Almost Free Algebras

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1 Introduction

In this paper we adopt the terminology of universal algebra. So by a free algebra we will mean that a variety (i.e., an equationally defined class of algebras) is given and the algebra is free in that variety. We will also assume that the language of any variety is countable.

In this paper the investigation of the almost free algebras is continued. An algebra is said to be almost free if "most" of its subalgebras of smaller

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cardinality are free. For some varieties, such as groups and abelian groups, every subalgebra of a free algebra is free. In those cases "most" is synonomous with "all". In general there are several choices for the definition of "most". In the singular case, if the notion of "most" is strong enough, then any almost free algebra of singular cardinality is free [9]. So we can concentrate on the regular case. In the regular case we will adopt the following definition. If κ is a regular uncountable cardinal and A is an algebra of cardinality κ then A is *almost free* if there is a sequence $(A_\alpha : \alpha < \kappa)$ of free subalgebras of A such that: for all α, $|A_\alpha| < \kappa$; if $\alpha < \beta$, then $A_\alpha \subseteq A_\beta$; and if δ is a limit ordinal then $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$. (In [3], such a chain is called a *κ-filtration.*) It should be noted that this definition is not the same as the definition in [9]. The definition there is sensitive to the truth of Chang's conjecture (see the discussion in [3] Notes to Chapter IV).

There are two sorts of almost free algebras; those which are essentially free and those which are essentially non-free. An algebra A is *essentially free* if $A * F$ is free for some free algebra F. Here $*$ denotes the free product. (Since our algebras will always be countably free the free product is well defined.) For example, in the variety of abelian groups of exponents six, a free group is a direct sum of copies of cyclic groups of order 6. The group $\bigoplus_{\aleph_1} C_3 \oplus \bigoplus_{\aleph_0} C_2$ is an almost free algebra of cardinality $\aleph_1$ which is not free but is essentially free. An algebra which is not essentially free is *essentially non-free.* In [2], the construction principle, abbreviated CP, is defined and it is shown that for any variety there is an essentially non-free almost free algebra of some cardinality if and only if there is an essentially non-free almost free algebra of cardinality $\aleph_1$ if and only if the construction principle holds in that variety. As well if V = L holds then each of the above equivalents is also equivalent to the existence of an essentially non-free almost free algebra in all non-weakly compact regular cardinalities.

In this paper we will investigate the essentially non-free spectrum of a variety. The *essentially non-free spectrum* is the class of uncountable cardinals κ in which there is an essentially non-free algebra of cardinality κ which is almost free. This class consists entirely of regular cardinals ([9]). In L, the essentially non-free spectrum of a variety is entirely determined by whether or not the construction principle holds. As we shall see the situation in ZFC may be more complicated.

For some varieties, such as groups, abelian groups or any variety of modules over a non-left perfect ring, the essentially non-free spectrum contains not only $\aleph_1$ but $\aleph_n$ for all $n > 0$. The reason for this being true in ZFC (rather than under some special set theoretic hypotheses) is that these varieties satisfy stronger versions of the construction principle. We conjecture that the hierarchy of construction principles is strict, i.e., that for each $n > 0$ there is a variety which satisfies the n-construction principle but not the $n+1$-construction principle. In this paper we will show that the 1-construction principle does not imply the 2-construction principle.

After these examples are given there still remains the question of whether these principles actually reflect the reason that there are essentially non-free κ-free algebras of cardinality κ. Of course, we can not hope to prove a theorem in ZFC, because of the situation in L (or more generally if there is a non-reflecting stationary subset of every regular non-weakly compact cardinal which consists of ordinals of cofinality ω). However we will prove that, assuming the consistency of some large cardinal hypothesis, it is consistent that a variety has an essentially non-free almost free algebra of cardinality $\aleph_n$ if and only if it satisfies the n-construction principle. (We will also show under milder hypotheses that it is consistent that the various classes are separated.)

Definition 1.1 A variety $\mathcal{V}$ of algebras satisfies the n-construction principle, CP_n, if there are countably generated free algebras $H \subseteq I \subseteq L$ and a partition of ω into n infinite blocks (i.e. sets) $s^1, \ldots, s^n$ so that

(1) H is freely generated by $\{h_m : m < \omega\}$, and for every subset $J \subseteq \omega$ if for some k, $J \cap s^k$ is finite then the algebra generated by $\{h_m : m \in J\}$ is a free factor of L; and
(2) $L = I * F(\omega)$ and H is not a free factor of L.

Here $F(\omega)$ is the free algebra on $\aleph_0$ generators, and H is a free factor of L, denoted $H|L$, means that there is a free algebra G so that $H * G = L$.

The construction principle of [2] is the principle we have called CP_1. The known constructions of an almost free algebra from CP_n seem to require the following set theoretic principle. (The definition that follows may be easier to understand if the reader keeps in mind that a λ-system is a generalization of a stationary set consisting of ordinals of cofinality ω.)

Definition 1.2 (1) A λ-*set* of height n is a subtree S of ${}^{\leq n}\lambda$ together with a cardinal λ_η for every $\eta \in S$ such that $\lambda_\emptyset = \lambda$, and:

(a) for all $\eta \in S$, η is a final node of S if and only if $\lambda_\eta = \aleph_0$;

(b) if $\eta \in S \backslash S_f$, then $\eta\hat{\ }\langle\beta\rangle \in S$ implies $\beta \in \lambda_\eta$, $\lambda_{\eta\hat{\ }\langle\beta\rangle} < \lambda_\eta$ and $E_\eta \stackrel{\text{def}}{=} \{\beta < \lambda_\eta : \eta\hat{\ }\langle\beta\rangle \in S\}$ is stationary in λ_η (where $S_f \stackrel{\text{def}}{=} \{\eta \in S : \eta$ is $\triangleleft$-maximal in $S\}$ is the family of final nodes of S).

