})$, i.e. satisfying it for some family $\mathbb{I}$ of ideals, complete enough, is preserved by $\aleph_1 - Sp_6(W)$-iterations, $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha \rangle$; that is if each $\underset{\sim}{\mathbb{Q}}_i$ is like that then $P_\alpha = \aleph_1 - Sp_6(W)$-Lim$(\bar{\mathbb{Q}})$ is like that, provided some mild condition holds (say $\mathbb{Q}_i$ is $UP^1(\mathbb{I}_i, \mathbf{W})$, $\underset{\sim}{\mathbb{I}}_i$ is $\underset{\sim}{\kappa}_i$-complete, $\mathbb{P}_i$ satisfies the κ_i-c.c.; we can even make $\underset{\sim}{\mathbb{I}}_i, \underset{\sim}{\kappa}_i$ to be just $\mathbb{P}_{i+1}$-names, see there). The proof is more similar to the proofs of preservation of properness and semi-properness than with the proofs in [Sh:b, XI], (=[Sh:f, XI]), [GiSh 191], [Sh:f, XV], and hopefully more transparent. The proof will be non-trivially shorter if we use just the particular case of the revised countable support (i.e., $\leq_{\rm vpr}$ is equality and $\leq_{\rm pr}$ is $\leq$). We give a sufficient condition for α not being collapsed by $\mathbb{P}_\alpha$ e.g. α is strongly inaccessible, $\beta < \alpha \Rightarrow |\mathbb{P}_\beta| < \alpha$ and: W stationary in α (so α is Mahlo) or $\leq_{\rm vpr}$ is equality and the iteration is suitable enough. Lastly, if e.g. α is the first strongly inaccessible, $i < \alpha \Rightarrow |\mathbb{P}_i| < \alpha$ we give a sufficient condition for α not being collapsed.]

[Here we make the condition more similar to semi-proper iteration, that is the demand is that for suitable models N (one on which "the right trees grow") above each $p \in \mathbb{Q} \cap N$ there is an $(N, \mathbb{Q})$-semi-generic q. There is some price though.
[?] However, if $\mathbb{Q}$ satisfies UP^0 and the κ-c.c., then $\mathbb{Q} * \underset{\sim}{\rm Levy}(\aleph_1, < \kappa)$ is appropriate in our iteration.]

[Here we deal with the parallel of "$\mathbb{Q}$ add no new real because it is $\mathbf{W}$-complete for some stationary $\mathbf{W} \subseteq \omega_1$".]

[We show that various forcing notions fall under our context. In particular (variants of) Namba forcing, shooting a club through a stationary $S \subseteq \{\delta < \lambda : {\rm cf}(\delta) = \aleph_0\}$ where $\lambda = {\rm cf}(\lambda) > \aleph_0$, and prove that the older condition from [Sh:f] implies the present one.]

§1 Preliminaries

1.1 Definition/Notation. 1) A forcing notion here, $\mathbb{P}$, is a nonempty set (abusing notation, it too is denoted by $\mathbb{P}$) and three partial orders $\le_0, \le_1, \le_2$ (more exactly quasi-orders and $\le^*_\ell = \le^{\mathbb{P}}_\ell$) and a minimal element $\emptyset_{\mathbb{P}} \in \mathbb{P}$ (so $\emptyset_{\mathbb{P}} \le_\ell p$ for $p \in \mathbb{P}$) and for $\ell = 0, 1$ we have $[p \le_\ell q \to p \le_{\ell+1} q]$. We call $p \in \mathbb{P}$ very pure if $\emptyset_{\mathbb{P}} \le_0 p$ and we call q a very pure extension of p if $p \le_0 q$. We call $p \in \mathbb{P}$ pure if $\emptyset_{\mathbb{P}} \le_1 p$ and we call q a pure extension of p if $p \le_1 q$. Let $\le$ be $\le_2$, let $\le_{\rm pr}$ be $\le_1$ and $\le_{\rm vpr}$ be $\le_0$.

We call $\mathbb{P}$ κ-vp-complete if: for any $<_{\rm vpr}$-increasing sequence $\langle p_\alpha : \alpha < \delta \rangle, \delta < \kappa$ with $p_0 = \emptyset_{\mathbb{P}}$ there is a $\le_{\rm vpr}$-upper bound p. We define vp-κ-complete similarly waiving $p_0 = \emptyset_{\mathbb{P}}$. We define $\kappa- \le_\ell$-complete and $\le_\ell -\kappa$-complete similarly.

The forcing relation, of course, refers to the partial order $\le$. We denote forcing notions by $\mathbb{P}, \mathbb{Q}, \mathbb{R}$. Let $\mathbb{P}^1 \subseteq \mathbb{P}^2$ mean $p \in \mathbb{P}^1 \Rightarrow p \in \mathbb{P}^2, \le^{\mathbb{P}^1}_\ell = \le^{\mathbb{P}^2}_\ell \restriction \mathbb{P}^1$ for $\ell = 0, 1, 2$ and let $\mathbb{P}^1 \subseteq_{\rm ic} \mathbb{P}^2$ means $\mathbb{P}^1 \subseteq \mathbb{P}^2$ and for $\ell \le 2$, if $p, q \in \mathbb{P}^1$ are $\le_\ell$-incompatible in $\mathbb{P}^1$, then they are $\le_\ell$-incompatible in $\mathbb{P}^2$. Let $\mathbb{P}^1 \lessdot \mathbb{P}^2$ means $\mathbb{P}^1 \subseteq_{\rm ic} \mathbb{P}^2$ & $(\mathbb{P}^1, \le) \lessdot (\mathbb{P}^2, \le)$.

2) $\bar{\mathbb{P}}$ denotes a $\lessdot$-increasing sequence of forcing notions. $\bar{\mathbb{Q}}$ denotes a sequence of the form $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha \rangle$ such that $\langle \mathbb{P}_i : i < \alpha \rangle$ is a $\lessdot$-increasing sequence. Usually $\underset{\sim}{\mathbb{Q}}_i$ is a $\mathbb{P}_i$-name, $\Vdash_{\mathbb{P}_i}$ "$\mathbb{P}_{i+1}/\mathbb{P}_i \cong \underset{\sim}{\mathbb{Q}}_i$".

3) Convention: If $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha \rangle, \mathbb{P}_i$ is $\lessdot$-increasing, we may write $\bar{\mathbb{Q}}$ instead of $\langle \mathbb{P}_i : i < \alpha \rangle$.

4) For a forcing notion $\mathbb{P}$ (as in part (1)) we define $\hat{\mathbb{P}}$:

(*a*) the set of elements of $\hat{\mathbb{P}}$ is

$$\Big\{ A : A \subseteq \mathbb{P}, \text{ and for some } p \in A \text{ (called a witness) we have}$$
$$(i) \quad (\forall q \in A)(\exists r)[q \le r \in A \ \& \ p \le_{\rm vpr} r]$$
$$(ii) \quad \text{there is an upper bound } p^* \in \mathbb{P} \text{ of } A \text{ such that } p \le_{\rm vpr} p^*$$
$$\text{moreover } (\forall p' \in A)(p \le_{\rm vpr} p' \Rightarrow p' \le_{vpr} p^*) \Big\}$$

(we call p^* an outer witness for A or for $A \in \hat{\mathbb{P}}$ if clause (ii) hold), and

(*b*) $\hat{\mathbb{P}}$ is ordered by: $A \le B$ iff: $A = B$ or $A = \emptyset$ or for some $q \in B, (\forall p \in A)(p \le q)$ and we call q a witness to $A \le B$

(*c*) we define the order $\le_\ell$ on $\hat{\mathbb{P}}$ by: $A \le_\ell B$ <u>iff</u> $A \le B$ and $A \ne B$ implies that for every witnesses p for A and every witness q for B we have $p \in A$ & $q \in$

B & $(\forall p')(p \le_{\rm vpr} p' \in A \to p' \le_\ell q)$; we call such a pair (p,q) a witness for $A \le_\ell B$. [See 1.10(5)]

(*d*) we stipulate sometime $\emptyset \le_\ell A$ for every $A \in \hat{\mathbb{P}}$ or $\emptyset = \emptyset_{\mathbb{P}}$ [Saharon].

5) CC($\mathbb{P}$) is the minimal regular uncountable cardinal θ such that $\mathbb{P}$ satisfies the θ-c.c. We may add

(*iii*) if $p' \in A$ satisfies clause $(i) + (ii)$ then there is $p'' \in A$ such that $p' \le_{\rm vpr} p''$ & $p \le_{\rm vpr} p''$.

1.2 Observation.: 1) For any forcing notion $\mathbb{P}$, (as in 1.1(1), of course), also $\hat{\mathbb{P}}$ is a forcing notion (in particular $\le^{\hat{\mathbb{P}}_\ell}$ is a quasi order for $\ell \le 2$) and $\mathbb{P} \subseteq_{\rm ic} \hat{\mathbb{P}}$ and $\mathbb{P} \lessdot \hat{\mathbb{P}}$ and $\mathbb{P}$ is $\le_{\rm vpr}$-dense in $\hat{\mathbb{P}}$ when we identify p and $\{p\}$.
2) If $A_i \le_\ell B$ for $i < i^*$ then $B \le_{\rm vpr} B^+$ where $B^+ = \bigcup\limits_{i<i^*} A_i \cup B$.
3) If $\ell \in \{0,1,2\}$ and $(\mathbb{P}, \le_\ell)$ is θ-complete (i.e., an increasing sequence of length $< \theta$ has an upper bound) then so is $(\hat{\mathbb{P}}, \le_\ell)$.

Proof. 1) Check.
2) Easy.
3) If $\delta < \theta, \langle A_i : i < \delta\rangle$ is $\le_\ell$-increasing let (p_i, q_i) witness $A_i \in \hat{\mathbb{P}}$. If $\langle p_i : i < \delta\rangle$ is eventually constant then $\langle A_i : i < \delta\rangle$ is eventually constant and A_j for j large enough can serve. If not, without loss of generality $(\forall i < \delta) p_i \ne p_{i+1}$ and let (p_i, q_i) witness $A_i \le_\ell A_{i+1}$. Clearly $\langle q_i : i < \delta\rangle$ has a $\le_\ell$-upper bound in $\mathbb{P}$, call it q. Now $\{q\}$ is as required. $\square_{1.2}$

1.3 Definition. Let $MAC(\mathbb{P})$ be the set of maximal antichains of the forcing notion $\mathbb{P}$.

1.4 Remark. 1) Note: $|MAC(\mathbb{P})| \le 2^{|\mathbb{P}|}, \mathbb{P}$ satisfies the $|\mathbb{P}|^+$-c.c. and if $\mathbb{P}$ satisfies the λ-c.c. then $|MAC(\mathbb{P})| \le |\mathbb{P}|^{<\lambda}$.
2) Note:

(*) if $\mathbb{Q}$ is a forcing notion, $\lambda = \lambda^{<\lambda} > |\mathbb{Q}| + \aleph_0, \Vdash_{\mathbb{Q}}$ "$(\forall \mu < \lambda)\mu^{\aleph_0} < \lambda$" and $\mathbb{Q}' = \mathbb{Q} *$ Levy$(\aleph_1, < \lambda)$ then $|MAC(\mathbb{Q}')| = |\mathbb{Q}'| = \lambda$.

1.5 Notation. Car is the class of cardinals.
IRCar is the class of (infinite) regular cardinals.
RCar = IRCar $\cup\{1\}$.
URCar is the class of uncountable regular cardinals.
$\mathscr{D}^{cb}_\lambda$ is the filter of co-bounded subsets of λ.
$\mathscr{D}_\lambda$ is the club filter on λ for λ regular uncountable.
$\eta^- = \eta \restriction (\ell g(\eta) - 1)$ for a finite sequence η of length > 0.

1.6 Notation. 1) $\mathscr{H}(\chi)$ is the family of sets with transitive closure of power $< \chi$; let $<^*_\chi$ be a well ordering of $\mathscr{H}(\chi)$.
2) Let W be a function from the set of strongly inaccessible cardinals to $\{0, 1, \frac{1}{2}\}$; if $\alpha \notin$ Dom(W) we understand $W(\alpha) = 0$ and let $\alpha \in W$ means $W(\alpha) = 1$.

1.7 Definition. 1) Assume $\bar{\mathbb{P}}$ is a $\lessdot$-increasing sequence of forcing notions. Let

$$\text{Gen}^r(\bar{\mathbb{P}}) =: \Big\{G : \text{for some (set) forcing notion } \mathbb{P}^* \text{ we have } \bigwedge_{i<\alpha} \mathbb{P}_i \lessdot \mathbb{P}^* \text{ and for some } G^* \subseteq \mathbb{P}^* \text{ generic over } \mathbf{V} \text{ we have } G = G^* \cap \bigcup_{i<\alpha} \mathbb{P}_i\Big\}.$$

2) If $\bar{\mathbb{Q}} = \langle \mathbb{P}_i : i < \alpha \rangle$ or $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha \rangle$ where $\mathbb{P}_i$ is a $\lessdot$-increasing we define a $\bar{\mathbb{Q}}$-name $\underset{\sim}{\tau}$ almost as we define $(\bigcup\limits_{i<\alpha} \mathbb{P}_i)$-names, but we do not use maximal antichains of $\bigcup\limits_{i<\alpha} \mathbb{P}_i$, that is:

(∗) $\underset{\sim}{\tau}$ is a function, Dom$(\underset{\sim}{\tau}) \subseteq \bigcup\limits_{i<\alpha} \mathbb{P}_i$ and for every $G \in$ Gen$^r(\bar{\mathbb{Q}})$, $\underset{\sim}{\tau}[G]$ is defined iff Dom$(\underset{\sim}{\tau}) \cap G \neq \emptyset$ and then $\underset{\sim}{\tau}[G] \in \mathbf{V}[G]$ [from where "every $G \ldots$" is taken? E.g., $\mathbf{V}$ is countable, G any set from the true universe] and $\underset{\sim}{\tau}[G]$ is definable with parameters from $\mathbf{V}$ and the parameter $\bigcup\limits_{i<\alpha} \mathbb{P}_i \cap G$ (so $\underset{\sim}{\tau}$ is really a first-order formula with the variable $\bigcup\limits_{i<\alpha} \mathbb{P}_i \cap G$ and parameters from $\mathbf{V}$).

Now $\Vdash_{\bar{\mathbb{Q}}}$ has a natural meaning.
3) For $p \in \bar{\mathbb{Q}}$ (i.e. $p \in \bigcup\limits_{i<\alpha} \mathbb{P}_i$) and $\bar{\mathbb{Q}}$-names $\underset{\sim}{\tau}_0, \dots, \underset{\sim}{\tau}_{n-1}$ we let $\{\underset{\sim}{\tau}_0, \dots, \underset{\sim}{\tau}_{n-1}\}$ be the name for the set that contains exactly those $\underset{\sim}{\tau}_i[\bar{\mathbb{Q}}]$ that are defined. We let $p \Vdash$ "$\underset{\sim}{\tau} = x$" if for every G such that $p \in G \in \text{Gen}^r(\bar{\mathbb{Q}})$ we have $\underset{\sim}{\tau}[G] = x$. If $\beta < \alpha$ and $G_\beta \subseteq \mathbb{P}_\beta$, we let $\underset{\sim}{\tau}[G_\beta] = x$ means that for some $p \in G_\beta$ we have $p \Vdash_{\bar{\mathbb{Q}}}$ "$\underset{\sim}{\tau} = x$", so possibly no $p \in \bigcup\limits_{i<\alpha} \mathbb{P}_i$ forces a value to $\underset{\sim}{\tau}$ and no such p forces $\underset{\sim}{\tau}$ is not definable.
4) We say a $\bar{\mathbb{Q}}$-name $\underset{\sim}{x}$ is full <u>if</u> $\underset{\sim}{x}[G]$ is well defined for every $G \in \text{Gen}^r(\bar{\mathbb{Q}})$.
5) A simple $\bar{\mathbb{Q}}$-named[1] $[j, \beta)$-ordinal $\underset{\sim}{\zeta}$ is a $\bar{\mathbb{Q}}$-name $\underset{\sim}{\zeta}$ such that: if $G \in \text{Gen}^r(\bar{\mathbb{Q}})$ and $\underset{\sim}{\zeta}[G] = \xi$ then $j \le \xi < \beta$ and $(\exists p \in G \cap \mathbb{P}_{\xi \cap \alpha}) p \Vdash_{\bar{\mathbb{Q}}}$ "$\underset{\sim}{\zeta} = \xi$" (where $\alpha = \ell g(\bar{\mathbb{Q}})$); however, we allowed $\underset{\sim}{\zeta}[G]$ to be undefined. If we omit "$[j, \beta)$" we mean $[0, \ell g(\bar{\mathbb{Q}})) = [0, \alpha)$. If we omit "simple", we mean replacing $(\exists p \in G \cap \mathbb{P}_{\xi \cap \alpha})$ by $(\exists p \in G \cap \mathbb{P}_{(\xi+1) \cap \alpha})$ (this is used in [Sh:f, Ch.X,§1], we shall only remark on it here).
6) A simple $\bar{\mathbb{Q}}$-named[2] $[j, \beta)$-ordinal $\underset{\sim}{\zeta}$ is a simple $\bar{\mathbb{Q}}$-named[2] $[j, \beta)$-ordinal of depth Υ for some ordinal Υ, where this is defined below by induction on Υ. In all cases it is a $\bar{\mathbb{Q}}$-name of an ordinal from the interval $[j, \beta)$ so may be undefined, i.e., we allow non full such names.

<u>Case 1</u>: $\Upsilon = 0$.

This is an ordinal $\in [j, \beta)$, or is "undefined" (in the full case this is forbidden).

<u>Case 2</u>: $\Upsilon > 0$.

For some $\gamma < \ell g(\bar{\mathbb{Q}}) \cap \beta$ and maximal antichain $\mathscr{I} = \{p_\varepsilon : \varepsilon < \varepsilon^*\}$ of $\mathbb{P}_\gamma$, there is a sequence $\langle \underset{\sim}{\zeta}_\varepsilon : \varepsilon < \varepsilon^* \rangle$ such that $\underset{\sim}{\zeta}_\varepsilon$ is a simple $\bar{\mathbb{Q}}$-named[2] $[\text{Max}\{j, \gamma\}, \beta)$-ordinal of depth $\Upsilon_\varepsilon < \Upsilon$ and: $\underset{\sim}{\zeta}[G_\xi] = \xi$ iff $\xi \ge \gamma$ and for some ε we have $p_\varepsilon \in G_\xi \cap \mathbb{P}_\gamma$ and $\underset{\sim}{\zeta}_\varepsilon[G_\xi] = \xi$ (including the case: not defined). If we omit "$[j, \beta)$" we mean $[0, \ell g(\bar{\mathbb{Q}})) = [0, \alpha)$.
7) If we omit "simple" in (6) we mean that in case 2, $\underset{\sim}{\zeta}_i$ is a not necessarily simple $\bar{\mathbb{Q}}$-name[2] and $\mathscr{I} \subseteq \mathbb{P}_{\gamma+1}$.
8) We say $\bar{\mathbb{P}}$ is W-continuous or $(\bar{\mathbb{P}}, W)$ is continuous when for every $\delta \in W \cap \ell g(\bar{\mathbb{P}})$ if $(\forall i < \delta)$ [density $(\mathbb{P}_i) < \delta$, or just $\mathbb{P}_i$ satisfies the cf(δ)-c.c.], <u>then</u> $\mathbb{P}_\delta = \bigcup\limits_{i<\delta} \mathbb{P}_i$.

We say $\bar{\mathbb{P}}$ is W-smooth or $(\bar{\mathbb{P}}, W)$ is smooth if $\delta \in W \Rightarrow \mathbb{P}_\delta = \bigcup_{i<\delta} \mathbb{P}_i$. We say $\underset{\sim}{\zeta}$ is a simple$^\ell$ $(\bar{\mathbb{Q}}, W)$-named $[j, \beta)$-ordinal <u>if</u> it is a simple$^\ell$ $\bar{\mathbb{Q}}$-named ordinal and $\delta \in W \cap (\ell g(\bar{\mathbb{Q}}) + 1) \Rightarrow (\exists \alpha < \delta)(\Vdash \underset{\sim}{\zeta} \notin [\alpha, \delta))$.

1.8 Claim. *1) Assume that $\bar{\mathbb{Q}}$ is W-continuous. If $\underset{\sim}{\zeta}$ is a simple $\bar{\mathbb{Q}}$-named^2 $[0, \gamma)$-ordinal, $\gamma \in W$ is regular and $\beta < \gamma$ implies density$(\mathbb{P}_\beta) < \gamma$ or just ($\mathbb{P}_\beta$ satisfies the $\mathrm{cf}(\gamma)$)-c.c.), <u>then</u> for some $\beta < \gamma, \underset{\sim}{\zeta}$ is a simple $\bar{\mathbb{Q}} \restriction \beta$-named^2 $[0, \beta)$-ordinal.*

2) If $\bar{\mathbb{Q}}$ is W-continuous and $\gamma \in W \Rightarrow \gamma$ regular and $\underset{\sim}{\zeta}$ is a simple $\bar{\mathbb{Q}}$-named^2 $[0, \alpha)$-ordinal <u>then</u> $\underset{\sim}{\zeta}$ is a simple $(\bar{\mathbb{Q}}, W)$-named^2 $[0, \alpha)$-ordinal.

3) If $\underset{\sim}{\zeta}$ is a simple $\bar{\mathbb{Q}}$-named^2 $[0, \alpha)$-ordinal <u>then</u> there is a full simple $\bar{\mathbb{Q}}$-named^2 $[0, \alpha)$-ordinal $\underset{\sim}{\zeta}'$ such that $\Vdash_{\bar{\mathbb{Q}}}$ "if $\underset{\sim}{\zeta}$ is well defined then it is equal to $\underset{\sim}{\zeta}'$".

Proof. By induction on the depth of $\underset{\sim}{\zeta}$. $\square_{1.8}$

1.9 Remark. 1) We can restrict in the definition of $\mathrm{Gen}^r(\bar{\mathbb{Q}})$ to $\mathbb{P}^*$ in some class K, and get a K-variant of our notions.
2) Note: even if in 1.7(1) we ask $\mathrm{Dom}(\underset{\sim}{\tau})$ to be a maximal antichain of $\bigcup_{i<\delta} \mathbb{P}_i$ it will not be meaningful as in the appropriate $\mathbb{P}_\delta$, we have $\bigwedge_{i<\delta} \mathbb{P}_i \lessdot \mathbb{P}_\delta$ but not necessarily $\bigcup_{i<\delta} \mathbb{P}_i \lessdot P_\delta$ hence it will not in general be a maximal antichain.
3) Note that in the simple case we wrote $\mathbb{P}_{\xi \cap \alpha}$ not $\mathbb{P}_{(\xi+1) \cap \alpha}$. Compare this with the remark [Sh:f, Ch.XIV,1.1B]. Here in the main case we use full simple $\bar{\mathbb{Q}}$-named^2 ordinals, though we shall remark on the affect of the non-simple case; as a result we will not have a general associativity law, but the definition of $\mathrm{Sp}_3 - \mathrm{Lim}_\kappa(\bar{\mathbb{Q}})$ will be somewhat simplified. As said earlier, we can interchange decisions on this matter. Of course, also [Sh:f, Ch.XV] can be represented with this iteration.
4) The "name^1" is necessary for the $\kappa > \aleph_1$ case, but "name^2" is preferable for $\kappa = \aleph_1$, so we could have concentrated on name^1 for $\kappa > \aleph_1$, name^2 for $\kappa = \aleph_1$, but actually we concentrated on simple, name^2 for $\kappa = \aleph_1$; see 1.15(B) below.

1.10 Fact. 1) For $\bar{\mathbb{P}} = \langle \mathbb{P}_i : i < \ell g(\bar{\mathbb{P}}) \rangle$, a $\lessdot$-increasing sequence of forcing notions, $\ell \in \{1, 2\}$ and simple $\bar{\mathbb{P}}$-named$^\ell$ $[j, \beta)$-ordinal $\underset{\sim}{\zeta}$ and $p \in \bigcup_{i<\alpha} \mathbb{P}_i$ there are ξ, q and q_1

such that $p \le q \in \bigcup_{i<\ell g(\bar{\mathbb{P}})} \mathbb{P}_i$ and: either $q \Vdash_{\bar{\mathbb{P}}}$ "$q_1 \in \underset{\sim}{G}$", $q_1 \in \mathbb{P}_\xi, \xi < \alpha, [p \in \mathbb{P}_\xi \Rightarrow q = q_1]$ and $q_1 \Vdash_{\bar{\mathbb{P}}}$ "$\underset{\sim}{\zeta} = \xi$" or $q \Vdash_{\bar{\mathbb{P}}}$ "$\underset{\sim}{\zeta}$ is not defined" (and even $p \Vdash_{\bar{\mathbb{P}}}$ "$\underset{\sim}{\zeta}$ is not defined").

2) For $\bar{\mathbb{P}}$ and $\ell \in \{1,2\}$ as above, and simple $\bar{\mathbb{P}}$-named$^\ell$ $[j,\beta)$-ordinals $\underset{\sim}{\zeta}, \underset{\sim}{\xi}$, also $\max\{\underset{\sim}{\zeta}, \underset{\sim}{\xi}\}$ and $\min\{\underset{\sim}{\zeta}, \underset{\sim}{\xi}\}$ are simple $\bar{\mathbb{Q}}$-named$^\ell$ $[j,\beta)$-ordinals (naturally defined, so max $\{\underset{\sim}{\zeta}, \underset{\sim}{\xi}\}[G]$ is defined iff a $\underset{\sim}{\zeta}[G], \underset{\sim}{\xi}[G]$ are defined, and $\min\{\underset{\sim}{\zeta}, \underset{\sim}{\xi}\}[G]$ is defined iff $\underset{\sim}{\zeta}[G]$ is defined or $\underset{\sim}{\xi}[G]$ is defined). If $\underset{\sim}{\zeta}, \underset{\sim}{\xi}$ are full then so are $\max\{\underset{\sim}{\zeta}, \underset{\sim}{\xi}\}$ and $\min\{\underset{\sim}{\zeta}, \underset{\sim}{\xi}\}$.

3) For $\bar{\mathbb{P}}$ and ℓ as above, $n < \omega$ and simple $\bar{\mathbb{P}}$-named$^\ell$ ordinals $\underset{\sim}{\xi}_1, \dots, \underset{\sim}{\xi}_n$ and $p \in \bigcup_{i<\ell g(\bar{\mathbb{P}})} \mathbb{P}_i$ there are $\zeta < \alpha$ and $q \in \mathbb{P}_\zeta$ such that, first: $p \le q$ or at least $q \Vdash_{\mathbb{P}_\zeta}$ "$p \in \mathbb{P}_i / \underset{\sim}{G}_{\mathbb{P}_\zeta}$ for some $i < \ell g(\bar{\mathbb{P}})$" and second: for some $\ell \in \{1, \dots, n\}$ we have $q \Vdash_{\bar{\mathbb{P}}}$ "$\zeta = \underset{\sim}{\xi}_\ell = \max\{\underset{\sim}{\xi}_1, \dots, \underset{\sim}{\xi}_n\}$" or $q \Vdash_{\bar{\mathbb{P}}}$ " $\max\{\underset{\sim}{\xi}_1, \dots, \underset{\sim}{\xi}_n\}$ not defined". Similarly for min.

4) The same holds for simple $(\bar{\mathbb{Q}}, W)$-names$^\ell$ and we can omit simple.

5) If $\underset{\sim}{\zeta}_i$ is a simple $\bar{\mathbb{Q}}$-named$^1[\beta_i, \gamma_i)$-ordinal for $i < i^*$ then[1] $\sup\{\underset{\sim}{\zeta}_i : i < i^*\}$ is simple $\bar{\mathbb{Q}}$-named^1 $[\mathrm{Min}_i\, \beta_i, \sup_i \gamma_i)$-ordinal.

6) Similarly for $\{\underset{\sim}{\zeta}_i : i\}$ and when we omit "simple".

7A) A simple $\bar{\mathbb{P}}$-name$^\ell[j,\beta)$-ordinal

- (a) ζ is a $\bar{\mathbb{P}}$-named$^\ell[j,\beta)$-ordinal;
- (b) if $j_2 \le j_1 < \beta_1 \le \beta_2$ then any [simple] $\bar{\mathbb{P}}$-named $[j_1, \beta_1)$-ordinal is a [simple] $\bar{\mathbb{P}}$-named$^\ell[j_2, \beta_2)$-ordinal;
- (c) a [simple] $\bar{\mathbb{P}}$-named$^2[j,\beta)$ ordinal is a [simple] $\bar{\mathbb{P}}$-named$^1[j,\beta)$-ordinal
- (d) if $\beta \le \alpha' \le \ell g(\bar{\mathbb{P}})$ then any [simple] $\bar{\mathbb{P}}$-named$^\ell[j,\beta)$-ordinal is a $(\bar{\mathbb{P}} \restriction \alpha')$-named$^\ell[j,\beta)$-ordinal.

Proof. Straight.

1.11 Discussion. We have in defining our iteration several possible variants, some of our particular choices are not important: we can make it like revised countable

[1]this seems lacking for "name^2".

support as in [Sh:f, Ch.X,§1] or like $\aleph_1$-RS in [Sh:f, Ch.XIV,§1], or as in [Sh:f, Ch.XIV,§2] (as here); for most uses $\kappa = \aleph_1$ and we could restrict ourselves to $\le_{\rm pr} = \le_{\rm vpr}$ as equality; but in [\GoSh:511] we need the three partial orders.

So below we have finite support for non-pure, countable for pure and Easton for very pure.

1.12 Definition. 1) For a forcing notion $\mathbb{P}$ (as in 1.1) let $\mathbb{P}^{[\rm pr]}$ be defined like $\mathbb{P}$ except that we make $\emptyset_{\mathbb{P}} \le_{\rm pr} p$ for every $p \in \mathbb{P}$.
2) For a forcing notion $\mathbb{P}$ (as in 1.1) let $\mathbb{P}^{[\rm vp]}$ be defined like $\mathbb{P}$ except that we make $\emptyset_{\mathbb{P}} \le_{\rm vp} p$ (and $\emptyset \le_{\rm pr} \mathbb{P}$, of course) for every $p \in \mathbb{P}$.

1.13 Fact. 1) For a forcing notion $\mathbb{P}$ and $x \in \{\text{pr,vp}\}, \mathbb{P}^{[x]}$ is also a forcing notion, and they are equivalent as forcing notions.
2) For $x \in \{\text{pr,vp}^r\}$ the operations $\mathbb{P} \mapsto \hat{\mathbb{P}}$ and $\mathbb{P} \mapsto \mathbb{P}^{[x]}$ commute.
3) If $(x_1, x_2, x_3) \in \{(\text{pr,pr,pr}),(\text{vpr,pr,vpr}),(\text{vpr,vpr,pr})),(\text{vpr,vpr,vpr})\}$ then $\mathbb{P}^{[x_3]} = \mathbb{P}^{[x_1][x_2]}$.
4) θ-completeness is preserved in the natural cases.

1.14 Discussion. 1) Why do we bother with $\mathbb{P}^{[\rm pr]}, \mathbb{P}^{[\rm vpr]}$? If in the iteration defined below we use only $\underset{\sim}{Q}_i^{[\rm pr]}, \underset{\sim}{\mathbb{Q}}_i^{[\rm vp]}$, we get a variant of the definition without the need to repeat it. We may want that: if $\ell g(\bar{\mathbb{Q}}) = \lambda$ inaccessible and $i < \kappa \Rightarrow |\mathbb{P}_i| < \lambda$ then $\bigcup\limits_{i<\lambda} \mathbb{P}_i = \mathbb{P}_\lambda$ (here as done in [Sh:f, Ch.XIV,§2] we can just impose it).
Some other restrictions are for simplicity only.
2) Below the case $e = 6$ is the main one. [Saharon]

1.15 Definition/Claim. We define and prove the following by induction on α. Below $\kappa = \aleph_1, e = \partial$ so we can omit them (they are meaningful in §11)

- (A) [Definition] $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{Q}_i : i < \alpha \rangle$ is a $\kappa - \text{Sp}_e$-iteration for W or $\kappa - \text{Sp}_e(W)$-iteration (if W is absent we mean $\{\beta \le \alpha : \beta \text{ strongly inaccessible}\}$); α is called the length of $\bar{\mathbb{Q}}, \ell g(\bar{\mathbb{Q}})$.
- (B) [Definition] A simple $(\bar{\mathbb{Q}}, W)$-named$_e$ ordinal $\underset{\sim}{\zeta}$ and $\underset{\sim}{\zeta} \restriction [\alpha, \beta)$.
- (C) [Definition] A simple $(\bar{\mathbb{Q}}, W)$-named$_e$ atomic condition $\underset{\sim}{q}$ (or atomic $[j, \beta)$-condition where $j \le \beta \le \alpha$); also we define $\underset{\sim}{q} \restriction \xi, \underset{\sim}{q} \restriction \{\varepsilon\}, \underset{\sim}{q} \restriction [\xi, \zeta)$ for a simple $\bar{\mathbb{Q}}$-named$_e$ atomic condition $\underset{\sim}{q}$ and ordinals $\varepsilon < \alpha, \xi \le \zeta \le \alpha$ (or

simple $\bar{\mathbb{Q}}$-named$_e$ ordinals $\underset{\sim}{\xi}, \underset{\sim}{\zeta}$ instead ξ, ζ). We may add pure/very pure as adjectives to the condition.

(D) [Claim] Assume $\underset{\sim}{\zeta}$ is a simple $(\bar{\mathbb{Q}}, W)$-named$_e[j,\beta)$-ordinal. Then for any $\xi, \underset{\sim}{\zeta} \restriction \xi$ is a simple $(\bar{\mathbb{Q}}, W)$-named $[j, \min\{\beta,\xi\})$-ordinal and $\Vdash_{\bar{\mathbb{Q}}}$ "if $\underset{\sim}{\zeta} < \xi$ then $\underset{\sim}{\zeta} = \underset{\sim}{\zeta} \restriction \xi$; if $\underset{\sim}{\zeta} \geq \xi$ or $\underset{\sim}{\zeta}$ is undefined, then $\underset{\sim}{\zeta} \restriction \xi$ is undefined", also $\underset{\sim}{\zeta} \restriction \xi$ is a simple $(\bar{\mathbb{Q}} \restriction \xi, W)$-named ordinal.
Similarly $\underset{\sim}{\zeta} \restriction [\xi_1, \xi_2)$. If $\underset{\sim}{\xi}_1 \leq \underset{\sim}{\xi}_2 \leq \alpha$ are simple $(\bar{\mathbb{Q}}, W)$-named $[\alpha_1, \alpha_2)$-ordinals (the $\underset{\sim}{\xi}_1 \leq \underset{\sim}{\xi}_2$ means $\Vdash_{\bar{\mathbb{Q}}}$ "$\underset{\sim}{\xi}_1 \leq \underset{\sim}{\xi}_2$"), then $\underset{\sim}{\zeta} \restriction [\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_2)$ is a simple $(\bar{\mathbb{Q}}, W)$-named $[\alpha_1, \alpha_2)$-ordinal and $\Vdash_{\bar{\mathbb{Q}}}$ "if $\underset{\sim}{\zeta} \in [\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_2)$, then $\underset{\sim}{\zeta} = \underset{\sim}{\zeta} \restriction [\underset{\sim}{\xi}_2, \underset{\sim}{\xi}_2)$ otherwise $\underset{\sim}{\zeta} \restriction [\xi_1, \xi_2)$ is undefined. If in addition $\beta = \mathrm{Min}\{\alpha, \alpha_2, \ell g(\bar{\mathbb{Q}})\}$ and $\beta \leq \gamma, \alpha_1' \leq \alpha_1$ then $\underset{\sim}{\zeta} \restriction [\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_2)$ is a simple $(\bar{\mathbb{Q}} \restriction \beta, W)$-named $[\alpha_1', \gamma)$-ordinal. Also if $n < \omega$, for $\ell \in \{1, \ldots, n\}, \underset{\sim}{\xi}_\ell$ is a simple $\bar{\mathbb{Q}}$-named $[\beta_1, \beta_2)$-ordinal then $\mathrm{Max}\{\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n\}$ is a simple $\bar{\mathbb{Q}}$-named $[\beta_1, \beta_2)$-ordinal. Similarly for Min.

(E) [Claim] If q is a simple $(\bar{\mathbb{Q}}, W)$-named atomic $[j,\beta)$-condition, $\xi < \alpha$, then $\underset{\sim}{q} \restriction \xi$ is a simple $(\bar{\mathbb{Q}} \restriction \xi, W)$-named atomic $[j, \min\{\beta, \xi\})$-condition and $\underset{\sim}{q} \restriction \{\xi\}$ is a $\mathbb{P}_\xi$-name of a member of $\underset{\sim}{\mathbb{Q}}_\xi$ or undefined (and then it may be assigned the value $\emptyset_{\mathbb{Q}_\xi}$, the minimal member of $\underset{\sim}{\mathbb{Q}}_\xi$). If q is a simple $(\bar{\mathbb{Q}}, W)$-named atomic condition, $\underset{\sim}{\xi} \leq \underset{\sim}{\zeta} \leq \alpha$ are simple $(\bar{\mathbb{Q}}, W)$-named ordinals then $\underset{\sim}{q} \restriction [\underset{\sim}{\xi}, \underset{\sim}{\zeta})$ is a simple $(\bar{\mathbb{Q}}, W)$-named ordinal. Also $\underset{\sim}{q} \restriction \{\zeta\} = \underset{\sim}{q} \restriction [\zeta, \zeta+1)$, and if $\underset{\sim}{q}$ is a simple $(\bar{\mathbb{Q}}, W)$-named $[\zeta, \xi)$-ordinal, $\zeta' < \xi'$ and $\Vdash_{\bar{\mathbb{Q}}}$ "$\underset{\sim}{\zeta} \in [\zeta', \xi')$", then it is a simple $(\bar{\mathbb{Q}}, W)$-named $[\zeta', \xi')$-ordinal. Also "pure" and "very pure" are preserved by restriction.

(F) [Definition] The $\kappa - \mathrm{Sp}_e(W)$-limit of $\bar{\mathbb{Q}}, \mathrm{Sp}_e(W)\text{-}\mathrm{Lim}_\kappa(\bar{\mathbb{Q}})$, denoted by $\mathbb{P}_\alpha$ for $\bar{\mathbb{Q}}$ as in clause (A) in particular of length α, and $p \restriction \xi$ and $\mathrm{Dom}(p)$ for $p \in \mathrm{Sp}_e(W)\text{-}\mathrm{Lim}_\kappa(\bar{\mathbb{Q}}), \xi$ an ordinal $\leq \alpha$; (similarly for a simple $(\bar{\mathbb{Q}}, W)$-named $[0, \ell g(\bar{\mathbb{Q}}))$ ordinal $\underset{\sim}{\xi}$, etc. We also define $\mathbb{P}_{\underset{\sim}{\zeta}}$ for $\underset{\sim}{\zeta}$ a $(\bar{\mathbb{Q}}, W)$-named ordinal.

(G) [Theorem] $\mathrm{Sp}_e(W)\text{-}\mathrm{Lim}_\kappa(\bar{\mathbb{Q}})$ is a forcing notion (in the sense of 1.1(1)).

(H) [Theorem] Assume $\beta < \alpha = \ell g(\bar{\mathbb{Q}})$ or more generally, $\underset{\sim}{\beta}$ is a full simple

$(\bar{\mathbb{Q}}, W)$-named ordinal (see end of clause (F) above). Then $\mathbb{P}_\beta \subseteq_{\rm ic}$ $\mathrm{Sp}_e(W)$-$\mathrm{Lim}_\kappa(\bar{\mathbb{Q}})$ (so a submodel with the three partial orders, even compatibilities are preserved) and $[p \in \mathbb{P}_\beta \Rightarrow p \restriction \beta = p]$ and $[\mathbb{P}_\alpha \models$ "$p \le_\ell q$" $\Rightarrow \mathbb{P}_\beta \models$ "$p \restriction \beta \le_\ell q \restriction \beta$"] (for $\ell = 0,1,2$, of course) and $\mathbb{P}_\alpha \models$ "$p \restriction \beta \le_\ell p$". Also $q \in \mathbb{P}_\beta, p \in$ $\mathrm{Sp}_e(W)$-$\mathrm{Lim}_\kappa(\bar{\mathbb{Q}})$ are compatible iff $q, p \restriction \beta$ are compatible in $\mathbb{P}_\beta$. In fact, if $q \in \mathbb{P}_\beta, \mathbb{P}_\beta \models$ "$p \restriction \beta \le q$" then $q \cup (p \restriction [\beta, \alpha))$ belongs to $\mathbb{P}_\alpha$ and is a least upper bound of p, q and if $\mathbb{P}_\beta \models$ "$p \restriction \beta \le_\ell q$" even a $\le_\ell$-least upper bound of q. Hence $\mathbb{P}_\beta \lessdot (\mathrm{Sp}_e(W)$-$\mathrm{Lim}_\kappa(\bar{q}))$.

(I) [Claim] The set of $p \in \mathbb{P}_\alpha$ such that for every $\beta < \alpha$ we have $\Vdash_{\mathbb{P}_\beta}$ "$p \restriction \{\beta\}$ is a singleton or empty", is a dense subset of $\mathbb{P}_\alpha$. Also we can replace $\underset{\sim}{\mathbb{Q}}_\beta$ by $\hat{\mathbb{Q}}_\beta$ and the set of "old" $p \in \mathbb{P}_\alpha$ is a dense subset of the new (but actually do not use this).

(J) [Claim] If α is strongly inaccessible, $\zeta < \alpha \Rightarrow |\mathbb{P}_\zeta| < \alpha$ or just $\zeta < \alpha \Rightarrow CC(\mathbb{P}_\zeta) \le \alpha$ and $\alpha \in W$, <u>then</u> $\mathbb{P}_\alpha = \bigcup\limits_{\zeta<\alpha} \mathbb{P}_\zeta$.

<u>Proof and Definition</u>:

(A) $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha \rangle$ is a $\kappa - \mathrm{Sp}_e(W)$-iteration if $\bar{\mathbb{Q}} \restriction \beta$ is a $\kappa - \mathrm{Sp}_e(W)$-iteration for every $\beta < \alpha$, and if $\alpha = \beta + 1$, <u>then</u> $\mathbb{P}_\beta =$ $\mathrm{Sp}_e(W)$-$\mathrm{Lim}_\kappa(\bar{\mathbb{Q}} \restriction \beta)$ and $\underset{\sim}{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name of a forcing notion as in 1.1(1) here.

(B) We say $\underset{\sim}{\zeta}$ is a simple $(\bar{\mathbb{Q}}, W)$-named$_e$ $[j, \beta)$-ordinal <u>if</u> $\underset{\sim}{\zeta}$ is a simple $(\bar{\mathbb{Q}}, W)$-named^2 $[j, \beta)$-ordinal.

(C) We say $\underset{\sim}{q}$ is a simple $(\bar{\mathbb{Q}}, W)$-named atomic $[j, \beta)$-condition when: $\underset{\sim}{q}$ is a $\bar{\mathbb{Q}}$-name, and for some $\underset{\sim}{\zeta} = \underset{\sim}{\zeta}_{\underset{\sim}{q}}$, a simple $(\bar{\mathbb{Q}}, W)$-named $[j, \beta)$-ordinal, we have $\Vdash_{\bar{\mathbb{Q}}}$ "$\underset{\sim}{\zeta}$ has a value iff $\underset{\sim}{q}$ has, and if they have then $j \le \underset{\sim}{\zeta} <$ $\min\{\beta, \ell g(\bar{\mathbb{Q}})\}$ and $\underset{\sim}{q} \in \underset{\sim}{\mathbb{Q}}_{\underset{\sim}{\zeta}}$". If we omit "$[j, \beta)$" we mean "$[0, \alpha)$". Now $\underset{\sim}{q} \restriction \xi$ will have a value iff $\underset{\sim}{\zeta}_{\underset{\sim}{q}}$ has a value $< \xi$ and then its value is the value of $\underset{\sim}{q}$. Lastly, $\underset{\sim}{q} \restriction \{\xi\}$ will have a value iff $\underset{\sim}{\zeta}_{\underset{\sim}{q}}$ has value ξ and then its value is the value of $\underset{\sim}{q}$. Similarly for $\underset{\sim}{q} \restriction [\zeta, \xi)$ and $\underset{\sim}{q} \restriction \underset{\sim}{\xi}, \underset{\sim}{q} \restriction \{\underset{\sim}{\xi}\}$, $\underset{\sim}{q} \restriction [\underset{\sim}{\zeta}, \underset{\sim}{\xi})$. We say $\underset{\sim}{q}$ is pure if $\Vdash_{\bar{\mathbb{Q}}}$ "for $\xi < \alpha$, if $\underset{\sim}{\zeta}_q = \xi$ and $\underset{\sim}{q} \in \underset{\sim}{\mathbb{Q}}_\xi$ then $\underset{\sim}{\mathbb{Q}}_\xi \models \emptyset_{\underset{\sim}{\mathbb{Q}}_\xi} \le_{\rm pr} \underset{\sim}{q}$". We say $\underset{\sim}{q}$ is very pure if $\Vdash_{\bar{\mathbb{Q}}}$ "for $\xi < \alpha$, if $\underset{\sim}{q} \in \underset{\sim}{\mathbb{Q}}_\xi$, then $\underset{\sim}{\mathbb{Q}}_\xi \models \emptyset_{\underset{\sim}{\mathbb{Q}}_i} \le_{\rm vpr} \underset{\sim}{q}$".

$(D), (E)$ Left to the reader.

(F) We are defining $\mathrm{Sp}_e(W)\text{-}\mathrm{Lim}_\kappa(\bar{\mathbb{Q}})$ (where $\bar{\mathbb{Q}} = \langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha\rangle$, of course).

It is a quadruple $\mathbb{P}_\alpha = (\mathbb{P}_\alpha, \le, \le_{\mathrm{pr}}, \le_{\mathrm{vpr}})$ where

(a) $\mathbb{P}_\alpha$ is the set of $p = \{\underset{\sim}{q}_i : i < i^*\}$ satisfying for some witness $\underset{\sim}{\bar{\zeta}}$:

(i) each $\underset{\sim}{q}_i$ is a simple $(\bar{\mathbb{Q}}, W)$-named atomic condition,

and for every $\xi < \alpha$, we have
$\Vdash_{\mathbb{P}_\xi}$ "$p \restriction \{\xi\} =: \{\underset{\sim}{q}_i \restriction \{\xi\} : i < i^*\} \cup \{0_{\mathbb{Q}_\xi}\} \in \underset{\sim}{\hat{\mathbb{Q}}}_\xi$"

(ii) if $\alpha \in W$ is strongly inaccessible $> CC(\mathbb{P}_i) + \kappa$

for every $i < \alpha$, <u>then</u> $i^* < \alpha$

(iii) $\underset{\sim}{\bar{\zeta}} = \langle \underset{\sim}{\zeta}_\varepsilon : \varepsilon < j\rangle$ where $j < \kappa$ and

each $\underset{\sim}{\zeta}_\varepsilon$ is a simple $(\bar{\mathbb{Q}}, W)$-named $[0, \alpha)$-ordinal,

[the reader should think of $\{\underset{\sim}{\zeta}_\varepsilon : \varepsilon < j\}$

as the non-very-pure support of p]

(iv) for every $\xi < \alpha$ we have (we may replace
$\Vdash_{\bar{\mathbb{Q}}}$ by $\Vdash_{\mathbb{P}_\xi}$ as we use simple names)
$\Vdash_{\bar{\mathbb{Q}}}$ "if $(\forall \varepsilon < j)(\zeta_\varepsilon[\underset{\sim}{G} \cap \mathbb{P}_\xi]$ is $\neq \xi)$ (for example is

not well defined) <u>then</u> $\emptyset_{\mathbb{Q}_\xi} \le_{\mathrm{vpr}} p \restriction \{\xi\}$ in $\underset{\sim}{\hat{\mathbb{Q}}}_\xi$"

(v) if $\beta < \alpha$ then $p \restriction \beta =: \{\underset{\sim}{q}_i \restriction \beta : i < i^*\}$ belongs to $\mathbb{P}_\beta$

(vi) if $\alpha \in W$ is strongly inaccessible $> CC(\mathbb{P}_i) + \kappa$
for every $i < \alpha$ <u>then</u> for some $\beta < \alpha$
every $\underset{\sim}{\zeta}_i$ is a simple $(\bar{\mathbb{Q}}, W)$-named $[0, \beta)$-ordinal;

needed, e.g., in 6.12 (note: this demand follows by 1.8)

(b) for $p \in \mathrm{Sp}_e(W)\text{-}\mathrm{Lim}_\kappa(\bar{\mathbb{Q}})$ and $\xi < \ell g(\bar{\mathbb{Q}})$ we let:

$$p \restriction \xi =: \{r \restriction \xi : r \in p\}$$

$$p \restriction \{\xi\} =: \{r \restriction \{\xi\} : r \in p\}$$

we define similarly $p \restriction [\zeta, \xi), p \restriction \{\underset{\sim}{\zeta}\}, p \restriction [\underset{\sim}{\zeta}, \underset{\sim}{\xi})$.

(c) $\mathbb{P}_\alpha \models$ "$p^1 \le_{\mathrm{vpr}} p^2$" iff for every $\xi < \alpha$ we have (letting
$p^\ell = \{\underset{\sim}{q}^\ell_i : i < i^\ell(*)\}$ for $\ell = 1, 2$):

$$p^2 \restriction \xi \Vdash_{\mathbb{P}_\xi} \text{“}\underset{\sim}{\hat{\mathbb{Q}}}_\xi \models p^1 \restriction \{\xi\} \le_{\mathrm{vpr}} p^2 \restriction \{\xi\}\text{”}$$

(d) $\mathbb{P}_\alpha \models$ "$p^1 \leq_{\rm pr} p^2$" <u>iff</u>

(i) for every $\xi < \ell g(\bar{\mathbb{Q}})$, we have[2]
$p^2 \restriction \xi \Vdash_{\mathbb{P}_\xi}$ "then $\hat{\mathbb{Q}}_\xi \models p^1 \restriction \{\xi\} \leq_{\rm pr} p^2 \restriction \{\xi\}$"

(ii) for some ordinal $j < \kappa$ and simple $(\bar{\mathbb{Q}}, W)$-named $[0, \alpha)$-ordinals $\underset{\sim}{\zeta}_\varepsilon$ for $\varepsilon < j$, for every $\xi < \ell g(\bar{\mathbb{Q}})$ we have:
$p^2 \restriction \xi \Vdash_{\mathbb{P}_\xi}$ "if for no $\varepsilon < j$ do we have $\underset{\sim}{\zeta}_\varepsilon[\underset{\sim}{G}_{\mathbb{P}_\xi}] = \xi$, then $p^1 \restriction \{\xi\} \leq_{\rm vpr} p^2 \restriction \{\xi\}$ in $\hat{\underset{\sim}{Q}}_\xi$"; we call $\langle \underset{\sim}{\zeta}_\varepsilon : \varepsilon < j\rangle$ a witness.

(e) $\mathbb{P}_\alpha \models p^1 \leq p^2$ iff

(i) for every $\xi < \ell g(\bar{\mathbb{Q}})$ we have:

$$p^2 \Vdash_{\mathbb{P}_\xi} \text{"}\hat{\underset{\sim}{\mathbb{Q}}}_\xi \models p^1 \restriction \{\xi\} \leq p^2 \restriction \{\xi\}\text{"}$$

(ii) as in the definition of $\leq_{pr}$; clause (ii)

(iii) for some $n < \omega$ and simple $(\bar{\mathbb{Q}}, W)$-named ordinals $\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n$ we have:
for each $\xi < \ell g(\bar{\mathbb{Q}})$ we have
$p_2 \restriction \xi \Vdash_{\mathbb{P}_\xi}$ "if $\xi \neq \underset{\sim}{\xi}_\ell[\underset{\sim}{G}_{\mathbb{P}_\xi}]$ for
$\ell = 1, \ldots, n$ and
$\neg(\emptyset_{\underset{\sim}{\mathbb{Q}}_\xi} \leq_{\rm vp} p^1 \restriction \{\xi\})]$
then: $\hat{\underset{\sim}{\mathbb{Q}}}_\xi \models p^1 \restriction \{\xi\} \leq_{\rm pr} p^2 \restriction \{\xi\}$"

note that the truth value of $\zeta = \underset{\sim}{\xi}_\ell$ is a $\mathbb{P}_\zeta$-name so this is well defined.
We then (i.e. if $(i) + (ii) + (iii)$) say: $p_1 \leq p_2$ over $\{\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n\}$.

(f) Lastly, for $p \in \mathbb{P}_\alpha$ we let $\mathrm{Dom}(p) = \mathrm{Dom}_{\rm vp}(p) = \{\underset{\sim}{\zeta}_{\underset{\sim}{q}} : \underset{\sim}{q} \in p\}$ and $\mathrm{Dom}_{\rm pr}(p) = \{\underset{\sim}{\zeta}_\varepsilon : \varepsilon < j\}$ where $\bar{\zeta} = \langle \underset{\sim}{\zeta}_\varepsilon : \varepsilon < j\rangle$ is as in clause (F)(a) above (we can make it part of p).

[2] recall 1.1(4)(d) we can omit "$p^1 \restriction \{xi\} \neq \emptyset \Rightarrow$"

(g) We still have to define $\mathbb{P}_{\underset{\sim}{\beta}}$ for $\underset{\sim}{\beta}$ a full simple $(\bar{\mathbb{Q}}, W)$-named ordinal, it is $\{p : p = \{\underset{\sim}{q}_i : i < i^*\}$ and $\Vdash_{\bar{\mathbb{Q}}}$ "$\zeta_{\underset{\sim}{q}_i} < \underset{\sim}{\beta}$" that is if $\xi < \alpha$ then $\Vdash_{\mathbb{P}_\xi}$ "if $\underset{\sim}{\zeta}_{\underset{\sim}{q}_i}[G_{\mathbb{P}_\xi}] = \xi$ and $\underset{\sim}{\beta}[G_{\mathbb{P}_\xi}]$ is well defined then it is $> \xi$"$\}$.

(h) We call $p \in \mathbb{P}_\alpha$ full if it has a witness $\langle \underset{\sim}{\zeta}_\varepsilon : \varepsilon < j \rangle$ with each $\underset{\sim}{\zeta}_\varepsilon$ full.

(G) Let us check Definition 1.1(1) for $\mathbb{P}_\alpha =:$ $\mathrm{Sp}_e(W)\text{-}\mathrm{Lim}_\kappa(\bar{\mathbb{Q}})$:
Proof of $\leq^{\mathbb{P}_\alpha}$ is a partial order.
Suppose $p_0 \leq p_1 \leq p_2$. Let $n^\ell, \underset{\sim}{\xi}^\ell_1, \ldots, \underset{\sim}{\xi}^\ell_{n^\ell}$ and $j^\ell\,(< \kappa)$ $\underset{\sim}{\zeta}^\ell_\varepsilon$ (for $\varepsilon < j^\ell$) appear in the definition of $p_\ell \leq p_{\ell+1}$. Let $n = n^0 + n^1$, and

$$\underset{\sim}{\xi}_i = \begin{cases} \underset{\sim}{\xi}^0_i & \text{if} \quad 1 \leq i \leq n^1 \\ \underset{\sim}{\xi}^1_{i-n} & \text{if} \quad n^1 < i \leq n^1 + n^2. \end{cases}$$

Let $j = j^0 + j^1$ and

$$\underset{\sim}{\zeta}_\varepsilon = \begin{cases} \underset{\sim}{\zeta}^0_\varepsilon & \underline{\text{if}} \quad \varepsilon < j^0 \\ \underset{\sim}{\zeta}^1_{\varepsilon - j^0} & \underline{\text{if}} \quad \varepsilon \in [j^0, j^0 + j^1). \end{cases}$$

Let us check the three clauses of (e) of part (D).

Clause (i):
Let $\xi < \ell g(\bar{\mathbb{Q}})$ so for $\ell = 0, 1$

$$p_{\ell+1} \restriction \xi \Vdash_{\mathbb{P}_\xi} \text{“}p_\ell \restriction \{\xi\} \leq p_{\ell+1} \restriction \{\xi\} \text{ in } \hat{\mathbb{Q}}_\xi\text{”}.$$

As $\mathbb{P}_\xi \models$ "$p_1 \restriction \xi \leq p_2 \restriction \xi$" (by the induction hypothesis, clause (H)) clearly $p_2 \restriction \xi$ forces both assertions. As $\hat{\underset{\sim}{\mathbb{Q}}}_\xi$ is a partial order (under $\leq$) the conclusion follows.

Clause (ii):
Let $\xi < \ell g(\bar{\mathbb{Q}})$, so similarly $p^2 \restriction \xi \Vdash_{\mathbb{P}_\xi}$ "if $\xi \neq \underset{\sim}{\xi}^\ell_m$ for $m = 1, \ldots, n^\ell$ and $\ell = 0, 1$ (i.e., $\xi^\ell_m[\underset{\sim}{G}_{\mathbb{P}_\xi}]$ is $\neq \xi$ or is not well defined), $\underline{\text{then}}$ $p_0 \restriction \{\xi\} \leq_{\mathrm{pr}} p_1 \restriction \{\xi\}$ in $\hat{\underset{\sim}{\mathbb{Q}}}_\varepsilon$ and $p_1 \restriction \{\xi\} \leq_{\mathrm{pr}} p_2 \restriction \{\xi\}$ in $\hat{\underset{\sim}{\mathbb{Q}}}_\xi$" from which the result follows.

Clause (iii):
Lastly, for $\xi < \alpha$ we have $p_2 \restriction \xi \Vdash_{\mathbb{P}_\xi}$ "if $\xi \notin \{\underset{\sim}{\zeta}^\ell_\varepsilon[G_{\mathbb{P}_\zeta}] : \underset{\sim}{\zeta}^\ell_\varepsilon[G_{\mathbb{P}_\zeta}]$ well defined, $\varepsilon < j^\ell$ and $\ell \in \{0,1\}\}$ then $p_0 \restriction \{\xi\} \le_{\rm vpr} p_1 \restriction \{\xi\} \le_{\rm vpr} p_2 \restriction \{\xi\}$ hence $p_0 \restriction \{\xi\} \le_{\rm vpr} p_2 \restriction \{\xi\}$".

So we have proved the three conditions needed for $p_0 \le p_2$ by the definition above so really $p_0 \le p_2$ holds, so $\le$ is a partial order.

Proof of $\le_{\rm pr}$ is a partial order.
Similar proof.

Proof of $\le_{\rm vpr}$ is a partial order.
Similar proof just easier.

Proof of $p \le_{\rm pr} q \Rightarrow p \le q$:
By the definition; easy.

Proof of $p \le_{\rm vpr} q \Rightarrow p \le_{\rm pr} q$:
By the definition, check.

So in 1.1(1) all the requirements on $\mathbb{P}_\alpha$ holds.

$(H),(I),(J)$ We leave the checking to the reader (actually we prove (I) in the proof of 1.20 below). $\square_{1.15}$

1.16 Fact. 1) If $\bar{\mathbb{Q}}$ is a $\kappa - {\rm Sp}_e$-iteration and for each $i < \ell g(\bar{\mathbb{Q}})$ we have it is forced (i.e., $\Vdash_{\mathbb{P}_i}$) that $\le^{\mathbb{Q}_i}_{\rm pr} = \le^{\mathbb{Q}_i}$ and $\le^{\mathbb{Q}_i}_{\rm vpr}$ is equality, then $\bar{\mathbb{Q}}$ is a variant of κ-RS iteration (as in [Sh:f, Ch.XIV,§1]), i.e. they are the same if we use there only simple $\bar{\mathbb{Q}}$-named ordinals (or allow here non-simple ones so the version here is exactly as in [Sh:f, Ch.XIV,2.6]).