(2) A λ-*system* of height n is a λ-set of height n together with a set B_η for each $\eta \in S$ such that $B_\emptyset = \emptyset$, and for all $\eta \in S \setminus S_f$:

(a) for all $\beta \in E_\eta$, $\lambda_{\eta\hat{\ }\langle\beta\rangle} \leq |B_{\eta\hat{\ }\langle\beta\rangle}| < \lambda_\eta$;

(b) $\{B_{\eta\hat{\ }\langle\beta\rangle} : \beta \in E_\eta\}$ is an increasing continuous chain of sets, i.e., if $\beta < \beta'$ are in E_η, then $B_{\eta\hat{\ }\langle\beta\rangle} \subseteq B_{\eta\hat{\ }\langle\beta'\rangle}$; and if σ is a limit point of E_η (i.e. $\sigma = \sup(\sigma \cap E_\eta) \in E_\eta$), then $B_{\eta\hat{\ }\langle\sigma\rangle} = \cup\{B_{\eta\hat{\ }\langle\beta\rangle} : \beta < \sigma, \beta \in E_\eta\}$.

(4) For any λ-system $\Lambda = (S, \lambda_\eta, B_\eta : \eta \in S)$, and any $\eta \in S$, let $\bar{B}_\eta = \cup\{B_{\eta\restriction m} : m \leq \ell(\eta)\}$. Say that a family $\mathcal{S}$ of countable sets is *based on* Λ if $\mathcal{S}$ is indexed by S_f, and for every $\eta \in S_f$, $s_\eta \subseteq \bar{B}_\eta$.

A family $\mathcal{S}$ of countable sets is *free* if there is a transversal of $\mathcal{S}$, i.e., a one-one function f from $\mathcal{S}$ to $\cup\mathcal{S}$ so that for all $s \in \mathcal{S}$ $f(s) \in s$. A family of countable sets is, *almost free* if every subfamily of lesser cardinality is free.

Shelah, [10], showed that the existence of an almost free abelian group of cardinality κ is equivalent to the existence of an almost free family of countable sets of cardinality κ. The proof goes through λ-systems. In [3], the following theorem is proved (although not explicitly stated, see the proof of theorem VII.3A.13).

Theorem 1.1 *If a variety satisfies* CP_n *and* λ *is a regular cardinal such that there is a* λ*-system,* Λ*, of height* n *and an almost free family of countable sets based on* Λ*, then there is an essentially non-free algebra of cardinality* λ *which is almost free.*

Conjecture: The converse of the theorem above is true. I.e. for each regular cardinal $\lambda > \aleph_0$ and every variety the following two conditions are equivalent:

(α) for some $n < \omega$ the variety satisfies the principle CP_n and there exists a λ-system Λ of height n and an almost free family of countable sets based on Λ

(β) there exists an essentially non free almost free algebra (for the variety) of cardinality λ.

Since $\{n :$ the variety satisfies $\mathrm{CP}_n\}$ is an initial segment of ω we could conclude that it is a theorem of ZFC that there are at most $\aleph_0$ essentially non-free spectra.

Although we will not discuss essentially free algebras in this paper, these algebras can be profitably investigated. The *essentially free spectrum* of a variety is defined as the set of cardinals κ so that there is an almost free non-free algebra of cardinality κ which is essentially free. The conjecture is that the essentially free spectrum of a variety is either empty or consists of the class of successor cardinals. For those varieties for which CP_1 does not hold, i.e., the essentially non-free spectrum is empty, the conjecture is true [6]. It is always true that the essentially free spectrum of a variety is contained in the class of successor cardinals. (A paper which essentially verifies it is in preparation.)

A notion related to being almost free is being κ-free where κ is an uncountable cardinal. An algebra is κ-free if "most" subalgebras of cardinality less than κ are free. There are various choices for the definition of "most" and the relations among them are not clear. For a regular cardinal κ we will say that A is *κ-free* if there is a closed unbounded set in $\mathcal{P}_\kappa(A)$ (the set of subsets of A of cardinality less than κ) consisting of free algebras. Note that an almost free algebra of cardinality κ is κ-free. One important associated notion is that of being $\mathrm{L}_{\infty\kappa}$-free; i.e., being $\mathrm{L}_{\infty\kappa}$-equivalent to a free algebra. A basic theorem is that:

THEOREM 1.2 *If an algebra is κ^+-free, then it is $\mathrm{L}_{\infty\kappa}$-free.*

PROOF. See [9] 2.6(B). (Note κ^+-free in the sense here implies $E^{\kappa^+}_{\kappa^+}$-free as defined there.) □

We will use some of the notions associated with the $\mathrm{L}_{\infty\kappa}$-free algebras. Suppose κ is a cardinal and A is an algebra (in some fixed variety). A subalgebra B which is $<\kappa$-generated is said to be *κ-pure* if Player I has a winning strategy in the following game of length ω.

> Players I and II alternately choose an increasing chain $B = B_0 \subseteq B_1 \subseteq \ldots \subseteq B_n \subseteq \ldots$ of subalgebras of A each of which is $<\kappa$-generated. Player I wins a play of the game if for all n, B_{2n} is free and B_{2n} is a free factor of B_{2n+2}.

If $B \subseteq A$ is not $< \kappa$-generated (but $B \subseteq A$ are free) we just ask B_{n+1} to be $< \kappa$-generated over B_n for each n (used e.g. in the proof of 4.8).

The choice of the term κ-pure is taken from abelian group theory. The following theorem sums up various useful facts. Some of the results are obvious others are taken from [5].

THEOREM 1.3

(1) *An algebra is* $\mathrm{L}_{\infty\kappa}$*-free if and only if every subset of cardinality less than* κ *is contained in a* $<\kappa$*-generated algebra which is* κ*-pure.*

(2) *If* F *is a free algebra then a subalgebra is* κ*-pure if and only if it is a free factor.*

(3) *In any* $\mathrm{L}_{\infty\kappa}$*-free algebra the set of* κ*-pure subalgebras is* κ*-directed under the relation of being a free factor.*

(4) *If* $\kappa < \lambda$ *and* A *is* $\mathrm{L}_{\infty\lambda}$*-free, then any* κ*-pure subalgebra is also* λ*-pure.*

Notice that part (2) of the theorem above implies that for κ, if A is $\mathrm{L}_{\infty\kappa}$-free then there is a formula of $\mathrm{L}_{\infty\kappa}$ which defines the κ-pure subalgebras (of A, but the formula depends on A).

We will use elementary submodels of appropriate set theoretic universes on many occasions. We say that a cardinal χ *is large enough* if $(\mathrm{H}(\chi), \in)$ contains as elements everything which we are discussing. If A and B are free algebras which are subalgebras of some third algebra C, then by $A + B$, we denote the algebra generated by $A \cup B$ and define B/A to be *free* if any (equivalently, some) free basis of A can be extended to a free basis of $A + B$. Similarly for κ a regular uncountable cardinal, if $A + B$ is κ-generated over A then we say that B/A is *almost free* if there is a sequence $(B_\alpha : \alpha < \kappa)$ so that: for all α, $|B_\alpha| < \kappa$; for all $\alpha < \beta$, $B_\alpha \subseteq B_\beta$; if δ is a limit ordinal then $B_\delta = \bigcup_{\alpha<\delta} B_\alpha$; $A + B = A + \bigcup_{\alpha<\kappa} B_\alpha$; and for all α, B_α/A is free. The notions of *essentially non-free, strongly* κ*-free* etc. for pairs are defined analogously. The following lemma is useful.

LEMMA 1.4 *Suppose* $A \subseteq B$ *are free algebras and* $N \prec (\mathrm{H}(\chi), \in)$*, where* χ *is large enough. If* $A, B \in N$ *and* $(B \cap N)/(A \cap N)$ *is free then* $(B \cap N)/A$ *is free.*

PROOF. Let $X \in N$ be a free basis of A. By elementariness, $A \cap N$ is freely generated by $X \cap N$. Choose Y so that $(X \cap N) \cup Y$ is a free basis for $B \cap N$. We now claim that $X \cup Y$ freely generate the algebra they generate (namely $A + (B \cap N)$). Suppose not. Then there are finite sets $Y_1 \subseteq Y$ and $X_1 \subseteq X$ such that $Y_1 \cup X_1$ satisfy an equation which is not a law of the variety. By elementariness, we can find $X_2 \subseteq X \cap N$ so that $Y_1 \cup X_2$ satisfies the same equation. This is a contradiction. □

2 CP_1 does not imply CP_2

In this section we will present an example of a variety which satisfies CP_1 but not CP_2. The strategy for producing the example is quite simple. We write down laws which say that the variety we are defining satisfies CP_1 and then prove that it does not satisfy CP_2. We believe the same strategy will work for getting an example which satisfies CP_n but not CP_{n+1}. However there are features in the proof that the strategy works for the case $n = 1$ which do not generalize.

The variety we will build will be generated by projection algebras. A *projection algebra* is an algebra in which all the functions are projections on some coordinate. If a variety is generated by (a set of) projection algebras, then it is not necessarily true that every algebra in the variety is a projection algebra. For example, there may be a binary fuction f which in one algebra is projection on the first coordinate and in another is projection on the second coordinate.

In a variety generated by projection algebras there is a very simple characterization of the free algebras. It is standard that a free algebra in a variety is a subalgebra of a direct product of generators of the algebra which is generated by tuples so that for any equation between terms there is, if possible, a coordinate in which the equation fails for the tuples (see for example Theorem 11.11 of [1]). In a variety generated by projection algebras the free algebra on κ generators is the subalgebra of the product of the various projection algebras on κ generators which is generated by κ elements which differ pairwise in each coordinate.

THEOREM 2.1 *There is a variety satisfying* CP_1 *but not* CP_2*.*

PROOF. To begin we fix various sets of constant symbols: $\{c_{m,n}: m, n < \omega\}$ and $\{d_n: n < \omega\}$. The intention is to define an algebra I such that for all m, $\{c_{m,n}: n < \omega\} \cup \{d_n: n < m\}$ will be a set of free generators for an algebra in our variety. In particular, $\{c_{0n}: n < \omega\}$ will be a set of free generators. We intend that H will be the algebra generated by $\{d_n: n < \omega\}$ and I will be the whole algebra (in the definition of CP_1). We define the language and some equations by induction. We have to add enough function symbols so that for each m, $\{c_{m,n} : n, \omega\} \cup \{d_n : n < m\}$ generates the whole I, it suffices that it generates all $c \in \{c_{k,n} : k, n < \omega\} \cup \{d_n : n < \omega\}$. However, while doing this we have still to make each $\{c_{m,n} : n < \omega\} \cup \{d_n : n < m\}$ free. At each stage we will add a new function symbol to the language and consider a pair consisting of a constant symbol and a natural number m. (Note that the constants are not in the language of the variety.) The enumeration of the pairs should be done in such a way that each pair consisting a constant symbol and a natural number is enumerated at some step. Since this is a routine enumeration we will not comment on it, but assume our enumeration has this property. Also at each stage we will commit ourselves to an equation.