Proof. Straightforward.

1.17 Claim. *1) In 1.15 in Definition (F), clause (a)(iii) we can demand that each $\underset{\sim}{\zeta}_\varepsilon$ is full (simple $(\bar{\mathbb{Q}}, W)$-named $[0,\alpha)$-ordinal) and similarly in (d)(ii), (=(e)(ii)) and (e)(iii).*
2) Suppose $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha\rangle$ is a $\kappa - {\rm Sp}_e(W)$-iteration (so $\mathbb{P}_\alpha = {\rm Sp}_e(W)$-${\rm Lim}_\kappa(\bar{\mathbb{Q}})$). If $p \le q$ in $\mathbb{P}_\alpha$, then there are $r, n < \omega$ and $\xi_1 < \ldots < \xi_n < \alpha$ such that:

(a) $r \in \mathbb{P}_\alpha$
(b) $q \le r$
(c) $p \le r$ *above* $\{\xi_1, \dots, \xi_n\}$.

3) If $p^1, p^2 \in \mathbb{P}_\alpha$ *and 1.15(F)(e) holds but we allow* $\underset{\sim}{n}$ *to be a full simple* $(\bar{\mathbb{Q}}, W)$*-named ordinal, then* $p^2 \Vdash_{\mathbb{P}_\alpha}$ "$p_1 \in \underset{\sim}{G}_{\mathbb{P}_\alpha}$".

Remark. In fact, in 1.17(1) we can have $r \restriction [\xi_n, \alpha) = q \restriction [\xi_n, \alpha)$.

Proof. 1) As increasing those sets $(\{\underset{\sim}{\zeta}_\varepsilon : \varepsilon < \zeta\}, \{\underset{\sim}{\xi}_1, \dots, \underset{\sim}{\xi}_n\}$, respectively) cause no harm.
2) We prove this by induction on α.

Case 1: $\alpha = 0$.
Trivial.

Case 2: $\alpha = \beta + 1$.
Apply the induction hypothesis to $\bar{\mathbb{Q}} \restriction \beta, p \restriction \beta, q \restriction \beta$ (clearly $\bar{\mathbb{Q}} \restriction \beta$ is an $\kappa - \mathrm{Sp}_e(W)$-iteration, $p \restriction \beta \in \mathbb{P}_\beta, q \restriction \beta \in \mathbb{P}_\beta$ and $\mathbb{P}_\beta \models$ "$p \le q$", by 1.15, clause (H)). So we can find $r', m < \omega$ and $\{\xi'_1, \dots, \xi'_m\}$ such that:

(a)′ $r' \in \mathbb{P}_\beta$
(b)′ $\mathbb{P}_\beta \models q \restriction \beta \le r'$
(c)′ $p \restriction \beta \le r'$ (in $\mathbb{P}_\beta$) above $\{\xi_1, \dots, \xi_m\}$
(d)′ $\xi'_1 < \dots < \xi'_m$.

Let $n =: m + 1$ and

$$\xi_\ell = \begin{cases} \xi'_\ell & \text{if} \quad \ell \in \{1, \dots, m\} \\ \beta & \text{if} \quad \ell = n \end{cases}$$

and lastly $r = r' \cup (q \restriction \{\beta\})$.

Case 3: α is a limit ordinal.
Let $p \le q$ (in $\mathbb{P}_\alpha$) above $\{\underset{\sim}{\xi}_1, \dots, \underset{\sim}{\xi}_n\}$. We choose by induction on $\ell \le n$, the objects $r_\ell, \beta_\ell, \xi^*_\ell$ such that:

(α) $r_\ell \in \mathbb{P}_{\beta_\ell}$

(β) $r_\ell \le r_{\ell+1}$

(γ) $q \restriction \beta_\ell \le r_\ell$

(δ) $\beta_\ell \le \beta_{\ell+1} < \alpha$

(ε) $\beta_0 = 0, r_0 = \emptyset_{\mathbb{P}_0}$

(ζ) for $\ell \in \{1,\dots,n\}$ we have: either $r_\ell \Vdash_{\mathbb{P}_{\beta_\ell}}$ "$\underset{\sim}{\xi}_\ell = \xi^*_\ell$" and $\xi^*_\ell \le \beta_\ell$ or $\beta_\ell = \beta_{\ell-1}$ & $r_\ell = r_{\ell-1}$ and $r_\ell \cup (q \restriction [\beta,\alpha)) \Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\xi}_\ell$ is not defined or is $\ge \alpha$".

Carrying the definition is straight: for $i = 0$ use clause (ε). For $\ell+1 \le n$ when the second possibility of clause (ζ) fails there is r', such that $r_\ell \cup (q \restriction [\beta,\alpha)) \le r' \in \mathbb{P}_\alpha$, and $r' \Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\xi}_{\ell+1}$ is defined and is $< \alpha$", so there are $r'', \xi^*_{\ell+1} < \alpha$ such that $r' \le r'' \in \mathbb{P}_\alpha$ and $r'' \Vdash$ "$\underset{\sim}{\xi}_{\ell+1} = \xi^*_{\ell+1}$" so as "$\xi^*_{\ell+1}$ is a simple $\bar{\mathbb{Q}}$-named ordinal" we know that $\xi^*_{\ell+1} < \alpha$ and $r'' \restriction \xi^*_{\ell+1} \Vdash_{\mathbb{P}_{\xi^*_{\ell+1}}}$ "$\underset{\sim}{\xi}_{\ell+1} = \xi^*_{\ell+1}$". Let $\beta_{\ell+1} =: \max\{\beta_\ell, \xi^*_{\ell+1}\}$, and $r_{\ell+1} =: r'' \restriction \beta_{\ell+1}$. So we have carried the induction.

We now apply the induction hypothesis to $\bar{\mathbb{Q}} \restriction \beta_n, p \restriction \beta_n, r_n$; it is applicable as $\beta_n < \alpha$, and $\mathbb{P}_{\beta_n} \models$ "$p \restriction \beta_n \le q \restriction \beta_n \le r_n$". So there are $m < \omega, \xi_1 < \ldots < \xi_m < \beta_n$ and r^* such that $\mathbb{P}_{\beta_n} \models$ "$r_n \le r^*$" and $p \le r^*$ (in $\mathbb{P}_{\beta_n}$) above $\{\xi_1,\dots,\xi_m\}$. Now let $r =: r^* \cup (q \restriction [\beta_n,\alpha))$, clearly $q \le r$ and $p \le r$ above $\{\xi_1,\dots,\xi_m,\beta_n\}$.

3) So $p^1, p^2, \{\xi_\ell : \ell = 1,\dots,n\}$ are given. Assume that $p^2 \le q_1 \in \mathbb{P}_\alpha$. We can find q_2 such that $q_1 \le_2 \in \mathbb{P}_\alpha$ and q_2 forces a value to $\underset{\sim}{n}$ say $n(*)$. Next we can find q_3 such that $q_2 \le q_3 \in \mathbb{P}_\alpha$ and q_3 forces values to $\underset{\sim}{\xi}_\ell\, (\ell = 1,\dots,n(*)\}$, say $\xi^0_1,\dots,\xi^0_{n(*)}$. Now repeating the proof of "$\le^{\mathbb{P}_\alpha}$ is a partial order" (in clause (H) of 1.15) with p^1, p^2, q_3 here standing for p_0, p_1, p_2 there we let $\mathbb{P}_\alpha \models p^1 \le q_3$. As we have assumed just $p^2 \le q_1 \in \mathbb{P}_\alpha$ and $q_1 \le q_3$ we are done. $\square_{1.17}$

1.18 Claim. *Let $\bar{\mathbb{Q}}$ be a $\kappa - \mathrm{Sp}_e(W)$-iteration of length α.*

0) If $\underset{\sim}{\zeta}$ is a simple $(\bar{\mathbb{Q}}, W)$-named ordinal <u>*then*</u> *for some ordinal γ we have: $\underset{\sim}{\zeta}$ is a simple $(\bar{\mathbb{Q}}, W)$-named $[0,\gamma)$-ordinal.*

1)

(i) If $\beta < \alpha$ and $\underset{\sim}{\zeta}$ is a $\mathbb{P}_\beta$-name of a [full] simple $(\bar{\mathbb{Q}}, W)$-named $[\beta,\alpha)$-ordinal <u>*then*</u> *for some [full] simple $(\bar{\mathbb{Q}}, W)$-named $[\beta,\alpha)$-ordinal $\underset{\sim}{\xi}$ we have*

$$\Vdash_{\bar{\mathbb{Q}}} \text{"}\underset{\sim}{\zeta} = \underset{\sim}{\xi}\text{"}$$

(*ii*) *if* $\beta \le \alpha, \beta \le \gamma_1 \le \gamma_2$ *and* $\underset{\sim}{\zeta}$ *is a* $\mathbb{P}_\beta$*-name of a [full] simple* $(\bar{\mathbb{Q}}, W)$*-named* $[\gamma_1, \gamma_2)$*-ordinal* <u>*then*</u> *for some [full] simple* $(\bar{\mathbb{Q}}, W)$*-named* $[\gamma_1, \gamma_2)$*-ordinal* $\underset{\sim}{\xi}$ *we have* $\Vdash_{\bar{\mathbb{Q}}}$ "$\underset{\sim}{\zeta} = \underset{\sim}{\xi}$".

2) The same holds if we replace "ordinal" by "atomic condition" (so in (ii) we should demand $\gamma_2 \le \alpha$*).*
3) If $\alpha \le \gamma$ *and* $\underset{\sim}{\beta}$ *is a full simple* $(\bar{\mathbb{Q}}, W)$*-named* $[0, \alpha)$*-ordinal, and for each* $\beta < \alpha, \underset{\sim}{\zeta}_\beta$ *is a [full] simple* $(\bar{\mathbb{Q}}, W)$*-named* $[\beta, \gamma)$*-ordinal* <u>*then*</u> *for some [full] simple* $(\bar{\mathbb{Q}}, W)$*-named* $[0, \gamma)$*-ordinal* $\underset{\sim}{\xi}$ *we have*

$$\Vdash_{\bar{\mathbb{Q}}} \text{"if } \underset{\sim}{\beta}[\underset{\sim}{G}] = \beta (\text{so } \beta < \alpha) \text{ then } \underset{\sim}{\xi}[\underset{\sim}{G}] = \zeta_\beta[\underset{\sim}{G}]\text{"}.$$

4) If $\underset{\sim}{\beta}$ *is a full simple* $(\bar{\mathbb{Q}}, W)$*-named* $[0, \alpha)$*-ordinal and for each* $\beta < \alpha, \underset{\sim}{p}_\beta$ *is a* $(\bar{\mathbb{Q}}, W)$*-named* $[\beta, \alpha)$*-atomic condition* <u>*then*</u> *for some* $(\bar{\mathbb{Q}}, W)$*-named atomic condition p we have*

$$\Vdash_{\bar{\mathbb{Q}}} \text{"if } \underset{\sim}{\beta}[\underset{\sim}{G}] = \beta \text{ then } \underset{\sim}{p}[\underset{\sim}{G}] = \underset{\sim}{p}_\beta[\underset{\sim}{G}]\text{"}.$$

Proof. Easy. $\square_{1.18}$

—> scite{1.16} ambiguous

1.19 Claim. *1) Suppose F is a function,* <u>*then*</u> *for every ordinal* α *there is* $\kappa - \mathrm{Sp}_e(W)$*-iteration* $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha^\dagger \rangle$*, such that:*

(*a*) *for every i we have* $\underset{\sim}{\mathbb{Q}}_i = F(\bar{\mathbb{Q}} \restriction i)$*,*

(*b*) $\alpha^\dagger \le \alpha$

(*c*) *either* $\alpha^\dagger = \alpha$ *or* $F(\bar{\mathbb{Q}})$ *is not an* $\mathrm{Sp}_e(W) - \mathrm{Lim}_\kappa(\bar{\mathbb{Q}})$*-name of a forcing notion.*

2) Suppose $\beta < \alpha, G_\beta \subseteq \mathbb{P}_\beta$ *is generic over* $\mathbf{V}$*,* <u>*then*</u> *in* $\mathbf{V}[G_\beta], \bar{\mathbb{Q}}/G_\beta = \langle \mathbb{P}_i/G_\beta, \underset{\sim}{\mathbb{Q}}_i : \beta \le i < \alpha \rangle$ *is an* $\kappa - \mathrm{Sp}_e$*-iteration and* $\kappa - \mathrm{Sp}_e(W) - \mathrm{Lim}(\bar{\mathbb{Q}}) = \mathbb{P}_\beta * (\mathrm{Lim}(\bar{\mathbb{Q}})/\underset{\sim}{G}_\beta)$ *(essentially; more exactly up to equivalence) where, of course,* $\mathbb{P}_i/G_\beta = \{p \in \mathbb{P}_i : p \restriction \beta \in G_\beta\}$*.*
3) If $\bar{\mathbb{Q}}$ *is an* $\kappa - \mathrm{Sp}_e(W)$*-iteration,* $p \in \mathrm{Sp}_e(W) - \mathrm{Lim}_\kappa(\bar{\mathbb{Q}}), \mathbb{P}'_i = \{q \in \mathbb{P}_i : q \ge p \restriction i\}$*,*

$\underset{\sim}{\mathbb{Q}}'_i = \{p \in \underset{\sim}{\mathbb{Q}}_i : p \geq p \restriction \{i\}\}$ *and* $\emptyset_{\underset{\sim}{\mathbb{Q}}'_i} = p \restriction \{i\}, \leq_\ell^{\underset{\sim}{\mathbb{Q}}'_i} = \leq_\ell^{\underset{\sim}{\mathbb{Q}}_i} \restriction \underset{\sim}{\mathbb{Q}}'_i$ *for* $\ell = 0,1,2$ *<u>then</u>* $\bar{\mathbb{Q}} = \langle \mathbb{P}'_i, \underset{\sim}{\mathbb{Q}}'_i : i < \ell g(\bar{\mathbb{Q}})\rangle$ *is (essentially) a* $\kappa - \mathrm{Sp}_e(W)$*-iteration (and* $\mathrm{Sp}_e(W) - \mathrm{Lim}_\kappa(\bar{\mathbb{Q}}')$ *is* $P'_{\ell g(\bar{\mathbb{Q}})}$*).*

Proof. Should be clear.

1.20 Claim. *Suppose*

(a) $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha\rangle$ *is a* $\kappa - \mathrm{Sp}_e(W)$*-iteration (and* $\mathbb{P}_\alpha = \mathrm{Sp}_e(W) - \mathrm{Lim}_\kappa(\bar{\mathbb{Q}})$)

(b) $\ell(*) \in \{0,1\}$

(c) $\Vdash_{\mathbb{P}_i}$ "$(\underset{\sim}{\mathbb{Q}}_i, \leq_{\ell(*)})$ *is a* θ*-complete" for each* $i < \alpha$.

<u>Then</u>:
1) $(\mathbb{P}_\alpha, \leq_{\ell(*)})$ *is* θ*-complete, i.e. if* $\delta < \theta, \langle p_i : i < \delta\rangle$ *is* $\leq_{\ell(*)}$*-increasing <u>then</u> it has an* $\leq_{\ell(*)}$*-upper bound provided that:*
$\theta \leq \kappa$ *<u>or</u>* $\ell(*) = 0$ & $\theta \leq \mathrm{Min}\{\delta : \delta \in W$ *is strongly inaccessible and* $(\forall \beta < \delta)(|\mathbb{P}_\beta| < \delta)\}$*.*
2) Moreover for $\beta < \alpha$ *we have* $(\mathbb{P}_\alpha/\mathbb{P}_\beta, \leq_{\ell(*)})$ *is* θ*-complete.*
3) In fact, we can get $\leq_{\ell(*)}$*-lub (provided that there are such lub's for each* $\underset{\sim}{\mathbb{Q}}_i$*.*

Remark. We deal with θ-complete rather than strategically θ-complete (here and later) just for simplicity presentation, as it does not matter much by [Sh:f, CH.XIV,2.4].

Proof. Straightforward but we elaborate.
1) So assume $\delta < \theta$ and $p_i \in \mathbb{P}_\alpha$ for $i < \delta$ and $[i < j < \delta \Rightarrow p_i \leq_{\ell(*)} p_j]$. Now it is enough to find $p \in \mathbb{P}_\alpha$ such that

$$i < \delta \Rightarrow p_i \leq_{\ell(*)} p.$$

Let $p_i = \{\underset{\sim}{q}^i_\gamma : \gamma < \gamma_i\}$ and for each $\gamma < \gamma_i, \underset{\sim}{q}^i_\gamma$ is a simple $(\bar{\mathbb{Q}}, W)$-named atomic condition, say $\Vdash_{\bar{\mathbb{Q}}}$ "$\underset{\sim}{q}^i_\gamma \in \underset{\sim}{\mathbb{Q}}_{\underset{\sim}{\zeta}^i_\gamma}$", where $\underset{\sim}{\zeta}^i_\gamma$ is a simple $(\bar{\mathbb{Q}}, W)$-named ordinal (which is $\zeta_{\underset{\sim}{q}^i_\gamma}$). Now for each $\beta < \alpha$ let $\underset{\sim}{<}^*_\beta$ be a $\mathbb{P}_\beta$-name of a well ordering of $\underset{\sim}{\mathbb{Q}}_\beta$. For each $i(*) < \delta, \gamma(*) < \gamma_i$ let $\underset{\sim}{r}^{i(*)}_{\gamma(*)}$ be the following simple $(\bar{\mathbb{Q}}, W)$-named atomic condition:

Let $\zeta < \alpha, G_\zeta \subseteq \mathbb{P}_\zeta$ generic over $\mathbf{V}$ and $\underset{\sim}{\zeta}^{i(*)}_{\gamma(*)}[G_\zeta] = \zeta$, now work in $\mathbf{V}[G_\zeta]$, let $w_\zeta = \{i < \delta :$ for some $\gamma < \gamma_i$ we have $\underset{\sim}{\zeta}^i_\gamma[G_\zeta] = \zeta\}$. We let $u^\zeta_i = \{\gamma < \gamma_i : \underset{\sim}{\zeta}^i_\gamma[G_\zeta] = \zeta\}$ for each $i \in w_\zeta$. (As p_i is $\leq_{\ell(*)}$-increasing, w_ζ is an end segment of δ and $i(*) \in w_\zeta, \gamma(*) \in u^\zeta_{i(*)}$). For $i \in w_\zeta$ let $q^*_{i,\zeta} = (p_i \restriction \{\zeta\})[G_\zeta]$. Now define $\underset{\sim}{r}^{i(*)}_{\gamma(*)}[G_\zeta]$ as follows.

<u>Case 1</u>: For some $j < \delta$ the sequence $\langle q^*_{i,\zeta} : i \in w_\zeta \backslash j\rangle$ is constant.

Let $\underset{\sim}{r}^{i(*)}_{\gamma(*)}[G_\zeta] = q_{\mathrm{Min}(w_\zeta \backslash j), i}$.

<u>Case 2</u>: Not Case 1, but for some $\ell < 2, j < \delta$ the sequence $\langle q_{i,\zeta} : i \in w_\zeta \backslash j\rangle$ is $\leq_\ell$-increasing, without loss of generality ℓ minimal (on all possible j) and then $\underset{\sim}{r}^{i(*)}_{\gamma(*)}[G_\zeta] \in \underset{\sim}{\mathbb{Q}}_\zeta[G_\zeta]$ is the $<^*_\zeta$-first $\leq_\ell$-upper bound of $\{\underset{\sim}{q}^*_{i,\zeta} : i \in w_\zeta \backslash j\}$ where $<^*_\zeta = \underset{\sim}{\leq}^*_\zeta[G_\zeta]$. It exists by 1.2(2).

<u>Case 3</u>: Neither Case 1 nor Case 2 $\underset{\sim}{r}^{i(*)}_{\gamma(*)}[G_\zeta]$ is $\emptyset_{\underset{\sim}{\mathbb{Q}}_\zeta}$.

Let $p = \{\underset{\sim}{r}^{i(*)}_{\gamma(*)} : i(*) < \delta$ and $\gamma < \gamma_{i(*)}\}$.

If $\ell(*) = 0$ (i.e., p_i is $\leq_{\mathrm{vpr}}$-increasing) and $\langle \underset{\sim}{\zeta}_j : j < j^*\rangle$ witness $p_0 \in \mathbb{P}_\alpha$, then it witnesses $p \in \mathbb{P}_\alpha$ and easily $i < \delta \Rightarrow p_i \leq_0 p$.

So assume $\ell(*) = 1$, that is p_i is $\leq_{\mathrm{pr}}$-increasing. For $i_0 < i_2 < \delta$ let $\{\underset{\sim}{\zeta}^{i_0,i_1}_j : j < j_{\ell_0,\ell_1}\}$ be a witness to $p_{i_0} \leq_{pr} p_{i_1}$ and let $\{\underset{\sim}{\zeta}^i_j : j < j_i\}$ witness $p_i \in \mathbb{P}$. Letting κ^- be the maximal cardinal $< \kappa$, clearly $\{\underset{\sim}{\zeta}^{i_0,i_1}_j : i_0 < i_1 < \delta$ and $j < j_{i_0,i_1}\}$ has cardinality κ^-, so we can order it as $\{\underset{\sim}{\zeta}^{i_0(\varepsilon),i_1(\varepsilon)}_{j(\varepsilon)} : \varepsilon < \kappa^-\}$ and some $\{\underset{\sim}{\zeta}^{i(\varepsilon)}_{j'(\varepsilon)} : \varepsilon < \kappa^-\}$ list $\{\underset{\sim}{\zeta}^i_j : i < \delta, j < j_i\}$. Now $\{\underset{\sim}{\zeta}^{i(\varepsilon)}_{j'(\varepsilon)} : \varepsilon < \kappa^-\}$ witness $p \in \mathbb{P}_\alpha$ and $\{\underset{\sim}{\zeta}^{i_0(\varepsilon),i_1(\varepsilon)}_{j(\varepsilon)} : \varepsilon < \kappa^-\}$ witness $p_i \leq_1 p_\delta$ for every $i < \delta$.

2), 3) Similar proof. $\square_{1.20}$

1.21 Definition. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha\rangle$ be an $\kappa - \mathrm{Sp}_e(W)$-iteration.

1) We say $\underset{\sim}{y}$ is a $(\bar{\mathbb{Q}}, W, \zeta)$-name if: $\underset{\sim}{y}$ is a $\mathbb{P}_\alpha$-name, $\underset{\sim}{\zeta}$ is a simple $\bar{\mathbb{Q}}$-named $[0,\alpha)$-ordinal, and: if $\beta < \alpha, G_{\mathbb{P}_\alpha} \subseteq \mathbb{P}_\alpha$ is generic over $\mathbf{V}$ and for some $r \in G_{\mathbb{P}_\alpha} \cap \mathbb{P}_\beta$ we have $r \Vdash_{\bar{\mathbb{Q}}}$ "$\underset{\sim}{\zeta} = \beta$", then $\underset{\sim}{y}[G_{\mathbb{P}_\alpha}] \in \mathbf{V}[G_{\mathbb{P}_\beta}]$ is well defined and depends only on $G_{\mathbb{P}_\alpha} \cap \mathbb{P}_\beta$ so we write $\underset{\sim}{y}[G_{\mathbb{P}_\alpha} \cap \mathbb{P}_\beta]$; and if $G_{\mathbb{P}_\alpha} \subseteq \mathbb{P}_\alpha$ is generic over $\mathbf{V}$ and $\underset{\sim}{\zeta}[G_{\mathbb{P}_\alpha}]$

not well defined then $\underset{\sim}{y}[G_{\mathbb{P}_\alpha}]$ is not well defined (do not arise if $\underset{\sim}{\zeta}$ is full).

2) If $p \in \mathbb{P}_\alpha$ and $G_{\mathbb{P}_\alpha} \subseteq \mathbb{P}_\alpha$ is generic over $\mathbf{V}$, (or just in $\mathrm{Gen}^r(\bar{\mathbb{Q}})$), <u>then</u> $p[G_{\mathbb{P}_\alpha}]$ is the following function, $\mathrm{Dom}(p[\underset{\sim}{G}_{\mathbb{P}_\alpha}]) = \{\underset{\sim}{\zeta}_{\underset{\sim}{q}}[G_{\mathbb{P}_\alpha}] : \underset{\sim}{q} \in p\}$ and $(p[G_{\mathbb{P}_\alpha}])(\varepsilon) = \{\underset{\sim}{q}[G_{\mathbb{P}_\alpha}] : \underset{\sim}{q} \in p$ and $\underset{\sim}{\zeta}_{\underset{\sim}{q}}[G_{\mathbb{P}_\alpha}] = \varepsilon\}$.

1.22 Claim. *Suppose*

(a) $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha \rangle$ *is a* $\kappa - \mathrm{Sp}_e(W)$*-iteration.*

(b) $p \in \mathbb{P}_\alpha$ *and* $\underset{\sim}{\zeta}$ *is a simple* $(\bar{\mathbb{Q}}, W)$*-named* $[0, \alpha)$*-ordinal*

(c) $\underset{\sim}{r}$ *is a* $(\bar{\mathbb{Q}}, W, \underset{\sim}{\zeta})$*-named member of* $\mathbb{P}_\alpha / \mathbb{P}_{\underset{\sim}{\zeta}}$.

<u>Then</u>:
1) There is $q \in \mathbb{P}_\alpha$ *satisfying* $p \le q$ *such that:*

(∗) *if* $\xi < \alpha, G_\xi \subseteq \mathbb{P}_\xi$ *generic over* $\mathbf{V}$*, then*
 (α) $\underset{\sim}{\zeta}[G_\xi] = \xi$
 implies $(p \restriction \xi)[G_\xi] = (q \restriction \xi)[G_\xi]$ *and*
 (β) $\underset{\sim}{\zeta}[G_\xi] = \xi$ *implies*
 $(q \restriction [\xi, \alpha))[G_\xi] = \underset{\sim}{r}[G_\xi]$.

2) If in addition (for any $\ell < 3$*) clause* $(c)^+$ *below, <u>then</u> we can in* (∗) *add* $p \le_\ell q$

$(c)^+$ $\underset{\sim}{r}$ *is a* $(\bar{\mathbb{Q}}, W, \underset{\sim}{r})$*-named member of* $\mathbb{P}_\alpha / \mathbb{P}_{\underset{\sim}{\zeta}}$ *which is* $\le_\ell$*-above* $p \restriction [\underset{\sim}{\zeta}, \alpha)$.

Proof. Straightforward.

Central here is pure decidability.

1.23 Definition. 1) A forcing notion $\mathbb{Q}$ has pure (θ_1, θ_2)-decidability if: for every $p \in \mathbb{Q}$ and $\mathbb{Q}$-name $\underset{\sim}{\gamma} < \theta_1$, there are $a \subseteq \theta_1, |a| < \theta_2$ (but $|a| > 0$) and $r \in \mathbb{Q}$ such that $p \le_{\mathrm{pr}} r$ and $r \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\gamma} \in a$" (for $\theta_1 = 2$, alternatively, $\underset{\sim}{\gamma}$ is a truth value). If we write "$\le \theta_2$" we mean $|a| \le |\theta_2|$.
2) A forcing notion $\mathbb{Q}$ has pure θ-decidability where θ is an ordinal <u>if</u>: for every

$p \in \mathbb{Q}$ and $\mathbb{Q}$-name $\underset{\sim}{\gamma} < \theta$ there are $\gamma < \theta$ and $r \in \mathbb{Q}$ such that $p \le_{\rm pr} r$ and $r \Vdash_{\mathbb{Q}}$ "if $\theta < \omega$ then $\underset{\sim}{\gamma} = \gamma$ and if $\theta \ge \omega$ is a limit ordinal then $\underset{\sim}{\gamma} < \gamma$".

1.24 Observation. 1) If $\aleph_0 > \theta_2 > 2$ then pure $(\theta_2, 2)$-decidability is equivalent to pure $(2,2)$-decidability.
2) If $\mathbb{Q}$ is purely semi-proper (see [Sh:f, X] or here xxx) or just $\mathbb{Q}$ satisfies $\mathrm{UP}^0(\mathbb{I}, \mathbf{W})$ (see §5) then $\mathbb{Q}$ has pure $(\aleph_1, \aleph_1)$-decidability.
3) If $\mathbb{Q}$ is purely proper, then $\mathbb{Q}$ has $(\lambda, \aleph_1)$-decidability for every λ.
4) If $\mathbb{Q}$ has the c.c.c. (and we let $\le_{\rm pr}$ be equality if not defined), then $\mathbb{Q}$ is purely proper.
5) If $\le_{\rm pr} = \le$, then $\mathbb{Q}$ has pure $(\lambda, 2)$-decidability for every λ.

Proof. Think of the definitions.

1.25 Definition. 1) A forcing notion $\mathbb{P}$ is purely proper for χ large enough (e.g., $\mathscr{P}(\mathbb{P}) \in \mathscr{H}(\chi)$ is enough) and N is an elementary submodel of $(H(\chi), \in)$ to which $\mathbb{P}$ belongs and $p \in N \cap \mathbb{P}$ then there is $(N, \mathbb{P})$-semi-generic satisfying $p \le_{\rm pr} q \in \mathbb{P}$, see below.
2) q is $(N, \mathbb{P})$-generic if $q \Vdash_{\mathbb{P}}$ "if $\underset{\sim}{\tau}$ which belongs to N is a $\mathbb{P}$-name of an ordinal then $\underset{\sim}{\tau}[G_{\mathbb{P}}] \in N \cap$ Ord.
3) A forcing notion $\mathbb{P}$ is purely semi-proper if in part 4) we replace $(N, \mathbb{P})$-generic by $(N, \mathbb{P})$-semi-generic.
4) q is $(N, \mathbb{P})$-semi-generic if $q \Vdash_{\mathbb{P}}$ "if $\underset{\sim}{\tau}$ which belongs to N, is a $\mathbb{P}$-name of a countable ordinal then $\underset{\sim}{\tau}[G_{\mathbb{P}}] \in N \cap \omega_1$".

1.26 Claim. *Let $\bar{\mathbb{Q}}$ be a $\kappa - \mathrm{Sp}_e(W)$-iteration.*
1) The property "$\mathbb{Q}$ has pure δ^*-decidability and pure $(2,2)$-decidability" is preserved by $\aleph_1 - \mathrm{Sp}_e(W)$-iterations if δ^* is a limit ordinal.
2) The property "$\mathbb{Q}$ has pure $(2,2)$-decidablity" is preserved by $\aleph_1 - \mathrm{Sp}_e(W)$-iterations.

1.27 Remark. 1) This is like [Sh:f, Ch.XIV,2.13] and is reasonable for iterations not adding reals. For getting rid of pure $(2,2)$-decidability at the expense of others, natural demands, see §5.
2) Is this not suitable for name[1] ordinals only? By UP help.
3) See proof of 6.8 for use of $\bigcup_n q_n \cup p^*$ and more cases phrase as a subclaim?

Proof. Let $\alpha = \ell g(\bar{\mathbb{Q}})$ and let $\leq^*_\chi$ be a well ordering of $\mathscr{H}(\chi)$, let $\underset{\sim}{\leq}^*_{\chi,\mathbb{P}_\beta}$ be a $\mathbb{P}_\beta$-name of a well ordering of $\mathscr{H}(\chi)^{\mathbf{V}^{\mathbb{P}_\beta}}$. Let $p \in \mathbb{P}_\alpha$ and $\underset{\sim}{\tau}$ be a $\mathbb{P}_\alpha$-name of an ordinal $< \theta, \theta \in \{2, \delta^*\}$ and let $\underset{\sim}{\bar{\zeta}}^0 = \langle \underset{\sim}{\zeta}^0_\varepsilon : \varepsilon < j \rangle$ be a witness for p (see 1.15(F), clause (a)) without loss of generality each $\underset{\sim}{\zeta}^0_\varepsilon$ is full. For each $\varepsilon < j$ and $\xi < \alpha$ below we shall $\underset{\sim}{r}^0_{\varepsilon,\xi}, \underset{\sim}{\gamma}^0_{\varepsilon,\xi}$ and $\underset{\sim}{\mathbf{t}}_{\varepsilon,\xi}$ such that $\underset{\sim}{r}^0_{\varepsilon,\xi}$ is a $\mathbb{P}_{\xi+1}$-name of a condition with domain $\subseteq (\xi+1, \alpha), \underset{\sim}{\gamma}^0_{\varepsilon,\xi}$ is a $\mathbb{P}_{\xi+1}$-name of an ordinal $< \theta$ and $\underset{\sim}{\mathbf{t}}_{\varepsilon,\xi}$ is a $\mathbb{P}_{\xi+1}$-name of a truth value, satisfying the following. Let $\xi < \alpha, G_{\mathbb{P}_{\xi+1}} \subseteq \mathbb{P}_{\xi+1}$ be generic over $\mathbf{V}$ and $\underset{\sim}{\zeta}^0_\varepsilon[G_{\xi+1}] = \xi$:

- $(*)_1$ if there are $r \in \mathbb{P}_\alpha / G_{\mathbb{P}_{\xi+1}}$ and $\gamma < \theta$ satisfying (a) + (b) below then $\gamma = \underset{\sim}{\gamma}^0_{\varepsilon,\xi}[G_{\mathbb{P}_{\xi+1}}], r = \underset{\sim}{r}^0_{\varepsilon,\xi}[G_{\mathbb{P}_{\xi+1}}]$ are such objects, first by the fixed well ordering $<^*_\chi$ and $\underset{\sim}{\mathbf{t}}_{\varepsilon,\zeta}[G_{\mathbb{P}_{\xi+1}}]$ is truth, and does not depend on ε; if there are no such γ, r we let $\underset{\sim}{r}^0_{\varepsilon,\zeta}[G_{\mathbb{P}_{\xi+1}}]$ be the empty condition, $\underset{\sim}{\gamma}^0_{\varepsilon,\zeta}[G_{\mathbb{P}_{\xi+1}}] = 0$ and $\underset{\sim}{\mathbf{t}}_{\varepsilon,\zeta}[G_{\mathbb{P}_{\xi+1}}]$ is false.
  - (a) $p \leq_{\rm pr} r \in \mathbb{P}_\alpha / G_{\mathbb{P}_{\xi+1}}, p \restriction (\xi+1) = r \restriction (\xi+1)$ and we have $r \Vdash_{\mathbb{P}_\alpha / G_{\mathbb{P}_{\xi+1}}}$ "if $\theta = 2$ then $\underset{\sim}{\tau} = \gamma$ and if $\theta \geq \aleph_0$ then $\underset{\sim}{\tau} < \gamma$"
  - (b) if $\xi_1 < \xi$ then no r' satisfies (a) with $\xi_1, G_{\mathbb{P}_{\xi+1}} \cap \mathbb{P}_{\xi_1+1}$.

So by **1.18**(4), there is in $\mathbb{P}_\alpha$ a condition $\underset{\sim}{r}^0_\varepsilon$ which is $\underset{\sim}{r}^0_{\varepsilon,\xi}$ if $\underset{\sim}{\zeta}^0_\varepsilon[G] = \xi, \mathbf{t}_{\varepsilon,\zeta}[G] =$

—> scite{1.16} ambiguous

truth, is well defined (the $\underset{\sim}{\beta}$ there is $\underset{\sim}{\zeta}^0_\varepsilon$ here!, hence it is full). So easily $p_1 = p \cup \{\underset{\sim}{r}^0_\varepsilon : \varepsilon < j\}$ belongs to $\mathbb{P}_\alpha$ and is a pure extension of p (using $<^*_\chi$, noting that for each $\xi + 1 \leq \alpha$ and $G_{P_{\xi+1}} \subseteq \mathbb{P}_{\xi+1}$ generic over $\mathbf{V}$, if $\underset{\sim}{\zeta}^0_{\xi+1}[G_{P_{\xi+1}}] = \xi = \underset{\sim}{\zeta}^0_{\varepsilon_1}[G_{P_{\xi+1}}]$ then $\underset{\sim}{r}^0_{\varepsilon_1}[G_{\mathbb{P}_{\xi+1}}] = \underset{\sim}{r}^0_{\varepsilon_2}[G_{\mathbb{P}_{\xi+1}}]$).

We now define $p_2 = p_1 \cup \{\underset{\sim}{r}^1_\varepsilon : \varepsilon < j\}$ where $\underset{\sim}{r}^1_\varepsilon$ is an atomic $\bar{\mathbb{Q}}$-named condition with $\zeta_{\underset{\sim}{r}^1_\varepsilon} = \underset{\sim}{\zeta}_\varepsilon$ defined as follows

- $(*)$ if $\beta < \alpha, G_{\mathbb{P}_\beta} \subseteq \mathbb{P}_\beta$ generic over $\mathbf{V}$ and $\underset{\sim}{\zeta}_\varepsilon[G_{\mathbb{P}_\beta}] = \beta$ then in $\mathbf{V}[G_{\mathbb{P}_\beta}]$ we have $r \in \underset{\sim}{\mathbb{Q}}_\beta[G]$ is $\underset{\sim}{\leq}^*_{\chi,\mathbb{P}_\beta}[G_{\mathbb{P}_\beta}]$-minimal such that
  - (i) $\hat{\mathbb{Q}}_\beta[G_{\mathbb{P}_\beta}] \models$ "$p_2 \restriction \{\beta\} \leq_{\rm pr} \{r\} \in \hat{\mathbb{Q}}_\beta[G_{\mathbb{P}_\beta}]$"

(ii) for some $r_1 \in \mathbb{P}_{\beta+1}$ and $\gamma < \theta$ we have: $r_1 \restriction \beta \in G_{\mathbb{P}_\beta}$ and $r_1 \restriction \beta = r$ and: $r_1 \Vdash_{\mathbb{P}_{\beta+1}}$ "$\theta = 2$ & $\underset{\sim}{\gamma}^0_{\varepsilon,\beta} = \gamma$ or $\theta \geq \aleph_0$ & $\underset{\sim}{\gamma}^0_{\varepsilon,\xi} < \gamma$ and $\underset{\sim}{\mathbf{t}}_{\varepsilon,\beta} =$ truth" or r_1 forces ($\Vdash_{\mathbb{P}_{\beta+1}}$) that $\mathbf{t}_{\varepsilon,\beta} =$ false.

Let us choose now $\beta \leq \alpha$ and r_1 with β minimal such that

$\otimes$ $r_1 \in \mathbb{P}_\beta$ and there are $q \in \mathbb{P}_\alpha$ and $\gamma < \theta$ such that $p_2 \leq_{\rm pr} q$ and $q \restriction \beta \leq r_1$ and $r_1 \cup (q \restriction [\beta, \alpha)) \Vdash$ "$\theta = 2, \underset{\sim}{\tau} = \gamma$ or $\theta \neq 2, \underset{\sim}{\tau} < \gamma$".

There is such β as $\beta = \alpha (= \ell g(\bar{\theta}))$ is O.K.

Case 1: $\beta = 0$.
We are done.

Case 2: β is limit.
Without loss of generality, by 1.17 for some $n < \omega$ and $\xi_1 < \ldots < \xi_n < \beta$ we have: $p_2 \restriction \beta \leq r_1$ above $\{\xi_1, \ldots, \xi_n\}$. If $n = 0$ we are done (as $\beta = 0$) so assume $n > 0$. Let $\beta' = \xi_n + 1, r' = r_1 \restriction (\xi_n + 1)$ and there is q' defined by $q' \restriction (\xi_n + 1) = q \restriction (\xi_n + 1), q' \restriction (\xi_n + 1, \beta)$ is $r_1 \restriction (\xi_n + 1, \beta)$ if $r_1 \restriction (\xi_n + 1) \in \underset{\sim}{G}_{\mathbb{P}_{\xi_n+1}}$ and is $r_1 \restriction (\xi_n + 1, \beta)$ otherwise and lastly $q' \restriction [\beta, \alpha) = q \restriction [\beta, \alpha)$. Now β', r', q' satisfies: $r' \in \mathbb{P}_{\xi_n+1} = \mathbb{P}_{\beta'}, p_2 \leq_{\rm pr} q', q' \restriction \beta' \leq r'$ and $r' \cup q \restriction [\beta', \alpha) \Vdash_{\mathbb{P}_\alpha}$ "$\theta = 2, \underset{\sim}{\tau} = \gamma$ or $\theta \neq 2, \underset{\sim}{\tau} < \gamma$" and $\beta' < \beta$. So we get a contradiction to the choice of β.

Case 3: $r_1 \Vdash$ "$\beta_0 \notin \{\underset{\sim}{\zeta}^0_\varepsilon : \varepsilon < j\}$" where $\beta = \beta_0 + 1$.

The proof is similar to the one of case 2 using $\beta' = \beta_0$.

Case 4: None of the above.
So by "neither case 1 nor case 2" we have $\beta = \beta_0 + 1$, and as we can increase r_1 without loss of generality r_1 forces $\beta_0 \in \{\underset{\sim}{\zeta}^0_\varepsilon : \varepsilon < j\}$, so without loss of generality $r_1 \Vdash$ "$\beta_0 = \underset{\sim}{\zeta}^0_\varepsilon$" where $\varepsilon < j$.

Let $r_1 \in G_\beta \subseteq \mathbb{P}_\beta$ with G_β generic over $\mathbf{V}$; let $G_{\beta'} = G_\beta \cap \mathbb{P}_{\beta'}$ for $\beta' \leq \beta$. So $\underset{\sim}{\zeta}^0_\varepsilon[G_{\beta_0}] = \beta_0$.

We first ask: is there $\varepsilon_1 < j$ such that $\underset{\sim}{\zeta}^0_{\varepsilon_1}[G_{\beta_0}]$ is well defined so call it ζ, (so necessarily $\zeta \leq \beta_0$) and $\underset{\sim}{\mathbf{t}}_{\varepsilon_1,\zeta}[G_\beta]$ is truth?
If yes, then we get a contradiction to the minimality of β as ζ can serve by the

choice of β_2, so assume not. Now considering $\underset{\sim}{r}^0_{\varepsilon,\xi}, \underset{\sim}{\gamma}^0_{\varepsilon,\xi}$, clause (b) holds and r_1, q exemplifies $\underset{\sim}{\mathbf{t}}_{\varepsilon,\xi}[G_p] =$ truth. $\square_{1.26}$

1.28 Remark. 1) You may ask why we do not use the $\underset{\sim}{\zeta}^*$ defined by $\underset{\sim}{\zeta}^*[G_{\xi+1}] = \xi+1$ if $\underset{\sim}{\mathbf{t}}_{\varepsilon,\zeta}[G_{\xi+1}] =$ truth for some $\varepsilon < j$? The reason is that (as for $e = 6, \kappa = \aleph_1$) this seems not to be a simple $\bar{\mathbb{Q}}$-named ordinal.
2) By the proof, if $\bar{\mathbb{Q}}$ is a κ-$\mathrm{Sp}_e(W)$-iteration, $\alpha \le \ell g(\bar{\mathbb{Q}}), p \in \mathbb{P}_\alpha$ and for each $\beta < \alpha, \underset{\sim}{\mathbf{t}}_\beta$ is a $\mathbb{P}_\beta$-name of a truth value, $\underset{\sim}{p}_\beta$ a $\mathbb{P}_\beta$-name of some $p \in \mathbb{P}_\alpha/\underset{\sim}{G}_{\mathbb{P}_\beta}$ such that $\Vdash_{\mathbb{P}_\beta}$ "$p \le_{pr} \underset{\sim}{p}_\beta, p \restriction \beta = \underset{\sim}{p}_\beta \restriction \beta$" <u>then</u> we can find q such that $p \le_{\rm pr} q \in \mathbb{P}_\alpha$ and: if $G_\beta \subseteq \mathbb{P}_\beta$ is generic over $V, \beta < \alpha, \underset{\sim}{\mathbf{t}}_\beta[G_\beta] =$ truth and $\gamma < \beta \Rightarrow$ "$\underset{\sim}{\mathbf{t}}_\gamma[G_\beta \cap \mathbb{P}_\gamma] =$ false then $q \Vdash_{\mathbb{P}_\alpha/G}$ "$\underset{\sim}{\mathbb{P}}_\beta \in \underset{\sim}{G}_{\mathbb{P}_\beta}$".

We now consider some variants of the λ-c.c.

1.29 Definition. 1) We say $\mathbb{P}$ satisfies the local $\underset{\sim}{\partial}$-c.c. <u>if</u> $\underset{\sim}{\kappa}$ is a $\mathbb{P}$-name and $\{p \in \mathbb{P} : \mathbb{P} \restriction \{q : p \le q \in \mathbb{P}\}$ satisfies the ∂'-c.c. and $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\partial} = \partial'$" for some $\kappa'\}$ is dense in $\mathbb{P}$.
2) We say $\mathbb{P}$ satisfies the local $\underset{\sim}{\partial}$-c.c. purely if the set above is dense in $(\mathbb{P}, \le_{\rm pr})$.
3) We say $\mathbb{P}$ satisfies lc.pr. $\underset{\sim}{\partial}$-c.c. if:

- (a) $\underset{\sim}{\kappa}$ is a $(\mathbb{P}, \le_{\rm pr})$-name, usually of a regular cardinal of $\mathbf{V}$
(could use just a partial function from $\mathbb{P}$ to cardinals such that $\underset{\sim}{\kappa}(p) = \kappa \wedge p \le_{\rm pr} q \Rightarrow \underset{\sim}{\kappa}(q) = \kappa$, but abusing notation we write $q \Vdash$ "$\underset{\sim}{\kappa} = \kappa$" if $\underset{\sim}{\kappa}(q) = \kappa$)
- (b) for every $p \in \mathbb{P}$ for some q, ∂ we have $p \le_{\rm pr} q, q \Vdash_{(\mathbb{P}, \le_{\rm pr})}$ "$\underset{\sim}{\partial} = \partial$" and $\mathbb{P}_{\ge q}$ satisfies the ∂-c.c.
(we could use: if $\underset{\sim}{\kappa}(p) = \kappa$ then $\mathbb{P}_{\ge p}$ i.e. $(\{q : p \le q \in \mathbb{P}\}, \le^{\mathbb{P}})$ satisfies the κ-c.c.)

4) If $\mathbb{P}$ satisfies the lc.pr. $\underset{\sim}{\partial}$-c.c. and $q \in \mathbb{P}$ let $\kappa^{\rm mcc}_{\underset{\sim}{\partial}}(q, \mathbb{P}) = \partial$ means $q \Vdash_{(\mathbb{P}, \le_{\rm pr})}$ "$\underset{\sim}{\kappa} = \kappa$" and $\mathbb{P}_{\ge q}$ satisfies the κ-c.c.
5) Let $\underset{\sim}{\partial}^{\rm mcc}(\mathbb{P})$ be minimal such that $\mathbb{P}$ satisfies the lc.pr. $\underset{\sim}{\kappa}$-c.c.; that is $\underset{\sim}{\partial}'(q, \mathbb{P}) =$ $\mathrm{Min}\{\kappa : \mathbb{P}_{\ge q}$ satisfies that κ-c.c.$\}$ and $\underset{\sim}{\partial}(q, \mathbb{P}) = \kappa$ if $(\forall r')(q \le_{\rm pr} r \to \underset{\sim}{\partial}'(r, \mathbb{P}) = \kappa)$ (see below) and let $\partial^{\rm mcc}(q, \mathbb{P}) = \partial$ mean $\kappa^{\rm mcc}_{\underset{\sim}{\partial}(\mathbb{P})}(q, \mathbb{P}) = \partial$ where $\underset{\sim}{\kappa} = \underset{\sim}{\kappa}_{\rm mcc}(\mathbb{P})$.

1.30 Claim. *1) For a forcing notion $\mathbb{P}$ (as in 1.1) the $(\mathbb{P}, \le_{\rm pr})$-name $\underset{\sim}{\partial}^{\rm mcc}(\mathbb{P})$ is well defined, so*
2) If $\mathbb{P}$ satisfies the lc.pr. $\underset{\sim}{\partial}$-c.c. and $p \in \mathbb{P}$ <u>then</u> for some q we have $p \le_{\rm pr} q$ and $\partial_{\rm mcc}(q, \mathbb{P})$ is well defined.

Proof. Straight.

1.31 Definition. 1) We say $\mathbb{Q}$ has strong pr. $(\underset{\sim}{\partial}_1, \underset{\sim}{\partial}_2)$-decidability when $\underset{\sim}{\kappa}, \underset{\sim}{\kappa}_2$ are $(\mathbb{Q}, \le_{\rm pr})$-names of regular cardinals of $\mathbf{V}$ and <u>if</u> $p \in \mathbb{Q}, p \Vdash_{(\mathbb{Q}, \le_{\rm pr})}$ "$\underset{\sim}{\partial}_1 = \theta_1$ and $\underset{\sim}{\kappa}_2 = \theta_2$" and $\underset{\sim}{\zeta}_\varepsilon$ is a $\mathbb{Q}$-name of an ordinal $< \theta_1$ for $\varepsilon < \varepsilon^* < \theta_2$ <u>then</u> for some $a \subseteq \theta_1$ of cardinality $< \theta_2$ and q such that $p \le_{\rm pr} q \in \mathbb{Q}$ we have $q \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\zeta}_\varepsilon \in a$ for $\varepsilon < \varepsilon^*$".
2) We say $\mathbb{Q}$ has strong pr. $\underset{\sim}{\partial}$-decidability <u>if</u> for any θ it has pr. $(\theta, \underset{\sim}{\kappa})$-decidability (i.e. each $\underset{\sim}{\zeta}_\varepsilon$ is a $\mathbb{Q}$-name of an ordinal $< \theta$).
3) We use "weak" instead of "strong" in parts (1), (2) if above we restrict ourselves to the case $\varepsilon^* = 1$.
4) We let $\underset{\sim}{\partial}^w_\otimes(\mathbb{P})$ is the minimal $(\mathbb{P}, \le_{\rm pr})$-name $\underset{\sim}{\kappa}$ of a regular cardinal from $\mathbf{V}$ such that $\mathbb{P}$ has weak pr. $\underset{\sim}{\kappa}$-decidability. Similarly $\underset{\sim}{\partial}^{\rm St}_\otimes(\mathbb{P})$ for strong pr. $\underset{\sim}{\kappa}$-decidability.

1.32 Note/Observation. If $\bar{\mathbb{Q}}$ is an $\kappa - {\rm Sp}_e(W)$-iteration, <u>then</u>

- (a) $\langle (\mathbb{P}_i, \le^{\mathbb{P}_i}_{\rm pr}) : i \le \ell g(\bar{\mathbb{Q}}) \rangle$ is a $\lessdot$-increasing sequence
- (b) if $i < j \le \ell g(\bar{\mathbb{Q}}), q \in \mathbb{P}_j, q \restriction i \le_{\rm pr} p \in \mathbb{P}_i$ then p, q has a $\le_{\rm pr}$-lub, $p \cup q \restriction [i, j)$.

Proof. Check.

1.33 Claim. *1) If $\mathbb{P}$ satisfies the lc.pr. $\underset{\sim}{\partial}$-c.c. and $\partial = \partial^{\rm mcc}_{\underset{\sim}{\partial}}(p, \mathbb{P}) \Rightarrow \Vdash_{\mathbb{P}}$ "∂ is regular" <u>then</u> $\mathbb{P}$ has a strong pr $\underset{\sim}{\partial}$-decidability.*
2) Let $\bar{\mathbb{Q}}$ be a $\kappa_1 - {\rm Sp}_e(W)$-iteration $e \in \{4\}$. If $\delta \le \ell g(\bar{\mathbb{Q}})$ is a limit ordinal, u an unbounded subset of δ, for $i \in u$ we have $\mathbb{P}_i$ has the strong pr. $\underset{\sim}{\partial}_i$-decidability <u>then</u> letting $\underset{\sim}{\partial} =$ Min$\{\partial : \partial$ a regular cardinal in $\mathbf{V}$ and $\partial \ge \underset{\sim}{\partial}_i$ for $i \in u\}$ we have

- (i) *$\underset{\sim}{\partial}$ is a $(\mathbb{P}_\delta, \le_{\rm pr})$-name of a regular cardinal of $\mathbf{V}$*
- (ii) *$\mathbb{P}_\delta$ has weak pr. $\underset{\sim}{\partial}$-decidability.*

3) Similarly for $\mathbb{P}_\delta/G_{\mathbb{P}_\alpha}$ *when* $\alpha < \delta$ *and even* $\mathbb{P}_\delta/\mathbb{P}_\alpha$ *where* $\underset{\sim}{\alpha}$ *is a simple* $\bar{\mathbb{Q}}$*-named* $[0,\delta)$*-ordinal.*
4) If $\mathbb{P}$ *satisfies the strong pr* $\underset{\sim}{\partial}$*-decidability and* $p \in \mathbb{P}, \kappa = \kappa_{\underset{\sim}{\kappa}}(p,\mathbb{P})$ *<u>then</u>* $p \Vdash$ *"*∂ *is a regular cardinal.*

Proof. 1) Trivial.
2),3) Similar to the proof of 1.26 [Saharon!]
4) Trivial. $\square_{1.33}$

<u>1.34 Convention</u>: Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i : j \le \alpha, i < \alpha\rangle$ be a κ-$\mathrm{Sp}_e(W)$-iteration.

1.35 Claim. *Assume* $(\bar{\mathbb{Q}}, W)$ *is smooth, (see Definition 1.7(8)).*
1) If $p^* \in \mathbb{P}_j$ *<u>then</u>* $\langle \mathbb{P}'_j, \underset{\sim}{\mathbb{Q}}'_i : j \le \alpha, i < \alpha\rangle$ *is a* κ_1*-*$Sp_e(W)$*-iteration where*
$\mathbb{P}'_j = \{p \in \mathbb{P}_j : p^* \restriction j \le p\}$,
$\mathbb{Q}'_j = \{p \in \underset{\sim}{\mathbb{Q}}_j :\Vdash_{\mathbb{P}_j}$ *"*$p^* \restriction \{j\} \le p$ *in* $\hat{\mathbb{Q}}_j$*".*

2) If $\gamma < \alpha$ *and* $G_j \subseteq \mathbb{P}_\gamma$ *is generic over* $\mathbf{V}$ *<u>then</u>* $\langle \mathbb{P}_{\gamma+j}/G_\gamma, \underset{\sim}{\mathbb{Q}}_{\gamma+i} : j \le \alpha - \gamma$ *and* $i < \alpha - \gamma\rangle$ *is a* $\aleph_1 - \mathrm{Sp}_e(W)$*-iteration.*

Proof. Straight.

§2 Tree of Models

We present here needed information on trees and tagged trees. On partition of tagged trees, see Rubin, Shelah [RuSh 117] and, [Sh:b], [Sh:f, XI,3.5,3.5A,3.7,XV,2.6,2.6A,2.6B] and [Sh 136, 2.4,2.5,p.111-113]; or the representation in [Sh:e, AP,§1]; on history see [RuSh 117] and [Sh:f].

2.1 Definition. A tagged tree is a pair $(T, \mathbf{I})$ such that:

(1) T is a ω-tree, which here means a nonempty set of finite sequences of ordinals such that if $\eta \in T$ then any initial segment of η belongs to T. T is ordered by initial segments; i.e., $\eta \triangleleft \nu$ iff η is a proper initial segment of ν.

(2) $\mathbf{I}$ is a partial function such that for every $\eta \in T \cap \mathrm{Dom}(\mathbf{I}) : \mathbf{I}(\eta)$ is an ideal of subsets of some set called the domain of $\mathbf{I}_\eta$, $\mathrm{Dom}(\mathbf{I}_\eta)$, and[3]

$$\mathrm{Suc}_T(\eta) =: \{\nu : \nu \text{ is an immediate successor of } \eta \text{ in } T\} \subseteq \mathrm{Dom}(\mathbf{I}_\eta),$$

Usually $\mathbf{I}_\eta$ is $\aleph_2$-complete.

(3) For every $\eta \in T$ we have $\mathrm{Suc}_T(\eta) \neq \emptyset$.

2.2 Convention. For any tagged tree $(T, \mathbf{I})$ we can define $\mathbf{I}^\dagger$ by:

$$\mathrm{Dom}(\mathbf{I}^\dagger) = \{\eta : \eta \in T \cap \mathrm{Dom}(\mathbf{I}) \text{ and } \mathrm{Suc}_T(\eta) \subseteq \mathrm{dom}(\mathbf{I}_\eta) \text{ and } \mathrm{Suc}_T(\eta) \notin \mathbf{I}_\eta\}$$

and

$$\mathbf{I}^\dagger_\eta = \big\{\{\alpha : \eta\hat{\ }\langle\alpha\rangle \in A\} : A \in \mathbf{I}_\eta\big\};$$

we sometimes, in an abuse of notation, do not distinguish between $\mathbf{I}$ and $\mathbf{I}^\dagger$; e.g. if $\mathbf{I}^\dagger_\eta$ is constantly I^*, we write I^* instead of $\mathbf{I}$. Also, if $\mathbf{I} = \mathbf{I}_x$, we may write $\mathbf{I}^x_\eta$ for $\mathbf{I}_x(\eta)$.

[3] in this section it is not unreasonable to demand equality but this is very problematic in Definition 4.2(1), clause (d)

2.3 Definition. 1) Let η be called a splitting point of $(T, \mathbf{I})$ if $\mathbf{I}_\eta$ is well defined and $\mathrm{Suc}_T(\eta) \notin \mathbf{I}_\eta$ (normally this follows but we may "forget" to decrease the domain of $\mathbf{I}$). Let $\mathrm{split}(T, \mathbf{I})$ be the set of splitting points of $(T, \mathbf{I})$. We usually consider trees where each ω-branch meets $\mathrm{split}(T, \mathbf{I})$ infinitely often (see Definition 2.5(6), i.e. are normal).
2) For $\eta \in T$ let $T^{[\eta]} =: \{\nu \in T : \nu = \eta \text{ or } \nu \triangleleft \eta \text{ or } \eta \triangleleft \nu\}$.
3) For a tree T, let $\lim(T)$ be the set of branches of T; i.e. all ω-sequences of ordinals, such that every finite initial segment of them is a member of $T : \lim(T) = \{s \in {}^\omega\mathrm{Ord} : (\forall n) s \restriction n \in T\}$. We call them also ω-branches.
4) A subset Z of a tree T is a front if: $\eta \neq \nu \in Z$ implies none of them is an initial segment of the other, and every $\eta \in \lim(T)$ has an initial segment which is a member of Z.

2.4 Definition. We now define orders between tagged trees:

(a) $(T_1, \mathbf{I}_1) \leq (T_2, \mathbf{I}_2)$ <u>if</u> $T_2 \subseteq T_1$, and $\mathrm{split}(T_2, \mathbf{I}_2) \subseteq \mathrm{split}(T_1, \mathbf{I}_1)$, and for every $\eta \in \mathrm{split}(T_2, \mathbf{I}_2)$ we have $\mathbf{I}_2(\eta) \restriction \mathrm{Suc}_{T_2}(\eta) = \mathbf{I}_1(\eta) \restriction \mathrm{Suc}_{T_2}(\eta)$ (where $I \restriction A = \{B : B \subseteq A \text{ and } B \in I\}$).
(So every splitting point of T_2 is a splitting point of T_1, and if we suppose $\eta \in T \Rightarrow \mathrm{Suc}_T(h) = \mathrm{Dom}(I_\eta)$ then $\mathbf{I}_2$ is completely determined by $\mathbf{I}_1$ and $\mathrm{split}(T_2, \mathbf{I}_2)$ and T_2.)