For the remainder of this proof we will let the index of the constant d_n be n and the index of the constant $c_{m,n}$ be $m + n$. At stage n we are given a constant t_n (so that $t_n \in \{c_{m,k}, d_k: k, m \in \omega\}$) and a natural number m_n, we now add a new function symbol f_n to the language where the arity of f_n is chosen to be greater than m_n plus the sum of the indices of t_n and all the constant symbols which which have appeared in the previous equations. (No great care has to be taken in the choice of the arity, it just has to increase quickly.) Now we commit ourselves to the new equation

$$(*) \qquad t_n = f_n(d_i(i < m_n), c_{m_n,j}(j < k_n))$$

where the arity of f_n is $m_n + k_n$.

The variety we want to construct has vocabulary τ, the set of function symbols we introduced above. We use a subsidiary vocabulary τ' which is $\tau \cup \{c_{m,n}, d_n: m, n < \omega\}$. Let K_0 be the family of τ'-algebras which are projection algebras satisfying the equations $(*)$ whose universe consists of $\{c_{0,n}: n < \omega\}$ such that for all m, the interpretations of $c_{m,n}$ $(n < \omega)$ and d_n $(n < m)$ are pairwise distinct. Let $\mathbf{K}$ be the class of τ-reducts of members of K_0. We will shortly prove that $\mathbf{K}$ is non-empty. If we assume this for the moment, then it is clear that the variety generated by $\mathbf{K}$ satisfies (1) in the definition of CP_1 with $\{d_n : n < \omega\}$ standing for $\{h_n : n < \omega\}$. More exactly in the direct product of the elements of $\mathbf{K}$, for all m, $\{c_{m,n} : n < \omega\} \cup \{d_n : n < m\}$ freely generates a subalgebra. The choice of equations guarantee that all the subalgebras are the same. The proof that (2) of the definition holds as well as the proof that the variety is non-trivial rests largely on the following claim.

Claim 2.1.1 *For all m there is an element of $\mathbf{K}$ so that in that algebra $c_{00} = d_m$ (with the d_n's distinct — see definition of $\mathbf{K}$, K_0)*

The proof of the claim is quite easy. We inductively define an equivalence relation $\equiv$ on the constants and an interpretation of the functions as projections. At stage n, we define an equivalence relation $\equiv_n$ on $\{c_{km}, d_m : m, k < \omega\}$, so that $\equiv_n$ is a subset of $\equiv_{n+1}$, all but finitely many equivalence classes of $\equiv_n$ are singletons and $\Sigma\{\mathrm{card}(A) - 1 : A$ an $\equiv_n$ -equivalence class$\} \leq 2n$. Moreover we demand that for each $l < \omega$ no two distinct members of $\{c_{l,m} : m < \omega\} \cup \{d_m : m < l\}$ are $\equiv_n$-equivalent. To begin we set $c_{0,0} \equiv_0 d_m$. At stage n, there are two possibilities, either t_n has already been set equivalent to one of $\{d_k : k < m_n\} \cup \{c_{m_n,i} : i < k_n\}$ or not. In the first case our assumption on the arity guarantees that we can make f_n a projection function and we put $\equiv_n = \equiv_{n-1}$. In the second case there is some element in $\{c_{m_n,j} : j < k_n\}$ which has not been set equivalent to any other element. In this case we choose such an element, set it equivalent to t_n and to some $c_{0,m}$ (for a suitable m) and let f_n be the appropriate projection. In the end let $\equiv$ be $x \equiv y$ if and only if $(\exists n)(x \equiv_n y)$. Let M be the τ'-algebra with the set of elements $\{c_{n,m}, d_n : n, m < \omega\}/\equiv$, functions f_n as chosen above (note that f_n respects $\equiv$ as it is a projection) and $c_{n,m}, d_n$ interpreted naturally. Note that by the equation $(*)$ for every m, $\{c_{m,n} : n < \omega\} \cup \{d_n : n < m\}$ lists the members.

It remains to verify that condition (2) is satisfied. Let I denote the free algebra generated by $\{c_{0,n} : n < \omega\}$. Suppose that (2) is not true. Choose elements $\{e_n : n < \omega\}$, so that $I * F(\aleph_0)$ is freely generated by the d_n's and the e_n's. So there is some m, so that $c_{0,0}$ is in the subalgebra generated by $\{d_n : n < m\} \cup \{e_n : n < \omega\}$. But if we turn to the projection algebra where $c_{0,0} = d_m$, we have a contradiction (see definition of $\mathbf{K}$).

Finally we need to see that our variety does not satisfy CP_2. We will prove the following claim which not only establishes the desired result but shows the limit of our method.

CLAIM 2.1.2 *If $\mathcal{V}$ is a variety which is generated by projection algebras then $\mathcal{V}$ does not satisfy* CP_2.

Suppose to the contrary that we had such a variety. Let I and $\{x_{in}: i < 2, n < \omega\}$ be an example of CP_2. Choose $\{y_n: n < \omega\}$ so that

$$\{x_{1n}: n < \omega\} \cup \{y_n: n < \omega\}$$

is a set of free generators for I (we rename $\{h_n : n \in s_i\}$ as $\{x_{in} : n < \omega\}$. Notice that if θ is any verbal congruence (see below) on I which does not identify all elements then the image of a set of free generators of a subalgebra will freely generate their image in the subvariety $\mathcal{V}/\theta$ defined by the law. (A *verbal congruence* is a congruence which is defined by adding new laws to the variety.) Fix a vocabulary.

We will show by induction on the complexity of terms τ that

⊗ if $\mathcal{V}$ is a variety generated by projection algebras and if $X \cup Y$ are free generators of an algebra $A \in \mathcal{V}$, $a = \tau(\ldots, x_i, \ldots, y_j, \ldots)$, $x_i \in X$, $y_j \in Y$, $a \in A$ and $X \cup \{a\}$ freely generates a subalgebra of A then a is in the subalgebra generated by Y.

The base case of the induction is trivial. Suppose that $a = f(t_0, \ldots, t_n)$. For $i \leq n$ let θ_i be the congruence on A generated by adding the law $f(z_0, \ldots, z_n) = z_i$ and let $\mathcal{V}_i$ be the subvariety satisfying this law. Since $\mathcal{V}$ is generated by projection algebras so is $\mathcal{V}_i$, for all i. Furthermore $\mathcal{V}$ is the join of these varieties. In A/θ_i, $a/\theta_i = t_i/\theta_i$. By the inductive hypothesis we can choose for each i, a term s_i using only the variables from Y so that A/θ_i satisfies that $t_i/\theta_i = s_i/\theta_i$. Hence each variety, $\mathcal{V}_i$ satisfies the law $f(t_0, \ldots, t_n) = f(s_0, \ldots, s_n)$. So $\mathcal{V}$ satisfies the law as well. We have shown that $a = f(s_0, \ldots, s_n)$, i.e., that a is in the subalgebra generated by Y. So ⊗ holds.

Applying the last claim we have, $\{x_{0n}: n < \omega\}$ is contained in the subalgebra generated by $\{y_n: n < \omega\}$. Call the latter subalgebra B. Let F denote a countably generated free algebra. Since $B * F$ is isomorphic over B and hence over $\{x_{0n}: n < \omega\}$ to I, and $\{x_{0n} : n < \omega\}$ generates a free factor of I, necessarily $\{x_{0n}: n < \omega\}$ freely generates a free factor of $B * F$. Hence $\{x_{in}: i < 2, n < \omega\}$ freely generates a free factor of $I * F$. Thus we have arrived at a contradiction. □

3 Miscellaneous

One natural question is for which cardinals κ is every κ-free algebra of cardinality κ free (no matter what the variety). By the singular compactness theorem ([9]) every singular cardinal is such a cardinal. As well it is known that if κ is a weakly compact cardinal then every κ-free algebra of cardinality κ is free. Some proofs of this fact use the fact that for weakly compact cardinals we can have many stationary sets reflecting in the same regular cardinal (see the proof of [3] IV.3.2 for example). It turns out that we only need to have single stationary sets reflecting. We say that a stationary subset E of a cardinal κ *reflects* if there is some limit ordinal $\alpha < \kappa$ so that $E \cap \alpha$ is stationary in α. The relevance of the following theorem comes from the fact that the consistency strength of a regular cardinal such that every stationary set reflects in a regular cardinal is strictly less than that of a weakly compact cardinal [7]. So the consistency strength of a regular cardinal κ so that every almost free algebra of cardinality κ is free is strictly less than that of the existence of a weakly compact cardinal.

We separate out the following lemma which will be useful in more than one setting.

LEMMA 3.1 *Suppose F is a free algebra and $G \subseteq H$ are such that H is a free factor of F and there are A, B free subalgebras of H so that $G \subseteq A$, $\mathrm{rank}(B) = \mathrm{card}(H) + \aleph_0$ and $A * B = H$. Then G is a free factor of F if and only if G is a free factor of H.*

PROOF. Obviously if G is a free factor of H then G is a free factor of F. Suppose now that G is a free factor of F. Since G is a free factor of F we can choose B_1 so that $|B| = |B_1|$ and G is a free factor of $A * B_1$. But since H is isomorphic over A to $A * B_1$, G is also a free factor of H. $\square$

Notice that in the hypothesis of the last lemma the existence of A and B is guaranteed if we assume that $|G| < |H|$.

In some varieties a union of an increasing chain of cofinality at least κ of κ-pure subalgebras is κ-pure as well. In general varieties this statement may not be true, in our later work we will need the following weaker result.

THEOREM 3.2 *Suppose κ is an inaccessible cardinal and E is a subset of κ such that every stationary subset of E reflects in a regular cardinal. If A is* *an almost free algebra of cardinality κ and there is a κ-filtration $(A_\alpha\colon \alpha < \kappa)$ of A such that for all $\alpha \notin E$, A_α is κ-pure then A is free.*

PROOF. Since κ is a limit cardinal, A is also $L_{\infty\kappa}$-free. Hence wlog E is a set of limit ordinals and each A_α is free (here we use that A is "almost free"); for all $\alpha < \beta$, $|A_\alpha| < |A_\beta|$ and for all $\alpha < \beta$, $A_{\alpha+1}$ is a free factor of $A_{\beta+1}$ and $\alpha \in \lambda \setminus E$ implies that A_α is κ-pure. Also wlog

$(*)_1$ if A_α is κ-pure in A then $A_{\alpha+1}/A_\alpha$ is free

$(*)_2$ if A_α is not κ-pure in A then $A_{\alpha+1}/A_\alpha$ is not essentially free.