(b) $(T_1, \mathbf{I}_1) \leq^* (T_2, \mathbf{I}_2)$ <u>if</u> $(T_1, \mathbf{I}_1) \leq (T_2, \mathbf{I}_2)$ and $\mathrm{split}(T_2, \mathbf{I}_2) = \mathrm{split}(T_1, \mathbf{I}_1) \cap T_2$.

(c) For any set A, $(T_1, \mathbf{I}_1) \leq^{\otimes}_A (T_2, \mathbf{I}_2)$ if $T_1 \supseteq T_2$ and $\eta \in A \cap T_2 \Rightarrow \mathrm{Suc}_{T_2}(\eta) = \mathrm{Suc}_{T_1}(\eta)$ and $\eta \in T_2 \cap \mathrm{split}(T_1, \mathbf{I}_1) \backslash A \Rightarrow \mathrm{Suc}_{T_2}(\eta) \neq \emptyset \bmod \mathbf{I}^1_\eta$.

(d) In (c) we may omit the subscript A when $A = T_2 \backslash \mathrm{split}(T_1, \mathbf{I}_1)$.

(e) $(T_1, \mathbf{I}_1) \leq^{\boxtimes}_\kappa (T_2, \mathbf{I}_2)$ <u>if</u> $T_2 \subseteq T_1$ is a subtree and if $\nu \triangleleft \eta \in \lim(T_2), \mathbf{I}_\nu$ is κ-complete, then for some $k \geq \ell g(\nu), \mathbf{I}^1_\nu \leq_{RK} \mathbf{I}^2_{\eta \restriction k}, \mathbf{I}^2_{\eta \restriction k}$ is κ-complete and $\eta \restriction k \in \mathrm{split}(T_2, \mathbf{I}_2)$. We can replace κ and κ-complete by φ and "satisfying φ", e.g. $\in \mathbb{I}$.

(f) $(T_1, \mathbf{I}_1) \leq^\kappa (T_2, \mathbf{I}_2)$ <u>when</u> $T_2 \subseteq T_1$, and if $\eta \in T_2 \cap \mathrm{split}(T_1, \mathbf{I}_1)$ and $\mathbf{I}^1_\eta$ is κ-complete then $\eta \in \mathrm{split}(T_2, \mathbf{I}_2)$ and $\mathbf{I}^2_\eta = \mathbf{I}^1_\eta$ and every $\eta \in \mathrm{split}(T_2, \mathbf{I}_2)$ is like that.

2.5 Definition. 1) For a set $\mathbb{I}$ of ideals, a tagged tree $(T, \mathbf{I})$ is an $\mathbb{I}$-tree if for every splitting point $\eta \in T$ we have $\mathbf{I}_\eta \in \mathbb{I}$ (up to an isomorphism).
2) For a tagged tree $(T, \mathbf{I})$ and set $\mathbb{I}$ of ideals let $\mathbf{I} \restriction \mathbb{I} = \mathbf{I} \restriction \{\eta \in \mathrm{Dom}(\mathbf{I})$ and $\mathbf{I}_\eta \in \mathbb{I}\}$.
3) Let in (2), $(T, \mathbf{I}) \restriction \mathbb{I} = (T, \mathbf{I} \restriction \mathbb{I})$.

4) A tagged tree $(T, \mathbf{I})$ is standard <u>if</u> for every non-splitting point $\eta \in T$, $|\mathrm{Suc}_T(\eta)| = 1$.
5) A tagged tree $(T, \mathbf{I})$ is full <u>if</u> every $\eta \in T$ is a splitting point.
6) A tagged tree $(T, \mathbf{I})$ is normal <u>if</u> for every $\eta \in \lim(T)$ for infinitely many $k < \omega$ we have $\eta \restriction k \in \mathrm{split}(T, \mathbf{I})$.

2.6 Remark. 1) Of course, the set $\lim(T)$ is not absolute; i.e., if $\mathbf{V}_1 \subseteq \mathbf{V}_2$ are two universes of set theory then in general $(\lim(T))^{\mathbf{V}_1}$ will be a proper subset of $(\lim(T))^{\mathbf{V}_2}$.
2) However, the notion of being a front is absolute:

- (a) Z a point <u>iff</u> Z contains a front and $\eta \neq \nu \in Z \Rightarrow \neg(\eta \trianglelefteq \nu)$
- (b) $\mathbf{V}_1 \models$ "Z contains a front of T", <u>iff</u> $Z \subseteq T$ and $\{\eta \in T : \forall k \leq \ell g(\eta)(\eta \restriction k \in Z\}$ has no ω-branch <u>iff</u> $Z \subseteq T$ there is a depth function $\rho : T \to$ Ord satisfying $\eta \triangleleft \nu$ & $\forall k \leq \ell g(\eta)[\eta \restriction k \notin Z] \to \rho(\eta) > \rho(\nu)$. (Levy absoluteness theorem) This function will also witness in $\mathbf{V}_2$ that Z is a front.

3) $Z \subseteq T$ contains a front <u>iff</u> Z meets every branch of T. So if $Z \subseteq T$ contains a front of T and $T' \subseteq T$ is a subtree, <u>then</u> $Z \cap T'$ contains a front of T'. This is absolute, too.

2.7 Definition. 1) An ideal I is λ-complete <u>if</u> any union of less than λ members of I is still a member of I.
2) A tagged tree $(T, \mathbf{I})$ is λ-complete if for each $\eta \in T \cap \mathrm{Dom}(\mathbf{I})$ or just $\eta \in \mathrm{split}(T, \mathbf{I})$, the ideal $\mathbf{I}_\eta$ is λ-complete.
3) A family $\mathbb{I}$ of ideals is λ-complete <u>if</u> each $I \in \mathbb{I}$ is λ-complete. We will only consider $\aleph_2$-complete families $\mathbb{I}$.
4) A family $\mathbb{I}$ is called restriction-closed <u>if</u>: $I \in \mathbb{I}, A \subseteq \mathrm{Dom}(I), A \notin I$ implies $I \restriction A = \{B \in I : B \subseteq A\}$ belongs to $\mathbb{I}$.
5) The restriction closure of $\mathbb{I}$, res-c$\ell(\mathbb{I})$ is $\{I \restriction A : I \in \mathbb{I}, A \subseteq \mathrm{Dom}(I), A \notin I\}$.
6) I is λ-indecomposable <u>if</u> for every $A \subseteq \mathrm{Dom}(I), A \notin I$ and $h : A \to \lambda$ there is $Y \subseteq \lambda, |Y| < \lambda$ such that $h^{-1}(Y) \notin I$. We say $\mathbf{I}$ (or we say $\mathbb{I}$) is λ-indecomposable <u>if</u> each $\mathbf{I}_\eta$ where $\eta \in \mathrm{split}(T, \mathbf{I})$ (or $I \in \mathbb{I}$) is λ-indecomposable.
7) I is strongly λ-indecomposable <u>if</u> for any $A_i \in I$ $(i < \lambda)$ and $A \subseteq \mathrm{Dom}(I), A \notin I$ we can find $B \subseteq A$ of cardinality $< \lambda$ such that for no $i < \lambda$ does A_i include B.
8) Let $\mathbb{I}^{[\kappa]} = \{I \in \mathbb{I} : I \text{ is } \kappa\text{-complete}\}$.

2.8 Fact. If λ is a regular cardinal and I is a strongly λ-indecomposable, <u>then</u> I is λ-indecomposable.

Proof. Given A, h as in 2.7(6), let $A_i = h^{-1}(\{j : j < i\})$ and $A'_i = h^{-1}(\{i\})$; if for some $i < \lambda, A_i \notin I$ we are done, otherwise by Definition 2.7(7) there is $B \subseteq A, |B| < \lambda$ such that: $i < \lambda \Rightarrow B \nsubseteq A_i$. But as λ is regular, $B \subseteq A, |B| < \lambda$ and $\langle A_i : i < \lambda\rangle$ is a $\subseteq$-increasing sequence of sets with union A, clearly for some $j < \lambda, B \subseteq A_j$, contradiction. $\square_{2.8}$

2.9 Definition. For a subset A of (an ω-tree) T we define by induction on the length of a sequence η, $\mathrm{res}_T(\eta, A)$ for each $\eta \in T$. Let $\mathrm{res}_T(\langle\rangle, A) = \langle\rangle$. Assume $\mathrm{res}_T(\eta, A)$ is already defined and we define $\mathrm{res}_T(\eta\hat{}\langle\alpha\rangle, A)$ for all members $\eta\hat{}\langle\alpha\rangle$ of $\mathrm{Suc}_T(\eta)$. If $\eta \in A$ then $\mathrm{res}_T(\eta\hat{}\langle\alpha\rangle, A) = \mathrm{res}_T(\eta, A)\hat{}\langle\alpha\rangle$, and if $\eta \notin A$ then $\mathrm{res}_T(\eta\hat{}\langle\alpha\rangle, A) = \mathrm{res}_T(\eta, A)\hat{}\langle 0\rangle$. If $\eta \in \lim(T)$, we let
$\mathrm{res}(\eta, A) = \bigcup_{k\in\omega} \mathrm{res}(\eta \restriction k, A)$.

2.10 Explanation. Thus $\mathrm{res}(T, A) =: \{\mathrm{res}_T(\eta, A) : \eta \in T\}$ is a tree obtained by projecting T; i.e., glueing together all members of $\mathrm{Suc}_T(\nu)$ whenever $\nu \notin A$.

We state now (Lemma 2.11 is [Sh:f, Ch.XI,5.3;p.559] and Lemma 2.12 is [Sh:f, XV,2.6;p.738] and Lemma 2.13 is [Sh:f, XI,3.5](2); p.546-7.

2.11 Lemma. *Let λ, μ be uncountable cardinals satisfying $\lambda^{<\mu} = \lambda$ and let $(T, \mathbf{I})$ be a tagged tree in which for each $\eta \in T$ either $|\mathrm{Suc}_T(\eta)| < \mu$ or $\mathbf{I}(\eta)$ is λ^+-complete. Then for every function $H : T \to \lambda$ there exists a subtree T' of T satisfying $(T, \mathbf{I}) \le^* (T', \mathbf{I})$ such that for $\eta^1, \eta^2 \in T'$ satisfying $\mathrm{res}_T(\eta^1, A) = \mathrm{res}_T(\eta^2, A)$ we have:*

- (*i*) $H(\eta^1) = H(\eta^2)$
- (*ii*) $\eta^1 \in A \Leftrightarrow \eta^2 \in A$,
- (*iii*) *if* $\eta \in T' \cap A$, *then* $\mathrm{Suc}_T(\eta) = \mathrm{Suc}_{T'}(\eta)$.

Proof. See [Sh:f, XV,2.6;p.738].

2.12 Lemma. *Let θ be an uncountable regular cardinal (the main case here is $\theta = \aleph_1$). Assume*

- (α) $\mathbb{I}$ *be a family of θ^+-complete ideals,*
- (β) $(T_0, \mathbf{I})$ *a tagged tree,*
- (γ) $A =: \{\eta \in T : |Suc_{T_0}(\eta)| \le \theta\}$,
- (δ) $[\eta \in T_0 \backslash A \Rightarrow \mathbf{I}_\eta \in \mathbb{I}\ \&\ \mathrm{Suc}_{T_0}(\eta) \notin \mathbf{I}_\eta]$

(ε $[\eta \in A \Rightarrow \mathrm{Suc}_{T_0}(\eta) \subseteq \{\eta\char`\^\langle i \rangle : i < \theta\}]$

(ζ) $H : T_0 \to \theta$ *and*

(η) $\bar{e} = \langle e_\eta : \eta \in A, |\mathrm{Suc}_{T_0}(\eta)| = \theta \rangle$ *is such that* e_η *is a club of* θ.

Then there is a club C of θ such that: for each $\delta \in C$ there is $T_\delta \subseteq T_0$ satisfying:

(a) T_δ *is a tree*

(b) *if* $\eta \in T_\delta, |\mathrm{Suc}_{T_0}(\eta)| < \theta$, *then* $\mathrm{Suc}_{T_\delta}(\eta) = \mathrm{Suc}_{T_0}(\eta)$ *and if* $\mathrm{Suc}(\eta) = \theta$, *then* $\mathrm{Suc}_{T_\delta}(\eta) = \{\eta\char`\^\langle i \rangle : i < \delta\}$ *and* $\delta \in e_\eta$

(c) $\eta \in T_\delta \backslash A$ *implies* $\mathrm{Suc}_{T_\delta}(\eta) \notin \mathbf{I}_\eta$

(d) *for every* $\eta \in T_\delta$ *we have* $H(\eta) < \delta$.

Proof. See [Sh:f, XV,2.6].

2.13 Lemma. *Let $(T, \mathbf{I})$ be a θ^+-complete $\mathbf{I}$-tree, and assume $\theta = \mathrm{cf}(\theta)$. If $\lim(T) = \bigcup_{i<\theta} B_i$ where B_i is a Borel subset of $\lim(T)$, then for some $i < \theta$ and $(T', \mathbf{I}')$ we have $(T, \mathbf{I}) \leq^* (T', \mathbf{I}')$ and $\lim(T') \subseteq B_i$.*

Proof. By [Sh:f, XI,3.5](2);p.546-7.

§3 Ideals and Partial Orders

3.1 Definition. 1) We call an ideal J non-atomic if $\{x\} \in J$ for every $x \in \mathrm{Dom}(J)$.
2) We call the ideal with domain $\{0\}$, which is $\{\emptyset\}$, the trivial ideal.

3.2 Definition. 1) For ideals J_1, J_2 we say

$$J_1 \leq_{\mathrm{RK}} J_2$$

if there is a function $h : \mathrm{Dom}(J_2) \to \mathrm{Dom}(J_1)$ such that

$$\text{for every } A \subseteq \mathrm{Dom}(J_2) \text{ we have} : A \neq \emptyset \bmod J_2 \Rightarrow h''(A) \neq \emptyset \bmod J_1$$

or equivalently,

$$J_2 \supseteq \{h^{-1}(A) : A \in J_1\}.$$

2) For families $\mathbb{I}_1, \mathbb{I}_2$ of ideals we say $\mathbb{I}_1 \leq_{\mathrm{RK}} \mathbb{I}_2$ if there is a function H witnessing it; i.e.,

- (i) H is a function from $\mathbb{I}_1$ into $\mathbb{I}_2$
- (ii) for every $J \in \mathbb{I}_1$ we have $J \leq_{\mathrm{RK}} H(J)$.

3) For families $\mathbb{I}_1, \mathbb{I}_2$ of ideals, $\mathbb{I}_1 \equiv_{\mathrm{RK}} \mathbb{I}_2$ if $\mathbb{I}_1 \leq_{\mathrm{RK}} \mathbb{I}_2$ & $\mathbb{I}_2 \leq_{\mathrm{RK}} \mathbb{I}_1$.

3.3 Claim. *1) If an ideal J is not non-atomic then $J \leq_{\mathrm{RK}}$ "the trivial ideal".*
2) $\leq_{\mathrm{RK}}$ is a partial quasi-order (among ideals and also among families of ideals).

Proof. Easy.

3.4 Definition. 1) For an (upward) directed quasi order[4] $L = (B, <)$ we define an ideal id_L:

$$\text{id}_L = \big\{A \subseteq B : \text{for some } y \in L \text{ we have } A \subseteq \{x \in L : \neg y \leq x\}\big\}.$$

Equivalently, the dual filter fil_L is generated by the "cones", where the cone of L defined by $y \in L$ is

$$L_y =: \{x \in L : y \leq x\}.$$

We call such an ideal a partial order ideal. We let $\text{Dom}(L) = \text{Dom}(\text{id}_L)(= B)$, but we may use L instead of $\text{Dom}(L)$ (like $\forall x \in L$).
2) For a partial order L let $\text{dens}(L) = \min\{|X| : X \subseteq \text{Dom}(L)$ is dense[5]; i.e. $(\forall a \in \text{Dom}(L))(\exists b \in X)[a \leq b])\}$ (this applies also to ideals considered as $(I, \subseteq)$).
3) For a family $\mathscr{L}$ of directed quasi orders let $\text{id}_{\mathscr{L}} = \{\text{id}_L : L \in \mathscr{L}\}$.

3.5 Fact. 1) id_L is λ-complete iff L is λ-directed.
2) $\text{dens}(L) = \text{dens}(\text{id}_{(L,<)}, \subseteq)$.
3) If $h : L_1 \to L_2$ preserves order (i.e. $\forall x, y \in L, (x \leq y \Rightarrow h(x) \leq h(y))$) and has cofinal (= dense) range (i.e. $(\forall x \in L_2)(\exists y \in L_1)[x \leq h(y)]$) then $\text{id}_{L_2} \leq_{\text{RK}} \text{id}_{L_1}$.
4) Let $h : L_1 \to L_2$. Now h exemplifies $\text{id}_{L_2} \leq_{\text{RK}} \text{id}_{L_1}$ iff for every $x_2 \in L_2$ there is $x_1 \in L_1$ such that: $(\forall y)[y \in L_1 \ \& \ x_1 \leq_{L_1} y \Rightarrow x_2 \leq_{L_2} h(y)]$; (equivalently for every $y \in L_1 : \neg(x_2 \leq_{L_2} h(y)) \Rightarrow \neg(x_1 \leq_{L_1} y)$; note that h is not necessarily order preserving).
5) If $L_1 \subseteq L_2$ and L_1 is a dense in L_2, then $\text{id}_{L_1} \equiv_{RK} \text{id}_{L_2}$.
6) If $\lambda = \text{density}(L)$ then id_L is λ-based; i.e. $X \subseteq \text{Dom}(\text{id}_L), X \notin \text{id}_L \Rightarrow (\exists Y \subseteq X)[|Y| \leq \lambda \ \& \ Y \notin \text{id}_L]$.

Proof. Straight. E.g.
3) If $A \subseteq L_1$ and $A \notin \text{id}_{L_1}$, then $(\forall x \in L_1)(\exists y)(x \leq_{L_1} y \in A)$ hence $(\forall x \in L_2)(\exists y, z \in L_1)(x \leq_{L_2} h(y) \ \& \ y \leq_{L_1} z \in A)$ hence $(\forall x \in L_2)(\exists z)(x \leq_{L_2} z \in h''(A))$ hence $h''(A) \notin \text{id}_{L_2}$ (and trivially $h''(A) \subseteq L_1$).
By Defintion 3.2(1) this shows $\text{id}_{L_2} \leq_{RK} \text{id}_{L_1}$.
4) Note: h exemplifies $\text{id}_{L_2} \leq_{RK} \text{id}_{L_1}$

iff

[4]no real difference if we ask partial order or just quasi orders; i.e., partial orders satisfy $x \leq y \ \& \ y \leq x \not\Rightarrow x = y$, quasi order not necessarily; note that in the case there is $x^* \in L$ such that $(\forall y \in L)(y \leq x^*)$ gives an ideal which is not non-atomic

[5]also called cofinal in this context

$$(\forall A \subseteq L_1)(A \neq \emptyset \text{ mod } \mathrm{id}_{L_1} \rightarrow h''(A) \neq \emptyset \text{ mod } \mathrm{id}_{L_2})$$

iff

$$(\forall A \subseteq L_1)(\forall x_2 \in L_2)[A \neq \emptyset \text{ mod } \mathrm{id}_{L_1} \rightarrow h''(A) \cap \{y \in L_2 : x_2 \leq_{L_2} y\} \neq \emptyset]$$

iff

$$(\forall x_2 \in L_2)(\forall A \subseteq L_1)(A \neq \emptyset \text{ mod } \mathrm{id}_{L_1} \rightarrow h''(A) \cap \{y \in L_2 : x_2 \leq_{L_2} y\} \neq \emptyset)$$

iff

$$(\forall x_2 \in L_2)(\{y \in L_1 : \neg(x_2 \leq_{L_2} h(y))\} = \emptyset \text{ mod } \mathrm{id}_{L_1}]$$

iff

$$(\forall x_2 \in L_2)(\exists x_1 \in L_1)(\forall y \in L_1)[\neg(x_2 \leq_{L_2} h(y)) \rightarrow \neg x_1 \leq_{L_1} y]$$

iff

$$(\forall x_2 \in L_2)(\exists x_1 \in L_1)(\forall y \in L_1)[x_1 \leq_{L_1} y \rightarrow x_2 \leq_{L_2} h(y)].$$

5) Letting h_1 be the identity map on L_1 by part (3) we get $\mathrm{id}_{L_2} \leq_{\mathrm{RK}} \mathrm{id}_{L_1}$; choose $h_2 : L_2 \rightarrow L_1$ which extends h_1, by part (4) we get $\mathrm{id}_{L_1} \leq_{\mathrm{RK}} \mathrm{id}_{L_2}$, together we are done.
6) Easily. $\square_{3.5}$

3.6 Fact. 1) For any ideal J (such that $(\mathrm{Dom}(J)) \notin J$), letting $J_1 = \mathrm{id}_{(J,\subseteq)}$, we have

- (i) J_1 is a partial order ideal
- (ii) $|\mathrm{Dom}(J_1)| = |J| \leq 2^{|\mathrm{Dom}(J)|}$
- (iii) $J \leq_{\mathrm{RK}} J_1$
- (iv) if J is λ-complete, then $(J, \subseteq)$ is λ-directed hence J_1 is λ-complete
- (v) $\mathrm{dens}(J, \subseteq) = \mathrm{dens}(J_1, \subseteq)$
- (vi) $\mathrm{dens}(J, \subseteq) \leq |J| \leq 2^{|\mathrm{Dom}(J)|}$.

2) For every ideal J and dense $X \subseteq J$ we can use $\mathrm{id}_{(X,\subseteq)}$ and get the same conclusions.
3) For every ideal J there is a directed order L such that:

$$J \leq_{\mathrm{RK}} \mathrm{id}_L, \ \mathrm{dens}(J) = \mathrm{dens}(L)$$

and: for every λ if J is λ-complete, then so is id_L.

Proof. Least trivial is (1)(iii), let $h : J \to \mathrm{Dom}(J)$ be such that $h(A) \in (\mathrm{Dom}(J))\backslash A$ (exists as $(\mathrm{Dom}(J)) \notin J$). Let $J_1 = \mathrm{id}_{(J,\subseteq)}$, so h is a function from $\mathrm{Dom}(J_1)$ into $\mathrm{Dom}(h)$ and we shall prove that h exemplifies the desired conclusion $J \leq_{\mathrm{RK}} J_1$ by Definition 3.2(1)

Assume toward contradicion that $X \subseteq \mathrm{Dom}(J_1) = J, X \notin J_1$ and $A =: h''(X)$ belongs to J. So $Y =: \{B \in J : \neg(A \subseteq B)\} \in \mathrm{id}_{(J,\subseteq)} = J_1$ (by the definition of $J_1 = \mathrm{id}_{(J,\subseteq)}$) hence (as $X \notin J_1$) for some $B \in X$ we have $B \notin Y$ hence by definition $A \subseteq B$, so $h(B) \in h''(X) = A$ contradicting the choice of $h(B)$ (as $A \subseteq B$). $\square_{3.6}$

3.7 Remark. So we can replace a family of ideals by a family of directed quasi orders without changing much the relevant invariants such as completeness or density as long as we do not mind adding "larger" ones in the appropriate sense.

3.8 Conclusion. For any family of ideals $\mathbb{I}$ there is a family of $\mathscr{L}$ of quasi order such that:

(*i*) $\mathbb{I} \leq_{\mathrm{RK}} \{\mathrm{id}_{(L,<)} : (L,<) \in \mathscr{L}\}$
(*ii*) $|\mathscr{L}| \leq |\mathbb{I}|$
(*iii*) $\sup\{|L| : (L,<) \in \mathscr{L}\} = \sup\{|J| : J \in \mathbb{I}\} \leq \sup\{2^{\mathrm{Dom}(J)} : J \in \mathbb{I}\}$
(*iv*) $\sup\{\mathrm{dens}(L,<) : (L,<) \in \mathscr{L}\} = \sup\{\mathrm{dens}(J,\subseteq) : J \in \mathbb{I}\}$
(*v*) $\mathbb{I}$ is λ-complete <u>iff</u> every $(L,<) \in \mathscr{L}$ is λ-directed.

Proof. Easy.

3.9 Definition. For a forcing notion $\mathbb{Q}$, satisfying the κ-c.c., a $\mathbb{Q}$-name $\underset{\sim}{L}$ of a quasi order with $Y \in \mathbf{V}, \Vdash_{\mathbb{Q}} Y = \mathrm{Dom}(\underset{\sim}{L}) \in \mathbf{V}$ for notational simplicity given (i.e., is not just a $\mathbb{Q}$-name); let $L^* = \mathrm{ap}_\kappa(\underset{\sim}{L}) = \mathrm{ap}_\kappa(\underset{\sim}{L}, \mathbb{Q})$ be the following quasi order

$$\mathrm{Dom}(L^*) = \{a : a \subseteq Y) \text{ and } |a| < \kappa\} \in \mathbf{V}$$

$$a \leq^* b \text{ iff } \Vdash_{\mathbb{Q}} \text{“}(\forall y \in a)(\exists x \in b)[\underset{\sim}{L} \models y < x]\text{”}$$

(this is a quasi-order only, i.e. maybe $a \leq^* b \leq^* a$ but $a \neq b$).

3.10 Claim. *1) For a forcing notion $\mathbb{Q}$ satisfying the κ-c.c. and a $\mathbb{Q}$-name $\underset{\sim}{L}$ of a λ-directed quasi order (with $\mathrm{Dom}(\underset{\sim}{L}) \in \mathbf{V}$ given, not just a $\mathbb{Q}$-name, for simplicity) such that $\lambda \geq \kappa$ we have:*

- (i) $\mathrm{ap}_\kappa(\underset{\sim}{L})$ *is λ-directed quasi order (in $\mathbf{V}$ and also in $\mathbf{V}^{\mathbb{Q}}$)*
- (ii) $|\mathrm{ap}_\kappa(\underset{\sim}{L})| \leq |\mathrm{Dom}(\underset{\sim}{L})|^{<\kappa}$
- (iii) $\Vdash_{\mathbb{Q}}$ “$\mathrm{id}_{\underset{\sim}{L}[G]} \leq_{\mathrm{RK}} \mathrm{id}_{\mathrm{ap}_\kappa(\underset{\sim}{L})}$”

2) For a forcing notion $\mathbb{Q}$ satisfying the μ-c.c., the local $\underset{\sim}{\kappa}$-c.c. and a $\mathbb{Q}$-name $\underset{\sim}{L}$ of a λ-directed quasi order such that $\lambda \geq \kappa$ we can find $\mathscr{L} \in \mathbf{V}$ such that

- (i) *$\mathscr{L}$ is a family of $< \mu$ λ-directed quasi orders (in $\mathbf{V}$ and also in $\mathbf{V}^{\mathbb{Q}}$)*
- (ii) *for each $L \in \mathscr{L}$ for some θ we have $|L| \leq \theta^{<\kappa}$ and* $\nVdash$ “$|\mathrm{Dom}(\underset{\sim}{L})| \neq \mu, \underset{\sim}{\kappa} = \kappa$”
- (iii) $\Vdash_{\mathbb{Q}}$ “$\mathrm{id}_{\underset{\sim}{L}[G]} \leq_{\mathrm{RK}} \mathrm{id}_L$ *for some $L \in \mathscr{L}$*”.

3) In (2), of course, we can replace $\mathbb{I}$ by a λ-complete $\leq_{\mathrm{RK}}$-upper bound.

Proof. 1) We leave (i), (ii) to the reader. We check (iii). Let $G \subseteq \mathbb{Q}$ be generic over $\mathbf{V}$, and in $\mathbf{V}[G]$ we define a function h from $\mathrm{ap}_\kappa(\underset{\sim}{L})$ to $\mathrm{Dom}(\underset{\sim}{L}[G])$ as follows: $h(a)$ will be an element of $\mathrm{Dom}(\underset{\sim}{L}[G])$ such that

$$(\forall x \in a)[\underset{\sim}{L}[G] \models \text{“}x \leq h(a)\text{”}].$$

It exists by the “λ-directed”, “$\lambda \geq \kappa$” assumptions. We can now easily verify the condition in 3.5(4).

2) Let $\{p_i : i < i^*\}$ be a maximal antichain of $\mathbb{Q}$ such that: $p_i \Vdash$ "$\underset{\sim}{\kappa} = \kappa_i$" and $\mathbb{Q}_{\geq p_i}$ satisfies the κ_i-c.c and without loss of generality $p_i \Vdash$ "$\mathrm{Dom}(\underset{\sim}{L}) = \mu_i$". Let $\mathbb{Q}_i = \mathbb{Q}_{\geq p_i}$ and $\underset{\sim}{L}_i$ be $\underset{\sim}{L}$ restricted to $\mathbb{Q}_i$ and lastly let $\mathscr{L} = \{\mathrm{ap}_{\kappa_i}(\underset{\sim}{L}_i, \mathbb{Q}_i) : i < i^*\}$, by part (1) we have $p_i \Vdash_{\mathbb{Q}}$ "$\mathrm{id}_{\underset{\sim}{L}[\underset{\sim}{G}_{\mathbb{Q}}]} \leq_{\mathrm{RK}} \mathrm{ap}_{\kappa_i}(\underset{\sim}{L}_i, \mathbb{Q}_i)$" by $\{p_i : i < i^*\}$ is a maximal antichain of $\mathbb{Q}$ hence $\Vdash_{\mathbb{Q}}$ "$\mathrm{id}_{\underset{\sim}{L}[\underset{\sim}{G}_{\mathbb{Q}}]} \leq_{\mathrm{RK}} \{\mathrm{ap}_{\kappa_i}(\underset{\sim}{L}_i, \mathbb{Q}_i) : i < i^*\}$.
A base $|\mathrm{ap}_{\kappa_i}(\underset{\sim}{L}_i, \mathbb{Q}_i)| \leq |\mathrm{Dom}(\underset{\sim}{L}_i)|^{<\kappa_i} = \mu_i^{<\kappa_i}$ and $\mathrm{ap}_{\kappa_i}(\underset{\sim}{L}_i, \mathbb{Q}_i)$ is λ-directed. Together we are done as necessarily $i^* < \mu$.
3) Easy by 3.13(3),(4) below. $\square_{3.10}$

3.11 Conclusion. 1) Suppose $\mathbb{Q}$ is a forcing notion satisfying the local κ-c.c., $\underset{\sim}{\mathbb{I}}_1$ a $\mathbb{Q}$-name of a family of λ-complete filters and $\lambda \geq \kappa$. Then there is, (in $\mathbf{V}$), a family $\mathbb{I}_2$ of λ-complete filters such that:

(*i*) $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\mathbb{I}}_1 \leq_{\mathrm{RK}} \mathbb{I}_2$"

(*ii*) if $\mathbb{Q}$ satisfies the μ-c.c. then $|\mathbb{I}_2| = |\underset{\sim}{\mathbb{I}}_1| + \mu$ i.e. $\Vdash_{\mathbb{Q}}$ "$|\mathbb{I}_2| = |\mathbb{I}_1|^V + \mu$"

(*iii*) $\sup\{|\mathrm{Dom}(J)| : J \in \mathbb{I}_2\} = \sup\{(2^\mu)^{<\kappa}$: some $q \in \mathbb{Q}$ forces that some $J \in \mathbb{I}_1$ has domain of power $\mu\}$.

2) If in $\mathbf{V}^{\mathbb{Q}}$ the set $\mathbb{I}_1$ has the form $\{\mathrm{id}_{(L,<)} : (L,<) \in \mathscr{L}_1\}$; i.e., is a family of quasi order ideals, then in (iii) we can have

(*iii*)′ $\sup\{|\mathrm{Dom}(J)| : J \in \mathbb{I}_2\} = \sup\{\mu^{<\kappa}$: some $q \in \mathbb{Q}$ force some $(L,<) \in \mathscr{L}$ has power $\mu\}$

(*iv*) $\mathbb{I}_2$ is a family of quasi order ideals.

Proof. Easy.

3.12 Remark. The aim of 3.10, 3.11 is the following: we will consider iterations $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha \rangle$ where $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ satisfies $\mathrm{UP}(\underset{\sim}{\mathbb{I}}_i)$", but $\underset{\sim}{\mathbb{I}}_i$ may not be a subset of the ground model $\mathbf{V}$. Now 3.11 gives us a good $\leq_{\mathrm{RK}}$-bound $\mathbb{I}'_i$ of $\mathbb{I}_i$ in $\mathbf{V}$, and we can prove (under suitable assumptions) that $\mathbb{P}_\alpha$ will satisfy the $\mathrm{UP}(\bigcup_{i<\alpha} \mathbb{I}'_i)$.

3.13 Definition. 1) For a family $\mathscr{L}$ of directed quasi order let the $(<\kappa)$-closure $c\ell_\kappa(\mathscr{L})$ be

$$\mathscr{L} \cup \left\{ \prod_{i<\alpha} L_i : L_i \in \mathscr{L} \text{ for } i < \alpha \text{ and } \alpha < \kappa \right\}$$

(the partial order on $\prod\limits_{i<\alpha} L_i$ is natural).
2) $\mathscr{L}$ is $(<\kappa)$-closed if for any $\alpha < \kappa$ and $L_i \in \mathscr{L}$ for $i < \alpha$ there is $L \in \mathscr{L}$ such that $i < \alpha \Rightarrow L_i \leq_{\mathrm{RK}} L$.

3.14 Claim. *1) If $\mathscr{L}$ is λ-complete, then $c\ell_\kappa(\mathscr{L})$ is λ-complete.*
2) $L_j \leq_{\mathrm{RK}} \prod\limits_{i<\alpha} L_i$ for $j < \alpha$.
3) If κ is regular, then $c\ell_\kappa(\mathscr{L})$ is κ-directed under $\leq_{\mathrm{RK}}$ and is $(<\kappa)$-closed.

Proof. Use 3.5(4).

§4 UP reintroduced

The reader may concentrate on the case $\mathbf{S} = \{\aleph_1\}$.

4.1 Convention. $\mathbb{I}$ will be a set of quasi order ideals, i.e. $\mathbb{I} = \mathbb{I}_{\mathscr{L}}$.

4.2 Definition. Fix $\mathbb{I}, \mathbf{S}, \mathbf{W}$ assuming

$(*)(a)$ $\mathbb{I}$ a family of $\aleph_2$-complete quasi-order ideals

(b) $\mathbf{S}$ is a set of regular cardinals to which $\aleph_1$ belongs

(c) $\mathbf{W} \subseteq \omega_1$ is stationary.

1) We say $\bar{N}$ is an $\mathbb{I}$-tagged tree of models (for χ or for (χ, x)) if there is an $\mathbb{I}$-tagged tree $(T, \mathbf{I})$ such that $\bar{N} = \langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ satisfies the following:

(a) for $\eta \in T$ we have $N_\eta \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ is a countable model

(b) $\mathbb{I} \in N_{\langle\rangle}$ and $x \in N_{\langle\rangle}$ (if x is present, we can use $\mathbb{I}$ as a predicate)

(c) $\eta \triangleleft \nu \in T$ implies $N_\eta \prec N_\nu$

(d) for $\eta \in T$ we have $\eta \in N_\eta$ and $\mathbf{I}_\eta \in N_\eta \cap \mathbb{I}$.

Whenever we have such an $\mathbb{I}$-tagged tree $\bar{N}$ of models, we write $N_\eta = \bigcup\limits_{k<\omega} N_{\eta \restriction k}$ for each $\eta \in \lim(T)$.

1A) $\bar{N} = \langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is a tagged tree of models if this occurs for some $\aleph_2$-complete $\mathbb{I}$. If x is not mentioned, we assume it contains all necessary information, in particular $\mathbb{I}, \mathbf{S}, \mathbf{W}$.

1B) In part (1) we say "weak $\mathbb{I}$-tagged tree of models" if we replace clause (d) by

$(d)^-(i)$ for $\eta \in T \cap \mathrm{Dom}(\mathbf{I})$ we have $\mathbf{I}_\eta \in N_\eta \cap \mathbb{I}$

(ii) if $\eta \in \mathrm{Split}(T, \mathbb{I})$ then for some one-to-one function $f \in N_\eta$ with domain $\supseteq \mathrm{Suc}_T(\eta)$ and range $\subseteq \mathrm{Dom}(\mathbf{I}_\eta)$ we have $\nu \in \mathrm{Suc}_T(\eta) \Rightarrow f(\nu) \in N_\nu$.

We say weaker if we omit clause $(d)^-(ii)$. In all the definitions below we can use this version (i.e., adding weak/weaker and replacing (d) by $(d)^-/(d)^-(i)$

2) We say $\bar{N}$ is truely $\mathbb{I}$-suitable (tagged tree of models) if clauses $(a) - (d)$ and:

(e) if $\eta \in T$ and $I \in \mathbb{I} \cap N_\eta$ then the set

$$\{\nu \in T^{[\eta]} : \nu \in \mathrm{split}(T) \text{ and } \mathbf{I}_\nu = I \in N_\eta\}$$

contains a front of $T^{[\eta]}$. So "$\bar N$ is truly $\mathbb{I}$-suitable tree (of models)" does not imply "$\bar N$ is an $\mathbb{I}$-tagged tree of models" as possibly $\mathbb{I} \notin N_{<>}$.
3) We say $\bar N$ is $\mathbb{I}$-suitable (a tagged tree of models) <u>if</u> clauses $(a) - (d)$ and:

- $(e)^-$ if $\eta \in T$ and $I \in \mathbb{I} \cap N$, <u>then</u> the set

$$\{\nu \in T^{[\nu]} : \nu \in \text{ split}(T) \text{ and } I \leq_{\text{RK}} \mathbf{I}_\nu \in N\}$$

contains a front of $T^{[\eta]}$.

4) We say $\bar N$ is λ-strictly $(\mathbb{I}, \mathbf{W})$-suitable <u>if</u> $\bar N$ is $\mathbb{I}$-suitable and in addition

- $(b)^+$ $\mathbb{I} \in N_{\langle\rangle}, \mathbf{W} \in N$ and $x \in N_{\langle\rangle}$
- (f) one of the following cases holds:
  - (i) $\lambda = \aleph_2$ and for some $\delta \in \mathbf{W}$, for all $\eta \in T$ we have: $N_\eta \cap \omega_1 = \delta$
  - (ii) $\lambda = \text{ cf}(\lambda) > \aleph_1, N_{<>} \cap \omega_1 \in \mathbf{W}$ and $\eta \triangleleft \nu \in T \Rightarrow N_\eta <_\lambda N_\nu$ (i.e., $N_\eta \cap \lambda$ is an initial segment of $N_\nu \cap \lambda$
    (note: if $\lambda = \mu^+$ this means $N_\eta \cap \mu = N_\nu \cap \mu$ so no contradiction with the case $\lambda = \aleph_2$)
  - (iii) $\lambda = \aleph_1$ and we demand nothing.

  If we omit λ, we mean $\lambda = \aleph_2$. So $\lambda = \text{ cf}(\lambda) > \aleph_1$ implies $\eta \in T \Rightarrow N_\eta \cap \omega_1 = N_{<>} \cap \omega_1 \in \mathbf{W}$. If we omit "strictly" we demand only $(b)^+$.

5) We say $\bar N$ is $\mathbf{S}$-strictly $(\mathbb{I}, \mathbf{W})$-suitable, <u>if</u> in addition to clauses (a) - (d),(e)$^-$ we have:

- $(b)^+$ $\mathbb{I} \in N_{\langle\rangle}, \mathbf{W} \in N_{\langle\rangle}, \mathbf{S} \in N_{\langle\rangle}$ and $x \in N_{\langle\rangle}$
- (g) for all $\nu \in T$ and $\lambda \in \mathbf{S} \cap N_\delta$ there is $\delta_\lambda, \nu < \lambda$ such that $(\forall \eta \in T)[\nu \trianglelefteq \eta \in T \Rightarrow \sup(N_\eta \cap \lambda) = \delta_\lambda]$.

6) We say $\bar N$ is λ^+-uniformly $(\mathbb{I}, \mathbf{W})$-suitable <u>if</u> it is $(\mathbb{I}, \mathbf{W})$-suitable and

- $(b)^+$ $\mathbb{I} \in N_{\langle\rangle}, \mathbf{W} \in N_{\langle\rangle}, \lambda \in N_{\langle\rangle}$ and $x \in N_{\langle\rangle}$
- $(f)'$ for some $a \in [\lambda]^{\aleph_0}$ for every $\eta \in \text{ lim}(T)$ we have

$$N_\eta \cap \lambda = a$$

and

$$\text{if } \lambda^+ = \aleph_2 \text{ then } a \in \mathbf{W}.$$

6A) We say $\bar{N}$ is $\mathbf{S}$-uniformly $(\mathbb{I}, \mathbf{W})$-suitable <u>if</u> it is $(\mathbb{I}, \mathbf{W})$-suitable and

- $(b)^+$ $\mathbb{I} \in N_{\langle\rangle}, \mathbf{W} \in N_{\langle\rangle}, \mathbf{S} \in N_{\langle\rangle}$ and $x \in N_{\langle\rangle}$
- $(g)'$ for every $\eta \in T, \lambda \in \mathbf{S} \cap N_\eta$ for some $a \in [\lambda]^{\aleph_0}$ for every ν satisfying $\eta \triangleleft \nu \in \lim(T)$ we have

$$N_\nu \cap \lambda = a.$$

7) In 4), 5) if we add "truely" if (e)$^-$ is replaced by (e).
8) If $\underset{\sim}{\mathbf{S}}$ is a $\mathbb{P}$-name then in the clauses above we mean $\mathbf{S}^* = \{\lambda : \Vdash_{\mathbb{P}} \text{“}\lambda \notin \underset{\sim}{\mathbf{S}}\text{”}\}$.
9) $\mathbb{I}^{[\lambda]} = \{I \in \mathbb{I} : I \text{ is } \lambda\text{-complete}\}$.

4.3 Definition. 1) We say $N \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ is $\underset{\sim}{\mathbf{S}}$-strictly or λ-strictly $(\mathbb{I}, \mathbf{W})$-suitable model if there is an $\underset{\sim}{\mathbf{S}}$-strictly or a λ-strictly $(\mathbb{I}, \mathbf{W})$-suitable $\bar{N}$ such that $N = N_{\langle\rangle}$, (see 4.2(5),(9), and see 4.2(8), it applies). We can add "truely".
2) We say N is $(\mathbb{I}, \mathbf{W})$-suitable if it is strictly $(\mathbb{I}, \mathbf{W})$-suitable, that is $\aleph_2$-strictly $(\mathbb{I}, \mathbf{W})$-suitable.

4.4 Definition. In Definitions 4.2, 4.3 we may omit $\mathbf{W}$ when it is ω_1, and may omit $\mathbf{S}$ when $\mathbf{S} = \{\aleph_2\}$, for 4.2(5) and 4.2(6), 4.2(6A). We may replace $\mathbf{S}$ by $*$ if $\mathbf{S} = U\text{Reg}$. Let $\eta \in (T, \mathbf{I})$ means $\eta \in T$ and we write T when $\mathbf{I}$ is clear.

4.5 Claim. *Assume*

- (i) $\underset{\sim}{\mathbf{S}}, \mathbb{P}, \mathbb{I}, \mathbf{W}, x \in \mathscr{H}(\chi)$ *and* $\mathbf{S}^*$ *is as in 4.2(8)*
  (i.e., $\mathbf{S}^* = \{\theta : \aleph_1 \leq \theta = \text{cf}(\theta) \leq |\mathbb{P}| \text{ and } \nVdash_{\mathbb{P}} \text{“}\theta \in \underset{\sim}{\mathbf{S}}\text{”}\} \cup \{\aleph_1\}$,
  e.g., $\underset{\sim}{\mathbf{S}} = \{\aleph_1\}$*), and*
- (ii) $\mathbb{I}$ *is a* $\aleph_2$*-complete or for each* $I \in \mathbb{I}, \kappa \in \mathbf{S}, I$ *is* κ*-indecomposable.*

<u>*Then*</u> *there is an* $\mathbf{S}$*-strictly, truely* $(\mathbb{I}, \mathbf{W})$*-suitable tree* $\bar{N}$ *with* $x \in N_{\langle\rangle}$.

Proof. We will construct this tree in three steps: first we find a suitable tree, then we thin it out to be a $\mathbf{S}^*$-uniformly suitable tree, then we blow up the models to make it $\mathbf{S}^*$-strict. For notational simplicity let $\mathbf{S}^* = \{\aleph_1\}$ (the reader can check the others).

First Step: An easy bookkeeping argument (to ensure 4.2(1)(e)) yields a truly $(\mathbb{I} \cup \{J^{\rm bd}_{\omega_1}\})$-suitable tree $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ satisfying $\nu \triangleleft \eta \Rightarrow \sup(N_\nu \cap \omega_1) < \sup(N_\eta \cap \omega_1)$ such that $N_\eta \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ and $x \in N_{<>}$; so for $\eta \in \lim(T)$ we let $N_\eta = \bigcup\limits_{\ell < \omega} N_{\eta \restriction \ell}$.

Moreover we can get that for all $\eta \in \lim(T)$, for each $I \in (\mathbb{I} \cap N_\eta) \cup \{J^{\rm bd}_{\omega_1}\}$, there are infinitely many k such that $\eta \restriction k \in \text{split}(T, \mathbf{I})$ and $\mathbf{I}_{\eta \restriction k} = I$ and $\text{Suc}_T(\eta \restriction k) = \{\eta\char`\^\langle x\rangle : x \in \text{Dom}(I)\}$.

Second Step: Define $H : T \rightarrow \omega_1$ by $H(\eta) = \sup(N_\eta \cap \omega_1) < \omega_1$. Apply 2.12 to get a subtree T' and a limit ordinal $\delta \in \mathbf{W} \subseteq \omega_1$ such that clauses (a) - (d) of 2.12 hold for δ. By clause (d) of 2.12, for all $\eta \in T', N_\eta \cap \omega_1 \subseteq \delta$. Let $\delta_0 < \delta_1 < \ldots \bigcup\limits_n \delta_n = \delta$, and let

$$T'' = \Big\{\eta \in T' : \text{for each } \forall k < \ell g(\eta), \text{ if } \text{Suc}_T(\eta \restriction k) = \{\eta \restriction k\char`\^\langle \alpha\rangle : \alpha < \omega_1\} \\ \text{so } \text{Suc}_{T'}(\eta \restriction k) = \{\eta \restriction k\char`\^\langle\alpha\rangle : \alpha < \delta\}) \\ \text{then } \eta(k) = \delta_k\Big\}.$$

Clearly T'' will be $\aleph_1$-uniformly suitable; i.e.
$\eta \in \lim(T) \Rightarrow N_{\eta, \ell} \cap \omega_1 = \delta$.

Third Step: For $\eta \in T_2$, let $N'_\eta =$ the Skolem Hull of $N_\eta \cup \delta$ in $(\mathscr{H}(\chi), \in, <^*_\chi)$. So $N'_\eta \cap \omega_1 \supseteq \delta$. Conversely, let $\nu \in \lim(T_2), \eta \triangleleft \nu$, then $N_\eta \cup \delta \subseteq N_\nu$, so $N'_\eta \subseteq N_\nu$ hence $N'_\eta \cap \omega \subseteq \delta$. So $N'_\eta \cap \omega_1 = \delta$, i.e. $\langle N'_\eta : \eta \in T\rangle$ is an $\aleph_1$-strict, $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-tree of models (see Definition 4.2(4)).

We claim that this tree is still truly suitable. Indeed, assume $\eta \in T_2, \nu \in \lim(T_2), \eta \trianglelefteq \nu$ and $I \in \mathbb{I} \cap N'_\eta$. Then for some $\alpha < \delta, I$ is in the Skolem hull of $N_\eta \cup \alpha$. Let $k < \omega$ be such that $\alpha \in N_{\nu \restriction k} \cap \omega_1$ and $k \leq \ell g(\eta)$. Then since $\langle N_\eta : \eta \in T_2\rangle$ was suitable, there is $\ell \geq k$ such that $\mathbf{I}_{\nu \restriction \ell} = I$. So $\langle N'_\eta : \eta \in T_2\rangle$ is also $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-suitable. $\square_{4.5}$

4.6 Fact. Assume $\mathbb{I}' \leq_{\rm RK} \mathbb{I}$, where $\mathbb{I}, \mathbb{I}'$ are families of ideals.
1) If $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is a $\mathbb{I}$-suitable tree and $\mathbb{I}' \in N_{\langle\rangle}$, then $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is also $\mathbb{I}'$-suitable.
2) If $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is $\mathbb{I}$-suitable and $\mathbb{I}' \in N_{\langle\rangle}$, then there is a tree $(T', \mathbf{I}')$ satisfying the following for some T'', f:

(a) $T'' \subseteq T$ and f is an isomorphism from T'' onto T', (i.e., is one to one onto preserving length and $\triangleleft$ and its negation) and $\eta \in T'' \Rightarrow \mathbf{I}''_\eta \le_{\rm RK} \mathbf{I}_{f(\eta)}, \mathbf{I}''_\eta \neq \mathbf{I}'_\eta$

(b) $\langle N_\eta : \eta \in (T', \mathbf{I}')\rangle$ is truely $\mathbb{I}'$-suitable when we let $I''_\eta = \mathbf{I}'_{f(\eta)}$

(c) $\mathrm{split}(T'', \mathbf{I}'') = T'' \cap \mathrm{split}(T, \mathbf{I})$ if $\mathbb{I}' \equiv_{\rm RK} \mathbb{I}$.

3) We can weaken the hypothesis to $\mathbb{I}' \le_{\rm RK} \mathbb{I} \cup \{\text{the trivial ideal}\}$. The same holds in similar situations.
4) In Definition 4.4, if $\mathbf{S} = \{\theta : \aleph_1 \le \theta = \mathrm{cf}(\theta) \le \lambda\}, \lambda = \mathrm{cf}(\lambda)$, then clause (f) of 4.2(4) (i.e., λ^+-strictly) and clause (g) of 4.2(5) (i.e. the demand concerning $\mathbf{S}$, i.e., $\mathbf{S}$-strictly) are equivalent. Similarly 4.2(6), 4.2(6A) are equivalent.
5) In part (2), $\langle N_\eta : \eta \in (T'', \mathbf{I})\rangle$ is a weak $\mathbb{I}'$-tagged tree, truely $\mathbb{I}'$-suitable; moreover it is enough to assume $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is a weak $\mathbb{I}$-suitable tree (see 5.2).
6) If $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is a weak $\mathbb{I}$-tagged tree then for some tree T' and tree isomorphic f from T' onto T letting $\mathbf{I}' = \langle \mathbf{I}_{f(\eta)} : \eta \in T'\rangle$ we have $\langle N_{f(\eta)} : \eta \in (T', \mathbf{I}')\rangle$ is a $\mathbb{I}$-tagged tree. All relevant properties are preserved. [Check, see 5.2.]

Proof. 1) Should be clear, as $\le_{\rm RK}$ is transitive (as a relation among ideals and also among families of ideals).
2) For every $\eta \in Y =: \{\eta \in T : (\exists I' \in \mathbb{I}')(I' \le \mathbf{I}_\eta)$ and $\eta \in \mathrm{split}(T, \mathbf{I})\}$ pick an ideal $\mathbf{I}''_\eta \in \mathbb{I} \cap N_\eta, \mathbf{I}''_\eta \le_{\rm RK} \mathbf{I}_\eta$ such that: for every $\nu \in T$, for every $I' \in \mathbb{I} \cap N_\nu$ the set $\{\eta \in T^{[\nu]} : I' = \mathbf{I}''_\eta$ and $\nu \triangleleft \eta$ and $\eta \in Y\}$ contains a front of $T^{[\nu]}$. This can be done using a bookkeeping argument.

Now define T' as follows. We choose by induction on n, a function f_n with domain $\subseteq T \cap {}^n\mathrm{Ord}$, such that $\eta \in \mathrm{Dom}(f_n) \Rightarrow f(\eta) \in N_\eta \cap {}^n\mathrm{Ord}$. Let f_0 be the identity on $\{<>\}$. Assume f_n has been defined and $\eta \in \mathrm{Dom}(f_n)$, and we shall define $f_{n+1} \restriction \mathrm{Suc}_T(\eta)$. If $\eta \in T \backslash Y$, then $\mathrm{Dom}(f_{n+1}) \cap \mathrm{Suc}_T(\eta) \subseteq \mathrm{Suc}_T(\eta)$ is a singleton $\{\nu_\eta\}$ and let $f_{n+1}(\nu_\eta) = f_n(\eta)\hat{\ }\langle(\nu_\eta(n)\rangle$. If $\eta \in T \cap Y$, then $\mathbf{I}'_\eta$ is already defined and it belongs to N_η. Let g_η be a witness for $\mathbf{I}'_\eta \le_{\rm RK} \mathbf{I}_\eta$ and stipulate $\mathrm{Dom}(\mathbf{I}_\eta) \supseteq \{x : \eta\hat{\ } < x > \in \mathrm{Suc}_T(\eta)\}$. Now g_η introduces an equivalence relation on $\mathrm{Dom}(\mathbf{I}_\eta)$. Let A_η be a selector set for this equivalence relation; i.e. $g_\eta \restriction A_\eta$ is 1-1 and has the same range as g_η. Note that we can choose g_η in N_η as $\mathbf{I}_\eta, \mathbf{I}'_\eta \in N_\eta$ (whereas $\mathrm{Suc}_T(\eta)$ does not necessarily belong to N_η) and then choose A_η and let $A'_\eta =: \{x \in A_\eta : (\exists y \in \mathrm{Suc}_T(\eta))[g_\eta(x) = g_\eta(y)]\}$ (so possibly $A_\eta \notin N_\eta$). Now for $x \in A'_n$ so $\eta\hat{\ } < x > \in \mathrm{Suc}_T(\eta)$ we let $f_{n+1}(\eta\hat{\ } < x >) = f_n(\eta)\hat{\ }\langle g_n(x)\rangle$ so $\mathrm{Dom}(f_{n+1}) \cap \mathrm{Suc}_T(\eta) = \{\eta\hat{\ } < x >: x \in A'_\eta\}$. Lastly, let $T' = \cup\{\mathrm{Rang}(f_n) : n < \omega\}, T'' = \cup\{\mathrm{Dom}(f_n) : n < \omega\}$ and for $\eta \in \mathrm{Dom}(f_n), n < \omega$ let $\mathbf{I}'_{f_n(\eta)} = \mathbf{I}''_\eta$ and $f = \cup\{f_n : n < \omega\}$. Now check.
3), 4) Left to the reader. $\square_{4.6}$

4.7 Definition. 1) Let $\chi > \aleph_0, \mathbb{I} \in \mathscr{H}(\chi)$ a set of ideals and $\mathbf{S} \in \mathscr{H}(\chi)$ a set of regular cardinals (or just limit ordinals) such that $\aleph_1 \in \mathbf{S}$. For N a countable elementary submodel of $\mathfrak{B} = (\mathscr{H}(\chi), \in)$ (or $\mathfrak{B}$ an expansion of $(\mathscr{H}(\chi), \in)$ with countable vocabulary) such that $\mathbb{I}, \mathbf{S} \in N$ we define $\text{Dp}(N) = \text{Dp}_{\mathbb{I}}(N) = \text{Dp}_{\mathbb{I}}(N, \mathfrak{B}) = \text{Dp}_{\mathbb{I}}(N, \mathbf{S}, \mathfrak{B}) \in \text{Ord} \cup \{\infty\}$, by defining when $\text{Dp}(N) \geq \alpha$ for an ordinal α, by induction on α:

$\text{Dp}(N) \geq \alpha$ iff N is as above and for every $I \in \mathbb{I} \cap N$

and for every $\beta < \alpha$ and $X \in I$ there is M satisfying :

(i) $\text{Dp}(M) \geq \beta$(hence $M \prec \mathfrak{B}$ is countable)

(ii) $N \prec M$

(iii) $\sup(M \cap \omega_1) = \sup(N \cap \omega_1)$ moreover

$\theta \in \mathbf{S} \cap N \Rightarrow \sup(N \cap \theta) = \sup(M \cap \theta)$

(iv) $M \cap \text{Dom}(I) \backslash X \neq \emptyset$.

2) We define $\text{Dp}'(N) = \text{Dp}'_{\mathbb{I}}(N)$ by defining: $\text{Dp}'(N) \geq \alpha$ iff N is as above and for every $J \in \mathbb{I} \cap N$ and $\beta < \alpha$ for some $I \in N \cap \mathbb{I}$ we have $J \leq_{\text{RK}} I$ and for every $X \in I$ there is M satisfying (i)-(iv) above.

4.8 Claim. *1) In Definition 4.7:*

(a) $\text{Dp}(N) \in \text{Ord} \cup \{\infty\}$ *if well defined*

(b) *if* $\text{Dp}(N) = \infty, I \in \mathbb{I} \cap N$ *then we can find* $Y \in I^+$ *(i.e.,* $Y \subseteq \text{Dom}(I), Y \notin I$*) and* $\bar{N} = \langle N_t : t \in Y \rangle$ *such that:*

(i) $\text{Dp}(N_t) = \infty$

(ii) $N \prec N_t$

(iii) $\sup(N \cap \omega_1) = \sup(N_t \cap \omega_1)$ *moreover* $\theta \in \mathbf{S} \cap N \Rightarrow \sup(N \cap \theta) = \sup(N_t \cap \theta)$.

2) If $\mathbb{I}_1 \leq_{\text{RK}} \mathbb{I}_2$ *and* $\{\mathbb{I}_1, \mathbb{I}_2\} \in N$ *and* $Dp_{\mathbb{I}_2}(N) \geq \alpha$ *then* $Dp_{\mathbb{I}_1}(N) \geq \alpha$.
3) $Dp_{\mathbb{I}}(N) = Dp'_{\mathbb{I}}(N)$.

Proof. Straightforward.

4.9 Claim. *1) Let $N \prec (\mathscr{H}(\chi), \in)$ be countable, $\mathbb{I} \in N, N \cap \omega_1 \in \mathscr{W}$. Then*

(a) *N is strictly $(\mathbb{I}, \mathscr{W})$-suitable iff $\mathrm{Dp}_{\mathbb{I}}(N) = \infty$*
(b) *N is $(\mathbb{I}, \mathscr{W})$-suitable iff $\mathrm{Dp}'_{\mathbb{I}}(N) = \infty$.*

2) Similarly with $\mathbf{S}$.

Proof. Easy.

4.10 Definition. 1) The forcing notion $\mathbb{P}$ satisfies $\mathrm{UP}^{\ell}_{\underset{\sim}{\lambda}}(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ (note if $\ell \neq 2$ we may omit $\underset{\sim}{\lambda}$) (adopting the conventions of 4.2(8); $\underset{\sim}{\lambda}$ is a purely decidable $\mathbb{P}$-name of a $\mathbf{V}$-cardinal) when: $\ell \in \{0,1,2\}$ and if χ is large enough and $\bar{N}$ is $\underset{\sim}{\mathbf{S}}$-strictly $(\mathbb{I}, \mathbf{W})$-suitable and $p \in N_{\langle\rangle} \cap \mathbb{P}$ and $P \in N_{\langle\rangle}$, of course, then there is $q \in \mathbb{P}$ such that $p \leq_{\mathrm{pr}} q \in \mathbb{P}$ and:

(a) if $\ell = 0$ then $q \Vdash$ "$N_{\langle\rangle}[\underset{\sim}{G}_{\mathbb{P}}] \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$ and $\sup(N_{<>}[\underset{\sim}{G}_{\mathbb{P}}] \cap \theta) = \sup(N_{<>} \cap \theta)$ if $\theta \in \underset{\sim}{S}$"
(b) if $\ell = 1$ then $q \Vdash$ "for some $\eta \in \lim(T)$ we have $N_\eta[\underset{\sim}{G}_{\mathbb{P}}] \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$ and for every $\theta \in \underset{\sim}{\mathbf{S}}$ we have $\sup(N_\eta[\underset{\sim}{G}_{\mathbb{P}}] \cap \theta) = \sup(N_\eta \cap \theta)$ where $N_\eta = \cup\{N_{\eta \restriction \ell} : \ell < \omega\}$ and η is not necessarily from $\mathbf{V}$"
(c) if $\ell = 2$ then $q \Vdash$ "$N_{\langle\rangle}[\underset{\sim}{G}_{\mathbb{P}}]$ is $(\underset{\sim}{\mathbf{S}} \backslash \underset{\sim}{\lambda}')$-strictly $(\mathbb{I}^{[\underset{\sim}{\lambda}]}, \mathbf{W})$-suitable and $\sup(N_{<>}[\underset{\sim}{G}_{\mathbb{P}}] \cap \theta) = \sup(N_{<>} \cap \theta)$ for every $\theta \in \underset{\sim}{S}$ (in particular $\aleph_1^{\mathbf{V}}$)" where $\underset{\sim}{\lambda}' = \aleph_2^{\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}}]}$.