Assume that A is not free. We will use the fact that

> If $\{A_\alpha\colon \alpha < \kappa\}$ is a filtration of A and B is a κ-pure subalgebra of cardinality less than κ then for a club C of κ, $\beta \in C$, $\mathrm{cf}\beta = \omega$ implies that A_β/B is free.

Let

$$E^* = \{\alpha < \kappa\colon A_\alpha \text{ is not a free factor of } A_{\alpha+1}\},$$

$$C^* = \{\alpha < \kappa : \alpha \text{ is a limit cardinal and } \beta < \alpha \Rightarrow |A_\alpha| < \alpha\}.$$

Now C^* is a club of κ and E^* is a stationary subset of κ (otherwise A is free). Choose λ a regular cardinal so that $|A_\lambda| = \lambda$ and $(E^* \cap C^*) \cap \lambda$ is stationary in λ. If A_λ is free then we can find a strictly increasing continuous sequence $\langle \alpha_i\colon i < \lambda\rangle$ such that $i < j$ implies $A_{\alpha_j}/A_{\alpha_i}$ is free. Let $C = \{i < \lambda\colon \alpha_i = i\}$. Since $(E^* \cap C^*) \cap \lambda$ is stationary there is $\beta \in E^* \cap C^* \cap \lambda \cap C$. So we can find β_n for $n < \omega$ such that $\beta_n \in C$, $\beta_0 = \beta$ and $\beta_n < \beta_{n+1}$.

Let $\beta_\omega = \cup\{\beta_n\colon n < \omega\}$. Then A_{β_ω}/A_β is free by the choice of C (and of the α_i's). Also $A_{\beta_{n+1}}/A_{\beta_0+1}$ is free. Together $A_{\beta_0+1}/A_{\beta_0}$ is essentially free, so by $(*)_2$ we know $A_{\beta_0+1}/A_{\beta_0}$ is free which contradicts our choice of E^* and $\beta_0 \in E^*$. Hence A_λ is not free. $\Box$

4 Getting CP_n

In this section we will deal with the problem of deducing CP_n from the existence of a κ-free algebra. We will need to deal with subalgebras of free algebras.

To handle certain technical details in this section we will deal with varieties in uncountable languages. Most things we have done so far transfer to this new situation if we replace of cardinality κ by κ-generated. One trick we will use is to pass from a pair of algebras B/A to a new algebra B^* by making the elements of A constants in the new variety. Recall that the notation B/A implies that B and A are subalgebras of an algebra C and both A and the subalgebra generated by $B \cup A$ are free.

Definition 4.1 Suppose $\mathcal{V}$ is a variety and A is a free algebra in the sense of $\mathcal{V}$. Let $\mathcal{V}_A$ denote the variety where we add constants for the elements of A and the equational diagram of A.

Notice that any element of $\mathcal{V}_A$ contains a homomorphic image of A. If B is any algebra which contains A we can view B as a $\mathcal{V}_A$ algebra.

Proposition 4.1 *Suppose A is a $\mathcal{V}$-free algebra and B is an algebra which contains A. Then for all κ, B is κ-free in $\mathcal{V}_A$ if and only if B/A is κ-free in $\mathcal{V}$.*

The following lemma is easy and lists some facts we will need.

Lemma 4.2 *Suppose $A \subseteq B$ and both A and B are free algebras on κ generators. Then the following are equivalent.*

(i) every subset of A of cardinality $< \kappa$ is contained in a subalgebra C which is a free factor of both A and B.

(ii) every free factor of A which is $<\kappa$-generated is also a free factor of B.

Proof. That (ii) implies (i) is obvious. Assume now that (i) holds and that C is a free factor of A which is $<\kappa$-generated. Let D be a $<\kappa$-generated free factor of both A and B which contains C. Since $A \cong_D B$, $A \cong_C B$. So C is a free factor of B. □

If A and B are free algebras which satisfy either (i) or (ii) above, we will write $A \prec_{\infty\kappa} B$. This notation is justified since for free algebras these conditions are equivalent to saying A is an $\mathrm{L}_{\infty\kappa}$-subalgebra of B. It is possible to give a simpler characterization of CP_n.

Theorem 4.3 *For any variety of algebras, CP_n is equivalent to the following statement. There are countable rank free algebras, $A \prec_{\infty\omega} B$ and countable rank free algebras A_k $(k < n)$ so that*

(i) *$A = *_{k<n} A_k$ and for all m, $*_{k\neq m} A_k$ is a free factor of B*

(ii) *if F is a countable rank free algebra, then A is not a free factor of $B * F$ (alternatively, B/A is essentially non-free).*

PROOF. CP_n clearly implies the statement above. Assume that A, B, A_k ($k < n$) are as above. We will show that $B * F$ together with A satisfies CP_n with $A, B, B * F$ here corresponding to to H, I, L there. It is enough to prove that (i) in the statement of CP_n holds. It suffices to show for all m that if C is a finite rank free factor of A_m then $*_{k\neq m} A_k * C$ is a free factor of $B * F$. Choose Y a complementary factor in B for $*_{k\neq m} A_k$. Choose now finite rank free factors D and E of $*_{k\neq m} A_k$ and Y respectively so that C is contained in $D * E$ and is a free factor of $D * C$. Clearly $D * E$ is a free factor of B and also of $D * Y$, hence we have $B * F \cong_{D*E} D * Y * F$. Now $C * D$ is a free factor of A, so as A is an $L_{\infty,\omega}$-submodel of $B * F$, all are countable generated, clearly $C * D$ is a free factor of $B * F$. By the last two sentences (as $C * D \subseteq D * E$) we have that $C * D$ is a free factor of $D * Y * F$. Also $*_{l\neq m} A_l$, $D * Y * F$ are freely amalgamated over D, D is a free factor of both and $D \subseteq D * C \subseteq D * Y * F$ are free. D is a free factor of $D * C$, $D * C$ is a free factor of $D * Y * F$; together $*_{l\neq m} * C$ is a free factor of $B * F$. So we have finished. $\square$

We next have to consider pairs (and tuples). The following two facts are standard and proved analogously to the results for algebras (rather than pairs).

LEMMA 4.4 *Suppose B/A is κ^+-free. Then it is strongly κ-free (i.e. $\mathrm{L}_{\infty\kappa}$-equivalent to a free algebra).*

COROLLARY 4.5 *Suppose κ is regular and $A \subseteq B$. If B/A is κ-free and $|B| = \kappa$ then $B = \cup_{\alpha<\kappa} B_\alpha$ (continuous) where $A = B_0$, $B_{\alpha+1}$ is countably generated over B_α and for all α, $B_\alpha \prec_{\infty\omega} B$.*

We now want to go from the existence of certain pairs to CP_n for various n. The difficulty is in suitably framing the induction hypothesis. We define the pair B/A to be $\aleph_0$-free if $A \prec_{\infty\omega} A + B$. In order to state our result exactly we will make an *ad hoc* definition.

DEFINITION 4.2 We say κ *implies* $\mathrm{CP}_{n,m}$ if: κ is regular, and for any variety $\mathcal{V}$ if $(*)_{\kappa,m}$ below holds then the variety satisfies CP_{n+m} where:

$(*)_{\kappa,m}$ there are free algebras (free here means in $\mathcal{V}$) $A, B, F_0, F_1, F_2, \ldots, F_m$ such that

- **(a)** all are free
- **(b)** all have dimension κ (i.e., a basis of cardinality κ)
- **(c)** A is a subalgebra of B
- **(d)** A is the free product of $F_0, F_1, F_2, \ldots, F_m$
- **(e)** if $m > 0$, for $k \in \{1, 2, \ldots, m\}$, B is free over the free product of $\{F_i : i \leq m$ but i is not equal to $k\}$
- **(f)** B/A is κ-free but not essentially free.

We say κ *implies* CP_n if for every m it implies $\mathrm{CP}_{n,m}$.

Remark: *1)* We can weaken the demand on the F_i to having the dimension be infinite. Note, as well, that there is no demand that the free product of $\{F_i : i \neq 0\}$ is a free factor of B.
2) Remember that if $\mathcal{V}$ satisfies CP_{n+1} then $\mathcal{V}$ satisfies CP_n.

Proposition 4.6 $\aleph_0$ *implies* CP_0.

Proof. Without loss of generality we can assume that B is isomorphic to $B * F$ over A where F is a countable rank free algebra. There are two cases to consider. First assume that the free product of $\{F_i : i \neq 0\}$ is a free factor of B. In which case by Theorem 4.3 we have an example of CP_{m+1} (hence $\mathcal{V}$ satisfies CP_m). Next assume that the free product of $\{F_i : i \neq 0\}$ is a not free factor of B. We claim that A^* and B are an example of CP_m, where A^* is the free product of $\{F_i : i \geq 1\}$. All that we have to check (by 4.3) is that for all k, such that $1 \leq k \leq m$, the free product of $\{F_i : i \geq 1, i \neq k\}$ is a free factor of B. But this is part of the hypothesis. $\square$

Note that in the definition we can allow to increase all dimensions to be just at least κ except that B should be generated by A together with a set of cardinality κ.

We will take elementary submodels of various set-theoretic universes and intersect them with an algebra.

PROPOSITION 4.7 *Suppose that A and B are free algebras and B/A is essentially non-free and B is κ-generated over A. If $N \prec (\mathrm{H}(\chi), \in)$, where $A, B \in N$, $\kappa + 1 \subseteq N$ and $\chi \geq |A| + \kappa$, then $(B \cap N)/(A \cap N)$ is essentially non-free. Furthermore if B/A is κ-free, then so is $(B \cap N)/(A \cap N)$.*

PROOF. First deal with the first assertion. Let $Y \in N$ be a free basis of A. (Note that such a Y must exist since $A \in N$.) So $A \cap N$ is freely generated by $Y \cap N$. Without loss of generality, we can assume that B is isomorphic over A to $B * F$ where F is a free algebra of rank κ. Under this assumption, $(B \cap N)/(A \cap N)$ is essentially non-free if and only if it is not free. Suppose that $(B \cap N)/(A \cap N)$ is free. Then we can find $Z \subseteq (N \cap B)$ so that $Z \cup (Y \cap N)$ is a basis of $N \cap B$.

We first claim that $Z \cup Y$ is a free basis for the algebra it generates. If not, then there is some finite $Z_1 \subseteq Z$ and finite $Y_1 \subseteq Y$ so that $Z_1 \cup Y_1$ is not a free basis for the algebra it generates. That is it satisfies some equation which is not a law of the variety. By elementariness, we can find $Y_2 \subseteq N \cap Y$ so that $Z_1 \cup Y_2$ satisfies the same law. This is a contradiction.

To finish the proof we must see that $Z \cup Y$ generates B. Choose $X \in N$ of cardinality κ so that $X \cup A$ generates B. Since $\kappa + 1 \subseteq N$, $X \subseteq N$. Hence $X \subseteq (B \cap N)$. As $Z \cup (Y \cap N)$ generates $B \cap N$ and Y generates A we conclude that $X \cup A$ (and hence B) is contained in the algebra generated by $Z \cup Y$.