2) If we omit ℓ we mean $\ell = 0$.
If $\mathbf{W} = \omega_1$ we may omit it. We write $*$ instead of $\underset{\sim}{\mathbf{S}}$ if $\underset{\sim}{\mathbf{S}} = \{\lambda : \mathbf{V}[\underset{\sim}{G}_{\mathbb{P}}] \models \lambda \in \mathrm{UReg}^{\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}}]}\}$. If we omit $\underset{\sim}{\mathbf{S}}$ we mean $\{\aleph_1^{\mathbf{V}}\}$.
3) The forcing notion $\mathbb{P}$ satisfies $\mathrm{UP}^4_{\underset{\sim}{\kappa}, \underset{\sim}{\lambda}}(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ when $\underset{\sim}{\kappa}, \underset{\sim}{\lambda}$ are $(\mathbb{P}, \leq_{\mathrm{pr}})$-names of regular $\mathbf{V}$-cardinals and for some x we have: if χ is large enough and $\bar{N} = \langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is an $\underset{\sim}{\mathbf{S}}$-strictly $(\mathbb{I}, \mathbf{W})$-suitable tree of models for (χ, x) and $p \in \mathbb{P} \cap N_{\langle\rangle}$ and $\langle \underset{\sim}{\kappa}, \underset{\sim}{\lambda}, \mathbb{P}, \mathbb{I}\rangle \in N_{\langle\rangle}$, then for some $q, \underset{\sim}{T}$ we have:

(a) $p \leq_{\mathrm{pr}} q \in \mathbb{P}$
(b) q is $(\bar{N} \restriction \underset{\sim}{T}, \underset{\sim}{\kappa}, \underset{\sim}{\lambda}, \mathbb{P})$-semi$_4$ generic (see below).

3A) If $\underset{\sim}{\kappa} = \infty$ we can replace $\underset{\sim}{T}$ by $\underset{\sim}{\eta}$ such that $\underset{\sim}{T} = \{\underset{\sim}{\eta} \restriction n : n < \omega\}$ (see below) so $\Vdash$ "$\underset{\sim}{\eta} \in \lim(T)$" and then in clause (b) write $N_{\underset{\sim}{\eta}}$ instead of $\bar{N} \restriction \underset{\sim}{T}$. We then may omit κ. We may $\underset{\sim}{\lambda}$ if $\underset{\sim}{\lambda} = \underset{\sim}{\kappa}(\mathbb{P})$, see Definition 1.29(5).

3B) We say that q is $(\bar{N} \restriction \underset{\sim}{T}, \underset{\sim}{\kappa}, \underset{\sim}{\lambda}, \mathbb{P})$-semi$_4$ generic where $\bar{N}, \underset{\sim}{T}, \underset{\sim}{\kappa}, \underset{\sim}{\lambda}$ is as above <u>if</u>:

$q \Vdash_{\mathbb{P}}$ "(*i*) $\underset{\sim}{T}$ is a subtree of T (so $\underset{\sim}{T} \subseteq T$ is closed under initial segments $<> \in \underset{\sim}{T}, \eta \in \underset{\sim}{T} \Rightarrow \text{Suc}_{\underset{\sim}{T}}(\eta) \neq \emptyset$)

(*ii*) $N_\eta[\underset{\sim}{G}_{\mathbb{P}}] \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$ for $\eta \in \underset{\sim}{T}$

(*iii*) $\bar{N}[\underset{\sim}{G}_{\mathbb{P}}] \restriction \underset{\sim}{T}$ has $(\underset{\sim}{\lambda}, \underset{\sim}{\lambda})$-covering which means: if $\underset{\sim}{\eta}$ is an ω-branch of $\underset{\sim}{T}$ and $y \in N_{\underset{\sim}{\eta}}[\underset{\sim}{G}_{\mathbb{P}}]$ is a set of $< \underset{\sim}{\lambda}[\underset{\sim}{G}_{\mathbb{P}}]$ ordinals (if $\underset{\sim}{\lambda}[\underset{\sim}{G}_{\mathbb{P}}]$ is not a cardinal, this means $\leq |\underset{\sim}{\lambda}[\underset{\sim}{G}_{\mathbb{P}}]|$) <u>then</u> for some $A \in \mathbf{V} \cap \bigcup_{\ell<\omega} N_{\underset{\sim}{\eta} \restriction \ell}$ we have $|A|^{\mathbf{V}} < \underset{\sim}{\lambda}[\underset{\sim}{G}_{\mathbb{P}}]$ and $y \subseteq A$

(*iv*) $\langle N_\eta[\underset{\sim}{G}_{\mathbb{P}}] : \eta \in (\underset{\sim}{T}, \mathbf{I})\rangle$ is a strictly $\mathbb{I}^{[\underset{\sim}{\kappa}[\underset{\sim}{G}_{\mathbb{P}}]]}$-suitable tree of models".

4) We define $\text{UP}^3_{\underset{\sim}{\kappa},\underset{\sim}{\lambda}}(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ similarly replacing clause (b) by the weaker

(*b*)$^-$ q is $(N_{\underset{\sim}{\eta}}, \underset{\sim}{\kappa}, \underset{\sim}{\lambda}, \mathbb{P})$-semi$_3$ -generic, (see below).

4A) If $\underset{\sim}{\kappa} = \infty$ we can replace $\underset{\sim}{T}$ by $\underset{\sim}{\eta}$ such that $\underset{\sim}{T} = \{\underset{\sim}{\eta} \restriction n : n < \omega\}$ so $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\eta} \in \lim(T)$" and replace $\bar{N} \restriction \underset{\sim}{T}$ by $N_{\underset{\sim}{\eta}}$. We may omit $\underset{\sim}{\lambda}$ if it is ∞ (but see 4.12(0)). We then may omit $\underset{\sim}{\kappa}$ if it is ∞, too.

4B) q is $(N, \underset{\sim}{\kappa}, \underset{\sim}{\lambda}, \mathbb{P})$-semi$_3$-generic is defined as in (4) only replacing clause (iii) in $\boxtimes$ of (3B) by

(*iii*)$^-$ $\bar{N}[\underset{\sim}{G}_{\mathbb{P}}] \restriction \underset{\sim}{T}$ has $(\underset{\sim}{\lambda}, 1)$-covering which means for every $y \in \mathbf{V} \cap N_{\underset{\sim}{\eta}}[\underset{\sim}{G}_{\mathbb{P}}]$ for some $A \in \mathbf{V} \cap N_{\underset{\sim}{\eta}}$ we have $|A|^V < \underset{\sim}{\lambda}[\underset{\sim}{G}_{\mathbb{P}}]$ and $y \in A$, recalling $N_{\underset{\sim}{\eta}}[\underset{\sim}{G}_{\mathbb{P}}] = \cup\{N_{\underset{\sim}{\eta} \restriction \ell}[\underset{\sim}{G}_{\mathbb{P}}] : \ell < \omega\}$".

5) We allow to use $\underset{\sim}{\mathbb{I}}$, a $\mathbb{P}$-name of an element of $\mathbf{V}$ as above <u>if</u>:

(a) it is decidable purely
(b) if $q \in \mathbb{P}$ decides $\underset{\sim}{\mathbb{I}} = \mathbb{I}$ then $\mathbb{P}_{\geq q}$ satisfies $\mathrm{UP}^\ell(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$.

6) We say that $\mathbb{P}$ satisfies the $\mathrm{UP}^5_{\underset{\sim}{\kappa},\underset{\sim}{\lambda}}(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ <u>iff</u>

(a) $\underset{\sim}{\kappa}$ and $\underset{\sim}{\lambda}$ are $\mathbb{P}$-names of $\mathbf{V}$-cardinals such that $\mathbb{P}$ satisfies the $\underset{\sim}{\kappa}$-c.c. purely locally
(b) $\underset{\sim}{\mathbb{I}}$ is a $\mathbb{P}$-name of a set which belongs to $\mathbf{V}$, it is a set of ideals and $\underset{\sim}{\mathbb{I}}$ is decidable purely
(c) is κ-complete if $\nVdash_{\mathbb{P}}$ "$\neg(\underset{\sim}{\kappa} = \kappa \ \& \ I \in \underset{\sim}{\mathbb{I}})$",
(d) if $p \in \mathbb{P}$ forces $\underset{\sim}{\mathbb{I}} = \mathbb{I}, \mathbb{I} \subseteq \mathbb{I}', \underset{\sim}{\lambda} = \lambda, \underset{\sim}{\kappa} = \kappa$ and $\mathbb{I}' \backslash \mathbb{I}$ is a set of λ-complete ideals <u>then</u> for some x
  ⊠ if $\bar N = \langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is strictly $\mathbb{I}'$-suitable, $x \in N_{<>}$, then for some $q, \underset{\sim}{T}$ we have: $p \leq_{\mathrm{pr}} q \in \mathbb{P}$ and q is $(\bar N \restriction \underset{\sim}{T}, \underset{\sim}{\kappa}, \underset{\sim}{\lambda}, \mathbb{P})$-semi$_5$-generic, (see below).

7) We define $\mathbb{P}$ satisfies $\mathrm{UP}^5_{\underset{\sim}{\kappa}}(\underset{\sim}{\mathbb{I}}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ as in part (6) but restrict ourselves to $\mathbb{I}' = \mathbb{I}$. [others?]
8) Assume $\bar N = \langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ and $\mathbb{P} \in N_{<>}$ satisfies $\mathrm{UP}^\ell_{\underset{\sim}{\kappa}}(\underset{\sim}{\mathbb{I}}, \mathbf{W})$ and $\langle \mathbb{P}, \underset{\sim}{\mathbb{I}}, \mathbf{W}, \dots\rangle \in N_{<>}$. We say q is $(\bar N, \underset{\sim}{\kappa}, \mathbb{P})$-semi$_5$-generic (for $\bar N$ when not understood from the context) if:

$$q \Vdash \text{"for some } \underset{\sim}{T} \text{ we have } (T, \underset{\sim}{\mathbb{I}}) \leq^{\underset{\sim}{\kappa}} (\underset{\sim}{T}', \mathbf{I}), \text{ see 2.4, clause (f) and}$$
$$\eta \in \underset{\sim}{T}' \Rightarrow N_\eta[\underset{\sim}{G}_{\mathbb{P}}] \cap \omega_1 = N_{<>} \cap \omega_1 \text{ and}$$
$$\eta \in T' \ \& \ \mu \in \underset{\sim}{\mathbf{S}} \Rightarrow \sup(N_{\underset{\sim}{\eta}}[G_{\mathbb{P}}] \cap \mu) = \sup(N_{\underset{\sim}{\eta}} \cap \mu)\text{"}.$$

9) We write $\mathrm{UP}^3(\underset{\sim}{\mathbb{I}}, \mathbf{S}, \mathbf{W})$ for $\mathrm{UP}^3_{\underset{\sim}{\kappa}}(\underset{\sim}{\mathbb{I}}, \mathbf{S}, \mathbf{W})$ where $\underset{\sim}{\kappa}$ is $\underset{\sim}{\kappa}(\mathbb{P})$ see 1.29(5), 1.30(1).

4.11 Definition. [?] We call $\mathbb{I}$ to be a name if it is a name of an old family of ideals purely decidable.

4.12 Claim. *0) In Definition 4.10(3)-(6), if $\underset{\sim}{\lambda} \ge \underset{\sim}{\kappa}(\mathbb{P})$, <u>then</u> the demand concerning $\underset{\sim}{\lambda}$ (i.e., clause (iii) of 4.10(3B) holds trivially (as increasing p purely, $p \Vdash$ "$\underset{\sim}{\lambda} = \lambda$" and $\mathbb{P}_{\geq p}$ satisfies the λ-c.c).*
1) If $\mathbb{Q}$ satisfies $UP^4_{\underset{\sim}{\kappa},\underset{\sim}{\lambda}}(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$, <u>then</u> it satisfies $\mathrm{UP}^3_{\underset{\sim}{\kappa},\underset{\sim}{\lambda}}(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$. If $\mathbb{Q}$ satisfies $\mathrm{UP}^3_{\underset{\sim}{\kappa},\underset{\sim}{\lambda}}(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$, <u>then</u> $\mathbb{Q}$ satisfies $\mathrm{UP}^1(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ and $\mathrm{UP}^2_{\underset{\sim}{\kappa}}(\mathbb{I}, \mathbf{W})$. If $\mathbb{Q}$ satisfies $\mathrm{UP}^2_{\underset{\sim}{\kappa}}(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ or $\mathrm{UP}^1(\mathbb{I}, \mathbf{S}, \mathbf{W})$ then it satisfies $\mathrm{UP}^0(\mathbb{I}, \mathbf{S}, \mathbf{W})$. If $\ell \in \{3, 4\}$ and $\mathbb{Q}$ satisfies $\mathrm{UP}^\ell_{\underset{\sim}{\kappa},\underset{\sim}{\lambda}}(\mathbb{I}, \underset{\sim}{\mathbb{I}}, \mathbf{W})$ and $\underset{\sim}{\kappa}_1 \ge \underset{\sim}{\kappa}, \underset{\sim}{\lambda}_1 \ge \underset{\sim}{\lambda}, [\ell = 4 \Rightarrow \underset{\sim}{\lambda}_1 = \underset{\sim}{\lambda}]$, <u>then</u> $\mathbb{Q}$ satisfies $\mathrm{UP}^\ell_{\underset{\sim}{\kappa}_1,\underset{\sim}{\lambda}_1}(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$.
1A) If $\mathbb{Q}$ satisfies $\mathrm{UP}^\ell(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$, <u>then</u> it satisfies $\mathrm{UP}^0(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ which implies "$\mathbb{Q}$ has pure $\aleph_1$-decidability", see Definition 1.23(2).
2) The forcing notion $\mathbb{Q}$ satisfies $\mathrm{UP}^\ell(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ <u>iff</u> its completion (i.e., $\mathbb{Q}^1$, or equivalently its completion to a complete Boolean algebra) satisfies it assuming $\le_{\rm pr} = \le$.
3) If $\mathbb{Q}$ satisfies $\mathrm{UP}(\mathbb{I}, *, \mathbf{W})$, (i.e., see 4.10(2)) and $\mathbb{I}$ is μ^+-complete (e.g., $\mathbb{I} = \emptyset$), <u>then</u> any "new" countable set of ordinals $< \mu$ is included in an "old" countable set of ordinals; i.e., one from $\mathbf{V}$.
4) $\mathbb{Q}$ satisfies $\mathrm{UP}(\emptyset, *)$ <u>iff</u> $\mathbb{Q}$ is purely proper (see Definition 1.25(1)).
5) $\mathbb{Q}$ satisfies $\mathrm{UP}(\emptyset, \{\aleph_1\})$ <u>iff</u> $\mathbb{Q}$ is purely semiproper (see Definition 1.25(2))[6].
6) If $\mathbb{Q}$ satisfies $\mathrm{UP}(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ and $\mathbb{I} \subseteq \mathbb{I}_1, \underset{\sim}{\mathbf{S}}_1 \subseteq \underset{\sim}{\mathbf{S}}$ and $\mathbf{W}_1 \subseteq \mathbf{W}$ <u>then</u> $\mathbb{Q}$ satisfies $\mathrm{UP}^\ell(\mathbb{I}_1, \underset{\sim}{\mathbf{S}}_1, \mathbf{W}_1)$.
7) In Definition 4.2, if $\mathbb{P}$ satisfies the κ-c.c. (e.g. $\kappa = |\mathbb{P}|^+$) <u>then</u>:

(a) we can replace $\underset{\sim}{\mathbf{S}}$ by any set $\underset{\sim}{\mathbf{S}}'$ of uncountable regular cardinals of $\mathbf{V}$, such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbf{S}} \cap \kappa = \underset{\sim}{\mathbf{S}}' \cap \kappa$".

8) In Definition 4.10 (in all the variants), <u>if</u> we demand "for χ large enough, for some $x \in \mathscr{H}(\chi)$, for every $\bar N$ such that $x \in N_{\langle\rangle}$ and ..." we get an equivalent definition.
9) In Definition 4.10 we can use weak $\mathbb{I}$-tagged trees, i.e. we get with this an equivalent definition.

Proof. 1), 2) Trivial.
3) Straightforward.
4) Use 4.5 below.
5),6) If $\mathbb{I} = \emptyset$, then every $N \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ is a $\mathbb{I}$-model.

[6] that is: if $\mathbb{Q} \in N \prec (\mathscr{H}(\chi), \in), N$ is countable, $p \in \mathbb{Q} \cap N$, <u>then</u> for some q we have $p \le_{pr} q$ and $q \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\tau} \in N$" for every $\mathbb{Q}$-name $\underset{\sim}{\tau} \in N$ of a countable ordinal

7) Easy.
8) Check the Definition.
9) As in [Sh:f].
10) The "weak" version allows more trees of models so apparently is a stronger condition, but by 4.8(4) it is equivalent. $\square_{4.12}$

4.13 Conclusion. If $\mathbb{P}$ satisfies $\mathrm{UP}^\ell(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ and $\underset{\sim}{\mathbf{S}}$ is as in 4.2$(*)(b)$ (or $\underset{\sim}{\mathbf{S}} = \{\aleph_1\}$) (recall that this notation implies $\mathbb{I}$ is $\aleph_2$-complete, $\aleph_1 \in \underset{\sim}{\mathbf{S}}$ and $\mathbf{W} \subseteq \omega_1$ stationary) then $\Vdash_P$ "$\mathbf{W}$ is stationary". Moreover, if $\mathbf{W}' \subseteq \mathbf{W}$ is stationary then also $\Vdash_P$ "$\mathbf{W}'$ is a stationary subset of ω_1".

Proof. The "moreover" fact is by 4.12(7) (i.e., monotonicity in $\mathbf{W}$).

Assume that $p \Vdash$ "$\underset{\sim}{C}$ is a club of ω_1 and $\underset{\sim}{C} \cap \mathbf{W} = \emptyset$". By 4.5 we can find an $\aleph_1$-strictly $(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$-suitable tree of models $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ with $\underset{\sim}{C}, p \in N_{\langle\rangle}$. Let $\delta = N_{<>} \cap \omega_1$, so $\delta \in \mathbf{W}$. By $\mathrm{UP}^\ell(\mathbb{I}, \underset{\sim}{\mathbf{S}}, \mathbf{W})$ we can find a condition q as in Definition 4.2 in particular $p \leq_{\mathrm{pr}} q$. Clearly $q \Vdash$ "$N_{\langle\rangle}[G] \cap \omega_1 = \delta$" and, trivially $p \Vdash_P$ "$\underset{\sim}{C}$ is unbounded in $N_{\langle\rangle}[G] \cap \omega_1$" hence $p \Vdash$ "$N_{\langle\rangle}[G] \cap \omega_1 \in \underset{\sim}{C}$". So $q \Vdash_Q$ "$\delta \in \underset{\sim}{C} \cap \mathbf{W}$", contradiction. $\square_{4.13}$

4.14 Remark. Usually we assume $\mathbb{I}, \mathbf{S}$ satisfies 4.2(*)(a) + (c), $\underset{\sim}{\mathbf{S}} = \{\aleph_1\}$ is the main case.

4.15 Remark. 1) From the proof of 4.5 we can conclude that in 4.2; we can replace "$\mathbf{S}$-strictly $(\mathbb{I}, \mathbf{W})$-suitable, $N_\eta \cap \omega_1 = \delta \in \mathbf{W}$" by "$(\mathbb{I}, \mathbf{S}, \mathbf{W})$-suitable", and then the condition q will be $N_{\langle\rangle}$-semi-generic.
2) As at present $\underset{\sim}{\mathbf{S}} = \{\aleph_1\}$ seem to suffice, we shall use only it for notational simplicity.

§5 An iteration theorem for UP

5.1 Claim. *1) If $\bar{N} = \langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ is a tagged tree of models for $(\chi, \langle x, \mathbb{P}\rangle), \mathbb{P}$ a forcing notion and $\mathbb{P} \in N_{\langle\rangle}$, <u>then</u> $\Vdash_{\mathbb{P}}$ "$\langle N_\eta[\underset{\sim}{G}_{\mathbb{P}}] : \eta \in (T, \mathbf{I})\rangle$ is a tagged tree of models for $(\chi, \langle x, \mathbb{P}, G\rangle)$".*
2) If in addition $\mathbb{P}$ satisfies the κ-c.c. and $\mathbb{I} \in N_{\langle\rangle}$ is κ-closed (see Definition 3.13(2)) and $\bar{N}$ is $\mathbb{I}$-suitable, <u>then</u>

(∗) $\Vdash_{\mathbb{P}}$ "$\langle N_\eta[\underset{\sim}{G}_{\mathbb{P}}] : \eta \in (T, \mathbf{I})\rangle$ *is $\mathbb{I}$-suitable*".

3) If in part (2) assume in addition that $\underset{\sim}{\kappa}, \underset{\sim}{\mathbb{I}}$ to be $\mathbb{P}$-names of objects from $\mathbf{V}$ such that $\underset{\sim}{\mathbb{I}}$ is purely decidable and $\mathbb{P}$ satisfies the local $\underset{\sim}{\kappa}$-c.c. purely, <u>then</u> for every $p \in N_{<>} \cap \mathbb{P}$ there is $q, p \le_{\rm pr} q \in N_{<>} \cap \mathbb{P}$ forcing (∗) above. If $\underset{\sim}{\mathbb{I}}$ is purely decidable and $\mathbb{P}$ is locally $\underset{\sim}{\kappa}$-c.c. purely then we can find q satisfying $p \le_{\rm pr} q \in N_{<>} \cap \mathbb{P}$ and forcing (∗).

Proof. 1) Straight.
2) So $\mathbb{P}$ satisfies the κ-c.c. and let $G \subseteq \mathbb{P}$ be generic over $\mathbf{V}$. Now from Definition 4.2(1), clearly $\langle N_\eta[G] : \eta \in (T, \mathbf{I})\rangle$ satisfies clauses (a) - (d), so it is enough to check clause $(e)^-$ of Definition 4.2(3). So let $I \in \mathbb{I} \cap N_\eta[G]$ where $\eta \in T$. Hence there is $\underset{\sim}{I} \in N_\eta$ such that $\underset{\sim}{I}$ is a $\mathbb{P}$-name and $\underset{\sim}{I}[G] = I$. Let $\mathbb{I}' = \{J \in \mathbb{I}$: for some $p \in \mathbb{P}$ we have $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{I} = J$"$\}$. So $\mathbb{I}'$ belongs to $\mathbf{V}$ and is a subset of $\mathbb{I}$ of cardinality $< \kappa$ and $\mathbb{I}' \in N_\eta$ hence there is $I^* \in \mathbb{I}$ such that $(\forall J)(J \in \mathbb{I}' \Rightarrow J \le_{RK} I^*)$, so without loss of generality $I^* \in N_\eta$, hence as N is $\mathbb{I}$-suitable clearly $\{\nu : \eta \triangleleft \nu \in T$ and $I^* \le_{RK} \mathbf{I}_\nu\}$ contains a front of $T[\eta]$. Hence in $\mathbf{V}[G]$, the set $\{\nu \in T : I \le_{\rm RK} \mathbf{I}_\nu\}$ contains a front of $T^{[\eta]}$ as required.
3) Left to the reader. $\square_{5.1}$

The point of the following claim is to get more from some UP^ℓ than seems on the surface; our aim is to help iterating.

5.2 Claim. *1) Assume $\ell \in \{3, 4\}$ and the forcing notion $\mathbb{Q}$ satisfies $\mathrm{UP}^\ell_{\underset{\sim}{\kappa},\underset{\sim}{\lambda}}(\underset{\sim}{\mathbb{I}}, \mathbf{W})$ and $\underset{\sim}{\kappa}, \underset{\sim}{\lambda}, \underset{\sim}{\mathbb{I}}, \underset{\sim}{\mathbb{I}}^+$ are $\mathbb{Q}$-names with pure decidability and $\Vdash$ "$\underset{\sim}{\mathbb{I}}^+ \backslash \underset{\sim}{\mathbb{I}}$ is $\underset{\sim}{\lambda}$-complete". <u>Then</u> the forcing notion $\mathbb{Q}$ satisfies the $\mathrm{UP}^\ell_{\underset{\sim}{\kappa},\underset{\sim}{\lambda}}(\underset{\sim}{\mathbb{I}}^+, \mathbf{W})$.*
2) Suppose

(a) $\mathbb{I}_0, \mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3$ *are sets of quasi-order ideals,* $\mathbb{I}_1 \subseteq \mathbb{I}_0 \subseteq \mathbb{I}_2, \mathbb{I}_3 = \mathbb{I}_1 \cup (\mathbb{I}_2 \backslash \mathbb{I}_0), \mathbb{I}_2 \backslash \mathbb{I}_0 = \mathbb{I}_3 \backslash I_1$ *is κ-closed, $\kappa \le \lambda$ and $\mathbb{I}_2 \backslash \mathbb{I}_0$ is λ-complete*

(b) $\bar{N} = \langle N_\eta : \eta \in (T^*, \mathbf{I}^*) \rangle$ *is a strict truely* $\mathbb{I}_2$*-suitable tree of models (for* χ *and* $x = \langle \mathbb{Q}, \mathbb{I}_0, \mathbb{I}_1, \mathbb{I}_2, \kappa, \lambda \rangle$

(c) $p \in N_{<>}, \ell \in \{3, 4\}$ *and* $\mathbb{Q}_{\geq p}$ *satisfies the* λ*-c.c.*

(d) $\varphi(-)$ *is a property with* $\bar{N}, G_{\mathbb{Q}}$ *as parameters (and possibly others)*

(e) *for any* T' *a subtree of* T^* *such that* $\langle N_\eta : \eta \in (T', \mathbf{I}) \rangle$ *is a truely* $\mathbb{I}_0$*-suitable tree of models there are* $q = q_{T'}, \underset{\sim}{T} = \underset{\sim}{T}(T')$ *such that*

  (i) $p \leq_{\rm pr} q \in \mathbb{Q}$

  (ii) $(T', \mathbf{I}^*) \leq (\underset{\sim}{T}, \mathbf{I}^*)$

  (iii) $q \Vdash_{\mathbb{Q}}$ "$\langle N_\eta[\underset{\sim}{G}_{\mathbb{Q}}] : \eta \in (\underset{\sim}{T}, \mathbf{I}) \rangle$ *is a truely* $\mathbb{I}_1$*-suitable tree of models and for every* $\eta \in \underset{\sim}{T}$ *we have*

    (α) $N_\eta[\underset{\sim}{G}_{\mathbb{Q}}] \cap \omega_1 = N_{<>} \cap \omega_1$ *and*

    (β) $\varphi[\eta]$

    (γ) *if* $\ell = 3, y \in N_\eta[\underset{\sim}{G}_{\mathbb{Q}}]$ *is a member of* $\mathbf{V}$ *then* $\{\nu : \eta \triangleleft \nu \in \underset{\sim}{T}$, *and for some* $A \in N_\nu, A$ *a set of cardinality* $< \lambda$ *we have* $y \in A\}$ *contains a front of* $T^{[\eta]}$

    (δ) *if* $\ell = 4$ *and* $y \in N_\eta[\underset{\sim}{G}_{\mathbb{Q}}]$ *is a set of* $< \lambda$ *members of* $\mathbf{V}$ *then* $\{\nu : \eta \triangleleft \nu \in \underset{\sim}{T}$ *and for some* $A \in N_\eta$ *a set of cardinality* $< \lambda$ *we have* $y \subseteq A\}$ *contains a front of* $\underset{\sim}{T}^{[\eta]}$.

Then there are $q, \underset{\sim}{T}$ *such that:*

$\circledast$(i) $p \leq_{\rm pr} q \in \mathbb{Q}$

(ii) $(T^*, \mathbf{I}^*) \leq (\underset{\sim}{T}, \mathbf{I}^*)$

(iii) $q \Vdash_{\mathbb{Q}}$ "$\langle N_\eta[\underset{\sim}{G}_{\mathbb{Q}}] : \eta \in (\underset{\sim}{T}, \mathbf{I}) \rangle$ *is a truely* $\mathbb{I}_3$*-suitable tree of models and for every* $\eta \in \underset{\sim}{T}$ *we have*

  (α) $N_\eta[\underset{\sim}{G}_{\mathbb{Q}}] \cap \omega_1 = N_{<>} \cap \omega$

  (β) $\models \varphi(\eta)$

  (γ), (δ) *as in clause (iii) of (e) above.*

3) In part (2), if $\mathbb{Q}$ *satisfies the* λ*-c.c., then we can omit* (γ), (δ) *in their two appearances as they follow.*

4) In part (2), we can replace "truely $\mathbb{I}_\ell$*-suitable" by "weakly* $\mathbb{I}_\ell$*-suitable".*

5.3 Remark. In part (2) clause (e) we can restrict T' to those needed.

Proof. 1) As in the definition of $\mathrm{UP}^{\ell}_{\underset{\sim}{\kappa},\underset{\sim}{\lambda}}(\mathbb{I}^+,\mathbf{W})$ let $\bar N = \langle N_\eta : \eta \in (T,\mathbb{I})\rangle$ be a strict $\mathbb{I}^+$-suitable tree of models for χ and $x \equiv \langle \mathbb{Q}, \underset{\sim}{\mathbb{I}}, \underset{\sim}{\mathbb{I}}^+, \mathbf{W}, \underset{\sim}{\kappa}, \underset{\sim}{\lambda}\rangle$ and $p \in N_{<>}$. We can find $p' \in N_{<>}, p \le_{\mathrm{pr}} p' \in \mathbb{Q}$ which forces $\underset{\sim}{\kappa}, \underset{\sim}{\lambda}, \underset{\sim}{\mathbb{I}}, \underset{\sim}{\mathbb{I}}^+$ to be say $\kappa, \lambda, \mathbb{I}, \mathbb{I}^+$ respectively. Now we can apply part (2) of the claim with $\bar N, \mathbb{I}, \mathbb{I}^{[\kappa]}, \mathbb{I}^+, \mathbb{I}^{[\kappa]} \cup (\mathbb{I}^+ \backslash \mathbb{I}), x = x$ here standing for $\bar N, \mathbb{I}_0, \mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3, \varphi$ there.
2) Let $\mathscr{T} = \{T : T$ is a subtree of T^* such that $\bar N \restriction T$ is a truely $\mathbb{I}_0$-suitable tree of models$\}$ or just $\mathscr{T} = \{T : T$ a subtree of T^* such that $(<>\in T, \eta \in T \Rightarrow \emptyset \ne \mathrm{Suc}_T(\eta) \subseteq \mathrm{Suc}_{T^*}(\eta), T$ closed under initial segments and): if $\eta \in \mathrm{split}(T^*, \mathbf{I}^*)$ & $\mathbf{I}_\eta \in \mathbb{I}_0$ then $\mathrm{Suc}_T(\eta) \in \mathbf{I}^+_\eta$ and if $\eta \notin \mathrm{split}(T^*, \mathbf{I}^*) \vee \mathbf{I}_\eta \notin \mathbb{I}_0$ then $|\mathrm{Suc}_T(\eta)| = 1\}$. For any $T \in \mathscr{T}$ by assumption (e) there are $q_T, \underset{\sim}{T}^{\mathbb{Q}}[T]$ as required there.

We shall show that some such q_T is as required. We define a $\mathbb{Q}$-name $\underset{\sim}{T}^{\oplus}$ as follows:
for $G_{\mathbb{Q}} \subseteq \mathbb{Q}$ generic over $\mathbf{V}$ we let

$$T^{\otimes} = \underset{\sim}{T}^{\otimes}[G_{\mathbb{Q}}] = \{\eta \in T^* : N_\eta[G] \cap \omega_1 = N_{<>} \cap \omega_1 \text{ and } \ell \le \ell g(\eta) \Rightarrow \varphi(\eta \restriction \ell)\}.$$

Clearly

$(*)_1$ $q_T \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{T}^{\mathbb{Q}}[T] \subseteq T^{\otimes}$" for every $T \in \mathscr{T}$.

Working in $\mathbf{V}[G_{\mathbb{Q}}]$ we define a depth function Dp function from $T^{\otimes}$ to Ord $\cup\{\infty\}$ by defining for any ordinal α when $Dp(\eta) \ge \alpha$ as follows:

⊠ $\mathrm{Dp}(\eta) \ge \alpha$ <u>iff</u> the following conditions hold:
- (α) $\eta \in T^{\otimes}$
- (β) for every $\beta < \alpha$ there is $\nu \in \mathrm{Suc}_{T^{\otimes}}(\eta)$ such that $Dp(\nu) \ge \beta$
- (γ) if $\beta < \alpha$ and $\ell \le \ell g(\eta)$ and $I \in N_\eta[G] \cap \mathbb{I}_3$ <u>then</u> for some ν with $\mathrm{Dp}(\nu) \ge \beta$ we have $\eta \trianglelefteq \nu \in T^{\otimes}$ & $\mathbf{I} = \mathbf{I}_\nu \in \mathbb{I}_3$ and $\{\nu \in \mathrm{Suc}_{T^{\otimes}}(\nu) : Dp(\nu) \ge \beta\} \ne \emptyset \mod \mathbf{I}_\nu$
- (δ) if $\beta < \alpha$ and $\underset{\sim}{y} \in N_\eta$ is a $\mathbb{Q}$-name of a member of $\mathbf{V}$ when $\ell = 3$ and is a set of cardinality $< \lambda$ when $\ell = 4$ <u>then</u> for some $\eta', \eta \trianglelefteq \eta'$ with $\mathrm{Dp}(\eta') \ge \beta$ and $A \in N_{\eta'}$ of cardinality $< \lambda$ we have $[\ell = 3 \Rightarrow \underset{\sim}{y}[\underset{\sim}{G}_{\mathbb{Q}}] \in A]$ and $[\ell = 4 \Rightarrow \underset{\sim}{y}[G_{\mathbb{Q}}] \subseteq A]$.

Clearly it is enough to show in $\mathbf{V}$ that for some $T \in \mathscr{T}$ we have $q_T \Vdash$ "$\mathrm{Dp}(<>) = \infty$". Note

$(*)_2$ if $\eta \triangleleft \nu \in T^{\otimes}$ then $\mathrm{Dp}(\eta) \geq \mathrm{Dp}(\nu)$.

Clearly in $\mathbf{V}^{\mathbb{Q}}$ we have

$(*)_3$ if $\eta \in T^{\otimes}$, $\mathrm{Dp}(\eta) < \infty$ and $\mathbf{I}_\eta \in \mathbb{I}_3$ then $\{\nu \in \mathrm{Suc}_{T^{\otimes}}(\eta) : \mathrm{Dp}(\eta) < \mathrm{Dp}(\nu)\} = \emptyset \bmod \mathbf{I}_\eta$

$(*)_4$ if $\eta \in T^{\otimes}$, $\mathrm{Dp}(\eta) < \infty$ and $\mathbf{I}_\eta \notin \mathbb{I}_3, \nu \in \mathrm{Suc}_{T^{\otimes}}(\eta)$ then $\mathrm{Dp}(\eta) \geq \mathrm{Dp}(\nu)$.

For each $\eta \in T^{\otimes}$ such that $\mathbf{I}_\eta \in \mathbb{I}_2 \backslash \mathbb{I}_0$ define the set A_η as follows:
First, $\mathscr{A}_\eta$ is the minimal family of sets satisfying

(i) if $\ell = 3$ and $\underset{\sim}{y} \in N_\eta$ is a $\mathbb{Q}$-name of a member of $\mathbf{V}$ <u>then</u> the set $A^3_{\eta,\underset{\sim}{y}} =: \{\rho \in \mathrm{Suc}_{T^*}(\eta) :$ for some set $A \in N_\rho \subseteq \mathbf{V}$ of cardinality $< \kappa, \underset{\sim}{y} \in A\}$ belongs to $\mathscr{A}_\eta$

(ii) if $\ell = 4$, parallely (using $A^4_{\eta,\underset{\sim}{y}}$), i.e., if $y \in N_\eta$ is a $\mathbb{Q}$-name of a family of $< \kappa$ members of $\mathbf{V}$ then the set $A^4_{\eta,\underset{\sim}{y}} =: \{\rho \in \mathrm{Suc}_{T^*}(\eta)$: for some set $A \in N_\rho \subseteq \mathbf{V}$ of cardinality $, \kappa$ we have $\underset{\sim}{y} \subseteq A\}$ belongs to $\mathscr{A}_\eta$

(iii) if $\ell \leq \ell g(\eta), \beta = \mathrm{Dp}(\eta \restriction \ell)$ then the set $A^*_{\eta,\ell} = \{\rho \in \mathrm{Suc}_{T^*}(\eta) : \mathrm{Dp}(\rho) \geq \beta\}$ belongs to $\mathscr{A}_\eta$.

Let $\mathscr{A}'_\eta = \{A \in \mathscr{A}_\eta : A = \emptyset \bmod \mathbf{I}_\eta\}$, note that $\mathscr{A}'_\eta$ is a countable family of members of $\mathbf{I}_\eta$ (more exactly the ideal $\mathbf{I}^{\mathbf{V}^{\mathbb{Q}}}_\eta$, which $\mathbf{I}_\eta$ generates in $\mathbf{V}^{\mathbb{Q}}$), and so actually we have defined a $\mathbb{Q}$-name $\underset{\sim}{A}_\eta = \cup\{A : A \in \mathscr{A}'_\eta\}$.
Now we can define in $\mathbf{V}$ a sequence $\langle B^*_\eta : \eta \in T^* \rangle$ such that

$(*)_5$ $B_\eta = \emptyset \bmod \mathbf{I}_\eta$ for $\eta \in T^*$

$(*)_6$ (i) if $\mathrm{Suc}_{T^*}(\eta) = \emptyset \bmod \mathbf{I}_\eta$ or $\mathbf{I}_\eta \notin \mathbb{I}_2 \backslash \mathbb{I}_0$ then $B^*_\eta = \emptyset$

(ii) if $\mathrm{Suc}_{T^*}(\eta) \neq \emptyset \bmod \mathbf{I}_\eta$ and $\mathbf{I}_\eta \in \mathbb{I}_2 \backslash \mathbb{I}_0$ <u>then</u> $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{A}_\eta \subseteq B^*_\eta$ if $\eta \in \underset{\sim}{T}^{\otimes}$".

This is possible as $\mathbb{Q}_{\geq p}$ satisfies the λ-c.c. and each $I \in \mathbb{I}_2 \backslash \mathbb{I}_0$ is λ-complete.
Now we define

$$T_0 = \{\eta \in T^* : \text{ for every } \ell < \ell g(\eta) \text{ we have } \eta \restriction (\ell + 1) \notin B^*_{\eta \restriction \ell}\}.$$

Clearly we can find $T_1 \in \mathscr{T}$ such that $T_1 \subseteq T_0$ (in particular by $(*)_6$(i)). So if $q_{T_1} \Vdash_{\mathbb{Q}}$ "Dp(<>) $= \infty$" then we are done so toward contradiction we assume that this fails. Hence, there is a subset $G_{\mathbb{Q}}$ of $\mathbb{Q}$ generic over $\mathbf{V}$ such that $q_{T_1} \in G_{\mathbb{Q}}$ and $\mathbf{V}[G_{\mathbb{Q}}] \models$ "Dp(<>) $< \infty$". As $q_{T_1} \in G_{\mathbb{Q}}$ clearly $\underset{\sim}{T}(T)[G] \subseteq T^{\otimes}[G_{\mathbb{Q}}]$.
By the choice of $G_{\mathbb{Q}}$ we have $Dp(<>) < \infty$ hence by $(*)_2$ we have $\eta \in \underset{\sim}{T}(T_1)[G_{\mathbb{Q}}] \Rightarrow \eta \in \underset{\sim}{T}^{\otimes}[G_{\mathbb{Q}}] \Rightarrow Dp(\eta) < \infty$.

Now we shall prove by induction on $\alpha \in$ Ord that $\eta \in \underset{\sim}{T}(T_1)[G_{\mathbb{Q}}] \Rightarrow$ Dp$(\eta) \geq \alpha$. For $\alpha = 0, \alpha$ limit we have no problem, so let $\alpha = \beta + 1$, assume toward contradiction Dp$(\eta) = \beta$ for some $\eta \in \underset{\sim}{T}(T_1)[G_{\mathbb{Q}}]$, hence by $(*)_2$ and the induction hypothesis we have

$\boxtimes_2$ $\eta \trianglelefteq \nu \in \underset{\sim}{T}(T_1)[G_{\mathbb{Q}}] \Rightarrow$ Dp$(\nu) = \beta$

$\boxtimes_3$ if $I \in (\mathbb{I}_2 \backslash \mathbb{I}_0) \cap N_\eta[G_{\mathbb{Q}}]$ then for some $\rho \in (\underset{\sim}{T}(T_1)[G_{\mathbb{Q}}], \eta \triangleleft \rho$ (in fact "many ρ's) and $J \in \mathbb{I}_3 \cap N_\rho$ we have $I = \mathbf{I}_\rho$
[why? by clause (e)(iii) of the assumption, i.e. choice of $q_{T_1}, \underset{\sim}{T}(T_1)$.]

$\boxtimes_4$ if $I \in \mathbb{I}_3 \cap N_\eta[G_{\mathbb{Q}}]$ then there is ν satisfying $\eta \triangleleft \nu \in \underset{\sim}{T}(T_1)[G_{\mathbb{Q}}]$ such that $I = \mathbf{I}_\nu$
[why? if $I \in \mathbb{I}_2 \backslash \mathbb{I}_1$ by $\boxtimes_3$, if $I \in \mathbb{I}_0$ use the choice of $\underset{\sim}{T}(T_1)$.]

Now

$\boxtimes_5$ if $I \in N_\eta \cap \mathbb{I}_3$ <u>then</u> for some ν satisfying $\eta \triangleleft \nu \in \underset{\sim}{T}(T_1)[G_{\mathbb{Q}}]$ we have $\mathbf{I}_\ell = \mathbf{I}_\nu$ and
$\{\rho \in \text{Suc}_{T^\otimes}(\nu) : \text{Dp}(\rho) \geq \beta\} \neq \emptyset \bmod \mathbf{I}_\nu$.

[Why true? We can choose ν such that $I = \mathbf{I}_\nu$ and $\eta \triangleleft \nu \in \underset{\sim}{T}(T)[G_{\mathbb{Q}}]$ and $\text{Suc}_{\underset{\sim}{T}(T_1)[G_{\mathbb{Q}}]}(\nu) \neq \emptyset \bmod I_\nu$ and choose $\rho' \in \text{Suc}_{\underset{\sim}{T}(T_1)[G_{\mathbb{Q}}]}(\nu) \subseteq \underset{\sim}{T}^{\otimes}[G_{\mathbb{Q}}]$. First assume $\mathbf{I}_\nu \in \mathbb{I}_3 \backslash \mathbb{I}_0$.
Now easily

$$\underset{\sim}{A}^*_{\nu,\ell g(\eta)}[G_{\mathbb{Q}}] = \emptyset \bmod \mathbf{I}_\nu \Rightarrow \underset{\sim}{A}^*_{\nu,\ell g(\eta)}[G_{\mathbb{Q}}] \subseteq B^*_\nu \Rightarrow \rho' \notin \underset{\sim}{A}_\nu[G_{\mathbb{Q}}]$$

by the definition of $\underset{\sim}{A}^*_{\nu,\ell g(\eta)}[G_{\mathbb{Q}}]$ and β we get

$$\rho' \in \{\rho \in \text{Suc}_{T^\otimes[G_{\mathbb{Q}}]}(\nu) : \text{Dp}(\rho) \geq \text{Dp}(\nu)(= \beta)\}$$

easy contradiction.

Next assume $\mathbf{I} \in \mathbb{I}_1$. Now $\mathrm{Suc}_{\underset{\sim}{T}(T_1)[G_{\mathbb{Q}}]}(\nu)$ is a set witnessing the requirement.]
Now for $\eta \in \underset{\sim}{T}(G_1)[G_{\mathbb{Q}}]$ we check the definition of $\mathrm{Dp}(\eta) \geq \beta + 1$: clause (α) holds as $\underset{\sim}{T}(T_1)[G_{\mathbb{Q}}] \subseteq \underset{\sim}{T}^{\otimes}[G_{\mathbb{Q}}]$, clause (β) holds by the induction hypothesis, clause (γ) holds by $\boxtimes_3 + \boxtimes_4 + \boxtimes_5$ and clause (δ) by the choice of $q_{T_1}, \underset{\sim}{T}(T_1)$. So $\mathrm{Dp}(\eta) \geq \beta + 1$ contradiction.
3),4) Similar to the proof of part (2). $\square_{5.2}$

We now can deduce more implications between the UP^{ℓ}-s.

5.4 Conclusion. Assume that $\underset{\sim}{\kappa}, \underset{\sim}{\lambda}$ are purely decided $\mathbb{Q}$-names and $\mathbb{I}^{[\underset{\sim}{\kappa}]}$ is $(< \underset{\sim}{\lambda})$-closed, (which just means: if $r \Vdash$ "$\underset{\sim}{\kappa} = \kappa, \underset{\sim}{\lambda} = \lambda$ and $\underset{\sim}{\mathbb{I}} = \mathbb{I}$" where $r \in \mathbb{P}$ then $\mathbb{I}^{[\kappa]}$ is λ-closed) and $\mathbb{Q}$ satisfies the local $\underset{\sim}{\lambda}$-c.c. purely.
1) If $\mathbb{Q}$ satisfies $\mathrm{UP}^1_{\underset{\sim}{\kappa},\underset{\sim}{\lambda}}(\mathbb{I}, \mathbf{W})$ <u>then</u> $\mathbb{Q}$ satisfies $\mathrm{UP}^2_{\underset{\sim}{\kappa}}(\underset{\sim}{\mathbb{I}}, \mathbf{W})$ and $\mathrm{UP}^4_{\underset{\sim}{\kappa},\underset{\sim}{\lambda}}(\underset{\sim}{\mathbb{I}}, \mathbf{W})$, $\mathrm{UP}^3_{\underset{\sim}{\kappa},\underset{\sim}{\lambda}}(\underset{\sim}{\mathbb{I}}, \mathbf{W})$.

2) $\mathrm{UP}^1(\underset{\sim}{\mathbb{I}}, \mathbf{W})$ implies $\mathrm{UP}^5_{\underset{\sim}{\kappa},\underset{\sim}{\lambda}}(\underset{\sim}{\mathbb{I}}, \mathbf{W})$ (for $\mathbb{Q}$) <u>if</u> $\underset{\sim}{\kappa} = \underset{\sim}{\lambda}$ is a $\mathbb{Q}$-name decidably pure, $\mathbb{Q}$ satisfies the local $\underset{\sim}{\kappa}$-c.c.
3) 4-5 fill!!!

Proof. Why? By Definition 4.10 and Lemma 5.2. $\square_{5.4}$

5.5 Definition. 1) We say $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\mathbb{I}}_i, \underset{\sim}{\kappa}_i, \underset{\sim}{\mathbf{S}}_i : i < \alpha \rangle$ is $\mathrm{UP}^{4,e}(\mathbf{W}, W)$-suitable iteration <u>if</u>:

- (a) $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : i < \alpha, j < \alpha \rangle$ is an $\aleph_1 - \mathrm{Sp}_e(W)$-iteration[7]
- (b) $\underset{\sim}{\mathbb{I}}_i$ is a $\mathbb{P}_i$-name of a set of quasi order ideals with domain a cardinal in $\mathbf{V}^{\mathbb{P}_i}$ for notational simplicity <u>or</u> even just a $\mathbb{P}_{i+1}$-name of such objects (i.e., $\Vdash_{\mathbb{P}_{i+1}}$ "$\underset{\sim}{\mathbb{I}} \in \mathbf{V}^{\mathbb{P}_i}$") such that in $\mathbf{V}^{\mathbb{P}_i}, \underset{\sim}{\mathbb{I}}_i/\underset{\sim}{G}_{\mathbb{P}_i}$ which is a $\underset{\sim}{\mathbb{Q}}_i$-name, is purely decidable
- (c) $\mathbf{W} \subseteq \omega_1$ is stationary
- (d) for each $i < \alpha$, we have: $\underset{\sim}{\kappa}_i$ is a $\mathbb{P}_i$-name of a regular uncountable cardinal of $\mathbf{V}^{\mathbb{P}_i}$, purely decidable
- (e) $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ satisfies $\mathrm{UP}^4_{\underset{\sim}{\kappa}_i,\underset{\sim}{\kappa}_{i+1}}(\underset{\sim}{\mathbb{I}}_i, \underset{\sim}{\mathbf{S}}_i, \mathbf{W})$ and $\underset{\sim}{\mathbb{I}}_i$ is $\underset{\sim}{\kappa}_i$-complete set of partial order ideals (from $\mathbf{V}$) and $(\underset{\sim}{\mathbb{Q}}_i, \leq_{\mathrm{vpr}})$ is $\aleph_1$-complete (see 1.1)"

[7] the reader can fix W as the class of strongly inaccessible cardinals

(f) or $i < j$ we have $\Vdash_{P_j}$ "$\underset{\sim}{\kappa}_i \le \underset{\sim}{\kappa}_j$"

(g) $\mathbb{P}_{i+1}$ satisfies the $\underset{\sim}{\kappa}_i$-c.c., or just for every $p \in \mathbb{P}_{i+1}$ there are κ', q such that

- (α) $p \le_{\rm pr} q \in \mathbb{P}_{i+1}$
- (β) $q \Vdash_{(\mathbb{P}_{i+1}, \le_{\rm pr})}$ "$\underset{\sim}{\kappa}_{i+1} = \kappa'$"
- (γ) $\kappa \le \kappa' \in$ UReg and $\underset{\sim}{\zeta}_\varepsilon$ is a $\mathbb{P}_i$-name of an ordinal for $\varepsilon < \varepsilon^* < \kappa'$ and $p', q \restriction i \le p' \in \mathbb{P}_i$ <u>then</u> there is $q', p' \le_{\rm pr} q' \in \mathbb{P}_i$ and set a of $< \kappa'$ ordinals such that $q' \Vdash_P$ "$\underset{\sim}{\zeta}_\varepsilon \in a$ for $\varepsilon < \varepsilon^*$"

(h) [?] for any $i < \alpha$ for some $n < \omega$, if $i < j < \alpha$ and $p \in \mathbb{P}_{j+1}$ and $\kappa^{\otimes}_{\underset{\sim}{\kappa}_j}(p, \mathbb{P}_{j+1}) = \kappa$ and $\underset{\sim}{\zeta}_\varepsilon$ is a $(\mathbb{P}_i)_{\ge p \restriction i}$-name of an ordinal for $\varepsilon < \varepsilon^* < \kappa$ <u>then</u> for some q and a we have: $p \restriction i \le_{\rm pr} q \in \mathbb{P}_i$ and $q \Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\zeta}_\varepsilon \in a$ for $\varepsilon < \varepsilon^*$" where a is a set of $< \kappa$ ordinals.

2) We may write $\mathrm{UP}^{4,e}$ instead $\mathrm{UP}^{4,e}(\omega_1$, the class of strongly inaccessibles). If we omit $\underset{\sim}{\mathbf{S}}_i$ we mean $\{\aleph_1\}$.

If we omit $\underset{\sim}{\mathbb{I}}_i$, we mean "some $\underset{\sim}{\mathbb{I}}$ as required" (note that the requirements on $\underset{\sim}{\mathbb{I}}_i$ are actually on each member so the family of candidates to being $\underset{\sim}{\mathbb{I}}$ is closed under union). If we omit e we mean $e = 6$. We may omit W if it is the class of strongly inaccessible cardinals.

If we omit $\underset{\sim}{\kappa}_i$ we mean some such $\mathbb{P}_{i+1}$-name. (Can we eliminate names? Well if we use the iteration as in [Sh:f, Ch.X,§1], (RCS) no, but if we waive associativity as done here, we can).

3) We defined $\mathrm{UP}^{3,e}(\mathbf{W}, W)$-suitable iterations similarly but replace clause (e) by:

$(e)_a$ as above replacing $\mathrm{UP}^4_{\underset{\sim}{\kappa}_i, \underset{\sim}{\kappa}_{i+1}}$ by $\mathrm{UP}^3_{\underset{\sim}{\kappa}_i, \underset{\sim}{\kappa}_{i+1}}$.

4) We define a $\mathrm{UP}^\ell(\mathbb{I}, \mathbf{W}, W)$-iterations as above but with $\underset{\sim}{\mathbb{I}}_i = \underset{\sim}{\mathbb{I}}^{[\underset{\sim}{\kappa}_i]}$ [Saharon like §6, straight in successor, in limit work it out, Question: $\underset{\sim}{\kappa}_i$ pure name?]]

We can also deal with strong preservation

5.6 Definition. We say that $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : i < \alpha, j \le \alpha \rangle$ is a weak $\mathrm{UP}^4(\mathbf{W}, W)$-iteration if

(a) $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha \rangle$ is a $\aleph_1 - \mathrm{Sp}_6(W)$-iteration, $\underset{\sim}{\kappa}_i$ is $(\mathbb{P}_i, \le^{\mathbb{P}_i}_{\rm pr})$-name such that $j < i \Rightarrow \underset{\sim}{\kappa}_j \le \underset{\sim}{\kappa}_i$

(b) if $i < j \leq \alpha$ and i is non-limit, $p \in \mathbb{P}_i$ and $p \Vdash_{(\mathbb{P}_i, \leq_{\rm pr})}$ "$\underset{\sim}{\kappa}_i = \kappa_i$" then $p \Vdash_{\mathbb{P}_i}$ "$\mathbb{P}_j / G_{\mathbb{P}_i}$ satisfies $\mathrm{UP}^4_{\underset{\sim}{\kappa}_i, \underset{\sim}{\kappa}_j}(\underset{\sim}{\mathbb{I}}, \mathbf{W})$ for some $\mathbb{P}_i$-name $\underset{\sim}{\mathbb{I}}$ of $\underset{\sim}{\kappa}_i$-complete ideals".

5.7 Lemma. *Assume that* $\mathbf{W} \subseteq \omega_1$ *is stationary and* $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\mathbb{I}}_i, \underset{\sim}{\kappa}_i : i < \alpha \rangle$ *is a* $UP^4(\mathbf{W}, W)$*-suitable iteration, and* $\mathbb{P}_\alpha = \mathrm{Sp}(W) - \mathrm{Lim}(\bar{\mathbb{Q}})$ *be the limit.*
For $j \leq \alpha$ *we define* $\underset{\sim}{\kappa}^*_j$, *a* $(\mathbb{P}_j, \leq_{\rm pr})$*-name of a member of* $\mathrm{UReg}^{\mathbf{V}}$: $\underset{\sim}{\kappa}^*_j =$ *Min*$\{\kappa \in \mathrm{UReg}^{\mathbf{V}} : i < j \Rightarrow \underset{\sim}{\kappa}_i \leq \kappa$ *and* $\kappa \geq \aleph_2$ *and* $\kappa \geq j\}$.
For simplicity we can assume

⊠ *there are* $\langle \kappa'_i : i < \alpha \rangle, \mathbb{P}_i$ *satisfies the* κ'_i*-c.c. and for each* $i, i + \omega \leq \alpha$ *for some* $\Vdash_{(\mathbb{P}_i, \leq_{\rm pr})}$ "$\underset{\sim}{\kappa}_i \leq \kappa'_i$", $\Vdash_{(\mathbb{P}_{i+1}, \leq_{\rm pr})}$ "$\kappa'_i \leq \underset{\sim}{\kappa}'_{i+1}$" *(and* $\kappa^*_\beta = \beta \Rightarrow \mathbb{P}_\beta = \bigcup_{i<\beta} \mathbb{P}_i$ *(i.e.* κ^*_β *is strongly inaccessible* $> |\mathbb{P}_i|$ *for* $i < \beta$ *or change support?)*

1) For each $\beta \leq \alpha, \mathbb{P}_\beta$ *satisfies* $UP^5_{\underset{\sim}{\kappa}^*_\beta}(\mathbb{I}'_\beta, \mathbf{W})$ *for some* κ^**-complete* $\mathbb{I}'_\beta \in \mathbf{V}$.
1A) If $\gamma \leq \beta \leq \alpha, p \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{\kappa}^*_\gamma = \kappa_\gamma$", $p \in G_{\mathbb{P}_\gamma}$ *then in* $\mathbf{V}[G_{\mathbb{P}_\gamma}]$ *the forcing notions* $\mathbb{P}_\beta / G_{\mathbb{P}_\gamma}$ *satisfies* $\mathrm{UP}^5_{\underset{\sim}{\kappa}_\gamma, \underset{\sim}{\kappa}_\beta}(\underset{\sim}{\mathbb{I}}'_{\beta,\delta}, \mathbf{W})$ *for some* κ_j*-complete* $\underset{\sim}{\mathbb{I}}_{\beta,\gamma} \in \mathbf{V}^{\mathbb{P}_\gamma}$ *(so this justifies the weak in 5.6).*
2) In fact each $I \in \mathbb{I}'_\beta$ *has domain of cardinality* $\leq (\sup_{\gamma<\beta}\{\lambda^\kappa : \nVdash_{\mathbb{P}_\gamma}$ "$\neg(\exists I \in \bigcup_{i<j} \underset{\sim}{\mathbb{I}}_i)(\lambda = |\mathrm{Dom}(I)|)$ *and* $\underset{\sim}{\kappa}_i > \kappa$ *for some* $i \leq \gamma$"$\}$ *and* $|\mathbb{I}'_\beta| \leq \sum_{\gamma<\beta}(\aleph_0 + |\mathbb{P}_\gamma| +$ *sup*$\{\lambda^\kappa : \nVdash_{\mathbb{P}_\gamma}$ "$\neg(|\bigcup_{i\leq\gamma} \underset{\sim}{\mathbb{I}}_i| \leq \lambda$ *and* $\kappa_i > \kappa$ *for some* $i \leq \gamma)$"$\}$. *Similarly for* $\underset{\sim}{\mathbb{I}}_{\gamma,\beta}$.
3) Similarly for weak $\mathrm{UP}^5(\mathbf{W}, W)$*-iterations.*

5.8 Remark. 1) We can also get the preservation version of this Lemma.
2) The reader can concentrate on the case that $\underset{\sim}{\kappa}'_\ell$'s are objects and not names.

Proof. 1) We prove this by induction on β, so without loss of generality $\beta = \alpha$. For each $\gamma < \alpha$ let $\mathscr{I}_\gamma =: \{q \in \mathbb{P}_{\gamma+1} : q$ forces a value to $\underset{\sim}{\kappa}_\gamma$, called $\kappa_{\gamma,q}$ and q forces $\underset{\sim}{\mathbb{I}}_\gamma$ to be equal to a $\mathbb{P}_\gamma$-name called $\underset{\sim}{\mathbb{I}}_{\gamma,q}$ and $q \restriction \gamma$ forces that $|\underset{\sim}{\mathbb{I}}_\gamma|$ is $\leq \mu_{\gamma,q}$ but no q' such that $q \restriction \gamma \leq_{\rm pr} q' \in \mathbb{P}_\gamma$ forces a smaller bound$\}$. Let $\mu_\gamma = \sup_{q \in \mathscr{I}_\gamma} \mu_{\gamma,q}$.

Let $q \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{\mathbb{I}}_\gamma = \{\underset{\sim}{I}_{\gamma,\zeta} : \zeta < \underset{\sim}{\zeta}_{\gamma,q} \le \mu_{\gamma,q}\}$" for $q \in \mathscr{I}_\gamma$ and let $\mathscr{I}_{\gamma,\zeta} = \{q \in \mathscr{I}_\gamma :$ $\mu_{\gamma,q} > \zeta$ and $q \Vdash$ "Dom$(\underset{\sim}{I}_{\gamma,\zeta})$ is $\le \lambda_{\gamma,q,\zeta}$" and no q' such that $q \restriction \gamma \le_{\rm pr} q' \in \mathbb{P}_\gamma$ forces a smaller bound$\}$ and let $\underset{\sim}{I}_{\gamma,\zeta}$ be $\mathrm{id}_{\underset{\sim}{L}_{\gamma,q,\zeta}}$, so $\underset{\sim}{L}_{\gamma,q,\zeta}$ is a $\mathbb{P}_\gamma$-name of a $\underset{\sim}{\kappa}_{\gamma,q}$-directed quasi order on some $\lambda' \le \lambda_{\gamma,q,\zeta}$ (but $\Vdash_{\mathbb{P}_\gamma}$ "if $|\underset{\sim}{\mathbb{I}}_\gamma| \le \zeta < \mu_\gamma$ then let $L_{\gamma,\zeta}$ be trivial"). We can assume $L_{\gamma,q,\zeta}$ is a quasi order on $\lambda_{\gamma,q,\zeta}$ (putting every $\beta \in \lambda_{\gamma,q,\zeta} \setminus$ Dom$(I_{\gamma,q,\zeta})$ at the bottom.

For $q \in \mathscr{I}_\gamma$ let $L^*_{\gamma,q,\zeta}$ be $\mathrm{ap}_{\kappa_{\gamma,q,q}}(\underset{\sim}{L}_{\gamma,\zeta})$ for the forcing notion $\mathbb{P}^{[q]}_\gamma = \{p \in \mathbb{P}_\gamma : q \restriction \gamma \le^{\mathbb{P}_\gamma}_{pr} p\}$ from Definition 3.9, so it is defined in $\mathbf{V}$. So by Claim 3.10

(i) $L^*_{\gamma,q,\zeta}$ is $\kappa_{\gamma,q}$-directed partial order on $[\lambda_{\gamma,q,\zeta}]^{<\kappa_{\gamma,q}}$

(ii) $|L^*_{\gamma,q,\zeta}| \le (\lambda_{\gamma,q,\zeta})^{<\kappa_{\gamma,q}}$

(iii) $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{I}_{\gamma,\zeta} = \mathrm{id}_{\underset{\sim}{L}_{\gamma,q,\zeta}} \le_{\rm RK} \mathrm{id}_{L^*_{\gamma,q,\zeta}}$".

Let $\kappa_\beta = \sup\{\kappa_{\gamma,q} : \gamma < \beta \text{ and } q \in \mathscr{I}_\gamma\}$.
Let $\mathbb{I}^*_\beta$ be the $(<\kappa_\beta)$-closure of $\{\mathrm{id}_{L^*_{\gamma,q,\zeta}} : \gamma < \beta, q \in \mathscr{I}_\gamma, \zeta < \mu_{\gamma,q}\}$ (see Definition 3.13(1)).

Let $\bar N = \langle N_\eta : \eta \in (T^*, \mathbf{I})\rangle$ be a strict truely $(\mathbb{I}^*_\alpha, \mathbf{W})$-suitable tree of models for (χ, x), x coding enough information (so $\bar{\mathbb{Q}}, \mathbb{I}^*_\alpha, \mathbf{W}, W \in N_{\langle\rangle}$); why truely? see **?**.
$\longrightarrow$ scite{4.X} undefined

Let $\mathscr{T}_{\bar N}$ be the set of quadruples $(\gamma, q, \underset{\sim}{\nu}, \underset{\sim}{T})$ such that:

$\bigotimes_1$ $\gamma \le \alpha, q \in \mathbb{P}_\gamma, \underset{\sim}{T}$ is a $\mathbb{P}_\gamma$-name of a subtree of T^*,
$q \Vdash_{(\mathbb{P}_\gamma, \le_{\rm pr})}$ "$\underset{\sim}{\kappa}^*_\gamma = \kappa_\gamma$" and
$q \Vdash_{\mathbb{P}_\gamma}$ "$\langle N_\eta[\underset{\sim}{G}_{\mathbb{P}_\gamma}] : \eta \in (\underset{\sim}{T}, \mathbf{I} \restriction \underset{\sim}{T})\rangle$
is strictly $((\mathbb{I}^*_\alpha)^{[\kappa_\gamma]}, \mathbf{W})$-suitable tree,
$N_{\langle\rangle}[\underset{\sim}{G}_{\mathbb{P}_\gamma}] \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$ and $\underset{\sim}{\nu} \in \underset{\sim}{T}$
and $\gamma, \kappa \in N_{\underset{\sim}{\nu}}[\underset{\sim}{G}_{\mathbb{P}_\gamma}]$
and $\bar N[\underset{\sim}{G}_{\mathbb{P}_\gamma}]$ has (κ)-covering".

Now $\mathscr{T}'_{\bar N}$ is defined similarly as the set of quadruples $(\underset{\sim}{\gamma}, q, \underset{\sim}{\nu}, \underset{\sim}{T})$ such that: $\underset{\sim}{\gamma}$ is a simple $(\bar Q, W)$-named $[0,\alpha)$-ordinal, $q \in P_{\underset{\sim}{\gamma}}, \underset{\sim}{\nu}$ a $\mathbb{P}_{\underset{\sim}{\gamma}}$-name and $\underset{\sim}{\gamma} \in N_{\underset{\sim}{\nu}}[G_{\mathbb{P}_{\underset{\sim}{\gamma}}}]$. (I.e. if $\zeta < \beta, G_{\mathbb{P}_\zeta} \subseteq \mathbb{P}_\zeta$ is generic over $\mathbf{V}$ and $\zeta = \underset{\sim}{\gamma}[G_{\mathbb{P}_\zeta}]$ then $r \in q \Rightarrow \underset{\sim}{\zeta}_r[G_{\mathbb{P}_\zeta}] < \zeta$, i.e. is well defined $< \zeta$ <u>or</u> is forced $(\Vdash_{\mathbb{P}_\alpha / G_{\mathbb{P}_\zeta}})$ to be not well defined), and $q \Vdash_{\mathbb{P}_{\underset{\sim}{\gamma}}}$ "$\underset{\sim}{\nu} \in$ $\lim(T)$".

We consider the statements, for $\gamma \le \beta < \alpha$

$\boxtimes_{\gamma,\beta}$ for any $(\gamma, q, \underset{\sim}{\eta}, \underset{\sim}{T}) \in \mathscr{T}_{\bar N}$ and $\underset{\sim}{\rho}$ a $\mathbb{P}_\gamma$-name such that $q \Vdash_{P_\gamma}$ "$\underset{\sim}{\eta} \triangleleft \underset{\sim}{\rho} \in \underset{\sim}{T}$ and $\gamma \in N_{\underset{\sim}{\rho}}[\underset{\sim}{G}_\gamma]$" and $\underset{\sim}{p}'$ a $\mathbb{P}_\gamma$-name such that $q \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{p}'[G_{\mathbb{P}_\gamma}] \in N_{\underset{\sim}{\rho}}[\underset{\sim}{G}_{\mathbb{P}_\gamma}] \cap \mathbb{P}_\beta / G_{\mathbb{P}_\gamma}$ and $(\underset{\sim}{p}'[G_{\mathbb{P}_\gamma}]) \restriction \gamma \le_{pr} q$" and $\underset{\sim}{p}'[G_{\mathbb{P}_\gamma}]$ forces ($\Vdash_{(P_\beta, \le_{\rm pr})}$) a value κ^*_γ to $\underset{\sim}{\kappa}_\gamma$ (usually redundant) there is $(\beta, q', \underset{\sim}{\rho}, \underset{\sim}{T}') \in \mathscr{T}_{\bar N}$ such that $\underset{\sim}{p}' \le_{pr} q'$ (i.e., $p \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{p}'[\underset{\sim}{G}_{\mathbb{P}_\gamma}] \le_{pr} q$") and $q' \restriction \gamma = q$ and $q' \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\rho} \in \underset{\sim}{T}' \subseteq \underset{\sim}{T}$".

For simple $(\bar{\mathbb{Q}}, W)$-names $[0, \alpha)$-ordinals $\underset{\sim}{\gamma} \le \underset{\sim}{\beta}$ we define

$\boxtimes_{\underset{\sim}{\gamma},\underset{\sim}{\beta}}$ similarly $(\forall \beta < \beta^*) \forall \gamma \le \beta (\square_{\gamma,\beta})$ and $\underset{\sim}{\gamma}, \underset{\sim}{\beta}$.

Observation: If $\forall < \beta^*, \forall \gamma \le \beta (\boxtimes_{\gamma,\beta})$ and $\underset{\sim}{\gamma}, \underset{\sim}{\beta}$ are simple $\bar Q$-named $[0, \beta)$-ordinals $\Vdash \underset{\sim}{\gamma} \le \underset{\sim}{\beta} < \beta^*$ then $\boxtimes_{\underset{\sim}{\gamma}^*, \underset{\sim}{\beta}^*}$ (defined naturally).

Proof. By induction on the depth of $\underset{\sim}{\beta}$ (see 6.8, fact A).

We prove by induction on $\beta \le \alpha$ that

(*a*) $\mathbb{P}_\beta$ has pure $\underset{\sim}{\kappa}_\beta$-covering; i.e. if $\underset{\sim}{\tau}$ is a $\mathbb{P}_\beta$-name of an ordinal $< \underset{\sim}{\kappa}_\beta$ and $p \in \mathbb{P}_\beta$ then for some q and a we have: $p \le_{\rm pr} q \in \mathbb{P}_\beta, a \in \mathbf{V}$ is a set of ordinals and $q \Vdash$ "$|a| < \underset{\sim}{\kappa}_\beta$ & $\underset{\sim}{\tau} \le a$" (even over $\mathbb{P}_\gamma$)

(*b*) $\mathbb{P}_\beta$ has pure $(\aleph_1, \aleph_1)$-decidability

(*c*) for every $\gamma \le \beta$ we have $\boxtimes_{\gamma,\beta}$ (but for 5.7(3) we have to restrict ourselves to non-limit γ).

Note that for $\gamma = \beta$ the statement in clause (c) is trivial hence we shall consider only $\gamma < \beta$.

Case 1: $\beta = 0$.
Trivial.

Case 2: β a successor ordinal.

Clauses (a), (b) follows easily from clause (c) so let us concentrate on clause (c). As trivially $\boxtimes_{\gamma_0,\gamma_1}$ & $\boxtimes_{\gamma_1,\gamma_2} \Rightarrow \boxtimes_{\gamma_0,\gamma_2}$, clearly without loss of generality $\beta = \gamma + 1$.

Let $G_{\mathbb{P}_\gamma}$ be such that $q \in G_{\mathbb{P}_\gamma} \subseteq \mathbb{P}_\gamma$ and $G_{\mathbb{P}_\gamma}$ generic over $\mathbf{V}$.

Let $T' = \{\nu : \rho\hat{}\nu \in \underset{\sim}{T}[G_{\mathbb{P}_\gamma}]\}, \bar{N}' = \langle N'_\nu : \nu \in (T', \mathbf{I}')\rangle$ where $N'_\nu = N_{\rho\hat{}\nu}[G_{\mathbb{P}_\gamma}], \mathbf{I}'_\nu = \mathbf{I}^*_{\rho\hat{}\nu}$.
By 5.2 applied to $\bar{N}'$ we can find $\underset{\sim}{p}', \underset{\sim}{T}''$ as required.

Case 3: β a limit ordinal.
[Saharon note: it would be if $\underset{\sim}{\kappa}$ is a $(\mathbb{P}_\beta, \leq_{\rm pr^+})$-name, $\gamma < \beta, G_\gamma \subseteq \mathbb{P}_\gamma$ generic we should define $\underset{\sim}{\kappa}/G_\gamma$ anyhow we can use real names but then $\underset{\sim}{\kappa}^*_\gamma$ is just a $(\mathbb{P}_\gamma, \leq_{\rm pr})$-name. But if $\underset{\sim}{\kappa}_\beta$ are real cardinals no problem. But see clause (g) of definition of UP".]

Proof of Clause (a). If we use the c.c. version: easier, hardest case is $\kappa^*_\beta = \beta$, so β strongly inaccessible.
Note that we have to prove the weak version. If the property fails for $\beta, p, \underset{\sim}{\tau}$ (so $p \Vdash_{P_\beta}$ "$\underset{\sim}{\tau}$ an ordinal", etc.), then by the induction hypothesis β is minimal so definable in $(\mathscr{H}(\chi), \in)$ from $\bar{\mathbb{Q}}$ hence necessarily $\beta \in N_{<>}$ and without loss of generality $p, \underset{\sim}{\tau} \in N_{<>}$. Also without loss of generality $p \Vdash_{(\mathbb{P}_\beta, \leq_{\rm pr})}$ "$\underset{\sim}{\kappa}^*_\beta = \kappa^*_\beta$" for some $\kappa^*_\beta \in$ UReg V.

We shall now choose p_1 as in the proof of 1.26. Let $\langle \underset{\sim}{\zeta}_\varepsilon : \varepsilon < j\rangle$ be a witness for p (see 1.15 clause (F), in particular $(a)(iii)$ of (F)), so without loss of generality $j \leq \omega$. For each $\varepsilon < j$ and $\xi < \beta$, let $\underset{\sim}{a}_{\varepsilon,\xi}$ be a $\mathbb{P}_{\xi+1}$-name of a set of $< \kappa^*_\beta$ ordinals and let $\underset{\sim}{r}_{\varepsilon,\xi}$ be a $\mathbb{P}_{\xi+1}$-name of a member of $\mathbb{P}_\beta/G_{\xi+1}$ with domain $\subseteq [\xi+1, \beta)$ such that if $G_{\xi+1} \subseteq \mathbb{P}_{\xi+1}$ is generic over $\mathbf{V}$ and $\underset{\sim}{\zeta}_\varepsilon[G_{\xi+1} \cap \mathbb{P}_\xi] = \xi$, then $r = \underset{\sim}{r}_\varepsilon[\underset{\sim}{G}_{\xi+1}]$ satisfies:

- (a) if possible $p \leq_{\rm pr} (p \restriction (\xi+1)) \cup r \in \mathbb{P}_\beta$ and $(p \restriction (\xi+1)) \cup r \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\tau} \in a$" where $a = \underset{\sim}{a}_{\varepsilon,\zeta}[G_{\xi+1}] \in \mathbf{V}[G_{\xi+1}]$ is a set of $< \kappa^*_\beta$ ordinals and $\underset{\sim}{\mathbf{t}}_{\varepsilon,\zeta}[G_{\xi+1}] =$ true
- (b) if not possible $r = \underset{\sim}{r}_{\varepsilon,\xi}[G_{\xi+1}]$ is the empty function and $\underset{\sim}{a}_{\varepsilon,\xi}[G_{\xi+1}] = \emptyset$ and $\underset{\sim}{\mathbf{t}}_{\varepsilon,\xi}[G_{\xi+1}] =$ false.
  Also we can demand that:
- (c) $\underset{\sim}{\zeta}_{\varepsilon_1}[G_\xi] = \xi = \underset{\sim}{\zeta}_{\varepsilon_2}[G_\xi]$ then $\underset{\sim}{r}_{\varepsilon_1,\xi}[G_{\xi+1} \cap \mathbb{P}_\xi] = \underset{\sim}{r}_{\varepsilon_2,\xi}[G_\xi]$.

Let $\underset{\sim}{r}_\varepsilon[G_\beta] = r$ iff for some ξ we have $\underset{\sim}{\zeta}_\varepsilon[G_\beta \cap \mathbb{P}_\xi] = \xi$ and $r = \underset{\sim}{r}_{\varepsilon,\xi}[G_\beta]$, similarly we define $\underset{\sim}{a}_\varepsilon$. Let $p_1 = p \cup \bigcup\{\underset{\sim}{r}_\varepsilon : \varepsilon < j\}$. Clearly $p \leq_{\rm pr} p_1 \in \mathbb{P}_\beta$.

Next define p_2 as in 1.26: (recall (g) + (h) of Definition 5.4). I.e. for each $\varepsilon < j, \xi < \beta, G_\xi \subseteq \mathbb{P}_\xi$ generic over $\mathbf{V}, p \restriction \xi \in G_\xi, \underset{\sim}{r}_\varepsilon[G_\xi] = \xi$ there are $\underset{\sim}{r}'_{\varepsilon,\xi}, a'_{\varepsilon,\xi} \in \mathbf{V}[G_\beta]$ such that $\hat{\underset{\sim}{\mathbb{Q}}}_\xi[G_\xi] \models$ "$p_1 \restriction \{\xi\} \leq_{\text{pr}} r^1_{\varepsilon,\xi}$" and $r'_{\varepsilon,\xi} \Vdash_{\hat{\underset{\sim}{\mathbb{Q}}}_\xi}$ "$\underset{\sim}{a}_{\varepsilon,\xi} \subseteq a'_{\varepsilon,\xi}$". So really without loss of generality we have $\mathbb{P}_\xi$-names $\underset{\sim}{r}'_{\varepsilon,\xi}, \underset{\sim}{a}'_{\varepsilon,\xi}$ such that $\underset{\sim}{\zeta}_{\varepsilon_1}[G_\xi] = \xi = \underset{\sim}{\zeta}_{\varepsilon_2}[G_\xi]$ implies $\underset{\sim}{r}'_{\varepsilon_1,\xi}[G_\xi] = r'_{\varepsilon_2,\xi}[G_\xi]$ and $\underset{\sim}{a}'_{\varepsilon_1,\xi}[G_\xi] = \underset{\sim}{a}'_{\varepsilon_1,\xi}[G_\xi]$. We define $\underset{\sim}{r}'_\varepsilon, \underset{\sim}{a}'_\varepsilon$ by: $\underset{\sim}{r}'_\varepsilon[G_\beta] = r$ iff for some $\xi < \beta$ we have $\underset{\sim}{\zeta}_\varepsilon[G_\beta \cap \mathbb{P}_\xi] = \xi$ and $r = \underset{\sim}{r}_{\varepsilon,\xi}[G_\beta \cap \mathbb{P}_\xi]$ and similarly $\underset{\sim}{a}'_\varepsilon$. Now let $p_2 = p_1 \cup \{\underset{\sim}{r}'_\varepsilon : \varepsilon < j\}$ so $p_1 \leq_{\text{pr}} p_2$. We can finish as in the proof of 1.26 [fill]!!!

<u>Clause (b)</u>:

As in the proof of clause (a) without loss of generality we have $\beta, p, \underset{\sim}{\zeta} \in N_{<>}$. We define also p_1 as in the proof of clause (a) trying to force a countable bound for $\underset{\sim}{\zeta}$. Let $\langle \underset{\sim}{\zeta}^*_\varepsilon : \varepsilon < \omega \rangle$ be a witness for $p \in \mathbb{P}_\beta$ (see Definition 1.15, clause (F) in particular (a)(iii) of (F) and without loss of generality $\langle \underset{\sim}{\zeta}^*_\varepsilon : n < \omega \rangle$ belong to $N_{<>}$). We now choose by induction on n a quadruple $(\underset{\sim}{\gamma}_n, \underset{\sim}{q}_n, \underset{\sim}{\nu}_n, \underset{\sim}{t}_n)$ such that

(*i*) $\underset{\sim}{\gamma}_n$ is a simple $(\bar{\mathbb{Q}} \restriction \beta, W)$-named $[0, \beta)$-ordinal

(*ii*) $\underset{\sim}{\gamma}_0 = 0, \underset{\sim}{\gamma}_n < \underset{\sim}{\gamma}_{n+1}$

(*iii*) $\zeta_{\underset{\sim}{\zeta}^*_n} + 1 \leq \underset{\sim}{\gamma}_{n+1}$

(*iv*) $(\underset{\sim}{\gamma}_n, \underset{\sim}{q}_n, \underset{\sim}{\nu}_n, \underset{\sim}{T}_n) \in \mathscr{T}'_{\bar{N}}$

(*v*) $\underset{\sim}{q}_n = \underset{\sim}{q}_{n+1} \restriction \underset{\sim}{\gamma}_n$

(*vi*) $p \restriction \underset{\sim}{\gamma}_n \leq_{\text{pr}} \underset{\sim}{q}_n$.

No problem to carry it by the observation above.

Let $p_2 = \bigcup_{n<\omega} \underset{\sim}{q}_n \cup p$, clearly $p \leq_{\text{pr}} p_2, \underset{\sim}{q}_n \leq p_2$, and so it is enough to prove $p_2 \Vdash$ "$\underset{\sim}{\zeta} < N_{<>} \cap \omega_1$". So let $p_2 \in G_\beta$ with G_β a subset of $\mathbb{P}_\beta$ generic over $\mathbf{V}$. So there is $p_2^+ \in G_\beta$ satisfying $p \leq p_2 \leq p_2^+$ such that $p_2^+ \Vdash$ "$\underset{\sim}{\zeta} = \gamma^* < \omega_1$" and so $p \leq p_2^+$ so without loss of generality $p \leq p_2^+$ above $\xi_0 < \ldots < \xi_{m-1}$, using 1.17. There is n such that $[\underset{\sim}{\gamma}_n[G_\beta], \bigcup_{\ell<\omega} \underset{\sim}{\gamma}_\ell[G_\beta])$ is disjoint to $\{\xi_0, \ldots, \xi_{m-1}\}$, hence for some $\varepsilon < \omega$, letting $\xi = \underset{\sim}{\zeta}_\varepsilon[G_\beta]$ defining $\underset{\sim}{r}^0_{\varepsilon,\xi}[G_\beta \cap \mathbb{P}_{\xi+1}]$ we get $\underset{\sim}{\mathbf{t}}^0_{\varepsilon,\xi}$ = truth (see? 1.26) and $\xi < \underset{\sim}{\gamma}_n[G_\beta]$.

Now consider $N' = N_{<>}[G_\beta \cap \mathbb{P}_{\xi+1}]$ we know $N' \cap \omega_1 = N_{<>} \cap \omega_2$ (by clause (iv) above), and in it we have $p \restriction (\xi+1) \cup \underset{\sim}{r}^0_{\varepsilon,\xi}[G_\beta \cap p_{\xi+1}]$ forces a bound to $\underset{\sim}{\tau}$, but the condition is $\leq_{\rm pr} p_1 \leq_{\rm pr} p_2$, and it belongs to N', so the value is $< N_{<>} \cap \omega_1$ and we are done.

Clause (c):

By 5.2 it suffices to prove

$\bigotimes_2$ there are $r, \underset{\sim}{\eta}$ such that:

$\underset{\sim}{\eta}$ is a $\mathbb{P}_\beta$-name, $r \in \mathbb{P}_\beta, r \restriction \gamma = q \restriction \gamma, p' \leq_{pr} q$ and $r \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\eta} \in \lim(\underset{\sim}{T})$ and $N_{\underset{\sim}{\eta}\restriction\ell}[\underset{\sim}{G}_{\mathbb{P}_\beta}] \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$ and for every $y \in \mathbf{V} \cap \bigcup_{\ell<\omega} N_{\underset{\sim}{\eta}\restriction\ell}[G_{\mathbb{P}_\beta}]$ for some $A \in \mathbf{V} \cap \bigcup_{\ell<\omega} N_{\underset{\sim}{\eta}\restriction\ell}$ we have $|A|^{\mathbf{V}} < \underset{\sim}{\kappa}^*_\beta[G_{\mathbb{P}_\beta}]$ and $y \in A$",

We shall choose by induction on $n < \omega, \underset{\sim}{\gamma}_n, q_n, \underset{\sim}{\rho}_n, \underset{\sim}{T}_n, \underset{\sim}{k}_n, \underset{\sim}{p}_n$ such that:

(a) $(\underset{\sim}{\gamma}_n, q_n, \underset{\sim}{\rho}_n, \underset{\sim}{T}_n) \in \mathscr{T}'_{\bar{N}}$
(so $\underset{\sim}{\gamma}_n$ is a simple $\bar{\mathbb{Q}}$-named ordinal)

(b) $\underset{\sim}{k}_n$ is a $\mathbb{P}_{\underset{\sim}{\gamma}_n}$-name of a natural number

(c) $\underset{\sim}{\rho}_n$ is a $\mathbb{P}_{\underset{\sim}{\gamma}_n}$-name

(d) $q_n \Vdash_{\mathbb{P}_{\underset{\sim}{\gamma}_n}}$ "$\underset{\sim}{\rho}_n \in \underset{\sim}{T}_n \cap {}^{\underset{\sim}{k}_n}\mathrm{Ord}$"

(e) $\underset{\sim}{\gamma}_0 = \gamma$ and $\Vdash_{\bar{\mathbb{Q}}}$ "$\underset{\sim}{\gamma}_n < \underset{\sim}{\gamma}_{n+1} < \beta$ and $\underset{\sim}{\gamma}_{n+1}$ non-limt"
i.e., if $\zeta < \beta$ and $G_{\mathbb{P}_\gamma} \subseteq \mathbb{P}_\zeta$ is generic over $\mathbf{V}$ and $\zeta = \underset{\sim}{\gamma}_n[G_{\mathbb{P}_\zeta}]$ then $r \in q_n \Rightarrow \underset{\sim}{\zeta}_n[G_\zeta] < \zeta$ (i.e., is well defined $< \zeta$ or is forced to be not well defined),

(f) $q_{n+1} \restriction \underset{\sim}{\gamma}_n = q_n$

(g) $q_{n+1} \Vdash_{\mathbb{P}_{\underset{\sim}{\gamma}_{n+1}}}$ "$\underset{\sim}{\rho}_n \triangleleft \underset{\sim}{\rho}_{n+1}$, so $\underset{\sim}{k}_n < \underset{\sim}{k}_{n+1}$ and $\underset{\sim}{T}_{n+1} \subseteq \underset{\sim}{T}_n$"

(h) $\underset{\sim}{p}_n$ is a $\mathbb{P}_{\underset{\sim}{\gamma}_n}$-name, $p_0 = p, \underset{\sim}{p}_n \restriction \underset{\sim}{\gamma}_n \leq_{\rm pr} q_n$ and
$q_n \Vdash_{\mathbb{P}_{\underset{\sim}{\gamma}_n}}$ "$\underset{\sim}{p}_n \in N_{\underset{\sim}{\rho}_n}[G_{\mathbb{P}_{\underset{\sim}{\gamma}_n}}] \cap \mathbb{P}_\beta$ and $p_n \restriction \underset{\sim}{\gamma}_n \in G_{\mathbb{P}_{\underset{\sim}{\gamma}_n}}$"

(i) $q_n \Vdash_{\mathbb{P}_{\gamma_n}}$ "$\underset{\sim}{p}_n \leq^{\mathbb{P}_\beta}_{\rm pr} \underset{\sim}{p}_{n+1} \in N_{\rho_{n+1}}[G_{\mathbb{P}_{\gamma_n}}] \cap \mathbb{P}_\beta$"

(j) letting $\langle \underset{\sim}{\tau}_{\nu,\ell} : \ell < \omega \rangle$ list the $\mathbb{P}_\beta$-names of ordinals from N_ν: for $m, \ell \leq n$ we have:

$q_n \Vdash_{\mathbb{P}_{\gamma_n}}$ "$\underset{\sim}{p}_{n+1}$ force that: $a)$ if $\underset{\sim}{\tau}_{\rho_n \restriction m,\ell}$ is a countable ordinal, $m \leq \underset{\sim}{k}_n$

then it is smaller than some $\underset{\sim}{\tau}'_{\rho_n \restriction m,\ell} \in N_{\rho_{n+1}}[G_{\mathbb{P}_{\gamma_{n+1}}}]$,

a P_{γ_n}-name of a countable ordinal

$b)$ for some $A \in \mathbf{V} \cap N_{\eta_{n+1} \restriction \underset{\sim}{k}_{n+1}}$,

$|A|^{\mathbf{V}} < \kappa^*_\beta$ and $\underset{\sim}{\tau}_{\rho_n \restriction m,\ell} \in A$".

The induction is straight (later we shall show that $\bigcup_{n<\omega} q_n$ and $\underset{\sim}{\eta} = \bigcup_{n<\omega} \underset{\sim}{\rho}_n$ are as required in $\bigotimes_2$) by clauses (a) + (b) proved above.

* * *

Because we need and have $(*)_1$ or $(*)_2 + (*)_3$ below:

$(*)_1$ Assume $\leq_{pr}, \leq_{vpr}$ are equal to $\leq$ (i.e., $\Vdash_{\mathbb{P}_\beta}$ " $\leq^{\mathbb{Q}_\beta}_{vpr}$ is $\leq^{\mathbb{Q}_\beta}$ " for each $\beta < \alpha$), if $p \in \mathbb{P}_\beta, \gamma < \beta, \underset{\sim}{\tau}$ a $\mathbb{P}_\beta$-name of an ordinal <u>then</u> there are $p', \underset{\sim}{\tau}'$ such that:

(i) $\underset{\sim}{\tau}'$ is a $\mathbb{P}_\gamma$-name of an ordinal

(ii) $p \leq_{\rm pr} p' \in \mathbb{P}_\beta$ and $p \restriction \gamma = p' \restriction \gamma$

(iii) $p' \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\tau} = \underset{\sim}{\tau}'$".

[why? straight by **1.18**].

$\longrightarrow$ scite{1.16} ambiguous

[Saharon: maybe below we are stuck with $\underset{\sim}{\zeta} \in [\gamma, \beta)$, but this suffices - need to change?]

$(*)_2$ <u>old proof of clause (b)</u>: if $p \in \mathbb{P}_\beta, \gamma < \beta, \underset{\sim}{\tau}$ is a $\mathbb{P}_\beta$-name of a countable ordinal, <u>then</u> there are p', τ' such that

(i) τ' is a $\mathbb{P}_\gamma$-name of a countable ordinal

(ii) $p \le_{pr} p' \in \mathbb{P}_\beta$ and $p \restriction \gamma = p' \restriction \gamma$

(iii) $p' \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\tau} \le \tau'$",

[why $(*)_2$? let $\underset{\sim}{\zeta}$ be the following simple $\bar{\mathbb{Q}}$-named $[\gamma, \beta)$-ordinal: for $G_\zeta \subseteq \mathbb{P}_\zeta$ is generic over $\mathbf{V}$ for $\zeta \in [\gamma, \beta)$ we let $\underset{\sim}{\zeta}[G_\zeta] = \zeta$ if

(a) $p \restriction \zeta \notin G_\zeta$ <u>or</u>: for some $p' \in \mathbb{P}_\beta$ we have $p' \restriction \zeta = p \restriction \zeta$ and $\mathbb{P} \models p \le_{\rm pr} p'$ and $p' \Vdash_{\mathbb{P}_\beta / G_\zeta}$ "$\underset{\sim}{\tau} < \tau^*$" for some countable ordinal τ^*

(b) for no $\xi \in [\gamma, \zeta)$ does clause (a) hold for $\xi, G_\zeta \cap \mathbb{P}_\xi$.

Now if for some $\gamma \in [\alpha, \beta)$ we have $p \Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\zeta} = \gamma$" we are done. Also $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\zeta}[\underset{\sim}{G}_{\mathbb{P}_\alpha}]$ is well defined" as if $p \in G_\alpha \subseteq \mathbb{P}_\alpha$ and G_α is generic over $\mathbf{V}$, then for some $q \in G_\alpha$ and countable ordinal τ^* we have $q \Vdash$ "$\underset{\sim}{\tau} = \tau^*$. By the definition of $\aleph_1 - \mathrm{Sp}_e(W)$-iteration for some $\zeta \in [\gamma, \beta)$ we have $\xi \in [\zeta, \beta) \Rightarrow [p \restriction \{\xi\} \le_{pr}^{\underset{\sim}{\mathbb{Q}}_\xi} q \restriction \{\xi\}$ or $e = 4$ & $p \restriction \{\xi\}$ not defined[?]]. Define p' by: $p' \restriction \underset{\sim}{\zeta} = p \restriction \underset{\sim}{\zeta}$, and for $\xi \in [\underset{\sim}{\zeta}, \beta)$ we let $p' \restriction \{\xi\}$ be $q \restriction \{\xi\}$ if: $p \restriction \{\xi\} \le_{pr}^{\underset{\sim}{\mathbb{Q}}_\xi} q \restriction \{\xi\}$ or $e = 4$ & $p \restriction \{\xi\}$ not defined. We shall show that p' is as required, hence really $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\zeta} \in [\gamma, \beta)$ is well defined". So there is a $\mathbb{P}_{\underset{\sim}{\zeta}}$-name of $\underset{\sim}{p}'$ as appearing in the definition of $\underset{\sim}{\zeta}$ and it is, essentially, a member of $\mathbb{P}_\beta$. Now as we have finite apure support, the proof of "$\underset{\sim}{\zeta}[\underset{\sim}{G}_{\mathbb{P}_\alpha}]$ is well defined" gives $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\zeta}$ is not a limit ordinal $> \alpha$". Lastly $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\zeta}$ is not a successor ordinal $> \gamma$" is proved by the property of each $\underset{\sim}{\mathbb{Q}}_\xi$.]

$(*)_3$ <u>old proof of clause (a)</u>: if $p \in \mathbb{P}_\beta, \gamma < \beta, \underset{\sim}{\tau}$ a $\mathbb{P}_\beta$-name of a set of $< \kappa$ ordinals, <u>then</u> there are $p', \underset{\sim}{\tau}'$ such that:

(i) $\underset{\sim}{\tau}'$ is a $\mathbb{P}_\gamma$-name of a set of $< \kappa$ ordinals

(ii) $p \le_{pr} p' \in \mathbb{P}_\beta$

(iii) $p' \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\tau} \subseteq \underset{\sim}{\tau}'$.

[Why? Similar to the proof of $(*)_2$; note that it is automatic if the $\underset{\sim}{\kappa}_i$'s increase fast enough].

Finishing the induction we let $\underset{\sim}{\eta} = \bigcup_{n<\omega} \underset{\sim}{\rho}_n$ and we define $q_\omega \restriction \underset{\sim}{\gamma}_n = q_n$, $q_\omega \restriction [\bigcup_{n<\omega} \underset{\sim}{\gamma}_n, \beta)$ is defined as $\leq_{vpr}$-upper bound of $\langle \underset{\sim}{p}_m \restriction [\bigcup_{n<\omega} \gamma_n, \beta) : m < \omega \rangle$.

More formally, let γ^*, β and $G_{\gamma^*} \subseteq \mathbb{P}_{\gamma^*}$ be such that: G_{γ^*} is generic over $\mathbf{V}, \gamma^* = \bigcup_{n<\omega} \gamma^*_n, \gamma^*_n = \underset{\sim}{\gamma}_n[G_{\gamma^*}]$, let $p'_n = \underset{\sim}{p}_n[G_{\gamma^*} \cap \mathbb{P}_{\gamma^*_n}]$, let $p'_n \restriction [\gamma^*, \alpha) = \{\underset{\sim}{r}^n_\zeta : \zeta < \zeta^*_n\}$ where $\underset{\sim}{r}^n_\zeta$ is a simple $[\gamma^*, \alpha)$-named atomic condition.
Now we define $\underset{\sim}{s}^n_\zeta$, a simple $[\gamma^*, \alpha)$-named atomic condition as follows:

(a) $\zeta_{\underset{\sim}{s}^n_\zeta} = \zeta_{\underset{\sim}{r}^n_\zeta}$

(b) if $\zeta \in [\gamma^*, \alpha), G_{\gamma^*} \subseteq G_\gamma \subseteq \mathbb{P}_\gamma, G_\gamma$ generic over $V, \zeta_{\underset{\sim}{r}^n_\zeta}[G_\zeta] = \zeta$ then $\underset{\sim}{s}^n_\zeta[G_\zeta]$ is the $<^{*V[G_\zeta]}$-first elements of $\underset{\sim}{\mathbb{Q}}_\zeta[G_\zeta]$ which satisfies the following:

(∗)(α) $(\underset{\sim}{p}_n \restriction \{\zeta\}) \leq_{vpr} s$

(β) if $\langle \emptyset_{\underset{\sim}{\mathbb{Q}}_\zeta[G_\zeta]} \rangle \hat{} \langle (\underset{\sim}{p}_m \restriction \{\xi\})[G_\zeta] : m \in (n, \omega) \rangle$ has a $\leq_{vpr}$-upper bound then $\underset{\sim}{s}^n_\zeta[G_\zeta]$ is such upper bound.

Now actually such $\leq_{\rm vpr}$-upper bound actually exists, and q_ω is as required. $\square_{5.7}$

5.9 Claim. *Suppose $\mathbf{W} \subseteq \omega_1$ is stationary and $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \mathbb{I}_i, \underset{\sim}{\kappa}_i : i < \alpha \rangle$ is a $\mathrm{UP}^\ell(\mathbf{W}, W)$-suitable iteration (where $\ell \in \{4, 5\}$), and $\mathbb{P}_\alpha = \mathrm{Sp}_6(W) - \mathrm{Lim}(\bar{\mathbb{Q}})$.*

Each of the following is a sufficient condition for "$\bigcup_{\beta<\alpha} \mathbb{P}_\beta$ is a dense subset of $\mathbb{P}_\alpha$":

$(A)_\alpha$ *$\beta \in W$ is*[8] *strongly inaccessible and $\bigwedge_{\beta<\alpha}$ density $(\mathbb{P}_\beta) < \alpha$*

$(B)_\alpha$ *for every $i, \leq^{\mathbb{Q}_i}_{\rm vpr}$ is equality and $\mathbf{V} \models$ "$cf(\alpha) = \aleph_1$" or at least for some $\beta < \alpha$ we have $\Vdash_{\mathbb{P}_\beta}$ "$cf(\alpha) = \aleph_1$".*

Proof.
<u>Case 1</u>: $(A)_\alpha$
Straight by the definition of $\kappa - \mathrm{Sp}_6(W)$-iteration (see 1.15).

[8] remember W is a parameter in the definition of $\kappa - Sp_e(W)$- iteration

Case 2: $(B)_\alpha$
Follows by 5.7. $\square_{5.9}$

5.10 Conclusion. Assume $\mathbf{W} \subseteq \omega_1$ is stationary and $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\mathbb{I}}_i, \underset{\sim}{\kappa}_i : i < \alpha \rangle$ is a $\mathrm{UP}^4(\mathbf{W}, W)$-iteration with $\mathrm{Sp}_e(W)$-limit $\mathbb{P}_\alpha$.

If $\{\beta < \alpha : \bar{\mathbb{Q}} \restriction \beta$ satisfies $(A)_\beta \vee (B)_\beta$ from 5.9$\}$ is a stationary subset of α and $\beta < \alpha \Rightarrow \mathbb{P}_\beta$ satisfies the cf(α)-c.c. (e.g. has cardinality or at least density $<$ cf(α)), then $\mathbb{P}_\alpha$ satisfies the cf(α)-c.c.

Proof. Straight.

We may like to iterate up to e.g. the first inaccessible (we may below weaken $|\mathbb{P}_\beta| < \alpha$ to $\mathbb{P}_\beta$ satisfies the α-c.c. if $\mathbb{P}_\alpha = \bigcup\limits_{\beta<\alpha} \mathbb{P}_\beta$).

5.11 Claim. *[See 6.12]?? Assume*

(a) $\mathbf{W} \subseteq \omega_1$ *is stationary and* $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\mathbb{I}}_i, \underset{\sim}{\kappa}_i : i < \alpha \rangle$ *is a* $\mathrm{UP}^4(\mathbf{W}, W)$-*iteration* ($y \in \{a, b\}$) *with* $\mathrm{Sp}_6(W)$-*limit* $\mathbb{P}_\alpha$

(b) $\Vdash_{\mathbb{P}_\beta}$ *"if in* $\underset{\sim}{\mathbb{Q}}_\beta, \emptyset_{\underset{\sim}{\mathbb{Q}}_\beta} \neq p \leq_{\mathrm{vpr}} q$*, then* $p = q$ *and* $\emptyset^+_{\underset{\sim}{\mathbb{Q}}_\beta} \in \mathbb{Q}_\beta$ *is* $\neq \emptyset_{\underset{\sim}{\mathbb{Q}}_\beta}$ *but* $\emptyset_{\underset{\sim}{\mathbb{Q}}_\beta} \neq p \in \underset{\sim}{\mathbb{Q}}_\beta \Rightarrow \emptyset^+_{\underset{\sim}{\mathbb{Q}}_\beta} \leq_{\mathrm{pr}} p')$" *[?]*

(c) $S \subseteq \{\delta < \alpha : \mathrm{cf}(\delta) = \aleph_1\}$ *is stationary; for a club* E *of* $\kappa, \delta \in E \cap S$ & $\delta \leq \beta$ *implies* $\Vdash_{\mathbb{P}_\beta}$ "$(\{r \in \underset{\sim}{\mathbb{Q}}_\beta : \emptyset_{\underset{\sim}{\mathbb{Q}}_\beta} \leq_{\mathrm{vpr}} r\}, \leq_{\mathrm{vpr}})$ *is* δ^+-*directed (question directed above* $p, \emptyset <_{\mathrm{vpr}} p$

(d) $\alpha \notin W$ *is strongly inaccessible and:* $\beta < \alpha \Rightarrow \mathbb{P}_\beta < \alpha$.

Then:

(α) *forcing with* $\mathbb{P}_\alpha$ *does not collapse* α

(β) *any function from any* $\alpha(*) < \alpha$ *to ordinals in* $\mathbf{V}^{\mathbb{P}_\alpha}$ *belongs to some* $\mathbf{V}^{\mathbb{P}_\beta}$.

Proof. Clearly clause (α) follows from clause (β), so we shall prove just clause (β). If $W \cap \alpha$ is stationary, then by 5.10 we are done, so assume not and let E be a club of α disjoint to W, without loss of generality $\beta < \delta \in E \Rightarrow$ density$(\mathbb{P}_\beta) < \delta$. Suppose $p \in \mathbb{P}_\alpha$ and $p \Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{f} : \alpha(*) \rightarrow$ Ord is not in any $\mathbf{V}^{\mathbb{P}_\beta}$ for $\beta < \alpha$" where

$\alpha(*) < \alpha$.
We choose by induction on $\zeta < \alpha$ the tuple $(p_\zeta, \alpha_\zeta, \gamma_\zeta, \beta_\zeta, q_\zeta)$ such that:

- (a) $\beta_\zeta < \alpha$ is increasing continuous in ζ and $\beta_{\zeta+1} > \text{Min}(E \backslash B_\zeta)$
- (b) [?] $p_\zeta \in \mathbb{P}_{\beta_\zeta}$ is such that $\beta(*) \leq \beta < \alpha_\zeta \Rightarrow p_\zeta \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$p_\zeta \restriction \{\beta\} \neq \emptyset_{\underset{\sim}{\mathbb{Q}}_\beta}$ or $\underset{\sim}{\mathbb{Q}}_\beta = \{\underset{\sim}{\emptyset}_\beta\}$"
- (c) for $\xi < \zeta, p_\zeta \restriction \beta_\xi = p_\xi$
- (d) $p_\zeta \leq q_\zeta \in \mathbb{P}_\alpha$
- (e) $q_\zeta \Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{f}(\alpha_\zeta) = \gamma_\zeta$" but there is no $q' \in \mathbb{P}_{\beta_\zeta}$ compatible with p_ζ which forces this and $\alpha_\zeta < \alpha(*)$, of course
- (f) if $\text{cf}(\zeta) = \aleph_1$, then for some $\beta'_\zeta < \beta_\zeta$,
  $\gamma \in [\beta'_\zeta, \beta_\zeta) \Rightarrow q_\zeta \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$\emptyset_{\underset{\sim}{\mathbb{Q}}_\gamma} \leq_{\text{vpr}} = q_\zeta \restriction \{\gamma\}$"
- (g) $q_\zeta \in \mathbb{P}_{\beta_{\zeta+1}}$ and for every $\beta \in [\beta_\zeta, \alpha)$ we have

$p_{\zeta+1} \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "if $\emptyset_{\underset{\sim}{\mathbb{Q}}_\beta} <_{\text{vpr}} q_\zeta \restriction \{\beta\}$ in $\hat{\underset{\sim}{\mathbb{Q}}}_\beta$ then

$p_{\zeta+1} \restriction \{\beta\} = q_\zeta \restriction \{\beta\}$;
if not [?] then $r \in \underset{\sim}{\mathbb{Q}}_\beta$ such that $\emptyset_{\underset{\sim}{\mathbb{Q}}_\beta} <_{\text{vpr}} r$ and $\emptyset_{\underset{\sim}{\mathbb{Q}}_\beta}$ if there is none"

(see clause (b) in the assumptions of 5.11).

Having carried the definition, for some stationary $W' \subseteq S \subseteq \{\delta < \alpha : \text{cf}(\delta) = \aleph_1\}$ and $\gamma^* < \alpha$ and β' we have: $\zeta \in W' \Rightarrow \gamma_\zeta = \gamma^*$ & $\beta'_\zeta \leq \beta' < \zeta$.
As $|\mathbb{P}_{\gamma^*}| < \alpha = \text{cf}(\alpha)$, without loss of generality $\zeta \in W' \Rightarrow q_\zeta \restriction \gamma_\zeta = q^*$. Now choose $\xi < \zeta$ in W' then q_ζ, q_ξ are compatible and an easy contradiction to clause (c) (with q_ξ here playing the role of q' there). $\square_{5.11}$

Now we can refine 1.19 to the iteration theorem of this section.

5.12 Claim. *1) Suppose* $\mathbf{W} \subseteq \omega_1$ *be stationary,* F *is a function,* <u>*then*</u> *for every ordinal* α *there is* $\text{UP}^4(\mathbf{W})$*-iteration* $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \kappa_i : i < \alpha^\dagger \rangle$*, such that:*

- (a) *for every* i *we have* $\underset{\sim}{\mathbb{Q}}_i = F(\bar{\mathbb{Q}} \restriction i)$ *and* $\kappa_i = \kappa_{cc}(\mathbb{P}_i * \mathbb{Q}_i)$
- (b) $\alpha^\dagger \leq \alpha$
- (c) *either* $\alpha^\dagger = \alpha$ *or the following fails:*
  - $(*)$ $F(\bar{\mathbb{Q}})$ *is an* $(\text{Sp}_e(W) - \text{Lim}(\bar{\mathbb{Q}}))$*-name of a forcing notion forced to satisfy* $\text{UP}^4(\mathbb{I}, \mathbf{W})$ *for some* $\mathbb{I}$ κ*-complete set of ideals, where* κ *is minimal such that* $\text{Sp}_e(W) - \text{Lim}(\bar{\mathbb{Q}})$ *satisfies the* κ*-c.c*

(*d*) $\mathrm{Sp}_e(W) - \mathrm{Lim}(\bar{\mathbb{Q}})$ *does not collapse* $\aleph_1$ *and preserve stationary subsets of* $\mathbf{W}$ *(in fact it satisfies* $\mathrm{UP}^1(\mathbf{W})$.

Proof. Straight.

§6 Preservation of UP^0

Here we present alternatives to §5, i.e. to $\mathrm{UP}^4(\mathbf{W})$-iterations. In UP^6-iteration $\langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \kappa_j : k \le \alpha, i < \alpha\rangle$ the demands are weak but the κ_{i+j} may be large and it is quite similar to semi-properness (but for fewer models). In UP^6-iteration we carry with us trees.
Recall [?]

6.1 Definition. 1) We say q is $(N, \kappa, \mathbb{Q})$-semi-generic <u>if</u> q is $(N, \mathbb{Q})$-semi-generic and $q \Vdash$ "if $y \in N[\underset{\sim}{G}] \cap \mathbf{V}$ then for some $A \in N, |A|^{\mathbf{V}} < \kappa$ and $y \in A$".
2) Similarly with $\underset{\sim}{\kappa}$ instead of κ.

Remark. In 6.1(1) without loss of generality $y \in N[\underset{\sim}{G}] \cap \mathrm{Ord}$.

6.2 Lemma. *Suppose*

- (A) $\mathbb{Q}$ *is a forcing notion*
- (B) $\mathbb{I} \in N$ *is a family of ideals,* $\mathbb{I}$ *is* κ*-closed,* $\lambda \ge \aleph_2$ *(and it is natural but not needed to assume that* $\mathbb{I}$ *is* λ*-complete)*
- (C) N *is* λ*-strictly* $(\mathbb{I}, \mathbf{W})$*-suitable for* (χ, x) *as witnessed by* $\bar{N} = \langle N_\eta : \eta \in (T, \mathbf{I})\rangle, \mathbb{Q} \in N$ *(so* $N \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ *is countable,* $N = N_{<>}$ *and, of course,* x *codes* $\langle \mathbb{Q}, \mathbb{I}, \mathbf{W}\rangle$*), see Definition 4.2)*
- (D) q *is* $(N, \kappa, \mathbb{Q})$*-semi-generic*
- (E) *at least one of the following holds:*
  - (α) $|\mathrm{MAC}(\mathbb{Q})| < \lambda$
  - (β) q *is* $(\bar{N}, \kappa, \mathbb{Q})$*-semi-generic, i.e.* $(N_\eta, \kappa, \mathbb{Q})$*-semi-generic for every* $\underset{\sim}{\eta} \in T$
  - (γ) $q \Vdash$ "$\underset{\sim}{T} \subseteq T$ *is a subtree and every* $\eta \in \underset{\sim}{T}$ *for some* $\nu, \eta \trianglelefteq \nu \in \mathrm{split}(\underset{\sim}{T}, \mathbf{I}), \mathbf{I}_\eta \le_{RK} \mathbf{I}_\nu$ *and* $\eta \in \underset{\sim}{T} \Rightarrow N_\eta[\underset{\sim}{G}_{\mathbb{Q}}] \cap \omega_1 = N_\eta \cap \omega_1$".

<u>*Then*</u> $q \models$ "$N[\underset{\sim}{G}_{\mathbb{Q}}]$ *is* λ*-strictly* $(\mathbb{I}, \mathbf{W})$*-suitable for* $(\chi, \langle x, \underset{\sim}{G}_{\mathbb{Q}}\rangle)$" *(in fact, if* $\langle N_\eta : \eta \in (T, \mathbb{I})\rangle$ *was a witness for* N <u>*then*</u> $(\langle N_\eta[\underset{\sim}{G}] : \eta \in (T, \mathbf{I})\rangle$ *is a witness for* $N[\underset{\sim}{G}]$ *being* $(\mathbb{I}, \mathbf{W})$*-suitable for* (χ, x)*).*

Proof. Let $G \subseteq \mathbb{Q}$ be generic over $\mathbf{V}$ such that $q \in G$.

In $\mathbf{V}^{\mathbb{Q}}$, i.e., in $\mathbf{V}[G]$, clearly $N_\eta[G] \prec (\mathscr{H}(\chi)^{\mathbf{V}^{\mathbb{Q}}}, \in, <^*_\chi)$ (see [Sh:f, III,2.11,p.104], $N_\eta[G]$ is countable (trivially) and $N_{\eta\restriction k}[G] \prec N_\eta[G]$ for $k \leq \ell g(\eta)$.

As q is $(N, \kappa, \mathbb{Q})$-semi-generic and $N = N_{\langle\rangle}$, clearly q is $(N_{\langle\rangle}, \mathbb{Q})$-semi-generic. First assume $|\mathrm{MAC}(\mathbb{Q})| < \lambda$: now $\mathbb{Q} \in N$ hence $\mathrm{MAC}(\mathbb{Q}) \in N$ hence $|\mathrm{MAC}(\mathbb{Q})| \in N$, and as $N_{\langle\rangle} cap \lambda <_\lambda N_\eta \cap \lambda$ clearly $N_\eta \cap |\mathrm{MAC}(\mathbb{Q})| = N_{\langle\rangle} \cap |\mathrm{MAC}(\mathbb{Q})|$ hence also $N_\eta \cap$ MAC$(\mathbb{Q}) = N_{\langle\rangle} \cap$ MAC$(\mathbb{Q})$ hence every $\mathbb{Q}$-name $\underset{\sim}{\tau} \in N_\eta$ of an ordinal belongs to $N_{\langle\rangle}$ (essentially). Hence q is $(N_\eta, \kappa, \mathbb{Q})$-semi-generic, so $q \Vdash_{\mathbb{Q}}$ "$N_\eta[G] \cap \omega_1 = N_\eta \cap \omega_1 = N \cap \omega_1$", (even $N_{\eta\restriction\ell}[G] <_\lambda N_\eta[G]$). So q is $(N_\eta, \kappa, \mathbb{Q})$-semi-generic for any $\eta \in T$ and even $\eta \in (\lim T)^{\mathbf{V}}$ or $\eta \in (\lim T)^{\mathbf{V}^{\mathbb{Q}}}$.

This almost shows that $\langle N_\eta[G] : \eta \in (T, \mathbf{I})\rangle$ is a witness to $N[G]$ being λ-strictly $(\mathbb{I}, \mathbf{W})$-suitable for (χ, x).

The missing point is clause $(e)^-$ of Definition 4.2, that is, that there may be $\eta \in T$, and $J \in \mathbb{I} \cap N_\eta[G]$ such that $J \notin N_\eta$. So there is a $\mathbb{Q}$-name $\underset{\sim}{\tau} \in N_\eta$ satisfying $J = \underset{\sim}{\tau}[G]$; choose a $\underset{\sim}{\tau}$ like that with $\min\{|Y| : Y \in N_\eta$ and $\underset{\sim}{\tau}[G] \in Y\}$ minimal, and let the set Y be $\{J_i : i < \alpha\}$ (without loss of generality $Y \subseteq \mathbb{I}$), so without loss of generality $\langle J_i : i < \alpha\rangle$ belongs to N_η: by the minimality of $|Y| = |\alpha|$ and q being $(N_\eta, \kappa, \mathbb{Q})$-semi-generic we have $\alpha < \kappa$.

So as $\{J_i : i < \alpha\} \in N_\eta \cap \mathbb{I}$ and $\mathbb{I}$ is κ-closed there is $J \in N_\eta \cap \mathbb{I}$ such that $\bigwedge_{i<\alpha} J_i \leq_{\mathrm{RK}} J$ hence the set $\{\nu : \eta \lhd \nu \in$ split(T) and $J \leq_{\mathrm{RK}} \mathbf{I}_\nu\}$ is a front of $T^{[\eta]}$. So $\langle N_\eta[G] : \eta \in (T, \mathbb{I})\rangle$ is a witness for "$N[G]$ is λ-strictly $\mathbb{I}$-suitable" for $(\lambda, \chi, \langle x, G\rangle)$ (see Definition 4.2) so by 4.6 we know that $N[G]$ is $\mathbb{I}$-suitable.
Now if the second phrase of (E) holds, the proof is similar and if the third holds, we can prove that without loss of generality the second holds. $\square_{6.2}$

The proof suggests some definitions, but we first consider:

6.3 Definition. 1) For a forcing notion $\mathbb{Q}$, family $\mathbb{I}$ of ideals and cardinal and $\mathbb{Q}$-names $\underset{\sim}{\lambda}$ of a cardinal κ and stationary $\mathbf{W} \subseteq \omega_1$, we say $\mathbb{Q}$ satisfies $\mathrm{UP}^6(\mathbb{I}, \kappa, \underset{\sim}{\lambda}, \mathbf{W})$ or $\mathrm{UP}_{\kappa,\underset{\sim}{\lambda}}(\mathbb{I}, \mathbf{W})$ <u>if</u>:

(∗) for every χ regular large enough, $p \in \mathbb{Q}$ and N a strictly $(\mathbb{I}^{[\kappa]}, \mathbf{W})$-suitable model for χ satisfying $\{p, \mathbb{Q}, \kappa, \underset{\sim}{\lambda}\} \in N$, <u>there is</u> q satisfying $p \leq_{\mathrm{pr}} q \in \mathbb{Q}$ such that q is $(N, \mathbb{Q})$-semi-generic and $q \Vdash_{\mathbb{Q}}$ "$N[\underset{\sim}{G}_{\mathbb{Q}}]$ is strictly $\mathbb{I}^{[\underset{\sim}{\lambda}]}$-suitable model for χ, λ.

1A) We say "q is $(N, \underset{\sim}{\mathbb{I}}, \mathbb{Q})$-semi$_6$ generic" if q is $(N, \mathbb{Q})$-semi genric and $q \Vdash$ "$N[\underset{\sim}{G}_{\mathbb{Q}}]$ is strictly $\underset{\sim}{\mathbb{I}}$-suitable".

2) In part (1), if we omit λ, we mean $\lambda =$ Max$\{\lambda : \mathbb{I}$ is λ-complete$\}$, so λ is regular $> \aleph_1$.
3) For $\mathbb{Q}, \mathbb{I}, \mathbf{W}, \kappa, \underset{\sim}{\lambda}$ as in part (1) and $\underset{\sim}{\theta}$ a $\mathbb{Q}$-name of a cardinal; we say that $\mathbb{Q}$ satisfies $\mathrm{UP}^6_{\kappa,\underset{\sim}{\lambda},\underset{\sim}{\theta}}(\mathbb{I}, \mathbf{W})$ <u>if</u>:

(∗∗) if $\mathbb{I}^+$ is a set of partial orders ideal extending $\mathbb{I}, \chi$ large enough, $p \in \mathbb{Q} \cap N$ and N a strictly $(\mathbb{I}, \mathbf{W})$-suitable model for $\chi, \{p, \mathbb{Q}, \kappa, \underset{\sim}{\lambda}, \underset{\sim}{\theta}\} \in N$, <u>then</u> there is q satisfying $p \le_{pr} q \in \mathbb{Q}$ such that q is $(N, \mathbb{I}^{[\underset{\sim}{\lambda}]} \cup (\mathbb{I}^+ \backslash \mathbb{I})^{[\underset{\sim}{\theta}]}, \mathbb{Q})$-semi$_6$-generic (see (1A) above).

4) We say $\mathbb{Q}$ satisfies $\mathrm{UP}^5_{\kappa,\underset{\sim}{\lambda}}(\mathbb{I}, \mathbf{W})$ <u>if</u>: for any $(\mathbb{I}^{[\kappa]}, \mathbf{W})$-suitable tree $\langle N_\eta : \eta \in (T, \mathbb{I})\rangle$ of models and $p \in N_{<>}$ there are $q, \underset{\sim}{T}$ such that $p \le_{\mathrm{pr}} q \in \mathbb{Q}, q \Vdash$ "$\underset{\sim}{T} \subseteq T$ is a subtree and $\langle N_\eta[G_P] : \eta \in \underset{\sim}{T}\rangle$ is a $(\mathbb{I}^{[\lambda]}, \mathbf{W})$-suitable tree of models and $N_\eta[G_{\mathbb{Q}}] \cap \omega_1 = N_{<>} \cap \omega_1$. [Saharon but **?!**]
$\longrightarrow$ scite{4.X} undefined

Some variants of this Definition are equivalent by the following claim.

6.4 Claim. *1) If κ is regular uncountable and $\mathbb{Q}$ satisfies the κ-c.c. <u>then</u>: $q \in \mathbb{Q}$ is $(N, \kappa, \mathbb{Q})$-semi-generic <u>iff</u> $q \in \mathbb{Q}$ is $(N, \mathbb{Q})$-semi-generic.*
2) If N is $(\mathbb{I}, \mathbf{W})$-suitable for (χ, x) (see Definition 4.3), $\mathbb{I}$ is λ-complete, λ is regular and $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$, <u>then</u> there is a λ-strictly $(\mathbb{I}, \mathbf{W})$-suitable N' for (χ, x) such that $N \prec N'$ and $N' \cap \omega_1 = N \cap \omega_1$.
3) If q is $(N, \kappa, \mathbb{Q})$-semi generic, $|\mathrm{MAC}(\mathbb{Q})| < \lambda, \mathbb{I} \in N$ is κ-closed λ-complete and N is $\mathbb{I}$-suitable <u>then</u> q is $(N, \mathbb{I}^{[\lambda]}, \mathbb{Q})$-semi$_6$ generic.
4) If $|\mathrm{MAC}(\mathbb{Q})| < \lambda$ and $\mathbb{Q}$ is $\mathrm{UP}^0(\mathbb{I}^{[\kappa]}, \mathbf{W})$ <u>then</u> $\mathbb{Q}$ is $\mathrm{UP}^6_{\kappa,\lambda}(\mathbb{I}, \mathbf{W})$. Similarly for λ $\mathbb{Q}$-name such that for a dense set of p we have $p \Vdash \underset{\sim}{\lambda} = \lambda$ and $\mathrm{MAC}(\mathbb{Q}_{\ge p}) < \lambda$.

Proof. Straight.
1) Reflect.
2) Use the partition theorem 2.11.
3) By 6.2 using possibility (A).
4) By 6.2, too. $\square_{6.4}$

6.5 Claim. *1) If $\mathbb{Q}$ satisfies $\mathrm{UP}^6_{\kappa,\underset{\sim}{\lambda}}(\mathbb{I}, \mathbf{W})$ and $\mathbf{W}_1 \subseteq \mathbf{W}, \kappa_1 \le \kappa$ and $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\lambda} \le \underset{\sim}{\lambda}_1$", <u>then</u> $\mathbb{Q}$ satisfies $\mathrm{UP}^6_{\kappa_1,\underset{\sim}{\lambda}_1}(\mathbb{I}, \mathbf{W}_1)$.*

2) If $\mathbb{Q}_0$ *is a forcing notion satisfying* $\mathrm{UP}^6_{\kappa_0,\underset{\sim}{\kappa}_1}(\mathbb{I},\mathbf{W})$ *and* $\underset{\sim}{\mathbb{Q}}_1$ *is a* $\mathbb{Q}$*-name of a forcing notion satisfying* $\mathrm{UP}^6_{\underset{\sim}{\kappa}_1,\underset{\sim}{\kappa}_2}(\mathbb{I},\mathbf{W}_1)$*, then* $\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1$ *is a forcing notion satisfying* $\mathrm{UP}^6_{\kappa_0,\underset{\sim}{\kappa}_2}(\mathbb{I},\mathbf{W}_1)$.

3) If $\mathbb{Q}$ *satisfies* $\mathrm{UP}^6_{\kappa,\underset{\sim}{\lambda},\underset{\sim}{\theta}}(\mathbb{I},\mathbf{W}_1)$ *and* $\Vdash_{\mathbb{Q}} \underset{\sim}{\theta} \le \theta$ *and* $\mathbb{I} \subseteq \mathbb{I}^+$ *an* $\mathbb{I}^+ \backslash \mathbb{I}$ *is* θ*-complete then* $\mathbb{Q}$ *satisfies* $\mathrm{UP}^6_{\kappa,\underset{\sim}{\lambda}}(\mathbb{I}',\mathbf{W}_1)$.

6.6 Definition. 1) We say that $\bar{\mathbb{Q}} = \langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : j \le \alpha \text{ and } i < \alpha\rangle$ is a $\mathrm{UP}^{6,e}(\mathbb{I},\mathbf{W},W)$-suitable iteration (with $e = 6$ if not mentioned explicitly) if:

- (a) $\langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i : j < \alpha, i < \alpha\rangle$ is an $\aleph_1 - \mathrm{Sp}_e(W)$-iteration
- (b) $\mathbb{I}$ is a set of partial order ideals such that $\mathbb{I}^{[\kappa]}$ is κ-closed for any regular $\kappa, \aleph_2 \le \kappa \le |\mathbb{P}_\alpha|^+$
- (c) $\mathbf{W} \subseteq \omega_1$ is stationary
- (d) for each $i < \alpha$ we have: $\underset{\sim}{\kappa}_i$ is a $\mathbb{P}_i$-name of a regular cardinal $> \aleph_1$ in $\mathbf{V}$ [purity?],
- (e) (i) for $i < j$ we have $\Vdash_{\mathbb{P}_j}$ "$\underset{\sim}{\kappa}_i \le \underset{\sim}{\kappa}_j$"
  - (ii) if $\delta \le \alpha$ is a limit ordinal, then $\mathbb{P}_\delta$ satisfies the local $\underset{\sim}{\kappa}_\delta$-c.c. purely [and (?) if $\nVdash \underset{\sim}{\kappa}_\delta \ne \kappa$ then some κ-complete ideal on κ belongs to $\mathbb{I}$]
- (f) $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ satisfies $\mathrm{UP}^6_{\underset{\sim}{\kappa}_i,\underset{\sim}{\kappa}_{i+1}}(\underset{\sim}{\mathbb{I}}_i,\mathbf{W})$ and $(\underset{\sim}{\mathbb{Q}}_i, \le_{\mathrm{vpr}})$ is $\aleph_1$-complete".

2) We say $\langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : j \le \alpha, i < \alpha\rangle$ is a $\mathrm{UP}^4(\mathbb{I},\mathbf{W})$-iteration if:

$(a) - (e)$ as above
(f) $\mathbb{Q}_i$ satisfies $\mathrm{UP}^5_{\underset{\sim}{\kappa}_i,\underset{\sim}{\kappa}_{i+1}}(\mathbb{I},\mathbf{W})$.

6.7 Definition. 1) We say that $\langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : j \le \alpha \text{ and } i < \alpha\rangle$ is a weak $\mathrm{UP}^{6,e}(\mathbb{I},\mathbf{W})$-iteration if (when $e = 6$ we may omit it):

- (a) $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha\rangle$ is an $\aleph_1$-$\mathrm{SP}_e(W)$-iteration (and, of course, $\mathbb{P}_\alpha = \mathrm{Sp}_e(W)$-$\mathrm{Lim}_\kappa(\bar{\mathbb{Q}})$)
- (b) $\mathbb{I}$ is a set of partial order ideals

(c) $\mathbf{W} \subseteq \omega_1$ stationary

(d) $\underset{\sim}{\kappa}_j$ is a $\mathbb{P}_j$-name of a member of $\mathrm{RCar}^V \backslash \omega_2$, increasing with j

(e) for $i < j \leq \alpha, i$ nonlimit we have

$$\Vdash_{P_i} \text{“}\mathbb{P}_j/\mathbb{P}_i \text{ satisfies the } \mathrm{UP}_{\underset{\sim}{\kappa}_i,\underset{\sim}{\kappa}_j}(\mathbb{I},\mathbf{W})\text{”}$$

(f) $\Vdash_{\mathbb{P}_i}$ “$(\underset{\sim}{\mathbb{Q}}_i, \leq_{\mathrm{vpr}})$ is $\aleph_1$-complete”.

2) We say $\bar{\mathbb{Q}} = \langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : j \leq \alpha, i < \alpha \rangle$ is a $\mathrm{UP}^5(\mathbb{I}, \mathbf{W})$-iteration if:

$(a) - (d), (f)$ as above

(g) if $i < j \leq \alpha, i$ non limit we have
$\Vdash_{\mathbb{P}_i}$ “$\mathbb{P}_j/\underset{\sim}{G}_{\mathbb{P}_i}$ satisfies $\mathrm{UP}^5_{\underset{\sim}{\kappa}_i,\underset{\sim}{\kappa}_j}(\mathbb{I},\mathbf{W})$.

6.8 Claim. *1) If $\langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : j \leq \alpha$ and $i < \alpha \rangle$ is a $\mathrm{UP}^6(\mathbb{I}, \mathbf{W}, W)$-iteration, then it is a weak $\mathrm{UP}^6(\mathbb{I}, \mathbf{W}, W)$-iteration, moreover, in clause (e) also limit i is O.K.*
2) Assume $\bar{\mathbb{Q}} = \langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : j \leq \alpha, i < \alpha \rangle$ is an $\aleph_1 - \mathrm{Sp}_e(W)$-iteration

(a) *if α is a limit ordinal and $\beta < \alpha \Rightarrow \bar{\mathbb{Q}} \restriction \beta = \langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : j \leq \beta, i < \beta \rangle$ is a weak $\mathrm{UP}^6(\mathbb{I}, \mathbf{W})$-iteration and $\underset{\sim}{\kappa}_\alpha =$ sup$\{\kappa_j : j < \alpha\}$, then $\bar{\mathbb{Q}}$ is a weak $\mathrm{UP}^6(\mathbb{I}, \mathbf{W})$-iteration*

(b) *if $\alpha = \beta + 1, \bar{\mathbb{Q}} \restriction \beta$ is a weak $\mathrm{UP}^6(\mathbb{I}, \mathbf{W})$-iteration and in $\mathbf{V}^{\mathbb{P}_\beta}, \mathbb{Q}_\beta$ satisfies $\mathrm{UP}^6_{\underset{\sim}{\kappa}_\beta,\underset{\sim}{\kappa}_{\beta+1}}(\mathbb{I}, \mathbf{W})$ then $\bar{\mathbb{Q}}$ is a weak $\mathrm{UP}^6(\mathbb{I}, \mathbf{W})$-iteration.*
[Saharon - compare with 6.9 and 1.27(2)]

Proof. Let $\gamma \leq \beta \leq \alpha$ and we need

$\boxtimes_{\gamma,\beta}$ Assume $G_\gamma \subseteq \mathbb{P}_\gamma$ is generic over $\mathbf{V}, \kappa_i = \underset{\sim}{\kappa}_i[G_i]$ and let $N \in \mathbf{V}[G_\gamma], N$ is strictly $\mathbb{I}^{[\kappa]}$-suitable, $N \cap \omega_1 \in \mathbf{W}$ (so $N \prec (\mathscr{H}(\chi), \in)$ is countable) and $p \in \mathbb{P}_\beta/G_\gamma$ and $p \in \mathbb{P}_\beta \cap N$ and $\bar{\mathbb{Q}}, \beta, \gamma \in N$. Then we can find q satisfying $p \leq_{\mathrm{pr}} q \in \mathbb{P}_\beta/G_\gamma$ and $q \Vdash_{\mathbb{P}_\beta/G_\gamma}$ “$N[\underset{\sim}{G}_\beta] \cap \omega_1 = N \cap \omega_1$ and $N[\underset{\sim}{G}_\beta]$ is $\mathbb{I}^{[\underset{\sim}{\kappa}_\beta]}$-suitable”.
Without loss of generality [??] p forces a value to $\underset{\sim}{\kappa}_\beta$ moreover $\kappa_\beta = \kappa_{\underset{\sim}{\kappa}_\beta}(p, \mathbb{P}_\beta)]$.

Naturally, we prove this by induction on β (for all γ). The case $\gamma = \beta$ holds trivially so assume $\gamma < \beta$. If $\beta = 0$, we have nothing to prove. If β is a successor ordinal say $\gamma_1 + 1$ so $\gamma \leq \gamma_1$, now we use first $\boxtimes_{\gamma,\gamma_1}$ and then the demand on $\underset{\sim}{Q}_{\gamma_1}$ in definition 6.7, in clause (f).

So from now on we shall assume that β is a limit ordinal. As in the proof of 5.7 we can note

<u>Fact A</u>: If $\underset{\sim}{\gamma}_1 \leq \underset{\sim}{\gamma}_2$ are simple $\bar{\mathbb{Q}}$-named $[0,\beta)$-ordinals then $\boxtimes_{\underset{\sim}{\gamma}_1,\underset{\sim}{\gamma}_2}$ holds.

<u>Proof of the fact</u>: Here we use "$e = 6$" rather than "$e = 4$". On $\mathbb{P}_{\underset{\sim}{\zeta}}$ see Definition 1.15(F)(g). We prove it by induction on the depth of $\underset{\sim}{\zeta}_2$, see Definition 1.7(5). So we are given $G_{\underset{\sim}{\zeta}_2} \subseteq \mathbb{P}_{\underset{\sim}{\zeta}_1}$ generic over $\mathbf{V}$ and in particular let $\zeta_1 = \underset{\sim}{\zeta}_2[G_{\underset{\sim}{\zeta}_1}]$, (so it is simpler to say that $G_{\zeta_1} \subseteq \mathbb{P}_{\zeta_1}$ is generic over $\mathbf{V}, \underset{\sim}{\zeta}_1[G_{\zeta_1}] = \zeta_1$). Let $\kappa_1 = \underset{\sim}{\kappa}_{\zeta_1}[G_{\underset{\sim}{\zeta}_1}]$ and we are also given N which is strictly $\mathbb{I}^{[\kappa]}$-suitable, $p \in \mathbb{P}_{\underset{\sim}{\zeta}_2}/G_{\underset{\sim}{\zeta}_1}, p \in N, \{\bar{\mathbb{Q}}, \underset{\sim}{\zeta}_1, G_{\underset{\sim}{\zeta}_1}\} \in N, \underset{\sim}{\zeta}_2 \in N$. We have to find $q \in \mathbb{P}_{\underset{\sim}{\zeta}_2}/G_{\underset{\sim}{\zeta}_1}$ such that $p \leq_{\text{pr}} q, q$ is $(N, \mathbb{P}_{\underset{\sim}{\zeta}_2}/\mathbb{P}_{\underset{\sim}{\zeta}_1})$-generic and $q \Vdash N[G_{\mathbb{P}_{\underset{\sim}{\zeta}_2}/G_{\underset{\sim}{\zeta}_1}}]$ is strictly $\mathbb{I}^{[\kappa_2]}$-suitable.

If the depth of $\underset{\sim}{\zeta}_2$ is 0, then $\Vdash$ "$\underset{\sim}{\zeta}_2 = \zeta_2$" and we can use $\boxtimes_{\zeta_1,\zeta_2}$. So assume the depth of $\underset{\sim}{\zeta}_2$ is > 0, and so for some γ^* and a sequence $\langle \underset{\sim}{\zeta}_{2,\varepsilon} : \varepsilon < \varepsilon^* \rangle$ of simple $\bar{\mathbb{Q}}$-named $[\text{Max}\{\gamma^*, \gamma\}, \beta)$-ordinals and $\mathbb{P}_{\gamma^*}$-name $\underset{\sim}{\varepsilon}$ we have $\Vdash_{\bar{\mathbb{Q}}}$ "$\underset{\sim}{\zeta}_2 = \underset{\sim}{\zeta}_{2,\underset{\sim}{\varepsilon}}$". So without loss of generality $\{\gamma^*, \langle \underset{\sim}{\zeta}_{2,\varepsilon} : \varepsilon < \varepsilon^* \rangle, \underset{\sim}{\varepsilon}\} \in N$. Let $\underset{\sim}{\zeta}_1 = \text{Max}\{\gamma, \gamma^*, \underset{\sim}{\zeta}_1\}, \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\zeta}_1 \leq \underset{\sim}{\zeta}'_1 \leq \zeta_2$".

Now clearly $\underset{\sim}{\zeta}'_1$ is a simple $\bar{\mathbb{Q}}$-named $[0,\beta)$-ordinal, $\Vdash_{\bar{\mathbb{Q}}}$ "$\underset{\sim}{\zeta}_1 \leq \underset{\sim}{\zeta}'_1 \leq \underset{\sim}{\zeta}'_2$" and $\boxtimes_{\underset{\sim}{\zeta}_1,\underset{\sim}{\zeta}'_1}$ & $\boxtimes_{\underset{\sim}{\zeta}'_1,\underset{\sim}{\zeta}_2} \Rightarrow \boxtimes_{\underset{\sim}{\zeta}_1,\underset{\sim}{\zeta}_2}$ and $\boxtimes_{\underset{\sim}{\zeta}_1,\underset{\sim}{\zeta}_2}$ easily holds (by the cases proved above) so it is enough to prove $\boxtimes_{\underset{\sim}{\zeta}'_1,\underset{\sim}{\zeta}_2}$. This just means that without loss of generality $\zeta_1 \leq \gamma^*$ and even $\underset{\sim}{\zeta}_1 = \gamma^*$. Now $\underset{\sim}{\varepsilon}[G_{\underset{\sim}{\zeta}_1}] \in N$ so we use the induction hypothesis to get the desired q. $\square_{\text{fact}}$

<u>Fact B</u>: If $\underset{\sim}{\xi}$ is a simple $\bar{\mathbb{Q}}$-named $[0,\beta)$-ordinal, $p \in \mathbb{P}_\beta$ and $\underset{\sim}{\tau}$ is a $\mathbb{P}_\beta$-name of a countable ordinal, <u>then</u> there are $\underset{\sim}{\varepsilon}$ and q such that:

$(*)(i)$ $\mathbb{P}_\beta \models p_{\text{pr}} q$

(ii) $q \restriction \underset{\sim}{\xi} = p \restriction \underset{\sim}{\xi}$

(iii) $\underset{\sim}{\varepsilon}$ is a $\mathbb{P}_{\underset{\sim}{\xi}}$-name of a countable ordinal

(iv) $q \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\tau} < \underset{\sim}{\varepsilon}$".

Proof. Let $\langle \underset{\sim}{\zeta}_n : n < \omega \rangle$ be a witness for p, so each $\underset{\sim}{\zeta}_n$ is a simple $\bar{\mathbb{Q}}$-named $[0,\beta)$-ordinal. For each n we define a $P_{\underset{\sim}{\zeta}_n}$-name of $\mathbf{t}_n$ of a truth value and $\underset{\sim}{r}_n$ of a member of $\mathbb{P}_\beta, \underset{\sim}{r}_n = \underset{\sim}{r}_n \restriction [\underset{\sim}{\zeta}_n, \beta)$, as follows: if $G^n \subseteq \mathbb{P}_{\underset{\sim}{\zeta}_n}$ is generic over $\mathbf{V}$ and $\neg(\underset{\sim}{\zeta}_n[G^n] \geq \underset{\sim}{\xi}[G^n])$, and there are $q \in \mathbb{P}_\beta/G^n$ and $\varepsilon < \omega_1, \mathbb{P}_\beta \Vdash$ "$p \leq_{\rm pr} q$" and $q \Vdash_{\mathbb{P}_\beta/G^n}$ "$\underset{\sim}{\tau} < \varepsilon$", <u>then</u> $\underset{\sim}{\mathbf{t}}_n[G^n] =$ truth and then $\underset{\sim}{r}_n[G^n]$ is $q \restriction [\underset{\sim}{\zeta}_n, \beta)$ for some such q otherwise $\underset{\sim}{\mathbf{t}}_n[G^n] =$ false, $\underset{\sim}{r}[G^n] = \emptyset$. Let r'_n be the following $\mathbb{P}_{\underset{\sim}{\zeta}_n}$-name:
for $G^n \subseteq \mathbb{P}_{\underset{\sim}{\zeta}_n}$ <u>if</u>:

(a) $\underset{\sim}{\mathbf{t}}_n[G^n] =$ truth and

(b) for no $m < \omega$, for some $r \in G^n$ forces a value to $\underset{\sim}{\zeta}_m$, say $\zeta_m, \zeta_m < \zeta_n[G^n] \vee (\zeta_m = \underset{\sim}{\zeta}_n[G^n]\ \ \&\ \ m < n)$

<u>then</u> $\underset{\sim}{\zeta}'_n[G^n]$ is $\underset{\sim}{r}_n$, otherwise $\underset{\sim}{\zeta}'_n[G^n] = \emptyset$. Let $p^* = p \cup \bigcup\limits_n \underset{\sim}{r}'_n$, easily $q \in \mathbb{P}_\beta$ and $p \leq_{\rm pr} p^*$.
Let $\underset{\sim}{\xi}_n =$ Min$\{\underset{\sim}{\xi} + 1, \underset{\sim}{\zeta}_0 + 1, \ldots, \underset{\sim}{\zeta}_{n-1} + 1\}$, $\underset{\sim}{\xi}_n$ is a simple $\bar{\mathbb{Q}}$-named $[0,\beta)$-ordinal, $\underset{\sim}{\xi}_0 = \underset{\sim}{\xi}, \underset{\sim}{\zeta}_n < \underset{\sim}{\xi}_{n+1}, \underset{\sim}{\xi}_n \leq \underset{\sim}{\xi}_{n+1}$. Let N be a strictly $(\mathbb{I}, \mathbf{W})$-suitable model for χ such that $\{\bar{\mathbb{Q}}, p, \langle \underset{\sim}{\zeta}_n, \underset{\sim}{\xi}_n : n < \omega \rangle, \langle \underset{\sim}{r}_n, \underset{\sim}{r}'_n : n < \omega\}$ belongs to N, it exists by 4.5. Now we choose q_n by induction on $n < \omega$ such that

$$q_n \in \mathbb{P}_{\underset{\sim}{\zeta}_n} \text{ is } (N, \mathbb{I}^{[\kappa_{\underset{\sim}{\zeta}_n}]}, P_{\underset{\sim}{\zeta}_n}) - \text{semi}_6 \text{ generic}$$

$$p^* \restriction \zeta_n, \beta \leq_{\rm pr} q_n$$

$$q_{n+1} \restriction \underset{\sim}{\zeta}_n = q_n.$$

This is possible by Fact A. Now (see 1.26) $q^* =: \bigcup_{n<\omega} q_n \cup p^*$ belongs to $\mathbb{P}_\beta, p^* \le_{\rm pr} q^*$. It is enough to show that $q^* \Vdash \underset{\sim}{\tau} \in N \cap \omega_1$, assume not so there is $r, q^* \le r \in \mathbb{P}_\beta$ such that r forces a value $\varepsilon^* \in \omega_1 \backslash (N \cap \omega_1)$ to $\underset{\sim}{\tau}$.

By 1.17(1) without loss of generality $q^* \le r$ above $\{\Upsilon_1, \dots, \Upsilon_m\}$ for some $m < \omega$ and $\Upsilon_\ell < \beta$. There are r', k, ξ_k such that: $r \le r', \xi_k < \omega, r' \restriction [\xi_k, \beta) = r \restriction [\xi_k, \beta), r'$ forces $\underset{\sim}{\xi}_k$ is ξ_k and for each $\ell = 1, 2, \dots, m$ we have $\Upsilon_\ell < \xi_k$ or

$$\neg(\exists r')(\exists \Upsilon < \beta)(\exists k')[r \restriction \Upsilon \le r' \in \mathbb{P}_\Upsilon \ \& \ r' \Vdash \underset{\sim}{\xi}_{k'} = \Upsilon \ \& \ \Upsilon_\ell < \Upsilon].$$

Clearly r' forces that $(\forall n)[\underset{\sim}{\zeta}_n \ge \xi_k \to \underset{\sim}{\mathbf{t}}_n$ is truth] and we easily finish. $\square_{\text{Fact B}}$.

Now we do not just have to find q satisfying $\mathbb{P}_\beta / G_\gamma \models p \le_{\rm pr} q$ and q is $(N, \mathbb{P}_\beta/G_\gamma)$-semi-generic, but we need more in the $(N, \mathbb{I}^{[\underset{\sim}{\kappa}_\beta]} \restriction \mathbb{P}_\beta/G_\gamma)$-semi$_6$ generic. Now for $G_\beta \subseteq \mathbb{P}_\beta$ generic over $\mathbf{V}$, in $\mathbf{V}[G_\beta]$ for every countable elementary submodel M of $(\mathscr{H}(\chi)^{\mathbf{V}}, G_\beta \in, <^*), \langle \bar{\mathbb{Q}}, \gamma, \beta, \mathbb{I} \rangle \in M$, we have (in 4.8) defined $\mathrm{Dp}(M)$, an ordinal or ∞ and I_M such that

(A) $\mathrm{Dp}(M) = \infty$ iff $M[G_\beta] = \{\underset{\sim}{\tau}[G_\beta] : \underset{\sim}{\tau} \in M$ a $\mathbb{P}_\beta$-name$\}$ includes M, has the same countable ordinals, is $\prec (\mathscr{H}(\chi)^{\mathbf{V}[G_\beta]}, \in)$ and $M[G_\beta]$ is strictly $\mathbb{I}^{[\underset{\sim}{\kappa}_\beta[G_\beta]]}$-suitable

(B) if $\mathrm{Dp}(M) = \alpha < \infty$ then I_M is a member of $\mathbb{I} \cap M$ which is $\underset{\sim}{\kappa}_\beta[G_\beta]$-complete, and

$$Y_M = \{t \in \mathrm{Dom}(I) : \text{there is } N \text{ as above}, M \subseteq N, M \cap \omega_1 = N \cap \omega_1 \text{ and } t \in N \text{ and } \mathrm{Dp}(N) \ge \alpha\} = \emptyset \bmod I.$$

So we have $\mathbb{P}_\beta$-names $\underset{\sim}{\mathrm{Dp}}, \underset{\sim}{I}_M$.

Consider

$$\mathfrak{K} = \big\{(\zeta, G_\zeta, N) : \gamma \le \zeta < \beta, \zeta \text{ nonlimit}, G_\zeta \subseteq \mathbb{P}_\zeta \text{ generic over } \mathbf{V}, \text{ in } \mathbf{V}[G_\zeta], N[G_\zeta] \text{ is } \mathbb{I}^{[\underset{\sim}{\kappa}_\zeta[G_\zeta]]}\text{-suitable}\big\}.$$

We now define by induction $(\underset{\sim}{\zeta}_n, q_n, p_n, \underset{\sim}{N}_n)$ such that

(a) $\underset{\sim}{\zeta}_n$ is a simple $\bar{\mathbb{Q}}$-named $[\gamma, \beta)$-ordinal, (as $e = 6$, it is full)

(b) $\underset{\sim}{N}_n$ is a $\mathbb{P}_{\underset{\sim}{\zeta}_n}$-name, $q_n \in \mathbb{P}_{\underset{\sim}{\zeta}_n}, p_n$ is a $\mathbb{P}_{\underset{\sim}{\zeta}_m}$-name of a member of $\underset{\sim}{N}_n[G_{\underset{\sim}{\zeta}_n}] \cap (\mathbb{P}_\beta / \underset{\sim}{G}_{\zeta_n})$

(c) $\underset{\sim}{\zeta}_0 = \gamma, N_0 = N$

(d) if $G^n \subseteq \mathbb{P}_{\zeta_n}$ is generic over $\mathbf{V}, q_n \in G^n, G_\gamma \subseteq G^n, \underset{\sim}{\zeta}_n[G^n] = \zeta_n$ (so essentially G_n is just a generic subset of G_{ζ_n} over $\mathbb{V}$ such that $\underset{\sim}{\zeta}_n[G^n] = \zeta_n$), <u>then</u> $\underset{\sim}{N}_n[G^n]$ is a countable elementary submodel of $(\mathscr{H}(\chi)^{\mathbf{V}[G^n]}, \in)$ to which $\bar{\mathbb{Q}}, \gamma, \beta, \mathbb{I}$ belongs and is strictly $\mathbb{I}^{[\underset{\sim}{\kappa}_{\zeta_n}[G^n]]}$-suitable

(e) $\underset{\sim}{N}_n \subseteq N_{n+1}, \underset{\sim}{N}_n \cap \omega_1 = \underset{\sim}{N}_{n+1} \cap \omega_1, q_{n+1} \restriction \underset{\sim}{\zeta}_n = q_n, p_n \restriction \underset{\sim}{\zeta}_n \leq q_n, p_n \leq_{\rm pr} p_{n+1}$

(f) for G^n, ζ_n, N_n as in (d) and $I \in \mathbb{I} \cap N_n$ there is $k > n$ such that: if G^k, ζ_k, N_k are as in (d), $G^n \subseteq G^k$, <u>then</u> there is $t \in$ Dom$(I) \cap N_k \backslash \underset{\sim}{Y}_{(\underset{\sim}{N}_n[G^n])[\underset{\sim}{G}_\beta]}$ (i.e. forced to be there)
(recall p forces a value to $\underset{\sim}{\kappa}_\beta$) sn

(g) for G^n_n, ζ_n, N_n as in clause (d) above and $\underset{\sim}{\gamma} \in N_n$ a $\mathbb{P}_\beta$-name of a countable ordinal there is $k > n$ such that: if G^k, ζ_k, N_k are as in clause (d), $G^n \subseteq G^k$ then p_k forces ($\Vdash_{\mathbb{P}_\beta/G^k}$) to be $< N \cap \omega_1$.

No problem to carry the definition. Now

$(*)_a$ $q = \bigcup_{n<\omega} q_n \in \mathbb{P}_\beta$
Here we use the $q_{n+1} \restriction \underset{\sim}{\zeta}_n = q_n$ below $\bigcup_{n<\omega} \underset{\sim}{\zeta}_n$ and "$(\underset{\sim}{\mathbb{Q}}_i, \leq_{\rm vpr})$ is $\aleph_1$-complete" above $\bigcup_{n<\omega} \underset{\sim}{\zeta}_n$

$(*)_b$ for G^n, N_n, ζ_n as in clause (d), q is $(N_n, \mathbb{P}_\beta/G^n)$-semi generic and above $p \restriction \zeta_n$
[why? note clause (g)]

$(*)_c$ if $q \in G_\beta \subseteq \mathbb{P}_\beta, G_\beta$ is generic over $\mathbf{V}$ extending G_γ <u>then</u> in $\mathbf{V}[G_\beta]$, Dp$(\underset{\sim}{N}_n[G_\beta])$ is well defined

$(*)_d$ Dp$((N_0[G_\beta]))$ is ∞
(use clause (B) in the demands Dp and clause (f) above). We use $I^{[\kappa]}$ is κ-closed for the relevant α's.

So we are done. $\square_{6.8}$

6.9 Claim. *1) Assume that* $\bar{\mathbb{Q}} = \langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : j \le \alpha, i < \alpha \rangle$ *satisfies:*

- (α) *α is a limit ordinal*
- (β) *if $\beta < \alpha$ then $\langle \mathbb{P}_j, \underset{\sim}{Q}_i, \underset{\sim}{\kappa}_j : j \le \beta, i < \alpha \rangle$ is a weak* $\mathrm{UP}^6(\mathbb{I}, \mathbf{W}, W)$*-iteration*
- (γ) *$\mathbb{P}_\alpha$ is the $\aleph_1 - \mathrm{Sp}_6(W)$-limit of $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \alpha \rangle$*
- (δ) *$\underset{\sim}{\kappa}_\alpha$, a $\mathbb{P}_\alpha$-name is* $\sup\{\underset{\sim}{\kappa}_i : i < \alpha\}$.

Then $\bar{\mathbb{Q}}$ is a weak $\mathrm{UP}^6(\mathbb{I}, \mathbf{W}, W)$*-iteration.*
2) Assume that

- (α) $\langle \mathbb{P}_j, \mathbb{Q}_i, \underset{\sim}{\kappa}_i : j \le \alpha, i < \alpha \rangle$ *is a weak* $\mathrm{UP}^6(\mathbb{I}, \mathbf{W}, W)$*-iteration*
- (β) *in $\mathbf{V}^{\mathbb{P}}, \mathbb{Q}$ is a forcing notion satisfying* $\mathrm{UP}^6 \mathbb{I}, \kappa_\alpha, \underset{\sim}{\kappa}, \mathbf{W})$*, where κ_α is the interpretation of $\underset{\sim}{\kappa}_\alpha$) and let $\underset{\sim}{\mathbb{Q}}, \underset{\sim}{\kappa}$ be $\mathbb{P}_\alpha$-names of those objects.*

Then there is a $\mathrm{UP}^6(\mathbb{I}, \mathbf{W}, W)$*-iteration* $\langle P_i, \underset{\sim}{\mathbb{Q}}_j, \underset{\sim}{\kappa}_i : i \le \alpha+1, j < \alpha \rangle$ *with* $\underset{\sim}{\mathbb{Q}}_\alpha = \underset{\sim}{\mathbb{Q}}, \underset{\sim}{\kappa}_{\alpha+1} = \underset{\sim}{\kappa}$.

Proof. 1) By the proof of 6.8.
2) Straightforward.

6.10 Claim. *Assume*

- (a) $\bar{\mathbb{Q}} = \langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : j \le \alpha^*, i < \kappa \rangle$ *is a weak* $\mathrm{UP}^6(\mathbb{I}, \mathbf{W}, W)$*-iteration*
- (b) $\gamma < \beta \le \alpha^*$
- (c) *$G_\gamma \subseteq \mathbb{P}_\gamma$ is generic over V*
- (d) *N is a strictly $(\mathbb{I}, \mathbf{W})$-suitable model N for $(\chi, \langle \bar{\mathbb{Q}}, \gamma, \beta \rangle)$ in $\mathbf{V}[G_\gamma]$*
- (e) $p \in N \cap (\mathbb{P}_\beta / G_\gamma)$.

Then there is q such that:

- (α) $p \le_{\mathrm{pr}} q \in \mathbb{P}_\beta / G_\gamma$
- (β) $p \restriction \gamma = q \restriction \gamma$
- (γ) *q is $(N, \mathbb{P}_\beta / G_\gamma)$-semi generic*
- (δ) *q has a witness listing $\{\underset{\sim}{\zeta} \in N : \underset{\sim}{\zeta}$ a simple $\bar{\mathbb{Q}}$-named $[\gamma, \beta)$-ordinal$\}$.*

Proof. Like 6.8, forgetting to take care of the $I \in \mathbb{I} \cap \underset{\sim}{N}_n$ so we can use $\underset{\sim}{N} = N[\underset{\sim}{G}_{\mathbb{P}_{\varsigma_n}}]$.

$\square_{6.10}$

6.11 Claim. *Suppose F_f, F_c are functions (possibly classes), $\mathbf{W} \subseteq \omega_1$ is stationary, $\mathbb{I}$ is a class of ($\aleph_2$-complete) quasi order ideals, W a class of strongly inaccessible cardinals.*

Then for every ordinal α there is a unique $\bar{\mathbb{Q}} = \langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : j \leq \alpha^\dagger, i < \alpha \rangle$ such that:

(a) *$\bar{\mathbb{Q}}$ is a $\mathrm{UP}^6(\mathbb{I}, \mathbf{W}, W \cap (\alpha^\dagger + 1))$-iteration*
(b) *for every $i < \alpha^\dagger$ we have $\underset{\sim}{\mathbb{Q}}_i = F_f(\bar{\mathbb{Q}} \restriction i), \underset{\sim}{\kappa}_i = F_c(\bar{\mathbb{Q}} \restriction i)$*
(c) *$\alpha^\dagger \leq c$*
(d) *for limit $\beta \leq \alpha^\dagger$ we have $\underset{\sim}{\kappa}_\beta = \sup\{\underset{\sim}{\kappa}_\gamma : \gamma < \beta\}$*
(e) *if $\alpha^\dagger < \alpha$ then the following is impossible*
 (α) *$F_f(\bar{\mathbb{Q}} \restriction i)$ is a $\mathbb{P}_{\alpha^\dagger}$-name of a forcing notion*
 (β) *$F_c(\bar{\mathbb{Q}} \restriction i)$ is a $\mathbb{P}_{\alpha^\dagger}$-name of a $F_f(\bar{\mathbb{Q}} \restriction i)$-name of a $\mathbf{V}$-cardinal $\geq \aleph_1$*
 (γ) *$\Vdash_{\mathbb{P}_{\alpha^\dagger}}$ "$F_f(\bar{\mathbb{Q}} \restriction i)$ is $\mathrm{UP}^6_{\underset{\sim}{\kappa}_{\alpha^\dagger}, \underset{\sim}{\kappa}}(\mathbb{I}, \mathbf{W})$".*

Proof. Straight.

Next we give sufficient conditions for $\bigcup_{i<\delta} \mathbb{P}_i$ being a dense subset of $\mathbb{P}_j$

6.12 Claim. *Assume that $\langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : j \leq \alpha^*, i < \alpha^* \rangle$ is a $\mathrm{UP}^6(\mathbb{I}, \mathbf{W}, W)$-iteration.*
1) Assume

$(*)_1$ *$(\forall i < \alpha^*)[\Vdash_{\mathbb{P}_i}$ " $\leq^{\mathbb{Q}_i}_{\mathrm{vpr}}$ is equality"].*

If $\mathrm{cf}(\delta) = \aleph_1$ & $\delta \leq \alpha^*$ then $\bigcup_{i<\delta} \mathbb{P}_i$ is a dense subset of $\mathbb{P}_\delta$ even under $\leq_{\mathrm{pr}}$.
2) Assume

$(*)_2$ *$(\forall i < \alpha^*)(\Vdash_{\mathbb{P}_i}$ "$(\{r \in \underset{\sim}{\mathbb{Q}}_i : \emptyset_{\underset{\sim}{\mathbb{Q}}_i} \leq_{\mathrm{vpr}} r\}, \leq^{\underset{\sim}{\mathbb{Q}}_i}_{\mathrm{vpr}})$ is directed).*

If $\delta \le \alpha^*, \mathrm{cf}(\delta) = \aleph_1$ <u>*then*</u>

(a) $\bigcup_{i<\delta} \mathscr{D}_{\delta,i}$ *is a dense subset of* $\mathbb{P}_\delta$ *even under* $\le_{\rm pr}$ *where* $\mathscr{D}_{\delta,i} = \{p \in \mathbb{P}_\delta : \mathbb{P}_\delta \models$ "$p \restriction i \le_{\rm vpr} p$"$\}$

(b) *if* $i < \delta, \{p_0, p_1\} \subseteq \mathscr{D}_{\delta,i}$ *and* $p_0 \restriction i = p_j \restriction i$ *then* p_0, p_1 *has an upper bound* p *even in* $\mathscr{D}_{\delta,i}, p \restriction i = p_0 \restriction i = p_1 \restriction i$.

3) If $\delta \in W$ & $\delta \le \alpha^*$ & $(\forall i < \delta)$ *[*density$(\mathbb{P}_i) < \delta$*]* <u>*then*</u> $\mathbb{P}_\delta = \bigcup_{i<\delta} P_i$.

Proof. 1), 2) Let $p \in \mathbb{P}_\delta$, choose χ large enough. There is a strictly $(\mathbb{I}, \mathbf{W})$-suitable countable model $N \prec (\mathscr{H}(\chi), \in)$ to which $\{\bar{\mathbb{Q}}, \delta, p\}$ belongs. Applying 6.8 for $\gamma = 0, \beta = \delta$ (i.e. $\boxtimes_{\gamma,\beta}$ from the proof) we can find $q \in \mathbb{P}_\delta, p \le_{\rm pr} q$ which is $(N, \mathbb{I}^{[\underset{\sim}{\kappa}_\delta]}, \mathbb{P}_\delta)$-semi$_6$ generic and q has a witness $\subseteq \{\underset{\sim}{\zeta} \in N : \underset{\sim}{\zeta}$ a simple $\bar{\mathbb{Q}}$-named $[0,\delta)$-ordinal$\}$.
As $\mathrm{cf}(\delta) = \aleph_1$ there is an increasing continuous $\bar{\beta} = \langle \beta_\varepsilon : \varepsilon < \omega_1 \rangle$ with limit δ, without loss of generality $\bar{\beta} \in N$ so $q \Vdash$ "$N[\underset{\sim}{G}_{\mathbb{P}_\delta}] \cap \delta = \cup\{\beta_\varepsilon : \varepsilon \in N[\underset{\sim}{G}_{\mathbb{P}_\delta}] \cap \omega_1\}$" but $q \Vdash \cup\{\beta_\varepsilon : \varepsilon \in N[\underset{\sim}{G}_{\mathbb{P}_\delta}] \cap \omega_1\} = \cup\{\beta_\varepsilon : \varepsilon \in N \cap \omega_1\} = \beta_{N\cap\omega_1}$, so clearly $\mathbb{P}_\delta \models q \restriction \beta_{N\cap\omega_1} \le_{\rm vpr}$. For (1) it follows that $q \in P_{N\cap\delta}$ and we are done.
For (2) just reflect.
3) Straight.

6.13 Conclusion. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : j \le \alpha^*, i < \alpha^* \rangle$ be a $\mathrm{UP}^6(\mathbb{I}, \mathbf{W}, W)$-iteration and $\kappa = \mathrm{cf}(\kappa) \le \alpha^*$ and $(\forall i < \delta)$ (density$(\mathbb{P}_i) < \kappa$).
1) If $\{\theta < \kappa : \theta = \mathrm{cf}(\theta) \in W\}$ is stationary. <u>Then</u> $\mathbb{P}_\kappa$ satisfies the κ-c.c. (in fact a strong version and even under $\le^{\mathbb{P}_\kappa}_{\rm vpr}$.
2) If $(*)_1$ from 6.12(1), i.e. $\Vdash_{\mathbb{P}_i}$ " $\le^{\underset{\sim}{\mathbb{Q}}_i}_{\rm pr}$ is equality" for $i < \kappa$ <u>then</u> $\mathbb{P}_\kappa$ satisfies the κ-c.c. (in fact a strong version) even for $\le^{\mathbb{P}_\kappa}_{\rm vpr})(\theta \le_{\rm vpr})$ is $(2^{|\delta|})^+$-complete.
3) Assume $\kappa \notin W$ and $S \subseteq \{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_1\}$ is stationary and $i \ge \delta$ & $\delta \in S \Rightarrow \Vdash_{\mathbb{P}_i}$ "$(\{r : \emptyset_{\underset{\sim}{\mathbb{Q}}_i} \le_{\rm vpr} r\}$. <u>Then</u> forcing with $\mathbb{P}_\kappa$ does not add a function in ${}^\omega\mathrm{Ord}$ not in $\bigcup_{\beta<\kappa} \mathbf{V}^{\mathbb{P}_\beta}$, even any function in ${}^{\alpha(*)}\mathrm{Ord} \setminus \bigcup_{\beta<\kappa} \mathbf{V}^{\mathbb{P}_\beta}, \alpha(*) < \kappa$.

Proof. 1) Let $S = \{\theta < \kappa : \theta = \mathrm{cf}(\theta) \in W\}$ and let $\langle \mathbb{P}_\theta : \theta \in S \rangle$ be a sequence of members of $\mathbb{P}_\kappa$. So for each $\theta \in S$ for some $i(\theta) < \theta$ we have $p_\theta \restriction \theta \in \mathbb{P}_{i(\theta)}$ and we can find a pressing down function h on S such that $h(\theta_1) = h(\theta_2) \Rightarrow p_{\theta_1} \restriction \theta_1 = p_{\theta_2} \restriction \theta_2$. Clearly there is a club E or κ such that $\theta_i \in \theta_2 \cap s \ \& \ \theta_2 \in S \cap S \cap E \Rightarrow p_{\theta_1} \in P_{\theta_2}$.

Lastly, if $\theta_1, \theta_2 \in E \cap S$ and $f(\theta_1) = f(\theta_2)$ then $p_{\theta_1} \cup p_{\theta_2}$ is a common upper bound of $p_{\theta_1}, p_{\theta_2}$ (even a $\le_{\rm vpr}$ one).
2) Similar using $S = \{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_1$ and $\alpha < \delta \Rightarrow \mathrm{density}(\mathbb{P}_\alpha) < \delta\}$.
3) If $W \cap \kappa$ is stationary in κ use part (2), so let E be a club of κ disjoint to W. Assume toward contradiction that $p \Vdash_{\mathbb{P}_\kappa}$ "the function $\underset{\sim}{\tau} : \omega_1 \to {}^\kappa\mathrm{Ord}$ is not in $\bigcup\limits_{\beta<\kappa} \mathbf{V}^{\mathbb{P}_\beta}$", let S be as in part (2). We choose by induction on $j < \kappa, (p_j, \alpha_j)$ and if $\alpha_j \in$ Salso$(q_j, \varepsilon_j, \gamma_j, \beta_j)$ such that:

(*i*) $\alpha_j \in E$ is increasing continuous
(*ii*) $p_j \in \mathbb{P}_\kappa, p_0 = p$
(*iii*) $i < j \Rightarrow p_j \restriction \alpha_i = p_i$
(*iv*) $i < j \ \& \ \beta \in [\alpha_i, \alpha_j) \Rightarrow \Vdash_{\mathbb{P}_\beta}$ "$\emptyset \le_{\rm vpr} p_j \restriction \{\beta\}$ in $\hat{\underset{\sim}{\mathbb{Q}}}_\beta$"
(*v*) if $\alpha_i \in S, j < \alpha_i$ <u>then</u> for every $p_i \le q \in \mathscr{D}_{\alpha_i,j}$ there is $\varepsilon_i(q) < \omega_i$ such that: <u>if</u> there are ε, r such that $p_{i+1} \cup \le_{\rm vpr} r, r \restriction \alpha_i = q, \varepsilon < \omega_1, r$ forcing a value $\underset{\sim}{\tau}(\varepsilon)$ but for no $r, q \le r' \in \mathbb{P}_{\alpha_i}$, does $P_{i+1} \cup r'$ forces a value to $\underset{\sim}{\tau}(\varepsilon)$ then $\varepsilon = \varepsilon_i(q)$ satisfies this
(*vi*) $\alpha_i \in S$ then $p_{i+1} \le q_i, q_i \Vdash$ "$\underset{\sim}{\tau}(\varepsilon_i) = \gamma_i$", $\varepsilon_i < \alpha(*), \beta_i < \alpha_i, q_i \restriction \beta_i \le_{\rm vpr} q_i \restriction \alpha_i$
(*vii*) there is no $q, q_i \restriction \alpha_i \le q \in \mathbb{P}_{\alpha_i}$ such that $p_{i+1} \cup (q_i \restriction \alpha_i)$ forces a value to $\underset{\sim}{\tau}(\varepsilon_i)$.

For any j we choose (p_j, α_j).
For $j = 0$ let $p_0 = p$ and by 6.9 for some $\alpha_0 < \kappa, p_0 \in \mathbb{P}_{\alpha_0}$. For $j = i+1$, first choose p_j to satisfy clause (v) and then α_j such that $p_j, q \in \mathbb{P}_{\alpha_j}$. Lastly for j limit let $p_j = \bigcup\limits_{i<j} p_i, \alpha_j = \bigcup\limits_{i<j} \alpha_i$ and check. The contradiction is easy.

Let $G_{\alpha_i} \subseteq \mathbb{P}_{\alpha_i}$ be generic over $\mathbf{V}$ such that $p_i \in G_{\alpha_i}$. Clearly for some $\varepsilon < \alpha(*)$ for no $q \in \mathbb{P}_{\alpha_i}$ do we have $q \cup p_{i+1}$ forces a value to $\underset{\sim}{\tau}(\varepsilon)$ as otherwise $p_{\alpha_{i+1}} \Vdash_{\mathbb{P}_\kappa/G_{\alpha_i}} \underset{\sim}{\tau} \in \mathbf{V}[G_{\alpha_i}]$. Choose $\varepsilon_i < \alpha(*)$ as above, choose $q'_i \in G_{\alpha_i}$ which forces this choose $q_i \in \mathbb{P}_\kappa$ above $q'_i \cup p_{i+1}$ which forces a value to $\underset{\sim}{\tau}(\varepsilon_i)$ and without loss of generality there is $\gamma_i < \alpha_i$ such that $q_i \restriction \gamma_i \le_{\rm vpr} q_i \restriction \alpha_i$. Lastly let α_{i+1} be such that $\beta \in [\alpha_{i+1}, \kappa) \Rightarrow \Vdash_{P_\beta}$ "$\emptyset_{\underset{\sim}{\mathbb{Q}}_\beta} \le_{\rm vpr} p_{i+1} \restriction \{\beta\}$.

[Saharon: the role of W].

§6B On UP^2-iteration

6.1(?)Lemma. *Assume that* $\mathbf{W} \subseteq \omega_1$ *is stationary and* $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\mathbb{I}}_i, \underset{\sim}{\kappa}_i : i < \alpha \rangle$ *is a* $\mathrm{UP}^{2,e}(\mathbf{W}, W)$*-suitable iteration, and* $\mathbb{P}_\alpha = \mathrm{Sp}_e(W) - \mathrm{Lim}_\kappa(\bar{\mathbb{Q}})$ *be the limit and* $\underset{\sim}{\kappa}(\beta) = \sup\{\underset{\sim}{\kappa}_\gamma : \gamma < \beta\}$ *(this is a* $\mathbb{P}_\beta$*-name of a* V*-cardinal) and*

$$\kappa^-(\beta) = \mathrm{Sup}\{\kappa : \nVdash_{\mathbb{P}_\gamma} \text{“}\kappa \neq \underset{\sim}{\kappa}(\beta)\text{” for some } \gamma < \beta\}.$$

1) For each $\beta \leq \alpha, \mathbb{P}_\beta$ *satisfies* $\mathrm{UP}^{0,e}_{\underset{\sim}{\kappa}(\beta)}(\mathbb{I}'_\beta, \mathbf{W})$ *for some* $\aleph_2$*-complete* $\mathbb{I}'_\beta \in \mathbf{V}$.

2) In fact, $\mathbb{I}_\beta$ *is* κ^β*-complete where* $\kappa^\beta =$ *min*$\{\kappa :$ *for some* $\gamma < \beta$ *we have* $\nVdash_{\mathbb{P}_{\gamma+1}}$ "$\underset{\sim}{\kappa}_\gamma \neq \kappa$"$\}$*, and each* $I \in \mathbb{I}_\beta$ *has domain of cardinality*
$\leq (\sup_{\gamma<\beta}\{\lambda < \kappa_{\delta+1} : \nVdash_{\mathbb{P}_\gamma}$ "$\neg(\exists I \in \underset{\sim}{\mathbb{I}}_\gamma)(\lambda = |Dom(I)|)$"$\})$ *and*
$|\mathbb{I}'_\beta| \leq \sum_{\gamma<\beta}(\aleph_0 + |\mathbb{P}_\gamma| +$ *min*$\{\lambda : \Vdash_{\mathbb{P}_\gamma}$ "$|\underset{\sim}{\mathbb{I}}_\gamma| \leq \lambda)^{<\kappa}\}$.

3) Similarly for $\mathrm{UP}^{0,e}_a$ *and weak* $\mathrm{UP}^{0,e}_y(\mathbf{W})$*-iterations.*

6.2 Remark. We can also get the preservation version of this Lemma.

Proof. For each $\gamma < \alpha$ let $\mathscr{I}_\gamma =: \{q \in \mathbb{P}_{\gamma+1} : q$ forces a value to $\underset{\sim}{\kappa}_\gamma$, called $\kappa_{\gamma,q}$ and q forces $\underset{\sim}{\mathbb{I}}_\gamma$ to be equal to a $\mathbb{P}_\gamma$-name $\underset{\sim}{\mathbb{I}}_{\gamma,q}$ and $q \restriction \gamma$ forces a value to $|\mathbb{I}_\gamma|$ says $\mu_{\gamma,q}$ is this is purely decidable, if not, just an upper bound to it$\}$; let $\mathscr{I}'_\gamma \subseteq \mathscr{I}_\gamma$ a maximal antichain. Let $\mu_\gamma = \sup_{q \in \mathscr{I}'_\gamma} \mu_{\gamma,q}$.
Let $q \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{\mathbb{I}}_\gamma = \{\underset{\sim}{I}_{\gamma,\zeta} : \zeta < \mu_{\gamma,q}\}$" for $q \in \mathscr{I}_\gamma$ and let $\mathscr{I}_{\gamma,\zeta} = \{q \in \mathscr{I}_\gamma : \mu_{\gamma,q} > \zeta$ and $q \Vdash \mathrm{Dom}(\underset{\sim}{I}_{\gamma,\zeta})$ is $\lambda_{\gamma,q,\zeta}\}$ and let $\underset{\sim}{I}_{\gamma,\zeta}$ be $\mathrm{id}_{L_{\gamma,\zeta}}$, so $L_{\gamma,q,\zeta}$ is a $\mathbb{P}_\gamma$-name of a $\underset{\sim}{\kappa}_{\gamma,q}$-directed partial order on $\lambda_{\gamma,q,\zeta}$ (but $\Vdash_{\mathbb{P}_\gamma}$ "if $|\underset{\sim}{\mathbb{I}}_\gamma| \leq \zeta < \mu_\gamma$ then let $L_{\gamma,\zeta}$ be trivial").

For $q \in \mathscr{I}_\gamma$ let $L^*_{\gamma,q,\zeta}$ be $\mathrm{ap}_{\kappa_{\gamma,q}}(\underset{\sim}{L}_{\gamma,\zeta})$ for the forcing notions
$\mathbb{P}^{[q]}_\gamma = \{p \in \mathbb{P}_\gamma : q \restriction \gamma \leq^{\mathbb{P}_\gamma}_{pr} p\}$ from Definition 3.9. So by Claim 3.10

(*i*) $L^*_{\gamma,q,\zeta}$ is $\kappa_{\gamma,q}$-directed partial order on $[\lambda_{\gamma,q,\zeta}]^{<\kappa_{\gamma,q}}$
(*ii*) $|L^*_{\gamma,q,\zeta}| \leq (\lambda_{\gamma,q,\zeta})^{<\kappa_{\gamma,q}}$
(*iii*) $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{I}_{\gamma,\zeta} = \mathrm{id}_{\underset{\sim}{L}_{\gamma,\zeta}} \leq_{\mathrm{RK}} \mathrm{id}_{L^*_{\gamma,q,\zeta}}$".

Let $\kappa_\beta = \sup\{\kappa_{\gamma,q} : \gamma < \beta, q \in \mathscr{J}_\gamma\}$.
Let $\mathbb{I}^*_\beta$ be the $(< \kappa_\beta)$-closure of $\{\mathrm{id}_{L^*_{\gamma,q,\zeta}} : \gamma < \beta, q \in \mathscr{J}_\gamma, \zeta < \mu_{\gamma,q}\}$ (see Definition 3.13(1)).

Let N be $(\mathbb{I}^*_\alpha, \mathbf{W})$-suitable model for (χ, λ), x code enough informtion so for some $\bar{N}, N = N_{\langle\rangle}$ and $\bar{N} = \langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ be a strict $(\mathbb{I}^*_\alpha, \mathbf{W})$-suitable tree of models for (χ, x), x coding enough information (so $\bar{\mathbb{Q}}, \mathbb{I}^*_\alpha, \underset{\sim}{\mathbf{S}}, \mathbf{W} \in N_{\langle\rangle}$).

Let $\mathscr{T}_{\bar{N}}$ be the set of pairs (γ, p) such that:

$\bigotimes$ $\gamma \leq \alpha, p \in \mathbb{P}_\gamma$, and for some κ,
$p \Vdash_{P_\gamma}$ "$N[\underset{\sim}{G}_{P_\gamma}] \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$ and $\gamma \in N[G_{P_\gamma}]$".

$\mathscr{T}'_{\bar{N}}$ is defined similarly as the set of pairs $(\underset{\sim}{\gamma}, p)$ such that: $\underset{\sim}{\gamma}$ is a simple $(\bar{\mathbb{Q}}, W)$-named ordinal, $p \in \mathbb{P}_{\underset{\sim}{\gamma}}$. (I.e. if $\zeta < \beta, G_{\mathbb{P}_\gamma} \subseteq \mathbb{P}_\zeta$ is generic over $\mathbf{V}$ and $\zeta = \underset{\sim}{\gamma}_n[G_{\mathbb{P}_\zeta}]$ then $r \in q_n \Rightarrow \underset{\sim}{\zeta}_n[G_\zeta] < \zeta$, i.e. is well defined $< \zeta$ <u>or</u> is forced $(\Vdash_{\mathbb{P}_\alpha/G_\zeta})$ to be not well defined, and $p \Vdash_{\mathbb{P}_{\underset{\sim}{\gamma}}}$ "$\eta \in \lim T$").

We consider the statements, for $\gamma \leq \beta < \alpha$ (or restrict ourselves to γ non-limit)

$\boxtimes_{\gamma,\beta}$ for any $(\gamma, p) \in \mathscr{T}_{\bar{N}}$ and ρ such that
$p \Vdash_{\mathbb{P}_\gamma}$ "$\rho \triangleleft \underset{\sim}{\eta}$"
and $\underset{\sim}{p}'$ a $\mathbb{P}_\gamma$-name such that $p \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{p}'[G_{\mathbb{P}_\gamma}] \in N[\underset{\sim}{G}_{\mathbb{P}_\gamma}] \cap P_\beta/G_{\mathbb{P}_\gamma}$ and $(\underset{\sim}{p}'[G_{\mathbb{P}_\gamma}]) \restriction \gamma \leq_{pr} p$" <u>there is</u> $(\beta, q) \in \mathscr{T}$ such that $\underset{\sim}{p}' \leq_{pr} q$
(i.e. $p \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{p}'[\underset{\sim}{G}_{\mathbb{P}_\gamma}] \leq_{pr} q$") and $q \restriction \gamma = p$".

We prove by induction on $\beta \leq \alpha$ that $(\forall \gamma \leq \beta) \boxtimes_{\gamma,\beta}$ (but for 6.1(3), we use ($\forall$ non-limit $\gamma \leq \beta$) $\boxtimes_{\gamma,\beta}$), note that for $\gamma = \beta$ the statement is trivial hence we shall consider only $\gamma < \beta$.

<u>Case 1</u>: $\beta = 0$.
Trivial.

<u>Case 2</u>: β a successor ordinal (for part (3), β successor of non-limit ordinal).
As trivially $\boxtimes_{\gamma_0,\gamma_1}$ & $\boxtimes_{\gamma_1,\gamma_2} \Rightarrow \boxtimes_{\gamma_0,\gamma_2}$, clearly without loss of generality $\beta = \gamma + 1$.
Let $G_{\mathbb{P}_\gamma}$ be such that $p \in G_{\mathbb{P}_\gamma} \subseteq \mathbb{P}_\gamma$ and $G_{\mathbb{P}_\gamma}$ generic over V.
Let $\bar{N}' = \langle N_\nu[G_{\mathbb{P}_\gamma}] : \eta \in (T', \mathbf{I})\rangle$.
In $\mathbf{V}[G_{\mathbb{P}_\gamma}]$ we apply **?** for $\lambda = \aleph_2$ to $\bar{N}'$ and find $T' \subseteq T$ such that $\mathbf{V}[G_{\mathbb{P}_\gamma}] \models$
$\longrightarrow$ scite{6.2} undefined

"$\langle N'_\eta[G_{\mathbb{P}_\gamma}] : \eta \in (T'', \mathbf{I}')\rangle$ is strict $(\mathbb{I}^*_\gamma)^{[\kappa(\gamma)]}, \mathbf{W})$-suitable". So we can apply clause (f) of **?**.

$\longrightarrow$ scite{6.4A} undefined

Discussion: This is a question whether there is an $\mathbb{I}$-tree of model $\langle N_\eta : \eta \in (T, \mathbb{I})\rangle$ such that:

> if $\mathbb{I}_\eta$ is $\lambda(\mathbb{I}_\eta)$-complete, $\lambda(\mathbf{I}_\eta)$ regular, $\alpha < \lambda(\mathbf{I}_\eta) \Rightarrow |\alpha|^{\aleph_0} < \lambda(\mathbf{I}_\eta)$, then $\nu \in \text{Suc}_T(\eta) \Rightarrow N_\eta <_\lambda N_\nu$.

This would make 6.2? more effective.

Case 3: β is a limit ordinal.

We shall choose by induction on $n < \omega, \underset{\sim}{\gamma}_n, q_n, \underset{\sim}{p}_n$ such that:

(a) $(\underset{\sim}{\gamma}_n, q_n) \in \mathscr{T}'_{\bar{N}}$

(so $\underset{\sim}{\gamma}_n$ is a simple $(\bar{\mathbb{Q}}, W)$-named ordinal)

(b) $\underset{\sim}{\gamma}_0 = \gamma$ and $\Vdash_{\bar{\mathbb{Q}}}$ "$\underset{\sim}{\gamma}_n < \underset{\sim}{\gamma}_{n+1} < \beta$"

i.e. if $\zeta < \beta$ and $G_{\mathbb{P}_\gamma} \subseteq \mathbb{P}_\zeta$ is generic over V and $\zeta = \underset{\sim}{\gamma}_n[G_{\mathbb{P}_\zeta}]$ then

$r \in q_n \Rightarrow \underset{\sim}{\zeta}_n[G_\zeta] < \zeta$ (i.e. is well defined $< \zeta$ or is forced to be not well defined),

(c) $q_{n+1} \restriction \underset{\sim}{\gamma}_n = q_n$

(d) $\underset{\sim}{p}_n$ is a $\mathbb{P}_{\underset{\sim}{\gamma}_n}$-name, $p_0 = p, \underset{\sim}{p}_n \restriction \underset{\sim}{\gamma}_n \le_{pr} q_n$ and

$q_n \Vdash_{\mathbb{P}_{\underset{\sim}{\gamma}_n}}$ "$\underset{\sim}{p}_n \in N_{\underset{\sim}{\rho}_n}[G_{\mathbb{P}_{\underset{\sim}{\gamma}_n}}] \cap \mathbb{P}_\beta$ and $p_n \restriction \underset{\sim}{\gamma}_n \in G_{\mathbb{P}_{\underset{\sim}{\gamma}_n}}$"

(e) $q_n \Vdash_{\mathbb{P}_{\underset{\sim}{\gamma}_n}}$ "$\underset{\sim}{p}_n \le^{\mathbb{P}_\beta}_{pr} \underset{\sim}{p}_{n+1} \in N[G_{\mathbb{P}_{\gamma_n}}] \cap P_\beta$"

(f) letting $\langle \underset{\sim}{\tau}_\ell : \ell < \omega\rangle$ list the P_β-names of ordinals from N: for $m, \ell \le n$ we have:

$q_n \Vdash_{\mathbb{P}_{\underset{\sim}{\gamma}_n}}$ "$\underset{\sim}{p}_{n+1}$ force $(\Vdash_{\mathbb{P}_{\underset{\sim}{\gamma}_{n+1}}})$ that: if $\underset{\sim}{\tau}_\ell$ is a countable ordinal, then it is smaller than some $\underset{\sim}{\tau}'_\ell \in N[G_{\mathbb{P}_{\underset{\sim}{\gamma}_{n+1}}}]$, a $\mathbb{P}_{\underset{\sim}{\gamma}_n}$-name of a countable ordinal".

The induction is straight and $\bigcup\limits_{n<\omega} q_n$ are as required noting we need and have $(*)_1$ or $(*)_2$ below:

$(*)_1$ Assume $\leq_{\rm pr}, \leq_{\rm vpr}$ are equal to $\leq$ (i.e., $\Vdash_{\mathbb{P}_\beta}$ " $\leq^{\mathbb{Q}_\beta}_{\rm vpr}$ is $\leq^{\mathbb{Q}_\beta}$ " for each $\beta < \alpha$), if $p \in \mathbb{P}_\beta, \gamma < \beta, \underset{\sim}{\tau}$ a $\mathbb{P}_\beta$-name of an ordinal then there are $p', \underset{\sim}{\tau}'$ such that:

(i) $\underset{\sim}{\tau}'$ is a $\mathbb{P}_\gamma$-name of an ordinal

(ii) $p \leq_{\rm pr} p' \in \mathbb{P}_\beta$ and $p \restriction \gamma = p' \restriction \gamma$

(iii) $p' \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\tau} = \underset{\sim}{\tau}'$".

[why? straight by **1.18**].

$\Rightarrow$ scite{1.16} ambiguous

$(*)_2$ if $p \in \mathbb{P}_\beta, \gamma < \beta, \underset{\sim}{\tau}$ is a $\mathbb{P}_\beta$-name of a countable ordinal, then there are p', τ' such that

(i) τ' is a $\mathbb{P}_\gamma$-name of a countable ordinal

(ii) $p \leq_{\rm pr} p' \in \mathbb{P}_\beta$ and $p' \restriction \gamma = p \restriction \gamma$

(iii) $p' \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\tau} \leq \tau'$"

[why $(*)_2$? let $\underset{\sim}{\zeta}$ be the following simple $(\bar{\mathbb{Q}}, W)$-named $[\gamma, \beta)$-ordinal: $G_\zeta \subseteq \mathbb{P}_\zeta$ is generic over $\mathbf{V}$ for $\zeta \in [\gamma, \beta)$ we let $\underset{\sim}{\zeta}[G_\zeta] = \zeta$ if

(a) $p \restriction \zeta \notin G_\zeta$ or:
for some $p' \in \mathbb{P}_\beta$ we have $p' \restriction \zeta = p$ and $\mathbb{P}_\beta \models p \leq_{\rm pr} p'$
and $p' \Vdash_{\mathbb{P}_\beta / G_\zeta}$ "$\underset{\sim}{\tau} < \tau^*$" for some countable ordinal $\underset{\sim}{\tau}$

(b) for no $\xi \in [\gamma, \zeta)$ does clause (a) hold for $\xi, G_\zeta \cap \mathbb{P}_\xi$.

Now if $p \Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\zeta} = \gamma$" we are done. Also $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\zeta}[\underset{\sim}{G}_{\mathbb{P}_\alpha}]$ is well defined" as if $p \in G_\alpha \subseteq \mathbb{P}_\alpha$ and G_α is generic over $\mathbf{V}$, then for some $q \in G_\alpha$ and countable ordinal τ^* we have $q \Vdash$ "$\underset{\sim}{\tau} = \tau^*$. By the definition of $\aleph_1 - Sp_e(W)$-iteration for some $\zeta \in [\gamma, \beta)$ we have $\xi \in [\zeta, \beta) \Rightarrow [p \restriction \{\xi\} \leq^{\mathbb{Q}_\xi}_{\rm pr} q \restriction \{\xi\}$ or $e = 4$ & $p \restriction \{\xi\}$ not defined].
Define p' by: $p' \restriction \underset{\sim}{\zeta} = p \restriction \underset{\sim}{\zeta}$, and for $\xi \in [\zeta, \beta)$ we let $p' \restriction \{\xi\}$ be $q \restriction \{\xi\}$ if: $p \restriction \{\xi\} \leq^{\mathbb{Q}_\xi}_{\rm pr} q \restriction \{\xi\}$ or $e = 4$ & $p \restriction \{\xi\}$ not defined. Now p' is as required.

So there is a $\mathbb{P}_{\underset{\sim}{\zeta}}$-name of $\underset{\sim}{p}'$ as appearing in the definition of $\underset{\sim}{\zeta}$ and it is, essentially, a member of $\mathbb{P}_\beta$. Now as we have finite apure support, the proof of "$\underset{\sim}{\zeta}[\underset{\sim}{G}_{\mathbb{P}_\alpha}]$ is well defined" gives $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\zeta}$ is not a limit ordinal $> \gamma$". Lastly $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\zeta}$ is not a successor ordinal $> \gamma$" is proved by the property of each $\underset{\sim}{\mathbb{Q}}_\xi$.]

Finishing the induction we define $q_\omega \restriction \underset{\sim}{\gamma}_n = q_n, q_\omega \restriction [\bigcup\limits_{n<\omega} \underset{\sim}{\gamma}_n, \beta)$ is defined as $\leq_{vpr}$-upper bound of $\langle \underset{\sim}{p}_m \restriction [\bigcup\limits_{n<\omega} \gamma_n, \beta) : m < \omega \rangle$.

More formally, let γ^*, β and $G_{\gamma^*} \subseteq \mathbb{P}_{\gamma^*}$ be such that: G_{γ^*} is generic over $\mathbf{V}, \gamma^* = \bigcup\limits_{n<\omega} \gamma^*_n, \gamma^*_n = \underset{\sim}{\gamma}_n[G_{\gamma^*}]$, let $p'_n = \underset{\sim}{p}_n[G_{\gamma^*} \cap \mathbb{P}_{\gamma^*_n}]$, let $p'_n \restriction [\gamma^*, \alpha) = \{\underset{\sim}{r}^n_\zeta : \zeta < \zeta^*_n\}$ where $\underset{\sim}{r}^n_\zeta$ is a simple $[\gamma^*, \alpha)$-named atomic condition.
Now we define $\underset{\sim}{s}^n_\zeta$, a simple $(\bar{\mathbb{Q}}, W)$-named $[\gamma^*, \alpha)$-ordinal atomic condition as follows:

(a) $\zeta_{\underset{\sim}{s}^n_\zeta} = \zeta_{\underset{\sim}{r}^n_\zeta}$

(b) if $\zeta \in [\gamma^*, \alpha), G_{\gamma^*} \subseteq G_\gamma \subseteq \mathbb{P}_\gamma, G_\gamma$ generic over $\mathbf{V}, \zeta_{\underset{\sim}{r}^n_\zeta}[G_\zeta] = \zeta$ then $\underset{\sim}{s}^n_\zeta[G_\zeta]$ is the $<^{*\mathbf{V}[G_\zeta]}$-first elements of $\underset{\sim}{\mathbb{Q}}_\zeta[G_\zeta]$ which satisfies the following:

(∗)(a) $(\underset{\sim}{p}_n \restriction \{\zeta\})$

(b) if $\langle \emptyset_{\underset{\sim}{\mathbb{Q}}_\zeta[G_\zeta]}\rangle\hat{\ }\langle(\underset{\sim}{p}_m \restriction \{\xi\})[G_\zeta] : m \in (n, \omega)\rangle$ has a $\leq_{vpr}$-upper bound then $\underset{\sim}{s}^n_\zeta[G_\zeta]$ is such upper bound.

Now actually such $\leq_{vpr}$-upper bound actually exists, and q_ω is as required. $\square_{6.1}$

Now we can refine 1.19 to our iteration theorem.
Saharon revise.

6.3 Claim. *1) Suppose F is a function, $\mathbf{W} \subseteq \omega_1$ stationary, $\mathbb{I}$ a class of quasi order ideals, then for every ordinal α there is $UP^6(\mathbb{I}, \mathbf{W})$-iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_j, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_j : j \leq \alpha^\dagger, i < \alpha^\dagger\rangle$, such that:*

(a) *for every $i < \alpha^\dagger$ we have $\underset{\sim}{\mathbb{Q}}_i = F(\bar{Q} \restriction i)$,*

(b) $\alpha^\dagger \leq \alpha$

(c) *for* $\delta \leq \alpha^\dagger, \underset{\sim}{\kappa}_\delta$ *is as in clause (d) of Definition* **?**

--> *scite{6.3} undefined*

(d) *either* $\alpha^\dagger = \alpha$ *or* $F(\bar{\mathbb{Q}})$ *is not a pair* $(\underset{\sim}{\mathbb{Q}}, \underset{\sim}{\kappa})$ *such that:* $\underset{\sim}{\kappa}$ *is a* $\mathbb{P}_{\alpha^\dagger} * \underset{\sim}{\mathbb{Q}}$*-name of a cardinal from* $\mathbf{V}$ *and* $\Vdash_{\mathbb{P}_{\alpha^\dagger}}$ "$\underset{\sim}{\mathbb{Q}}$ *satisfies* $UP^6_{\underset{\sim}{\kappa}_{\alpha^\dagger}, \underset{\sim}{\kappa}}(\mathbb{I}, \mathbf{W})$.

2) Suppose $\beta < \alpha, G_\beta \subseteq \mathbb{P}_\beta$ *is generic over* $\mathbf{V}$*, then in* $\mathbf{V}[G_\beta], \bar{\mathbb{Q}}/G_\beta = \langle \mathbb{P}_i/G_\beta : \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\kappa}_i : \beta \leq i < \alpha \rangle$ *is an* $\mathrm{UP}^6(\mathbb{I}^{[\underset{\sim}{\kappa}_\beta[G_\beta]]}, \mathbf{W})$*-iteration.*

3) If $\bar{\mathbb{Q}}$ *is an* $\mathrm{UP}^6(\mathbb{I}, \mathbf{W})$*-iteration,* $p \in \mathrm{Sp}_e(W) - \mathrm{Lim}(\bar{\mathbb{Q}}), \mathbb{P}'_i = \{q \in \mathbb{P}_i : q \geq p \restriction i\}, \underset{\sim}{\mathbb{Q}}'_i = \{p \in \underset{\sim}{\mathbb{Q}}_i : p \geq p \restriction \{i\}\}$*,* <u>*then*</u> $\bar{\mathbb{Q}} = \langle \mathbb{P}'_i, \mathbb{Q}'_i : i < \ell g(\bar{\mathbb{Q}}) \rangle$ *is (essentially) an* $\mathrm{UP}^6(\mathbb{I}, \mathbf{W})$*-iteration.*

§7 No New Reals, Replacements for Completeness

Now we turn to "No New Reals", there are versions corresponding to [Sh:f, Ch.V,§1-§3] (**W**-complete), [Sh:f, Ch.V,§5-§7] ($\bigwedge\limits_{\alpha<\omega_1}$ α-proper $+\mathscr{D}$-completeness) and better [Sh:f, Ch.VIII, §4] (making the previous preserved) and in different directions [Sh:f, Ch.XVIII,§2] and [Sh 656].

We deal here with the first (here we are interested in the cases $\leq_{pr}=\leq$)

7.1 Definition. For $p \in \mathbb{Q}$ let $\mathbb{Q}^{\rm pr}_p = \{q \in \mathbb{Q} : p \leq_{\rm pr} q\}$. A point which may confuse is that the pure extension notion used in Definition 7.2, is not necessarily the one used seriously in the iteration. This is the reason for the case $e = 5$ in §1. [Saharon check: main question: do we really need the purity in the iteration for Nm$'$. For Nm it is not needed (as in [Sh:f, Ch.XI].]

7.2 Definition. 1) $\mathrm{UP}^4_{\rm com}(\mathbb{I}, \mathbf{W})$ is satisfied by the forcing notion $\mathbb{Q}$, <u>iff</u>: for any $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ a strict $(\mathbb{I}, \mathbf{W})$-suitable tree of models for $(\chi, x), x$ coding enough information, we have $(*)_1 \Rightarrow (*)_2$ where:

- $(*)_1$ for every $\eta, \nu \in T$, of the same length we have $(N_\eta, \mathbb{Q}) \cong (N_\nu, \mathbb{Q})$ and letting $h_{\eta,\nu}$ be the isomorphism from N_η onto N_ν we have $h_{\eta,\nu}(x) = x$ and $\ell < \ell g(\eta) \Rightarrow h_{\eta\restriction\ell,\nu\restriction\ell} \subseteq h_{\eta,\nu}$; (for $\eta, \nu \in \lim(T)$ let $h_{\nu,\eta} = \bigcup\limits_{\ell<\omega} h_{\nu\restriction\ell,\eta\restriction\ell}$)
- $(*)_2$ if $\eta^* \in \lim(T), p \in N_{\langle\rangle} \cap G$ and G_{η^*} is a $\leq_{pr}$-directed subset of $N_{\eta^*} \cap \mathbb{Q}^{pr}_p = \bigcup\limits_{\ell<\omega} N_{\eta^*\restriction\ell} \cap \mathbb{Q}^{pr}$, not disjoint to any dense subset of $\mathbb{Q}^{pr}_p \cap \bigcup\limits_{m<\omega} N_{\eta^*\restriction m}$ definable in $(\bigcup\limits_{m<\omega} N_{\eta^*\restriction m}, N_{\eta^*\restriction m}, \mathbf{I}_{\eta^*\restriction m})_{m<\omega}$, <u>then</u> there is $q \in \mathbb{Q}$ such that $p \leq_{pr} q$ and $q \Vdash_{\mathbb{Q}}$ "there is $\nu \in \lim(T)$ (in $\mathbf{V}^{\mathbb{Q}}$) such that $\bigcup\limits_{\ell<\omega} h_{\eta^*\restriction\ell,\nu\restriction\ell}(G \cap N_{\nu\restriction\ell})$ is a subset of $\underset{\sim}{G}_{\mathbb{Q}}$".

2) $\mathrm{UP}^4_{\rm stc}(\mathbb{I}, \mathbf{W})$ is satisfied by the forcing notion $\mathbb{Q}$ <u>iff</u> for any $\bar{N} = \langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ a strict $(\mathbb{I}, \mathbf{W})$-suitable tree of models for $(\chi, x), x$ coding enough information we have $(*)_1 \Rightarrow (*)_2$ where

- $(*)_1$ for every $\eta, \nu \in T$, of the same length we have $(N_\eta, \mathbb{Q}) \cong (N_\nu, \mathbb{Q})$ and letting $h_{\eta,\nu}$ be the isomorphism from N_η onto N_ν we have $h_{\eta,\nu}(x) = x$ and $\ell < \ell g(\eta) \Rightarrow h_{\eta\restriction\ell,\nu\restriction\ell} \subseteq h_{\eta,\nu}$; (for $\eta, \nu \in \lim(T)$ let $h_{\nu,\eta} = \bigcup\limits_{\ell<\omega} h_{\nu\restriction\ell,\eta\restriction\ell}$)

$(*)_2$ if $\eta^* \in \lim(T)$; $\rho \in N_{<>} \cap G$ then in the following game $\eth = \eth_{\mathbb{Q},p,\bar{N},\eta}$; player complete has a winning strategy. The plays last ω-moves, in the n-th move (we can incorporate the last into the game) where $\mathbf{t}_{\eta \restriction n} = T * x(\mathrm{Suc}(\eta^* \restriction n) \in \mathbf{I}^+_\eta)$ a condition $p_n \in P \cap N_{\eta^*}$ is chosen such that $p \leq p_0, [n > 0 \Rightarrow p_{n-1} \leq p_n]$. In the n-th move, the anti-completeness player chooses $q_n \in \mathbb{Q} \cap N_{\eta^*}$ such that $n = 0 \Rightarrow p = q_n$ and $n > 0 \Rightarrow p_{n-1} \leq_{\mathrm{pr}} q_n$ and the completeness player chooses $p_n \in \mathbb{Q} \cap N$ such that $q_n \leq p_n \in \mathbb{Q} \cap N_{\eta^*}$. In the end the completeness player wins iff one of the following occurs:

(α) for some $\mathbb{Q}$-name $\underset{\sim}{\tau} \in N_{\eta^*}$ of a countable ordinal, no p_n forces a value

(β) not (α) but there is q such that: $p \leq_{\mathrm{pr}} q \in \mathbb{Q}$ and $q \Vdash_{\mathbb{Q}}$ "there is $\eta \in \lim(T)$ such that $n < \omega \Rightarrow h_{\underset{\sim}{\eta} \restriction n, \eta^* \restriction n}(p_n) \in \underset{\sim}{G}_{\mathbb{Q}}$".

3) We define "$\mathbb{Q}$ satisfies $\mathrm{UP}^1_{\mathrm{com},\underset{\sim}{\kappa}}(\mathbb{I}, W)$ is defined as in (1) but we replace the conclusion of $(*)_2$ by: there is $q \in \mathbb{Q}$ such that $p \leq_{\mathrm{pr}} q$ and

$$\begin{aligned} q \Vdash_{\mathbb{Q}} \text{“there is } (T', \mathbf{I}) \text{ satisfying } (T, \mathbf{I}) \leq^{\underset{\sim}{\kappa}} (T', \mathbf{I}) \\ \text{(see 2.4(d)) such that for every} \\ \nu \in \lim(T') \text{ we have } h_{\nu,\eta^*}(G^*_\eta) \subseteq \underset{\sim}{G}_{\mathbb{Q}}\text{”.} \end{aligned}$$

4) Similarly $\mathrm{UP}^1_{\mathrm{stc},\underset{\sim}{\kappa}}(\mathbb{I}, W)$.

5) We say $\mathrm{UP}^4_{\mathrm{stc}}(\mathbb{I}, \mathbf{W})$ if letting $\mathfrak{p} = \langle (\underset{\sim}{\tau}_n, I_n, \eta^*(n) : I_{\eta^* \restriction n}, (\mathbf{t}_{\eta^* \restriction n}), n < \omega \rangle$ be such that $\{\underset{\sim}{\tau}_n : n < \omega\}$ list the $\mathbb{Q}$-names of ordinals in N_{η^*}, $\{I_n : n < \omega\}$ lists $\mathbb{I} \cap N_{\eta^*}$, the winning strategy on each stage depends just on $\theta, N_{<>} \cap \omega_1$ and in $\mathfrak{p}$ continuously.

7.3 Remark. 1) This property relates to the $\mathrm{UP}(\mathbb{I}, \mathbf{W})$ just as E-complete relates to E-proper (see [Sh:f, Ch.V,§1]).
2) Who satisfies this condition? See section 8, so $\mathbf{W}$-complete forcing notions, $\mathrm{Nm}'(D)$,$\mathrm{Nm}(D)$(D is $\aleph_2$-complete) $\mathrm{Nm}^{(\prime)}(T, \mathbf{I})$ (when $\mathbf{I}$ is $\aleph_2$-complete), and shooting a club through a stationary subset of some $\lambda = \mathrm{cf}(\lambda) > \aleph_1$ consisting of ordinals of cofinality $\aleph_0$ (and generally those satisfying the $\mathbb{I}$-condition from [Sh:f, Ch.XI]).

7.4 Claim. *If $\mathbb{Q}$ satisfies $\mathrm{UP}^1_{\mathrm{com}}(\mathbb{I}, \mathbf{W})$ or $\mathrm{UP}^1_{\mathrm{stc}}(\mathbb{I}, \mathbf{W})$ and $\mathbb{I}$ is $(2^{\aleph_0})^+$-complete, and $\mathbb{Q}$ has $(\omega_1, 2)$-pure decidability, then forcing by $\mathbb{Q}$ add no new real.*

Proof. Immediate.

7.5 Claim. *Suppose:*

(*a*) $\mathbb{Q}$ *is a forcing notion satisfying the* $\mathrm{UP}^1_{\mathrm{com}}(\mathbb{I}, \mathbf{W})$ *and the local* $\underset{\sim}{\kappa}$*-c.c. where* $\underset{\sim}{\kappa}$ *is a purely decidable* $\mathbb{Q}$*-name*

(*b*) $\bar{N} = \langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ *is a strict* $(\mathbb{I}, \mathbf{W})$*-suitable tree of models (for* χ *and* $x = \langle \mathbb{Q}, \underset{\sim}{\kappa}, \mathbb{I}, \mathbf{W})\rangle)$ *satisfying* $(*)_1$ *of Definition 7.2*

(*c*) *the family* $\underset{\sim}{\mathbb{I}}' =: \{I \in \mathbb{I} : I$ *is* $\underset{\sim}{\kappa}$*-complete*$\}$ *is* $(< \underset{\sim}{\kappa})$*-closed.*

Then $\mathbb{Q}$ *satisfies* $\mathrm{UP}^1_{\mathrm{stc},\underset{\sim}{\kappa}}(\mathbb{I}, \mathbf{W})$.

Proof. Let $(T^*, \mathbf{I}^*)$ and $\bar{N} = \langle N_\eta : \eta \in (T^*, \mathbf{I}^*)\rangle, \langle h_{\eta,\nu} : \eta, \nu \in T^* \cap {}^n\mathrm{Ord}$ for some $n\rangle$ be as in Definition 7.2.
Let $\eta^* \in \lim(T^*), p \in \mathbb{Q} \cap N_{<>}$ be given and we choose as our strategy for proving $\mathrm{UP}^1_{\mathrm{stc}}(\mathbb{I}, \mathbf{W})$ the same strategy that exists as $\mathrm{UP}^1_{\mathrm{stc}}(\mathbb{I}, \mathbf{W})$ and let $\langle p_n : n < \omega\rangle$ be a play as in Definition 7.2 in which the completeness player uses his winning strategy. Let $\mathscr{T} = \{T : (T^*, \mathbf{I}^*) \leq^* (T, \mathbf{I}^* \restriction T)\}$. As we can replace p by p' if $p \leq_{\mathrm{pr}} p' \in \mathbb{Q} \cap N_{<>}$, without loss of generality p forces a value to κ. So for every $T \in \mathscr{T}$ there are q and $\underset{\sim}{\eta}$ such that

$$p \leq_{\mathrm{pr}} q \in \mathbb{Q} \text{ and } q \Vdash_Q \text{“}\underset{\sim}{\eta} \in \lim \underset{\sim}{T} \text{ and } h = \bigcup h_{\underset{\sim}{\eta}} \restriction n, \eta^* \restriction n$$
$$\text{satisfies } n < \omega \Rightarrow h(\beta) \in \underset{\sim}{G}_{\mathbb{Q}}$$
$$\text{(hence } N_{\underset{\sim}{\eta} \restriction n} \cap \omega_1 = N_{<>} \cap \omega_1)\text{”}.$$

Remember that we can replace η^* by any $\eta^{**} \in \lim(T)$. Let

$$\underset{\sim}{T}^*[G_{\mathbb{Q}}] = \{\nu \in T^* : G_{\mathbb{Q}} \cap N_\nu \supseteq \{h_{\eta,\eta^* \restriction \ell g(\eta)}(r),$$
$$r \in \mathbb{Q} \cap N_{\eta^* \restriction (\ell g(\nu))} \text{ and } r \leq p_n \text{ for some } n\}$$

clearly it is a subtree. We continue as in the proof of 5.2. $\square_{7.5}$

7.6 Claim. *Suppose:*

(a) $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\mathbb{I}}_i, \underset{\sim}{\kappa}_i : i < \alpha \rangle$ *is a* $\mathrm{UP}^{1,e}(\mathbf{W}, W)$*-iteration with* $\mathrm{Sp}_e(W)$*-limit* $\mathbb{P}_\alpha$

(b) $\Vdash_{\mathbb{P}_i}$ "$\underset{\sim}{\mathbb{Q}}_i$ *satisfies* $\mathrm{UP}^1_{\mathrm{stc}}(\underset{\sim}{\mathbb{I}}_i, \mathbf{W})$

(c) $\underset{\sim}{\kappa}^-(\beta) = \mathrm{Min}\{\underset{\sim}{\kappa}_\gamma : \gamma < \gamma\}$,
$\kappa^+(\beta) = \mathrm{Sup}\{\kappa :$ *for some* $\gamma < \beta, \nVdash_{\mathbb{P}_\gamma} \underset{\sim}{\kappa}_\gamma \neq \kappa\}$,
$\kappa^-(\beta) = \mathrm{Min}\{\kappa :$ *for some* $\gamma < \beta$ *we have* $\nVdash_{\mathbb{P}_\gamma} \underset{\sim}{\kappa}_\gamma \neq \kappa\}$.

Then 1) for each $\beta \le \alpha, \mathbb{P}_\beta$ *satisfies* $\mathrm{UP}^1_{\mathrm{com}}(\mathbb{I}'_\beta, \mathbf{W})$ *for some* $\kappa^-(\beta)$*-complete set* $\mathbb{I}'_\beta$ *of (partial order) ideals.*
2) In fact, $\mathbb{I}_\beta$ *is* κ^β*-complete where*
$\kappa^\beta = \min\{\kappa :$ *for some* $\gamma < \beta$ *we have* $\nVdash_{\mathbb{P}_{\gamma+1}}$ "$\underset{\sim}{\kappa}_\gamma \neq \kappa$"$\}$, *and each* $I \in \mathbb{I}_\beta$
has domain of cardinality $\le (\sup\limits_{\gamma<\beta}\{\lambda < \kappa_{\delta+1} : \nVdash_{\mathbb{P}_\gamma}$ "$\neg(\exists I \in \underset{\sim}{\mathbb{I}}_\gamma)(\lambda = |\mathrm{Dom}(I)|)$"$\})$
and $|\mathbb{I}'_\beta| \le \sum\limits_{\gamma<\beta} (\aleph_0 + |\mathbb{P}_\gamma| + \min\{(\lambda : \Vdash_{\mathbb{P}_\gamma}$ "$|\underset{\sim}{\mathbb{I}}_\gamma| \le \lambda)^{<\kappa}\}$.

7.7 Remark. We can use this for iteration as in 5.11, the version with clauses (b), (d) or (d)$'$, $W \cap \alpha = \emptyset$. To prove $\mathbb{P}_\alpha$ does not add reals, it is enough to prove that for each $\beta < \alpha$, forcing with $\mathbb{P}_\beta$ does not add reals. By $\{p \in \mathbb{P}_\beta : (\forall \gamma < \beta) \Vdash_{\mathbb{P}_\gamma}$ "$\emptyset_{\underset{\sim}{\mathbb{Q}}_\gamma} < p \restriction \{\gamma\}$"$\}$ is $\le_{\mathrm{vpr}}$-dense. This should be useful in [\GoSh:511].

SAHARON: 1) Use less $\underset{\sim}{\kappa}$.
2) What requirements will resurrect $\le_{\mathrm{vpr}}$?

Proof. Similar to the one of 5.7.

For each $\gamma < \alpha$ let $\mathscr{I}_\gamma =: \{q \in \mathbb{P}_{\gamma+1} : q$ forces a value to $\underset{\sim}{\kappa}_\gamma$, called $\kappa_{\gamma,q}$ and q forces $\underset{\sim}{\mathbb{I}}_\gamma$ to be equal to a $\mathbb{P}_\gamma$-name $\underset{\sim}{\mathbb{I}}_{\gamma,q}$ and $q \restriction \gamma$ forces a value to $|\mathbb{I}_\gamma|$ says $\mu_{\gamma,q}\}$; let $\mathscr{I}'_\gamma \subseteq \mathscr{I}_\gamma$ be a maximal antichain. Let $\mu_\gamma = \sup\limits_{q \in \mathscr{I}'_\gamma} \mu_{\gamma,q}$.
Let $q \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{\mathbb{I}}_\gamma = \{\underset{\sim}{I}_{\gamma,\zeta} : \zeta < \mu_{\gamma,q}\}$" for $q \in \mathscr{I}_\gamma$ and let $\mathscr{I}_{\gamma,\zeta} = \{q \in \mathscr{I}_\gamma : \mu_{\gamma,q} > \zeta$ and $q \Vdash$ " $\mathrm{Dom}(\underset{\sim}{I}_{\gamma,\zeta})$ is $\lambda_{\gamma,q,\zeta}$" if this is purely decidable$\}$ and let $\underset{\sim}{I}_{\gamma,\zeta}$ be $\mathrm{id}_{L_{\gamma,\zeta}}$, so $L_{\gamma,q,\zeta}$ is a $\mathbb{P}_\gamma$-name of a $\underset{\sim}{\kappa}_{\gamma,q}$-directed partial order on $\lambda_{\gamma,q,\zeta}$ (but $\Vdash_{\mathbb{P}_\gamma}$ "if $|\underset{\sim}{\mathbb{I}}_\gamma| \le \zeta < \mu_\gamma$ then let $L_{\gamma,\zeta}$ be trivial").

For $q \in \mathscr{J}_\gamma$ let $L^*_{\gamma,q,\zeta}$ be $\mathrm{ap}_{\kappa_{\gamma,q}}(\underset{\sim}{L}_{\gamma,\zeta})$ for the forcing notions $\mathbb{P}^{[q]}_\gamma = \{p \in P_\gamma : q \restriction \gamma \leq^{\mathbb{P}_\gamma}_{pr} p\}$ from Definition 3.9. So by Claim 3.10

(*i*) $L^*_{\gamma,q,\zeta}$ is $\kappa_{\gamma,q}$-directed partial order on $[\lambda_{\gamma,q,\zeta}]^{<\kappa_{\gamma,q}}$

(*ii*) $|L^*_{\gamma,q,\zeta}| \leq (\lambda_{\gamma,q,\zeta})^{<\kappa_{\gamma,q}}$

(*iii*) $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{I}_{\gamma,\zeta} = \mathrm{id}_{\underset{\sim}{L}_{\gamma,\zeta}} \leq_{\mathrm{RK}} \mathrm{id}_{L^*_{\gamma,q,\zeta}}$".

Note: $\kappa^-(\beta) = \mathrm{Min}\{\kappa_{\gamma,q} : \gamma < \beta, q \in \mathscr{J}_\gamma\}$.
Let $\mathbb{I}^*_\beta$ be the $(< \kappa^-(\beta))$-closure of $\{\mathrm{id}_{L^*_{\gamma,q,\zeta}} : \gamma < \beta, q \in \mathscr{J}_\gamma, \zeta < \mu_{\gamma,q}\}$ (see Definition 3.13(1)).

Let $\bar{N} = \langle N_\eta : \eta \in (T^*, \mathbf{I})\rangle$ be a strict $(\mathbb{I}^*_\alpha, \mathbf{W})$-suitable tree of models for (χ, x), x coding enough information (so $\bar{\mathbb{Q}}, \mathbb{I}^*_\alpha, \mathbf{W}, W \in N_{\langle\rangle}$). For any $\gamma < \alpha$ and $G_\gamma \subseteq \mathbb{P}_\gamma$ generic over $\mathbf{V}$, $T \subseteq T^*$ and $\mathbf{I}^{[\kappa^-(\gamma)]}$-tree and $\nu \in T$ and $\eta^* \in \lim(T)$ and $p \in N_\nu[G_\gamma] \cap (\mathbb{P}_\alpha/G_\gamma)$ then let $\bar{N}_{\nu,T}[G] = \langle N_{\nu\char`^\rho}[G] : \nu\char`^\rho \in T\rangle$ then we can find a winning strategy $\mathbf{St}$ for the completeness player in the game $\eth = \eth_{\bar{N}_{\nu,T}}[G], p, \eta^*, \mathbb{P}_\alpha/G_\gamma$ of 7.2(2). Without loss of generality if $\eta^*_1, \eta^*_2 \in \lim(T)$ the isomorphism from $N_{\eta^*_1}[G]$ onto $N_{\eta^*_2}[G]$ commutes with the winning strategies; so the choice of η^* is not important. Of course, we have a name $\underset{\sim}{\mathbf{St}} = \mathbf{St}_{\underset{\sim}{\nu},\underset{\sim}{T},\underset{\sim}{\eta}^*}$.

Now fix $\eta^* \in \lim(T^*)$ and we define a strategy $\mathbf{St}$ for the game. For each simple $(\bar{\mathbb{Q}}, W)$-name of an $[0,\alpha)$-ordinal $\underset{\sim}{\gamma} \in N_{\eta^*}$, let $\langle(\tau^{\underset{\sim}{\gamma}}_n, \underset{\sim}{I}^\gamma_n, \eta^*(n), \mathbf{I}_{\eta^*\restriction n}, \mathbf{t}_{\eta^*\restriction m}) : n < \omega\rangle$ be as in 7.2(2) for $\underset{\sim}{\mathbb{Q}}_{\underset{\sim}{\gamma}}$.

We define $\mathbf{St}$ such that if $\bar{p} = \langle p_n : n < \omega\rangle$ is a play in which the completeness player uses his winning strategy then this holds for $\langle p_n(\underset{\sim}{\gamma}) : n < \omega\rangle$ for each γ, i.e.,

(∗) if $\gamma < \alpha, G_\gamma \subseteq \mathbb{P}_\gamma$ is generic over $\mathbf{V}$ and $p \restriction \gamma \in G$ and $\underset{\sim}{\gamma}[G_\gamma] = \gamma$ and $T \in \mathbf{V}[G_\gamma]$ is a subtree of $T^*, \mathbb{I}^{[\kappa^-(\gamma)]}$-large and $\eta^{**} \in \lim(T)^{V[G_\gamma]}$ and $p'_\gamma = (h^*_{\eta^{**},\eta^*}(p_n))(\gamma)[G_\gamma] \in \underset{\sim}{\mathbb{Q}}_\gamma[G_\gamma]$, <u>then</u> in $\langle p'_n : n < \omega\rangle$ the completeness player uses his winning strategy from above.

So fix such $\bar{p}^* = \langle p^*_n : n < \omega\rangle$, we would like to find q as in Definition 7.2(2).
Let $\mathscr{T}_{\bar{N}}$ be the set of quadruples $(\gamma, q, \underset{\sim}{\nu}, \underset{\sim}{T})$ such that:

$\bigotimes_1$ $\gamma \leq \alpha, q \in \mathbb{P}_\gamma, \mathbb{P}_\gamma \models p \restriction \gamma \leq_{\mathrm{pr}} q$ and
$q \Vdash_{\mathbb{P}_\gamma}$ "(α) $\underset{\sim}{\nu} \in \underset{\sim}{T} \subseteq T^*$, where $\underset{\sim}{\nu}, \underset{\sim}{T}$ are $\mathbb{P}_\gamma$-names
(β) $N_{\langle\rangle}[\underset{\sim}{G}_{\mathbb{P}_\gamma}] \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$

(γ) $\gamma \in \bigcup_{\ell<\omega} N_{\underset{\sim}{\nu}}[G_{\mathbb{P}_\gamma}]$

(δ) $\langle N_{\eta,\ell}[\underset{\sim}{G}_{\mathbb{P}_\gamma}] : \eta \in \underset{\sim}{T}$ is a strictly $(\mathbb{I}^*_\gamma)^{[\kappa(\gamma)]}$-suitable tree,

(ε) for every $\underset{\sim}{\eta} \in \lim(\underset{\sim}{T})$ we have

$\{h_{\underset{\sim}{\eta},\eta^*}(p^*_n) \restriction \gamma : n < \omega\}$ is a subset of $\underset{\sim}{G}_{\mathbb{P}_\gamma}$".

Now $\mathscr{T}'_{\bar N}$ is defined similarly as the set of quadruples $(\underset{\sim}{\gamma}, q, \underset{\sim}{\nu}, \underset{\sim}{T})$ such that: as in $\otimes_1$ but we have $\underset{\sim}{\gamma}$ is a simple $(\bar{\mathbb{Q}}, W)$-named ordinal, $q \in \mathbb{P}_{\underset{\sim}{\gamma}}$ and in clause (γ) $\Vdash \underset{\sim}{\gamma} \in N_{\underset{\sim}{\nu}}[G_{\mathbb{P}_{\underset{\sim}{\gamma}}}]$. (I.e., if $\zeta < \beta, G_{\mathbb{P}_\gamma} \subseteq \mathbb{P}_\zeta$ is generic over $\mathbf{V}$ and $\zeta = \underset{\sim}{\gamma}_n[G_{\mathbb{P}_\zeta}]$ then $r \in q_n \Rightarrow \underset{\sim}{\zeta}_n[G_\zeta] < \zeta$, i.e., is well defined $< \zeta$ <u>or</u> is forced $(\Vdash_{\mathbb{P}_\alpha/G_\zeta})$ to be not well defined, and $p \Vdash_{\mathbb{P}_{\underset{\sim}{\gamma}}}$ "$\eta \in \lim(T)$").

We consider the statements, for $\gamma \leq \beta < \alpha$

$\boxtimes_{\gamma,\beta}$ <u>for any</u> $(\gamma, p, \underset{\sim}{\eta}, \underset{\sim}{T}) \in \mathscr{T}_{\bar N}$ and $\underset{\sim}{\rho}$ such that
$p \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{\eta} \triangleleft \underset{\sim}{\rho} \in \underset{\sim}{T}$"
and $\underset{\sim}{p}'$ a $\mathbb{P}_\gamma$-name such that
$p \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{p}'[\underset{\sim}{G}_{\mathbb{P}_\gamma}] \in N_{\underset{\sim}{\rho}}[\underset{\sim}{G}_{\mathbb{P}_\gamma}] \cap P_\beta/\underset{\sim}{G}_{\mathbb{P}_\gamma}$ and
$(\underset{\sim}{p}'[G_{P_\gamma}]) \restriction \gamma \leq p$" <u>there is</u> $(\beta, q, \underset{\sim}{\nu}, \underset{\sim}{T}') \in \mathscr{T}$ such that
$\underset{\sim}{p}' \leq q$ (i.e., $p \Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{p}'[\underset{\sim}{G}_{\mathbb{P}_\gamma}] \leq q$") and $q \restriction \gamma = p$ and $\mathscr{T}' \subseteq \mathscr{T}$".

We prove by induction on $\beta \leq \alpha$ that $(\forall \gamma \leq \beta) \boxtimes_{\gamma,\beta}$ (or, for strong preservation), that ($\forall$ non-limit $\gamma \leq \beta)\boxtimes_{\gamma,\beta}$), note that for $\gamma = \beta$ the statement is trivial hence we shall consider only $\gamma < \beta$.

<u>Case 1</u>: $\beta = 0$.
Trivial.

<u>Case 2</u>: β a successor ordinal.
As trivially $\boxtimes_{\gamma_0,\gamma_1}$ & $\boxtimes_{\gamma_1,\gamma_2} \Rightarrow \boxtimes_{\gamma_0,\gamma_2}$, clearly without loss of generality $\beta = \gamma + 1$.
Let $G_{\mathbb{P}_\gamma}$ be such that $p \in G_{\mathbb{P}_\gamma} \subseteq \mathbb{P}_\gamma$ and $G_{\mathbb{P}_\gamma}$ generic over V.
Let $T' = \{\nu : \rho\hat{\ }\nu \in \underset{\sim}{T}[G_{\mathbb{P}_\gamma}]\}, \bar N' = \langle N'_\nu : \nu \in (T', \mathbf{I}')\rangle$ where $N'_\nu = N_{\rho\hat{\ }\nu}[G_{\mathbb{P}_\gamma}]$, $\mathbf{I}'_\nu = \mathbf{I}^*_{\rho\hat{\ }\nu}$.
By 7.5 applied to $\bar N'$ we can find $\underset{\sim}{p}', \underset{\sim}{T}''$ as required.

Case 3: β is a limit ordinal.

By 7.5 it suffices to prove $\bigotimes_2$ there are q and $\underset{\sim}{\eta}$ such that: $\underset{\sim}{\eta}$ is a $\mathbb{P}_\beta$-name, $q \in P_\beta, q \restriction \gamma = p \restriction \gamma, p \leq q$ and $q \Vdash$ "$\underset{\sim}{\eta} \in \lim(\underset{\sim}{T})$" and $\bigcup\limits_{\ell<\omega} N_{\underset{\sim}{\eta}\restriction\ell}[\underset{\sim}{G}_{\mathbb{P}_\beta}] \cap \omega_1 = N_{\langle\rangle} \cap \omega_1$ and $\{h_{\eta,\eta^*}(r) \restriction \beta : r \in G_{\eta^*}\} \subseteq \underset{\sim}{G}_{\mathbb{P}_\beta}$.

We should choose by induction on $n < \omega, \underset{\sim}{\gamma}_n, q_n, \underset{\sim}{\rho}_n, \underset{\sim}{\eta}_n, k_n$ such that:

(a) $(\underset{\sim}{\gamma}_n, q_n, \underset{\sim}{\eta}_n) \in \mathscr{T}'_{\bar{N}}$
(so $\underset{\sim}{\gamma}_n$ is a $\bar{\mathbb{Q}}$-named ordinal)

(b) $\underset{\sim}{k}_n$ is a $P_{\underset{\sim}{\gamma}_n}$-name of a natural number

(c) $\underset{\sim}{\rho}_n$ is a $\mathbb{P}_{\gamma_n}$-name (of a member of T)

(d) $q_n \Vdash_{\mathbb{P}_{\underset{\sim}{\gamma}_n}}$ "$\underset{\sim}{\eta}_n \restriction \underset{\sim}{k}_n = \underset{\sim}{\rho}_n$"

(e) $\underset{\sim}{\gamma}_0 = \gamma$ and $\Vdash_{\bar{\mathbb{Q}}}$ "$\underset{\sim}{\gamma}_n < \underset{\sim}{\gamma}_{n+1} < \beta$ and $\underset{\sim}{\gamma}_{n+1}$ non-limit"
i.e., if $\zeta < \beta$ and $G_{\mathbb{P}_\gamma} \subseteq \mathbb{P}_\zeta$ is generic over V and $\zeta = \underset{\sim}{\gamma}_n[G_{\mathbb{P}_\zeta}]$ then $r \in q_n \Rightarrow \underset{\sim}{\zeta}_n[G_\zeta] < \zeta$ (i.e. is well defined $< \zeta$ or is forced to be not well defined),

(f) $q_{n+1} \restriction \underset{\sim}{\gamma}_n = q_n$

(g) $q_{n+1} \Vdash_{\mathbb{P}_{\underset{\sim}{\gamma}_{n+1}}}$ "$\underset{\sim}{\rho}_n \triangleleft \underset{\sim}{\rho}_{n+1}$, so $\underset{\sim}{k}_n < \underset{\sim}{k}_{n+1}$".

Finishing the induction we let $\underset{\sim}{\eta} = \bigcup\limits_{n<\omega} \underset{\sim}{\rho}_n$ and we define $q_\omega \restriction \underset{\sim}{\gamma}_n = q_n$ and $q_\omega \in \mathbb{P} \bigcup\limits_{n<\omega} \underset{\sim}{\gamma}_n$.

We shall check that $\bigotimes_2$ holds which is straight.

7.8 Discussion. 1) As in §6 (not §5)?
2) The other NNR.
Like V and like XVIII.
A. Like XVIII,§2 - seem straight but check.
B. Like V,§6 - think.
3) Explain the specific choice for 7.2.
4) Think Ch.VI,§1, $\leq = \leq_{pr}$.
§3 not necessarily.

§8 Examples

Namba [Nm] defines $\mathrm{Nm}(J^{bd}_\lambda)$ (and also with ω ideals) as examples of forcing notion preserving $\aleph_1$ but changing the cofinality of some $\lambda = \mathrm{cf}(\lambda)$ to $\aleph_0$.
More [RuSh 117], [Sh:f, X,XI,XV,XIV,§5].

8.1 Definition. 1) For an ideal I on a cardinal λ, let the forcing notion $\mathrm{Nm}(I)$ be

$$\mathrm{Nm}(I) = \Big\{ T : T \subseteq {}^{\omega>}\lambda \text{ is non-empty, closed under initial segments and } (\forall \eta \in T)(\exists \nu)[\eta \triangleleft \nu \in T \ \& \ (\exists^{I^+} \alpha < \lambda)(\eta\char`^\langle \alpha \rangle \in T)] \Big\}$$

where $(\exists^{I^+} \alpha < \lambda)\mathrm{Pr}(\alpha)$ means $\{\alpha < \lambda : \mathrm{Pr}(\alpha)\} \in I^+$ and $I^+ = \{A \subseteq \lambda : A \notin I\}$ ordered by inverse inclusion and let $<_{\mathrm{pr}} = \leq$ and $\leq_{\mathrm{vpr}}$ be the equality $p \in \mathrm{Nm}(I)$ is normal if $\forall \eta \in p \Rightarrow |\mathrm{Suc}(\eta)| = 1 \vee \mathrm{Suc}_T(\eta) \neq \emptyset \mod I$.
2) For an ideal I and a cardinal λ, let the forcing notions $\mathrm{Nm}'(I)$ be

$$\mathrm{Nm}'(I) = \Big\{ T : T \subseteq {}^{\omega>}\lambda \text{ is non-empty, closed under initial segments and for some } n = n(T) < \omega \text{ we have: } (i)\ \ell \leq n \Rightarrow |T \cap {}^\ell\lambda| = 1 \quad (ii)\ \eta \in T \ \&\ \ell g(\eta) \geq n \Rightarrow (\exists^{I^+} \alpha < \lambda)[\eta\char`^\langle \alpha \rangle \in T] \Big\}$$

we call the $\eta \in T \cap {}^{n(T)}\lambda$ the trunk of T and denote it by $\mathrm{tr}(T)$)
<u>ordered</u> by inverse inclusion and let $\leq_{\mathrm{pr}} = \leq^*$ (see §2) and $\leq_{\mathrm{vpr}}$ be the equality.
3) Writing a filter D means the dual ideal.

8.2 Claim. *Let I be a κ-complete ideal on $\lambda, \lambda \geq \kappa \geq \aleph_2, I \in \mathbb{I}, \mathbb{I}$ is (restriction closed and) κ-complete.*
1) $\mathrm{Nm}(I)$ and $\mathrm{Nm}'(I)$ satisfies $\mathrm{UP}^1(\mathbb{I})$ and $\mathrm{UP}^4_{\lambda^+}(\mathbb{I})$ and $\mathrm{UP}^6_{\lambda^+}(\mathbb{I})$ so does not collapse $\aleph_1$.
) If I is uniform, then $\Vdash_{\mathrm{Nm}(I)}$ "$\mathrm{cf}(\lambda) = \aleph_0$" and $\Vdash_{\mathrm{Nm}'(I)}$ "$\mathrm{cf}(\lambda) = \aleph_0$", in fact if $[A \in I^+ \Rightarrow I \restriction A$ is λ'-decomposable] and λ' is regular, <u>then</u> the same holds for λ'.
3) $|\mathrm{Nm}(I)|, |\mathrm{Nm}'(I)| \leq 2^\lambda$.

4) If in addition $2^{\aleph_0} < \kappa$, then forcing with $\mathrm{Nm}(I)$ *does not add reals, moreover it satisfies the condition from 7.2,* $\mathrm{UP}^4_{\mathrm{com}}(\mathbb{I})$.
5) If in addition $2^{\aleph_0} < \kappa$ then forcing with $\mathrm{Nm}(I), \mathrm{Nm}'(I)$ *does not add reals; moreover, they satisfy the condition* $\mathrm{UP}^{4,+}_{\mathrm{stc}}(\mathbb{I})$.

Proof. 1) We will use the following fact about $\mathbb{Q} = \mathrm{Nm}(I)$ and $\mathbb{Q} = \mathrm{Nm}'(I)$:

(∗) If $p \in \mathbb{Q}, \underset{\sim}{\alpha}$ is a $\mathbb{Q}$-name of an ordinal,
then there is $q, p \leq_{pr} q$ such that the set
$\{\eta \in q :$ for some β we have $q^{[\eta]} \Vdash$ "$\underset{\sim}{\alpha} = \beta$"$\}$ contains a front.

This fact follows easily from 2.13 (let $H : p \to \{0, 1\}$ (i.e., $\mathrm{Dom}(H) = T^p$) be defined by $H(\eta) = 1$ iff $(\exists q)[p^{[\eta]} \leq_{pr} q \ \& \ q$ decides $\underset{\sim}{\alpha}]$, define $H(\eta) = \lim_{n<\omega}(H(\eta \restriction n))$ for $\eta \in \lim(p)$, and find q such that H is constant on $\lim(q)$). Let $Y = \{\eta \in T^q : H(\eta) = 1 \ \& \ (\forall \nu)(\nu \triangleleft \eta \to H(\eta) = 0]\}$, so Y is a front of q. For $\eta \in Y$ let q_η be such that $p^{[\eta]} \leq_{\mathrm{pr}} q_\eta$ and q_η forces a value to $\underset{\sim}{\alpha}$ let r be such that $T^r = \bigcup_{\eta \in Y} T^{q_\eta}$. So clearly r is as required, Y such a front.

Now let $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle$ be a strictly $\mathbb{I}$-suitable tree of models for χ, x satisfying $\{p, I, \mathbb{I}\} \in N_{\langle\rangle}$ where $p \in \mathbb{Q} \cap N_{<>}$ is a condition. We can now find a condition $q, p \leq_{pr} q$, a family $\langle p_\eta : \eta \in p\rangle$ of conditions and a function $f : q \to T$ satisfying the following:

(1) If $\eta \triangleleft \nu$ in q, then $f(\eta) \triangleleft f(\nu)$.
(2) For all η in q, $\mathrm{Suc}_T(f(\eta)) \neq 0 \bmod I$ and $\mathbf{I}_\eta = I$.
(3) For all η in q, $\mathrm{Suc}_q(\eta) \subseteq \mathrm{Suc}_T(f(\eta))$.
(4) For all η in $q, p_\eta \in N_{f(\eta)}, \mathrm{tr}(p_\eta) = \eta, p^{[\eta]} \leq_{pr} p_\eta$.
(5) For all η in $q, p_\eta \leq_{pr} q^{[\eta]}$.
(6) For all η in q, all names $\underset{\sim}{\alpha}$ in $N_{f(\eta)}$, the set
$\{\nu \in q : p_\nu$ decides $\alpha\}$ contains a front of $p^{[\eta]}$.

We can do this as follows: by induction on $\eta \in p$ we define $f(\eta), p_\eta$ and $\mathrm{Suc}_q(\eta)$. We can find $f(\eta)$ satisfying (2) + (3) because T is $\mathbb{I}$-suitable and $I \in \mathbb{I}$ and $\mathbb{I}$ is restriction closed. We choose p_η using a bookkeeping argument to take care of a case of (6), using (∗). Then we choose $\mathrm{Suc}_q(\eta)$ such that (3) are satisfied.

Lastly, let $q = \{\nu$: for some η, p_η is well defined and $\nu \trianglelefteq \eta\}$. Clearly $p \leq_{\mathrm{pr}} q \in \mathbb{Q}$.

Now let G be $\mathbb{Q}$-generic, $q \in G$. Now G defines a generic branch η through q. This induces a branch ν through $T : \nu = \bigcup_{n<\omega} f(\eta \restriction n)$. Let $\underset{\sim}{\alpha} \in N_{\nu \restriction k}$, then there is

ℓ such that $p_{\eta \restriction \ell} \Vdash$ "$\underset{\sim}{\alpha} = \beta$ and $\beta \in N_{f(\eta \restriction \ell)} \subseteq N_\nu$".

2) It is enough to prove the second version for any condition p, let $\mathbf{I}^p_\eta$ be I "mapped" to $\mathrm{Suc}_p(\eta)$.

For any condition p, for each $\eta \in p$ such that $\mathrm{Suc}_p(\eta) \neq \emptyset \bmod I$ let $h_\eta : \mathrm{Suc}_p(\eta) \rightarrow \lambda'$ be such that $(\forall \alpha < \lambda')[\{\nu \in \mathrm{Suc}_p(\eta) : h_\eta(\nu) < \alpha\} \in \mathbf{I}^p_\eta]$. Now letting $\underset{\sim}{\eta} \in {}^\omega\mathrm{Ord}, \underset{\sim}{\eta} \restriction \ell = {}^n\mathrm{Ord} \cap r$ for any $r \in \underset{\sim}{G}_{\mathbb{Q}}$ large enough, we let $\underset{\sim}{A} = \{h_{\underset{\sim}{\eta} \restriction \ell}(\underset{\sim}{\eta} \restriction (\ell+1)) : \ell < \omega\}$.

So easily $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{A} \subseteq \lambda'$ is unbounded.

3) Trivial.

4) Without loss of generality $\mathbb{I}$ is κ-complete (as we can decrease it). So by 7.4 it suffices to prove $\mathrm{UP}^1_{\mathrm{com}}(\mathbb{I}, \mathbf{W})$. So assume $\langle N_\eta : \eta \in (T, \mathbf{I})\rangle, h_{\eta,\nu}$ (for $\eta, \nu \in T \cup \lim(T), \ell g(\eta) = \ell g(\nu)$) are as in Definition 7.2, $(*)_1$ and $\eta^* \in \lim(T), G_{\eta^*}$ as in the assumption of $(*)_2$ there. Now choose inductively on $n < \omega, p_n$ and k_n such that: $p_0 = p, k_0 = 0, p_n \in G_{\eta^*}, p_n \in N_{\eta^* \restriction k_{n+1}}$, and $p_n < p_{n+1}, k_n < k_{n+1}, \eta_n$ is the trunk of $p_n, \eta_n \triangleleft \eta_{n+1}$, $\mathrm{Suc}_{p_n}(\eta_n) \neq \emptyset \bmod I$ (as in proof of part (1)) and $\mathrm{Suc}_T(\eta \restriction k_{n+1}) = \mathrm{Suc}_{p_n}(\eta_n)$ and $p_n^{[\eta_n \frown \langle \eta^*(k_{n+1})]} \leq_{pr} p_{n+1}$ and if $\underset{\sim}{\tau} \in N_{\eta^*}$ is a $\mathrm{Nm}(I)$-name of a countable ordinal then for some n, p_n decides its value.

5) The winning strategy of the completeness player is, given q_n, let $\nu = tr(T)$ and let n be minimal such that $q_n \in N_{\eta^* \restriction n}$ and $I \restriction \mathrm{Suc}_{q_n}(\nu) = I_{\eta^* \restriction n}$ and let $p_n = (q_n)^{[\nu \restriction \eta^*(n)]}$. $\square_{8.2}$

8.3 Definition. 1) We can consider an $\mathbb{I}$-suitable tree of models $\bar{N} = \langle N_\eta : \eta \in (T^*, \mathbb{I}^*)\rangle$, and let

$$a) \qquad \mathbb{Q}_{\bar{N}} = \Big\{T \subseteq T^* : T \text{ non-empty, closed under initial segments such that } \langle N_\eta : \eta \in (T, \mathbf{I} \restriction T)\rangle \text{ is an } \mathbb{I}\text{-suitable tree of models}\Big\}$$

ordered by inverse inclusion.

2) We can consider for any tagged tree $(T^*, \mathbf{I}^*)$

$$\mathbb{Q}^0_{(T^*,\mathbf{I}^*)} = \Big\{ T \subseteq T^* : T \text{ non-empty, closed under initial segments}$$
$$\text{such that for some } n = n(T),$$
$$(i) \qquad \ell \leq n \Rightarrow |T \cap {}^n\text{Ord}| = 1$$
$$(ii) \qquad \text{if } \eta \in T \ \ \& \ \ \ell g(\eta) \geq n \ \ \& \ \ \text{Suc}_{T^*}(\eta) \neq \emptyset \bmod \mathbf{I}_\eta$$
$$\text{then } \text{Suc}_T(\eta) \neq \emptyset \bmod \mathbf{I}_\eta \Big\}$$

is ordered by inverse inclusion.

$$\mathbb{Q}^1_{(T,\mathbf{I}^*)} = \Big\{ (T,\mathbf{I}) : (T^*,\mathbf{I}^*) \leq (T,\mathbf{I}), \text{ and for every } \eta \in \ \lim(T)$$
$$\text{we have } (\forall k)(\exists^\infty n)[\eta \restriction n \text{ is a splitting point of } (T,\mathbf{I})$$
$$\text{and } \mathbf{I}_{\eta \restriction k} \leq_{\text{RK}} \mathbf{I}_{\eta \restriction n} \Big\}$$

ordered by inverse inclusion. [Saharon" [$\mathbb{Q}^0 \neq \mathbb{Q}^2$?]

$$\mathbb{Q}^2_{(T,\mathbf{I}^*)} = \Big\{ (T,\mathbf{I}) : (T^*,\mathbf{I}^*)^{[\eta]} \leq^* (T,\mathbf{I}), \text{ for some } \eta \in T^* \Big\}$$

ordered by inverse inclusion.

8.4 Claim. *For the forcing notions defined in Definition 8.3 for $\mathbb{I}$ being $\aleph_2$-complete, of course, we have: if $P \in \{\mathbb{Q}_{\bar N}, \mathbb{Q}^0_{(T^*,\mathbf{I}^*)}, \mathbb{Q}^1_{(T,\mathbf{I}^*)}\}$, then*

- (a) *P satisfies $\text{UP}^1(\mathbb{I})$*
- (b) *if $I \in \mathbb{I} \Rightarrow |\text{Dom}(I)| < \lambda = \text{cf}(\lambda)$, then $|\mathbb{P}| \leq 2^{<\lambda}$ and even $\leq 2^\mu$ for some $\mu < \lambda$*
- (c) *if for λ regular*
  $(\forall I \in \mathbb{I})(\forall A \in (I)^+[I \restriction A$ is not λ-indecomposable]
  then $\Vdash_{\mathbb{P}}$ "$\text{cf}(\lambda) = \aleph_0$"
- (d) *if $\mathbb{P} = \mathbb{Q}^0_{(T^*,\mathbf{I}^*)}$ then $(\forall \eta \in \ lim(T^*))\, \exists^\infty n \bigwedge_{m \geq n} \forall A \in (\mathbf{I}^*_{\eta \restriction n})^+[\mathbf{I}_\eta \restriction A$ is not λ-indecomposable] then $\Vdash_{\mathbb{P}}$ "$\text{cf}(\lambda) = \aleph_0$"*
- (e) *if $\mathbb{I}$ is $(2^{\aleph_0})^+$-complete then forcing with $\mathbb{P}$ add no new reals, moreover it satisfies $\text{UP}^{4,+}_{\text{stc}}(\mathbb{I})$ and if $p \in \{\mathbb{Q}_{\bar N}, \mathbb{Q}^0_{(T^*,\mathbf{I}^*)}\}$ then it satisfies $\text{UP}^4_{\text{com}}(\mathbb{I})$.*

Proof. Left to the reader. $\square_{8.4}$

8.5 Definition. Let $\lambda = \text{cf}(\lambda) > \aleph_1, S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \aleph_0\}$ be stationary and

$$\text{club}_S(S) = \Big\{h : \text{for some non-limit } \alpha < \omega_1, \quad h \text{ is an increasing function from } \alpha \text{ to } S\Big\}$$

ordered by inverse inclusion, $\leq_{\text{pr}} = \leq, \leq_{vpr}$ is equality.

8.6 Claim. *For λ, S as in Definition 8.5 we have (for any I, I is an $\aleph_2$-complete ideal on λ extending J^{bd}_λ).*
1) Club(S) satisfies $\text{UP}^1(\{I\})$ of cardinality $\leq \lambda^{\aleph_0}$.
2) If I is $(2^{\aleph_0})^+$-complete and $I \in \mathbb{I}$, then Club(S) satisfies $\text{UP}^4_{\text{com}}(\mathbb{I})$, hence $\text{UP}^{4,+}_{\text{stc}}(\mathbb{I})$.

Proof. Left to the reader or follows from [Sh:f, XI,4.6] by 8.9 below. $\square_{8.6}$

8.7 Lemma. *Let $\bar{W} = \langle W_i : i < \omega_1 \rangle$ be a sequence of stationary subsets of $\{\alpha < \lambda : \text{cf}(\alpha) = \omega\}$ where $\lambda = \text{cf}(\lambda) > \aleph_0$ and let the forcing notion $\mathbb{P}[\bar{W}]$ be defined by*

$$\mathbb{P}[\bar{W}] =: \Big\{f : f \text{ is an increasing and continuous function from } \alpha + 1 \text{ into } W_0 \text{ for some } \alpha < \omega_1, \text{ such that for every } i \leq \alpha \text{ we have } f(i) \in W_i\Big\}$$

(ordered by inclusion). If $I \supseteq J^{bd}_\lambda$ be $\aleph_2$-complete, then $\mathbb{P}[\bar{W}]$ satisfies $\text{UP}^{4,+}(\mathbb{I})$ for any $\mathbb{I}$ such that $I \in \mathbb{I}$ and if $\mathbb{I}$ is $(2^{\aleph_0})^+$-complete it also satisfies $\text{UP}^4_{\text{com}}(\mathbb{I})$ hence $\text{UP}^{4,+}_{\text{stc}}(\mathbb{I})$.

Proof. Left to the reader or follows from [Sh:f, Ch.XI,4.6A] by 8.9 above. $\square_{8.7}$

Concerning **W**-completeness (see [Sh:f, Ch.V]):

8.8 Claim. *Assume* $\mathbf{W} \subseteq \omega_1$ *is stationary and* $\mathbb{Q}$ *is* $\mathbf{W}$*-complete forcing notion (i.e., if* χ *is large enough,* $\mathbb{Q} \in N \prec (\mathscr{H}(\chi), \in, <^*_\chi), N$ *countable,* $p_n \in \mathbb{Q} \cap N$ *is* $\leq_{\mathbb{Q}}$*-increasing and* $(\forall \mathscr{I} \in N)(\mathscr{I} \subseteq \mathbb{Q}$ *is dense* $\rightarrow \bigvee_n p_n \in \mathscr{I})$ *then* $\{p_n : n < \omega\}$ *has an upper bound in* $\mathbb{Q}$*).*

Then $\mathbb{Q}$ *satisfies* $\mathrm{UP}^{4,+}_{\mathrm{com}}(\mathbb{I}, \mathbf{W})$ *(i.e., for any* $\mathbb{I}$*).*

Proof. Trivial (see Definition **?**), for the ω-branch η^* of T there is $q, p \leq q, q$ is an
$\longrightarrow$ scite{7.1} undefined
upper bound of $\mathbb{Q}^*$ hence is $(\bigcup_{\ell<\omega} N_{\eta^* \restriction \ell}, \mathbb{Q})$-generic. $\square_{8.8}$

Comparing to [Sh:f] we have

8.9 Claim. *1) If* $\mathbb{Q}$ *satisfies the* $\mathbb{I}$*-condition of [Sh:f, Ch.XI,Def.2.6,2.7]* $\mathbb{I}$ *is* $(2^{\aleph_0})^+$*-complete, then* $\mathbb{Q}$ *satisfies* $\mathrm{UP}^4_{\mathrm{com}}(\mathbb{I})$*.*
2) If $\mathbb{Q}$ *satisfies condition* $\mathrm{UP}(\mathbb{I}, \mathbf{W})$ *of [Sh:f, Ch.XV, Definition 2.7A], then it satisfies* $\mathrm{UP}^4(\mathbb{I}, \mathbf{W})$ *here.*
3) If $\mathbb{Q}$ *is a proper or just semi-proper forcing notion, then* $\mathbb{Q}$ *satisfies* $\mathrm{UP}^6(\mathbb{I})$ *and all the* $UP^\ell(\mathbb{I})$*.*

Proof. Part (2) holds by [Sh:f, XV,2.11].
Part (1) follows by part (2) and [Sh:f, Ch.XV,2.11].
Part (3) is immediate by the definition (can use any fix branch). $\square_{8.9}$

§9 Reflection in $[\omega_2]^{\aleph_0}$

As an exercise we answer a question (and variants) of Jech[9].

9.1 Theorem. *Assume*

(*A*) *κ is large enough (supercompact or just Woodin).*

Then
1) there is a κ-c.c. semi-proper forcing notion $\mathbb{P}$ of cardinality κ such that in $\mathbf{V}^{\mathbb{P}}$

- (α) *$\aleph_2 = \aleph_1^V, \aleph_2 = \kappa$ and cardinal $\geq \kappa$ are the same as in $\mathbf{V}$ and $2^{\aleph_0} = \aleph_2$*
- (β) *every stationary subset of $[\omega_2]^{\leq \aleph_0}$ reflect (in a set of cardinality $\aleph_1$)*
- (γ) *D_{ω_1} is $\aleph_2$-saturated*
- (δ) *there is a projectively stationary $S \subseteq [\omega_2]^{\leq \aleph_0}$, see below such that there is no sequence $\langle a_i : i < \omega_1 \rangle$ increasing continuous, $a_i \in S, a_i \neq a_{i+1}$.*

2) Assume in addition

$\boxtimes$ *$\{\lambda : \lambda$ measurable$\}$ is not in the weakly compact ideal of κ.*

We can add to (1) the statement (on $\mathbf{V}^{\mathbb{P}}$)

(∗) *every stationary $S \subseteq S^2_0 = \{\delta < \omega_2 : cf(\delta) = \aleph_2\}$ contains a closed copy of ω_1.*

3) We may strengthen clause (δ) of (1) to "S is S^2_1-projectively stationary".

9.2 Definition. 1) We call $S \subseteq [\omega_2]^{\aleph_0}$ projectively stationary if:
for every club E of $[\omega_2]^{\aleph_0}$ and stationary co-stationary $W \subseteq \omega_1$ we can find a sequence $\langle a_i : i < \omega_1 \rangle$ increasing continuous, $a_i \in [\omega_2]^{\aleph_0}, a_i \in E$ and $\{i \in W : a_i \notin S\}$ is not a stationary subset of ω_1.
2) We say $S \subseteq [\omega_2]^{\aleph_0}$ is S^2_1-projectively stationary for $W \subseteq \omega_1$ stationary co-stationary, for stationarily many $\delta \in S^2_1$ if we let $\delta = \bigcup\limits_{i<\omega_1} a^\delta_i, a^\delta_i$ countable increasing continuous we have $\{i \in W : a^\delta_i \notin S\}$ non-stationary.

Proof. Like [Sh:f, Ch.XVI,2.4]'s proof.

1), 2) We define a RCS iteration $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : i \leq \kappa, j < \kappa \rangle$ such that:

[9] Done 10/97

$$|\mathbb{Q}_j| \le (2^{\aleph_2})^{\mathbf{V}^{\mathbb{P}_j}}$$

in $\mathbf{V}^{\mathbb{P}_j}$, $\mathbb{Q}_j$ is the disjoint union of the following (so the choice by which of them we force is generic), but if j is non-limit only (a) is allowed

- (a) $\mathbb{Q}^0_j =$ Levy$(\aleph_1, 2^{\aleph_1})*$ Cohen
- (b) $\mathbb{Q}^1_j$ = sealing all semi-proper maximal antichanges of D_{ω_1} provided that strong Chang conjecture holds in $\mathbf{V}^{\mathbb{P}_j}$ (true if j is measurable $> \aleph_0$)
- (c) if we like to have $(*)$ and in $\mathbf{V}^{\mathbb{P}_j}$, strong Chang conjecture holds then allow: $\mathbb{Q}^2_{j,S}$ where $S \subseteq S^2_0$ stationary not containing a closed copy of ω_1 and $\mathbb{Q}^2_{j,S}$ semi-proper where $\mathbb{Q}^2_{j,S}$ shoot an ω_1-increasingly continuous chain i.e.

$$\mathbb{Q}^2_{j,S} = \{(S,f) : S \subseteq S^2_0 \text{ stationary, Dom}(f) \text{ is a successor countable ordinal, } f \text{ is increasingly continuous into } S\}$$

$$(S, f_1) \le (S, f_2) \Leftrightarrow S_1 = S_2 \wedge f_1 \subseteq f_2.$$

So $\Vdash_{\mathbb{P}_j}$ "$\underset{\sim}{\mathbb{Q}}_j$ is semi-proper of cardinality $\le (2^{\aleph_1})^{V^{P_j}}$".

So by [Sh:f, Ch.XVI,§2,2.4,2.5]

- $\bigotimes_1$ for $i < j \le \kappa, \mathbb{P}_j/\mathbb{P}_i$ is semi-proper, so $\aleph_1$ is not collapsed
- $\bigotimes_2$ $\mathbb{P}_\kappa$ collapses every $\theta \in (\aleph_1, \kappa)$, satisfies the κ-c.c. and has cardinality κ
- $\bigotimes_3$ $\Vdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{D}_{\omega_1}$ is $\aleph_2$-saturated".

By preliminary forcing, without loss of generality there is $S_0 = \{\delta < \kappa : \delta \text{ strong limit, cf}(\delta) = \aleph_0\}$, stationary in κ, reflecting only in inaccessibles. Let $S_1 = \{\lambda < \kappa : \lambda \text{ is measurable}\}$ so we know S_1 is stationary. If we are proving the version with $(*)$, note that $\lambda \in S_1 \Rightarrow$ in $\mathbf{V}^{\mathbb{P}_\lambda}$, the strong Chang conjecture holds ([Sh:f, Ch.XIII,1.9]) hence $\mathbb{Q}^2_{\lambda,S}$ is semi-proper for every stationary $S \subseteq \{\delta < \lambda : \mathbf{V}^{\mathbb{P}_\lambda} \models \text{ cf}(\delta) = \aleph_0\}$. Also if $\boxtimes$ holds then

$$\Vdash_{\mathbb{P}_\kappa} \text{"}\underset{\sim}{S} \subseteq \{\delta < \lambda : \text{in } \mathbf{V}^{\mathbb{P}_\kappa}, \text{cf}(\delta) = \aleph_0\} \text{ is stationary } \Rightarrow$$
$$S'_1 = \{\lambda \in S_1 : \underset{\sim}{S} \restriction \lambda \text{ is a } \mathbb{P}_\lambda\text{-name of a stationary}$$
$$\text{subset of } \{\delta < \lambda : \text{cf}(\delta) = \aleph_0 \text{ in } \mathbf{V}^{\mathbb{P}_\lambda}\}\} \text{ is stationary".}$$

So

$\bigotimes_4$ if we are proving $(*)$, then in $\mathbf{V}^{\mathbb{P}_\kappa}$ every stationary $S \subseteq \{\delta < \kappa : \mathrm{cf}(\delta) = \aleph_0\}$ contains a close copy of ω_1.

Now we have to deal with the "projectively stationary". We can find function h, $\mathrm{Dom}(h) = S_0, h(\delta)$ is a $\mathbb{P}_\delta$-name of a stationary co-stationary subset of ω_1 (even $\mathbb{P}_\alpha$-name for some $\alpha < \delta$) such that: every such name appears stationarily often. Let $\langle \underset{\sim}{a}^\delta_i : i < \omega_1 \rangle$ be a $\mathbb{P}_\delta$-name such that

$$\Vdash_{\mathbb{P}_\delta} \text{“}\underset{\sim}{a}^\delta_i \subseteq \delta \text{ is countable unbounded in } \delta \text{ increasingly continuous in } i$$
$$\text{and } \delta = \bigcup_{i<\omega_1} \underset{\sim}{a}^\delta_i\text{”}.$$

Let

$$\underset{\sim}{\mathscr{W}}_\delta = \{a^\delta_i : i \in h(\delta)\}$$

$$\underset{\sim}{\mathscr{W}}_{<\alpha} = \cup\{\mathscr{W}_\delta : \delta \in \alpha \cap S_0\}$$

$$\underset{\sim}{\mathscr{W}} = \mathscr{W}_{<\kappa}.$$

So

$\bigotimes_5$ $\underset{\sim}{\mathscr{W}}$ is a $\mathbb{P}_\kappa$-name of a subset of $[\kappa]^{\aleph_0}$ and S_0 is stationary in $\mathbf{V}^{\mathbb{P}_\kappa}$ (as $\mathbb{P}_\kappa \models \kappa$-c.c.)

$\bigotimes_6$ $\Vdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{\mathscr{W}}$ is stationary.

[why? If $\Vdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{M}$ is a model with countable vocabulary and universe κ" <u>then</u> $\underset{\sim}{E} = \{\delta < \lambda : \underset{\sim}{M} \restriction \delta$ is a $\mathbb{P}_\delta$-name and is an elementary submodel of $M\}$ is a $\mathbb{P}_\kappa$-name of a club of κ hence contains a club E^* of κ from $\mathbf{V}$. So for a club $i < \omega_1, M \restriction a^\delta_i$ is an elementary submodel of M. But for stationarily many $i < \omega_1, a^\delta_i \in \mathscr{W}_\delta \subseteq \mathscr{W}$, so really $\mathscr{W}$ is stationary. If $\underset{\sim}{W}$ is a $\mathbb{P}_\kappa$-name of a stationary co-stationary subset of ω_1 then for some, even for stationarily many $\delta \in E^* \cap S_0$ we have $h(\delta) = \underset{\sim}{W}$ and so easily

$\bigotimes_7$ $\Vdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{\mathscr{W}}$ is projectively stationary".

Lastly, why would $\mathscr{W}$ contain no increasing ω_1-chains? Assume $p^* \Vdash$ "$\langle \underset{\sim}{a}_i : i < \omega_1\rangle$ is increasing continously and $\underset{\sim}{a}_i \in W$". So without loss of generality for some δ^* either

- (α) $p^* \Vdash$ "$\sup(a_i)$ is strictly increasing with limit δ^*" or
- (β) $p^* \Vdash$ "$\sup \underset{\sim}{a}_i$ is constantly δ^* for $i \geq i^*, i^* < \omega_1$ so without loss of generality $i^* = 0$".

Case A: The possibility (β) holds.
Necessarily $\delta^* \in S_0, p^* \Vdash_{\mathbb{P}}$ "$\{\underset{\sim}{a}_i : i < \omega_1\} \subseteq \mathscr{W}_{\delta^*}$" and as $\Vdash_{P_{\delta^*}}$ "$h(\delta^*)$ is co-stationary subset of ω_1" and $\mathbb{P}_\kappa/\mathbb{P}_{\delta^*}$ is semi-proper hence preserve stationarity of subsets of ω_1 we are done.

Case B: Possibility (α) holds and δ^* not strongly inaccessible. So $S_0 \cap \delta^*$ is not stationary in δ^* hence $\underset{\sim}{\mathscr{W}} \cap [\delta^*]^{\aleph_0} = \bigcup\limits_{\delta \in \delta^* \cap S_0} \mathscr{W}_\delta$ is not even stationary.

Case C: Possibility (α) holds and not case B, in $V^{P_{\delta^*}}$ strong Chang conjecture fails.

Then $\underset{\sim}{\mathbb{Q}}_\delta$ is Levy$(\aleph_1, 2^{\aleph_1}) *$ Cohen (as in clauses (b) and (c) in $\mathbf{V}^{\mathbb{P}_\delta}$ strong Chang conjecture holds), so as clearly in $\mathbf{V}^{\mathbb{P}_\delta}, 2^{\aleph_0} = \aleph_2$ (by the Cohen in (a), i.e. $\mathbb{Q}^0_j$), then in $\mathbf{V}^{\mathbb{P}_{\delta^*}}$ for every club E' of $[\delta^*]^{\aleph_0}$, we can find some $\delta < \delta^*$ and $2^{\aleph_0}$ members of $E' \cap [\delta]^{\aleph_0} \backslash \underset{\sim}{W}_{<\delta^*}$. So in $\mathbf{V}^{\mathbb{P}_{\delta^*}}, [\delta^*]^{\aleph_0} \backslash \underset{\sim}{\mathscr{W}}_{<\delta^*}$ is stationary and $\underset{\sim}{\mathbb{Q}}_\delta$ is proper so this holds in $\mathbf{V}^{\mathbb{P}_{\delta^*+1}}$. But $\mathbb{P}_\kappa/\mathbb{P}_{\delta^*+1}$ preserves stationarity of subsets of ω_1 hence in $\mathbf{V}^{\mathbb{P}_\kappa}[\delta^*]^{\aleph_0} < \mathscr{W} < \delta$ is stationary, so we are done.

Case D: Possibility (α) holds, not case B and in $\mathbf{V}^{\mathbb{P}_{\delta^*}}$ strong Chang conjecture holds.

Just note: in $\mathbf{V}^{\mathbb{P}_{\delta^*}}$, let $p \in \mathbb{Q}_{\delta^*}$, let χ large enough $N \prec (\mathscr{H}(\chi), \in)$ is countable to which $\bar{\mathbb{Q}}, \delta^*, G_{\mathbb{P}_{\delta^*}}, p, \langle \mathscr{W}_\delta : \delta \in S_0 \cap \delta^*\rangle$ belong, then we can find (see [Sh:f, Ch.XIII]) $T \subseteq {}^{\omega >}\omega_2$ closed under initial segments $T \cap N = \emptyset$, satisfying $(\forall \eta \in T)(\exists^{\aleph_2} \alpha)(\eta \char`^ \langle \alpha \rangle \in T)$ and $\langle N_\eta : \eta \in T\rangle$ such that

- (i) $N_{<>} = N$
- (ii) $N_\eta \prec (\mathscr{H}(\chi), \in)$ is countable
- (iii) $\eta \in N_\eta, N_\eta \cap \omega_1 = N \cap \omega_1$
- (iv) $\nu \triangleleft \eta \Rightarrow N_\eta \subseteq N_\nu$
- (v) if $\mathscr{I} = \{A_\zeta : \zeta < \zeta^*\}$ is a maximal antichain of $\mathscr{D}_{\omega_1}$ which is semi-proper and $\mathscr{I} \in N_\eta$ then for some $k < \omega, \eta \triangleleft \nu \in T \ \& \ \ell g(\nu) \geq k \Rightarrow N \cap \omega_1 \in \bigcup\limits_{\zeta \in N_\nu} A_\zeta$.

Let

$$E = \{\delta < \omega_2 : \text{if } \eta \in {}^{\omega>}\delta \text{ then } N_\eta \cap \omega_2 \text{ is a bounded subset of } \delta\}.$$

Now if $p \in \mathbb{Q}^0_\delta$ we do as in Case C. If $p \in \mathbb{Q}^1_{\delta^*}$, choose $\delta \in E$, $\mathrm{cf}(\delta) = \aleph_0$, and such that for every $\eta \in T \cap {}^{\omega>}\delta, \delta =$ otp$\{\beta < \delta : \eta\hat{\ }\langle\beta\rangle \in T\}$ and $\eta\hat{\ }\langle\alpha\rangle \in T \ \& \ \alpha < \delta \Rightarrow \sup(N_\alpha \cap \omega_2) < \delta\}$. Now we can by cardinality considerations ($2^{\aleph_0} > \aleph_1$) find $\eta \in \lim(T) \cap {}^\omega\delta$ such that letting $M = \bigcup\limits_{\ell<\omega} N_{\eta\restriction\ell}, M \cap \omega_2 = M \cap \delta \notin \mathscr{W}_{<\delta^*}$. So there is $q \in \mathbb{Q}_\delta$ which is $(M, \mathbb{Q}_\delta)$-generic, $p \le q$ (by the definition of $\mathbb{Q}^1_{\delta^*}$). Now q forces $\underset{\sim}{a}_{M\cap\omega_1} = a_{M\cap\omega_1}$ to be $M \cap \omega_2$ which is not in $\underset{\sim}{\mathscr{W}}_{<\delta^*}$.

Lastly if $q \in \mathbb{Q}^2_{j,S}$ (in $\mathbf{V}^{\mathbb{P}_{\delta^*}}$) as S does not reflect we can find $\delta \in E$ as above, $\delta \in S$, $\mathrm{cf}(\delta) = \aleph_0$ and choose η, M as above.
3) We may like to adapt the proof above.
We omit the choice of $\langle \underset{\sim}{a}^\delta_i : i < \omega_1\rangle$, <u>but</u> in $\underset{\sim}{\mathbb{Q}}_j$ if $j \in S_0$ we also choose a $\mathbb{P}_j$-name of a countable unbounded subset of $\delta, \underset{\sim}{a}_\delta$ and let $\underset{\sim}{\mathscr{W}}_\delta = \{\underset{\sim}{a}_\delta\}$ so $\mathbb{Q}_j$ is replaced by $\mathbb{Q}_j \times \{\underset{\sim}{a} : \underset{\sim}{a}$ a name as above$\}$. Now h_0 has domain $S^* = \{\delta : \delta$ strongly inaccessible, in $\mathbf{V}^{\mathbb{P}_\delta}$, strong Chang conjecture holds$\}$, $h_0(\delta)$ a $\mathbb{P}_\delta$-name of a stationary co-stationary subset of ω_1 and we add to clauses (a), (b), (c) above also

(d) define in $\mathbf{V}^{\mathbb{P}_\delta}$:

$$\mathbb{Q}^3_\delta = \Big\{\langle M_i : i \le j\rangle : \text{the ordinal } j \text{ is countable and } M_i \prec (\mathscr{H}((2^\delta)^+), \in) \text{ is countable increasing continuous, and: } \underline{\text{if}}\ M_i \cap \omega_1 \in h_0(\delta)\ \underline{\text{then}}\ M_i \in \mathscr{W}_{<\delta} \text{ and if } M_i \cap \omega_1 \notin h(\delta) \text{ then } M_i \notin \mathscr{W}_{<\delta}\Big\}.$$

Now again we use $\langle N_\eta : \eta \in T\rangle$ and choosing M it is enough to show that

$\boxtimes_1$ for some $\eta \in \lim T, \bigcup\limits_{i<\omega} N_{\eta\restriction\ell} \cap \omega_2 \in \mathscr{W}_{<\delta}$

$\boxtimes_2$ some $\eta \in \lim T, \bigcup\limits_{\ell<\omega} N_{\eta\restriction\ell} \cap \omega_2 \notin \mathscr{W}_{<\delta}$.

Now $\boxtimes_2$ is as before, $\boxtimes_1$ O.K. by the way $\underset{\sim}{\mathscr{W}}_{<\delta}$ is defined. $\boxtimes_{9.1}$

§10 Mixing finitary norms and ideals

We may consider replacing families of ideals by families of creatures see [RoSh 470] on creatures:
We hope it will gain something

10.1 Definition. : 1) A λ-creature $\mathfrak{c}$ consists of $(D^{\mathfrak{c}}, \leq, \mathrm{val}^{\mathfrak{c}}, \mathrm{nor}^{\mathfrak{c}}, \lambda^{\mathfrak{c}})$, where:

$\lambda^{\mathfrak{c}} = \lambda$

$D^{\mathfrak{c}}$ the domain,

$\leq$ a partial order on $D^{\mathfrak{c}}$,

$\mathrm{val}^{\mathfrak{c}}$ is a function from $D^{\mathfrak{c}}$ to $\mathscr{P}(\lambda)\backslash\{\emptyset\}$

$\mathrm{nor}^{\mathfrak{c}} : D^{\mathfrak{c}} \to \omega$ or to??

2) It is called simple if $\mathrm{nor}^{\mathfrak{c}}$ is always > 0 (without loss of generality constant, e.g. $\mathrm{Rang}(\mathrm{val}^{\mathfrak{c}}) = I^+, I$ an ideal on λ). A creature is a λ-creature for some λ.
$\mathbb{I}$ will be a set of creatures.

10.2 Definition. 1) An $\mathbb{I}$-tree is $(T, \mathbf{I}, \mathbf{d})$ such that:
for some ordinal $\alpha, T \subseteq {}^{\omega>}\alpha$ closed under initial segments, $\neq \emptyset$
$\mathbf{I}$ is a partial function $\mathrm{Dom}(\mathbf{I}) \subseteq T, \mathbf{I}_\eta \in \mathbb{I}$,
$\mathbf{d}$ has domain $\mathrm{Dom}(\mathbf{I}), \mathbf{d}(\eta) \in D^{\mathbf{I}_\eta}$
$\mathrm{val}^{\mathbf{I}_\eta}(\mathbf{d}(\eta)) = \{\alpha : \eta\char`^\langle\alpha\rangle \in T\}$.
2) Let $(T^*, \mathbf{I}^*, \mathbf{d}^*)$ be an $\mathbb{I}$-tree, such that

$(*)$ $(\forall\eta \in \lim T^*)[\lim\ \sup n < \omega\ \mathrm{nor}^{\mathbf{I}^*_{\eta\restriction n}}(\mathbf{d}^*_\eta) = \infty]$
and $\mathrm{Dom}(\mathbf{I}_\eta) = T^*$.

We define a forcing notion $\mathbb{Q} = \mathbb{Q}_{(T^*, \mathbf{I}^*, \mathbf{d}^*)}$:

$$\mathbb{Q} = \Big\{(T, \mathbf{I}, \mathbf{d}) : T \subseteq T^*, \mathbf{I} = \mathbf{I}^* \restriction T, \\ (T, \mathbf{I}, \mathbf{d}) \text{ an } \mathbb{I}\text{-tree}, \\ (\forall\eta \in T)(\mathbf{d}^*_\eta \leq^{\mathbf{I}_\eta} \mathbf{d}_\eta) \\ \text{and } ((\forall\eta \in \lim T^*) \lim\sup \mathrm{nor}^{\mathbf{I}^*_{\eta\restriction n}}(\mathbf{d}_\eta) = \infty)\Big\}.$$

Order: natural.

3) Let $(T^*, I^*, \mathbf{d}^*)$ be an $\mathbb{I}$-tree such that

(**) $(\forall \eta \in \lim T^*)(\forall n)(\forall^* \ell)(\mathrm{nor}^{I^*_{\eta \restriction n}}(\mathbf{d}^*_\eta) \geq n)$
(i.e. $\lim\inf = \infty$).

and define $\mathbb{Q}' = \mathbb{Q}'_{(T^*, \mathbf{I}^*, \mathbf{d}^*)}$ parallelly.
4) For $p \in \mathbb{Q}$ (or $p \in \mathbb{Q}'$) we write $p = (T^p, \mathbf{I}^p, \mathbf{d}^p)$. In this case for $\eta \in T^p$ we define $q = p^{[\eta]}$ by:
$T^{[q]} = \{\nu \in T^p : \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu\}, \mathbf{I}^q = \mathbf{I}^p \restriction T^q, \mathbf{d}^q = \mathbf{d}^p \restriction T^p$. Clearly $p \in \mathbb{Q} \Rightarrow p \leq q \in \mathbb{Q}$ and $p \in \mathbb{Q}' \Rightarrow p \leq q \in \mathbb{Q}'$.

10.3 Claim. *Let $(T, \mathbf{I}^*, \mathbf{d}^*)$ and $\mathbb{Q}, \mathbb{Q}'$ be as in 10.2. A sufficient condition for "$\aleph_1$ not collapsed" is:*

(a) *for $\mathbb{Q}$: (**) below*
(b) *for $\mathbb{Q}'$: (*) + (**) below where*

(*) *$\mathbb{I}$ has $\aleph_1$-bigness:*

$$(\forall \mathfrak{c} \in \mathbb{I})(\forall x \in D^{\mathfrak{c}})\Big[nor^{\mathfrak{c}}(x) > 0 \rightarrow (\forall h \in {}^{(\lambda^{\mathfrak{c}})}\omega_1)(\exists y) [x \leq^{\mathfrak{c}} y \ \& \ nor^{\mathfrak{c}}(y) \geq nor^{\mathfrak{c}}(x) - 1 \ \& \ (h \restriction val^{\mathfrak{c}}(y) \text{ is constant}] \Big]$$

(**) *$\mathbb{I}$ is $(\aleph_1, \aleph_1)$-indecomposable where $\mathbb{I}$ is (μ, κ)-indecomposable means:*

$\boxtimes_{\mathbb{I},\mu,\kappa}$ *if $\mathfrak{c} \in \mathbb{I}$ and $x \in D^{\mathfrak{c}}$ satisfies $nor^{\mathfrak{c}}(x) > 2$ and $A_\alpha \subseteq \lambda^{\mathfrak{c}}$ for $\alpha < \mu$ are such that $(\forall y)(x \leq y \in D^{\mathfrak{c}} \wedge val^{\mathfrak{c}}(y) \subseteq A_\alpha \rightarrow nor^{\mathfrak{c}}(y) + 2 \leq nor^{\mathfrak{c}}(x))$, then we can find $u \subseteq \lambda^{\mathfrak{c}}$ of cardinality $< \mu$ such that for every large enough $\alpha < \mu$ we have $u \nsubseteq A_\alpha$.*

Proof for $\mathbb{Q}$. Lets use given $p = (T, \mathbf{I}, \mathbf{d}) \in \mathbb{Q}$ and $\mathbb{Q}$-name $\underset{\sim}{\tau}$ such that $\Vdash \underset{\sim}{\tau} : \omega \rightarrow \omega_1$. Now we choose by induction on n, p^n, A_n such that:

(a) $p^n \leq p^{n+1}$
(b) $A_0, \ldots, A_n$ are fronts of p^n which means $(\forall \eta \in \lim^{p^n})(\exists! n)(\eta \restriction n \in A_\ell)$

(c) A_ℓ below $A_{\ell+1}$ which means
$(\forall \eta \in A_{\ell+1})(\exists \nu \triangleleft \eta)\nu \in A_\ell$
(so $A_n \subseteq T^{p^{n+1}}, (\forall \eta \in T^{p^n} \backslash T^{p^{n+1}})(\exists \nu \triangleleft \eta)(\nu \in A_n)$)

(d) $(\forall \nu \in A_n)(\forall \eta \in \text{ Suc}_{T^{p_n}}(\nu))(p_n^{[\eta]}$ forces a value to $\underset{\sim}{\tau}(n))$

(e) $\eta \in A_n \Rightarrow \text{ nor}^{\mathbf{I}^*_\eta}(\mathbf{d}^{p_n}_\eta) \geq n \ \& \ \mathbf{d}^{p_n}_\eta = \mathbf{d}^{p_{n+1}}_\eta$, it follows that $\ell < n \ \& \ \eta \in A_\ell \Rightarrow \mathbf{d}^{p_\ell}_\eta = \mathbf{d}^{p_n}_\eta \ \& \ \text{ Suc}_{T_{p_n}}(\eta) = \text{ Suc}_{T^{p_\ell}}(\eta)$.

So p^* is defined by $T^{p^*} = \bigcap\limits_{n<\omega} T^{p_n}, \mathbf{I}^{p^*} = \mathbf{I} \restriction T^{p^*}, \mathbf{d}^{p^*} = \bigcup\limits_{n} \{\mathbf{d}^{p_n} \restriction \{\eta \restriction \ell : \eta \in A_n$ and $\ell \leq \ell g(\eta)\} : n < \omega\}$ belong to $\mathbb{Q}$ and is an upper bound of $\{p_n : n < \omega\}$.

We define $h : T^{p^*} \to \omega_1$ as follows: if $\eta \in T^{p^*}, \nu \triangleleft \eta \trianglelefteq \nu', \nu \in A_{\ell-1}, \nu' \in A_\ell$ (if $\ell = 0$ omit ν', so just $\eta \trianglelefteq \nu'$), then $p^{*[\eta]}$ forces value to $\underset{\sim}{\tau} \restriction \ell$ call it $(\tau \restriction \ell)^{p^{*[\eta]}}$ and let

$$h(\eta) = \text{ Sup Rang}(\underset{\sim}{\tau} \restriction \ell)^{p^{*[\eta]}}.$$

For notational simplicity $A_n = T^{p^*} \cap {}^n\text{Ord}$.
We now define a game $\partial = \partial^\alpha_{T^{p^*}}$ for each $\alpha < \omega_1$:

A play of the game last ω moves, in the $(n-1)$-th move a member η_n of A_n is chosen such that $m < n \Rightarrow \eta_m \triangleleft \eta_n$, and fixing some $\eta_{-1} \in A_n$.
In the n-the move:

(a) the anti-decidability player chooses a set $A_n \subsetneq \text{ Suc}_{T^{p^*}}(\eta_{n-1})$ such that

$$B_n = \emptyset \vee (\text{nor}^{\mathbf{I}_\eta}(A_n) \leq n-2, n \geq 2)$$

$\boxtimes_1$ $B_n \neq \emptyset$ or $n \geq 3$ and for no d, satisfying $\mathbf{d}^{p^*}(\eta) \leq d \wedge \text{ nor}^{\mathbf{I}^*_2}(d) \geq n-2$ do we have $\eta\hat{\ }\langle\alpha\rangle \in A_n \Rightarrow \alpha \in \text{ val}^{\mathbf{I}^*_\eta}(d)$

(b) the decidability player chooses $\eta_n \in A_n$ such that $\neg(\exists \nu \in B_n)(\nu \trianglelefteq \eta_n)$ and $n \geq 1 \Rightarrow (\eta_n \restriction (\ell g(\eta_{n-1}) - 1)) \leq \alpha$.

Without loss of generality $A_0 = \{<>\}$.

If for some α and decidability player has a winning strategy, we can produce a condition as required.

If not, for every $\alpha < \omega_1$ the antidecidability player has a winning strategy $\mathbf{St}_\alpha$. For each $\eta \in T^{p^*}$ and $\alpha < \omega_1$, we consider the play of the game in which the antidecidability player has winning strategy $\mathbf{St}_\alpha$ and in some move n the decidability player chooses $\eta_n = \eta$. Reflecting there is no freedom left so there is at most one such play and n and let the antidecidability player choose set $B_{\eta,\alpha}$ as there (if no such game let $B_{\eta,\alpha} = \emptyset$).

Case 1: For some n and $\eta \in A_n$, we have: there is no countable $u \subseteq \mathrm{Suc}_{T^{p^*}}(\eta)$ such that for every large enough $\alpha < \omega_1, u \not\subseteq B_{\eta,\alpha}$ so by the assumption $(**)$ we get a contradiction.

Case 2: Not Case 1.

We can choose by induction on n, a countable subset $u_n \subseteq A_n$ such that: $u_0 = \{<>\}$

$$\begin{aligned}&\text{if } \eta \in A_n \text{ then for some } \alpha_\eta < \omega_1 \text{ for every } \alpha \in [\alpha_\eta, \omega_1),\\ &\qquad \text{if in the play in which the antidecidability player uses } \mathbf{St}_\alpha\\ &\qquad \text{and they arrive to } \eta, \text{ there is } \eta', \eta \triangleleft \eta' \in u_n + 1\\ &\qquad \text{which is a legal response of the decidability player.}\end{aligned}$$

mn Now let

$$\begin{aligned}\alpha^* = &\sup\{h(\eta)+1 : \text{ for some } \nu \in \bigcup_n u_n, \eta \triangleleft \nu, \eta \in \mathrm{Dom}(h)\} +\\ &\sup\{\alpha_\eta : \eta \in \bigcup_n u_n\}\end{aligned}$$

and we can find a play of ∂^{α^*} as above where the decidability player chooses η's from $\bigcup\limits_{n<\omega} u_n$. We get a contradiction.

Proof for $\mathbb{Q}'$.? We should make changes: in p^{n+1} we shrink $p_n^{[\eta]}$ for each $\eta \in T^{p^n} \cap {}^n\mathrm{Ord}$, to $q_\eta, p_n^{[\eta]} \leq_{pr} [\eta]$ and for each $\ell \leq n$, if possible, q_η forces a bound to $\underset{\sim}{\tau}(\ell)$ and, of course, $p_{n+1}^{[\eta]} = q_\eta$ for each such η and $T^{p^{n+1}} \cap {}^{\eta\geq}\mathrm{Ord} = T^{p^n} \cap {}^{n\geq}\mathrm{Ord}$ and $\mathbf{d}^{p^{n+1}} \restriction {}^{n\geq}\mathrm{Ord} = \mathbf{d}^{p^n} \restriction {}^{n\geq}\mathrm{Ord}$. So let $p^* = \bigcap\limits_n p_n$ be naturally defined, and we use 2-bigness to prove enough times q_η forces a bound.

Now we give details.

Proof for $\mathbb{Q}$. Given $p = (T, \mathbf{I}, \mathbf{d})$, for notational simplicity $\mathrm{tr}(p) = <>$ and $\mathrm{nor}^{\mathbf{d}^p_\eta}(\mathrm{Suc}_{T^p}(\eta)) > 2$ and $\mathbb{P}$-name $\underset{\sim}{\tau}$ such that $\Vdash \underset{\sim}{\tau} : \omega \to \omega_1$ we choose by induction on n, p^n such that:

- (a) $p^n \le p^{n+1}$, and p^n has trunk n
- (b) $A_0, \dots, A_n$ are fronts of p^n
- (c) A_ℓ below $A_{\ell+1}$ which means $(\forall \eta \in A_{\ell+1})(\exists \nu \triangleleft \eta)\nu \in A_\ell$ (so $A_n \subseteq T^{p^{n+1}}, (\forall \eta \in T^{p^n} \backslash T^{p^{n+1}})(\exists \nu \triangleleft \eta)(\nu \in A_n))$
- (d) $A_0 = \{<>\}[\eta \in A_n \wedge \eta \trianglelefteq \nu \in T^{p^n} \Rightarrow \ \mathrm{nor}^{\mathbf{I}_\nu}(\mathbf{d}^{p_n}_\nu) > n + 2]$
- (e) when $\eta \in A_n$ let $\ell = \ell_\eta \le n$ be maximal such that there are $\alpha_m < \omega_1, m < \ell$ and q satisfying $\mathrm{tr} q = \eta, p^{[\eta]}_n \le q \in P, q \Vdash$ " $\bigwedge\limits_{m<\ell} \underset{\sim}{\tau}(m) < \alpha'_n$ and $\eta \trianglelefteq \nu \in T^q \Rightarrow \ \mathrm{nor}^{\mathbf{d}^*_\nu}(\mathbf{d}^q_\nu) \ge n$ and we demand: $p^{[\eta]}_{n+1}$ satisfies the demand on q for some $\langle \alpha_m : m < \ell_\eta \rangle$, note possible $\ell_\eta = \nu$ then we are left with demand on norm.

So $p^*, T^{p^*} = \bigcap\limits_{n<\omega} T^{p_n}$ is an upper bound of $\{p_n : n < \omega\}$.

Clearly $p^* \in \mathbb{P}$ and $n < \omega \Rightarrow p_n \le p^*$. Let for $\eta \in T^{p^*}$, let $n(\eta) = \ \mathrm{Max}\{n :$ there is $\nu \triangleleft \eta, \nu \in A_n\}$ and $\nu_\eta \triangleleft \eta$ be in A_n and $\beta_\eta < \omega_1$ be minimal such that $p^{[\nu_\eta]}_{n+1}$ forces $\tau(0), \dots, \tau(\ell_{\nu_\eta} - 1) < \beta_\eta$. Using games as in the proof for $\mathbb{Q}'$ there is p^+ such that:

- (a) $p^* \le p^* \in \mathbb{P}$
- (b) $\rho \in T^{p^+} \Rightarrow \ \mathrm{nor}^{\mathbf{I}^*_\rho}(\mathbf{d}^{p^+}_\rho) \ge \ \mathrm{nor}^{\mathbf{I}^*_\rho}(\mathbf{d}^{p^*}_\rho) - 1$
- (c) $\beta^* = \sup\{\beta_\eta : \eta \in T^{p^+}\} < \omega_1$.

We continue as in [Sh:f, Ch.XIV,§5].

* * *

<u>Discussion</u>: We can continue to do iteration.
But more urgent: can $\mathbb{Q}, \mathbb{Q}'$ like this do anything not already covered by composition?

A natural thought is splitting or reaping numbers. We can think of the tree splitting in T^* as a list of the reals. BUT, what is the norm?

* * *

Not finished...check the better's theorem proof?

Assignment: [Sh:f, XIII,XVI] and here put together, so does the reflection $Pr_a(\lambda, f)$ works for ???

97/2/2 - Discussion:

Saying a creature is μ-complete meana that for pure extensions, increasing chains of length $< \mu$ have pure upper bounds? Probably pure means not changing the norm; maybe the $\aleph_1$-indecomposable is enough.
So the $\mathbb{I}$-th condition has a new meaning.

Question: Does the theorem here hold?

Question: Does this new context have real applications?

The first result to be discussed is moving from $\underset{\sim}{\mathbb{I}}$ to one in the ground model.

The second are 5.2, **?** preservation of N being suitable.
$\longrightarrow$ scite{6.2} undefined

§11 Variants of the iteration

As mentioned in §1, we can consider κ-RS iteration and variants of the Sp_e iteration.

11.1 Definition/Claim. Let κ be a successor cardinal or an infinite ordinal not a cardinal but an ordinal of power $|\kappa|, \kappa$ fix[10]. We define and prove the following by induction on α (here $e = \{3,4,5,6\}$). If $\kappa = \aleph_1$, we may omit it and this is the main case.

We repeat 1.15 with the following changes in the proof and definition:

(B) We say $\underset{\sim}{\zeta}$ is a simple $\bar{\mathbb{Q}}$-named$_e$ $[j,\beta)$-ordinal if

- $(*)_1$ $\underset{\sim}{\zeta}$ is a simple $\bar{\mathbb{Q}}$-named^1 $[j,\beta)$-ordinal and may restrict ourselves to $\kappa = \aleph_1 \Rightarrow e \in \{3,4\}$
- $(*)_2$ if $e \in \{5,6\}$ and $\kappa = \aleph_1$, <u>then</u> $\underset{\sim}{\zeta}$ is a simple $\bar{\mathbb{Q}}$-named^2 $[j,\beta)$-ordinal.

$(F)(a)$ in (v) replace the remark in the end by:
"if $e \in \{5,6\}, \alpha \in w$ then this demand follows by 1.8
and add:

- (vii) if $e = 3,5$ then for some $n < \omega$ and simple $\bar{\mathbb{Q}}$-named $[0, \ell g(\bar{\mathbb{Q}}))$-ordinals $\underset{\sim}{\xi}_1, \ldots, \underset{\sim}{\xi}_n$ we have, for every $\xi < \ell g(\bar{\mathbb{Q}}) \Vdash_{\mathbb{P}_\xi}$ "if for $\ell = 1, \ldots, n$ we have $\underset{\sim}{\xi}_\ell[\underset{\sim}{G}_{\mathbb{P}_\xi}] \neq \xi$ (for example $\underset{\sim}{\xi}_\ell[G_{\mathbb{P}_\xi}]$ not well defined) then $\emptyset_{\underset{\sim}{\mathbb{Q}}_\xi} \leq_{\mathrm{pr}} p \restriction \{\xi\}$ in $\hat{\underset{\sim}{\mathbb{Q}}}_\xi$"

$(F)(e)(iii)$ inside change $p_2 \restriction \xi \Vdash_{\mathbb{P}_\xi}$ "..." by
$p_2 \restriction \xi \Vdash_{\mathbb{P}_\xi}$ "if $\xi \neq \underset{\sim}{\xi}_\ell[\underset{\sim}{G}_{\mathbb{P}_\xi}]$ for

$\ell = 1, \ldots, n$ and
$[e = 4 \vee e = 6 \Rightarrow \neg(\emptyset_{\underset{\sim}{\mathbb{Q}}_\xi} \leq_{vp} p^1 \restriction \{\xi\})]$
then: $\hat{\underset{\sim}{\mathbb{Q}}}_\xi \models p^1 \restriction \{\xi\} \leq_{pr} p^2 \restriction \{\xi\}$".

[10] For κ inaccessible, see **?**.
—> scite{1.22} undefined

11.2 Claim. *1) As in 1.16 adding* $\kappa \neq \aleph_1 \vee e \in \{3,4\}$.
2) If $\bar{\mathbb{Q}}$ *is an* $\kappa - Sp_e(W)$*-iteration, and for each* i *the quasi-order* $\leq^{\mathbb{Q}_i}_{\text{pr}}$ *is equality hence* $\leq^{\mathbb{Q}_i}_{\text{vpr}}$ *is equality, then* $\bar{\mathbb{Q}}$ *is essentially a finite support iteration.*
[Saharon: maybe restrict yourself above the constantly function $\zeta \mapsto \emptyset_{\mathbb{Q}_\zeta}$*, so we have to use* $\kappa > \ell g(\bar{\mathbb{Q}})$*.]*

11.3 Claim. *1) Add*

(d) *if* $e = 3,5$ *then* r *is pure outside* $\{\xi_1, \dots, \xi_n\}$.

2) In the proof on "$\xi^*_{\ell \times 1}$*" ?? we say, i.e., simple*^{1} *if* $e \in \{3,4\}$ *and simple*^{2} *if* $e \in \{5,6\}$.

11.4 Claim. *5) If* $e \in \{3,4\}$ *and*[11] *for each* $\beta < \ell g(\bar{\mathbb{Q}})$, $\underset{\sim}{\mathbf{t}}_\beta$ *is a* $\mathbb{P}_\beta$*-name of a truth value, then there is a simple* $(\bar{\mathbb{Q}}, W)$*-named* $[0,\alpha)$*-ordinal* $\underset{\sim}{\zeta}$ *such that* $\underset{\sim}{\zeta}[G_\beta] = \beta$ *iff* $\underset{\sim}{\mathbf{t}}_\beta[G_\beta] =$ *truth and* $\gamma < \beta \Rightarrow \underset{\sim}{\mathbf{t}}_\gamma[G_\beta] =$ *false for any subset* G_β *of* $\mathbb{P}_\beta$ *generic over* $\mathbf{V}$.

We can deal with the parallel of hereditarily countable names. This is not used in later sections.

11.5 Definition. We define for an $\kappa - \text{Sp}_e(W)$-iteration $\bar{\mathbb{Q}}$, and cardinal μ (μ regular), when is a $(\bar{\mathbb{Q}}, W)$-name hereditarily $< \mu$, and in particular when a $(\bar{\mathbb{Q}}, W)$-named $[j,\alpha)$-ordinal is hereditarily $< \mu$ and a $(\bar{\mathbb{Q}}, W)$-named $[j,\alpha)$-atomic condition hereditarily $< \mu$, and which conditions of $\text{Sp}_e(W)\text{-Lim}_\kappa(\bar{\mathbb{Q}})$ are hereditarily $< \mu$. For simplicity we are assuming that the set of members of $\mathbb{Q}_i$ is in $\mathbf{V}$. This is done by induction on $\alpha = \ell g(\bar{\mathbb{Q}})$.

<u>First Case</u>: $\alpha = 0$.
Trivial.

<u>Second Case</u>: $\alpha > 0$.

(A) A $\bar{\mathbb{Q}}$-named $[j,\alpha)$-ordinal $\underset{\sim}{\xi}$ hereditarily $< \mu$ is a $(\bar{\mathbb{Q}}, W)$-named $[j,\alpha)$-ordinal which can be represented as follows: there is $\langle (p_i, \xi_i) : i < i^* \rangle, i^* < \mu$, each ξ_i an ordinal in $[j,\alpha), p_i \in \mathbb{P}_{\xi_i}$ is a member of $\mathbb{P}_{\xi_i}$ hereditarily $< \mu$ and for any $G \in \text{Gen}^r(\bar{\mathbb{Q}}), \underset{\sim}{\zeta}[G]$ is ζ iff for some i we have:

  (a) $p_i \in G, \zeta_i = \zeta$
  (b) if $p_j \in G$ then $\zeta_i < \zeta_j \vee (\zeta_i = \zeta_j \ \& \ i < j)$

[11] for the parallel for $e \in \{5,6\}, \kappa = \aleph_1$ we need pure decidability and restrict ourselves to "above p" for purely dense sets of $p - s$

(B) A $(\bar{\mathbb{Q}}, W)$-named $[j,\alpha)$-atomic condition $\underset{\sim}{q}$ hereditarily $< \mu$, is a $(\bar{\mathbb{Q}}, W)$-named $[j,\alpha)$-atomic condition which can be represented as follows: there is $\langle (p_i, \zeta_i, q_i) : i < i^* \rangle, i^* < \mu, \zeta_i \in [j,\alpha), p_i \in \mathbb{P}_{\zeta_i}, q_i \in \mathbf{V}$, and for any $G \in \text{Gen}^r(\bar{\mathbb{Q}}), \underset{\sim}{q}[G]$ is q iff for some i we have:

(a) $p_i \in G, q = q_i$, and $p_i \Vdash_{\mathbb{P}_{\zeta_i}}$ "$q \in \underset{\sim}{\mathbb{Q}}_{\zeta_i}$"

(b) if $p_j \in G$ then $\zeta_i < \zeta_j \vee (\zeta_i = \zeta_j \ \& \ i < j)$

(C) A member p of $\mathbb{P}_\alpha = Sp_e(W)\text{-Lim}_\kappa(\bar{\mathbb{Q}})$ is hereditarily $< \mu$ if each member of r is a $(\bar{\mathbb{Q}}, W)$-named atomic condition hereditarily $< \mu$

(D) A $(\bar{\mathbb{Q}}, W)$-name of a member of $\mathbf{V}$ hereditarily $< \kappa$ is defined as in clause (B), similarly for member $x \in \mathbf{V}^{\mathbb{P}_\alpha}$ such that $y \in$ transitive closure of $x \Rightarrow |y| < \mu$.

11.6 Concluding Remarks. 1) We have not really dealt with the case κ is inaccessible. The point is that in this case, we do not know a priori the length of the list of the members of a condition (which are atomic conditions). It is natural to work on it together with "decidability on bound on $\underset{\sim}{\alpha} < \kappa$ by pure extensions", see 1.23 below.
2) We can think of putting together [Sh:f, Ch.XIV] and [Sh 587].
3) We can ask: Does "Souslin forcing notions" help?

11.7 Claim. $e \in \{4,5\}$ *is O.K.*

11.8 Claim. *In the proof of 1.26, in case 3 add:*
(the point is that $e \in \{4,6\}$*). Instead* $e \in \{4,6\}$ *it is enough to assume:*

$\boxtimes_{\underset{\sim}{\mathbb{Q}}_\beta}$ *for every* $q', q'' \in \underset{\sim}{\mathbb{Q}}_{\beta_0}$ *we have*

$\emptyset_{\mathbb{Q}_{\beta_0}} \leq_{\text{vpr}} q' \leq q'' \Rightarrow q' \leq_{\text{pr}} q''$.

11.9 Remark. 1) Add:
but for $e = 4$ we could use appropriate $p_1 = p \cup \{\underset{\sim}{r}_1\}, \underset{\sim}{r}$ an atomic $(\bar{\mathbb{Q}}, W)$-named condition, $\underset{\sim}{\zeta}_{\underset{\sim}{r}} = \underset{\sim}{\zeta}$, see 1.7(5).
2) Holds for $e \in \{4,6\}$.

11.10 Definition. 0) Let $\bar{\mathbb{Q}}$ be a κ_1-$\mathrm{Sp}_e(W)$-iteration of length α. Let $\underset{\sim}{\zeta}$ denote a simple $\bar{\mathbb{Q}}$-named $[0,\alpha)$-ordinal <u>or</u> a simple $\bar{\mathbb{Q}}$-named$^2[0,\alpha)$-ordinals and Ξ a countable set of such objects.

1) For an atomic simple $\bar{\mathbb{Q}}$-named condition $r, r \restriction \underset{\sim}{\zeta}$ is defined by $r \restriction \underset{\sim}{\zeta}[G] = r^* \in \mathbb{P}_\zeta$ if $\underset{\sim}{\zeta}[G] \geq \underset{\sim}{\zeta}_r[G], r[G] = r^*$ and $\emptyset_{p_\zeta}$ otherwise.

2) For $q \in \mathbb{P}_\alpha, q \restriction \underset{\sim}{\zeta} = \{r \restriction \zeta : r \in q\}$ and $q \restriction \Xi = \bigcup\limits_{\underset{\sim}{\zeta} \in \Xi} q \restriction \zeta$.

3) $\mathbb{P}_{\underset{\sim}{\zeta}} = \{p \in \mathbb{P}_\alpha : p \restriction \underset{\sim}{\zeta} = p$, i.e., for every $G \subseteq \mathbb{P}_\alpha$ generic over $\mathbf{V}, p \restriction \underset{\sim}{\zeta}[G] = p[G]\}$

$$P_\Xi = \{p \in \mathbb{P}_\alpha : p \restriction \Xi = p\}$$

both with the order inherited from $\mathbb{P}_\alpha$.

11.11 Claim. *Let $\bar{\mathbb{Q}}$ be an $\aleph_1 - \mathrm{Sp}_e(W)$-iteration.*

1) If $\underset{\sim}{\zeta}_1$ is a simple $\bar{\mathbb{Q}}$-named $[\beta_1,\beta_2)$-ordinal, $\underset{\sim}{\zeta}_2$ is a simple $\bar{\mathbb{Q}}$-named^1 $[\beta_1,\beta_2)$-ordinal, <u>then</u> there is a simple $\bar{\mathbb{Q}}$-named $[\beta_1,\beta_2)$-ordinal $\underset{\sim}{\zeta}$ such that for $G \subseteq \mathbb{P}_\alpha$ is generic over $\mathbf{V}$:

- (a) *if $\underset{\sim}{\zeta}_1[G] = \underset{\sim}{\zeta}_1[G \cap \mathbb{P}_\xi] = \xi$ and $\mathrm{Min}\{\varepsilon$: some $p \in G \cap \mathbb{P}_\varepsilon$ decided to be ε or be undefined$\} > \varepsilon$ <u>then</u> $\underset{\sim}{\zeta}[G] = \underset{\sim}{\zeta}[G \cap \mathbb{P}_\xi] = \xi$*
- (b) *otherwise undefined.*

2) Let $\underset{\sim}{\zeta}$ be a simple $\bar{\mathbb{Q}}$-named^1 ordinal. For r an atomic $\bar{\mathbb{Q}}$-named condition $r \restriction \underset{\sim}{\zeta}$ is an atomic $\bar{\mathbb{Q}}$-named condition.

3) For $q \in \mathbb{P}_\alpha$ we have $q \restriction \underset{\sim}{\zeta} \in \mathbb{P}_\alpha$.

4) For $q_1, q_2 \in \mathbb{P}_\alpha, q_1 \leq q_2 \Rightarrow q_1 \restriction \underset{\sim}{\zeta} \leq q_2 \restriction \underset{\sim}{\zeta}$.

5) If $q \in \mathbb{P}_{\underset{\sim}{\zeta}}, p \in \mathbb{P}_\alpha, p \restriction \zeta \leq q$ <u>then</u> $p \cup q \in \mathbb{P}_\alpha$ is a lub of p and q.

6) $\mathbb{P}_{\underset{\sim}{\zeta}} \lessdot \mathbb{P}_\alpha$.

7) If $G \subseteq \mathbb{P}_{\underset{\sim}{\alpha}}$ is generic over $\mathbf{V}, \xi = \underset{\sim}{\zeta}[G]$ <u>then</u> $G \cap P_{\underset{\sim}{\zeta}}, G \cap \mathbb{P}_\xi$ are essentially the same.

8) The parallel statements with Ξ instead of $\underset{\sim}{\zeta}$.

Remark. In fact by part (1), part (6) follows from the parts (2)-(5).

11.12 Claim. *Let* $e = 4(2)$. *Assume* $\underset{\sim}{\zeta}_n$ *is a simple* $\bar{\mathbb{Q}}$*-named for* $n < \omega, \underset{\sim}{\zeta}_n < \underset{\sim}{\zeta}_{n+1}$ *and for every* $G \subseteq \mathbb{P}_\alpha$ *generic over* $\mathbf{V}$, *for some* $n, \varphi(n, G \cap \mathbb{P}_{\underset{\sim}{\zeta}_n[G]})$. *Then for some simple* $\bar{\mathbb{Q}}$*-name ordinal* $\underset{\sim}{\xi}$, *we have*

$$\Vdash_{\mathbb{P}_\alpha} \text{“for some } n, \underset{\sim}{\xi}[\underset{\sim}{G}_{\mathbb{P}_\alpha}] = \underset{\sim}{\zeta}_n[\underset{\sim}{G}_{\mathbb{P}_\alpha}] \text{ and } \varphi(n, \underset{\sim}{G}_{\mathbb{P}_\alpha} \cap \mathbb{P}_{\underset{\sim}{\zeta}_n[G_{\mathbb{P}_\alpha}]})\text{”}.$$

REFERENCES.

[GiSh 191] Moti Gitik and Saharon Shelah. On the **I**-condition. *Israel Journal of Mathematics*, **48**:148–158, 1984.

[RoSh 470] Andrzej Rosłanowski and Saharon Shelah. Norms on possibilities I: forcing with trees and creatures. *Memoirs of the American Mathematical Society*, **141**(671):xii + 167, 1999.

[RuSh 117] Matatyahu Rubin and Saharon Shelah. Combinatorial problems on trees: partitions, Δ-systems and large free subtrees. *Annals of Pure and Applied Logic*, **33**:43–81, 1987.

[Sh 656] Saharon Shelah. NNR Revisited. *Preprint.*

[Sh:e] Saharon Shelah. *Non–structure theory*, accepted. Oxford University Press.

[Sh:b] Saharon Shelah. *Proper forcing*, volume 940 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, xxix+496 pp, 1982.

[Sh 136] Saharon Shelah. Constructions of many complicated uncountable structures and Boolean algebras. *Israel Journal of Mathematics*, **45**:100–146, 1983.

[Sh:f] Saharon Shelah. *Proper and improper forcing*. Perspectives in Mathematical Logic. Springer, 1998.

[Sh 587] Saharon Shelah. Not collapsing cardinals $\leq \kappa$ in $(< \kappa)$–support iterations. *Israel Journal of Mathematics*, **136**:29–115, 2003.