The second statement is very simple to prove. If B/A is κ-free, choose in N, X of cardinality κ so that B is generated over A by X and a sequence $(X_\alpha \colon \alpha < \kappa)$ which witnesses that B/A is κ-free. Since each $X_\alpha \subseteq N$, $(B \cap N)/(A \cap N)$ is κ-free. □

THEOREM 4.8 *(1) $\aleph_1$ implies CP_1.*
(2) Suppose κ is a regular cardinal. Assume that for every variety $\mathcal{V}$ and free algebras A, B in $\mathcal{V}$, if B/A is κ-free essentially non-free of cardinality κ then there are: λ which implies $\mathrm{CP}_{n,m+1}$, χ large enough and $M \prec (\mathrm{H}(\chi), \in, <)$ of cardinality $< \kappa$, $M \cap \kappa$ an ordinal $\geq \lambda$ so that $A, B \in M$ and $B \cap M/A$ is λ-pure in B/A and there is an elementary submodel $N \prec (\mathrm{H}(\chi), \in, <)$ such that $A, B, M \in N$, $\lambda + 1 \subseteq N$, $|N| = \lambda$ and $N \cap B/A \cup (B \cap N)$ is almost free essentially non-free . Then κ implies $\mathrm{CP}_{n+1,m}$.
(3) Suppose κ is a regular cardinal. Assume that for every variety $\mathcal{V}$ and free algebras A, B in $\mathcal{V}$, if B/A is κ-free essentially non-free of cardinality κ then

there are: λ which implies CP_n*, χ large enough and $M \prec (\mathrm{H}(\chi), \in, <)$ of cardinality $< \kappa$, $M \cap \kappa$ an ordinal $\geq \lambda$, so that $A, B \in M$ and $B \cap M/A$ is λ-pure in B/A and there is an elementary submodel $N \prec (\mathrm{H}(\chi), \in, <)$ such that $A, B, M \in N$, $\lambda + 1 \subseteq N$, $|N| = \lambda$ and $N \cap B/A \cup (B \cap N)$ is almost free essentially non-free . Then κ implies* CP_{n+1}.

Proof. By Corollary 4.5 and Proposition 4.6, (1) is a special case of (2). Also part (3) follows from part (2), by the definitions. So we will concentrate on that case.

Consider an instance of checking that κ implies $\mathrm{CP}_{n+1,m}$, i.e., we are given $A, B, F_0, F_1, \ldots, F_m$ as in the definition of $(*)_{\kappa,m}$. Let M, χ, λ be as guaranteed in the assumption of the theorem and let N be an elementary submodel of the χ approximation to set theory to which $A, B, M, F_0, F_1, \ldots, F_m$ belong, N has cardinality λ, $\lambda + 1 \subseteq N$ and $(N \cap B)/(A \cup (B \cap M))$ is almost free essentially non free of dimension λ.

Note first that there is a filtration $(B_i : i < \kappa)$ of B such that for all i, B_i/A is free. We assume that the filtration is in M, so there is some i such that $M \cap B = B_i$. So in particular, $M \cap B/A$ is free. It is now easy to see that the algebra generated by $(M \cap B) \cup A$ is the free product over $A \cap M$ of A and $M \cap B$. More exactly it suffices to show that there any relation between elements is implied by the laws of the variety and what happens in A and $B \cap M$. Fix $Y \in M$ a set of free generators of A. As we have pointed out before $Y \cap M$ freely generates $A \cap M$. As well, since $B \cap M/A$ is free, for any finite set $C \subseteq B \cap M$, there is a finite subset $D \subseteq B \cap M$ so that $C \subseteq D + (A \cap M)$ and $D/A \cap M$ is free. Let $Z \subseteq M$ be such that $Z \cup (Y \cap M)$ is a set of free generators for $A + D$. To finish the proof that $A + B \cap M$ is the free product of A and $B \cap M$ over $A \cap M$, it suffices to see that $Y \cup Z$ freely generates $A + D$. The set obviously generates $A + D$. By way of obtaining a contradiction assume that a forbidden relation holds among some elements of Z and some elements of Y. Then since M is an elementary submodel of an approximation to set theory, the elements of Y can be taken to be in $Y \cap M$, which contradicts the choice of Z. Finally note that since $B \cap M/A$ is free, $B \cap M/A \cap M$ is essentially free.

Let A_0 be $A \cap M$ and let B_0 be $B \cap M$. Let A_1 be the subalgebra of B generated by $A \cup B_0$.

As each F_k $(k \leq m)$ is free and we can assume belongs to M, clearly

$$F_k^0 =^{df} F_k \cap M$$

is free of dimension λ and F_k is the free product of F_k^0 and some free F_k^1 which has dimension κ.

Without loss of generality, each F_k^1 belongs to N. Let B_1 be the subalgebra of B generated by $B_0 \cup F_0^1$. Since B_0/A_0 is essentially free, B_1/A_0 is free. Let F_{m+1}^1 be a free subalgebra of B_1 such that B_1 is the free product of A_0 and F_{m+1}^1.

Without loss of generality, $F_{m+1}^1 \in N$. For $k \leq m+1$, let F_k^* be $F_k^1 \cap N$ if $k > 0$ and A_0 if $k = 0$. Let B^* be $B \cap N$ and let A^* be the subalgebra of B generated by $[A \cup (B \cap M)] \cap N$.

That B^*, A^*, F_k^* (for $k \leq m+1$) are free should be clear, as well as the fact that A^* is the free product of $\{F_k^* : k \leq m+1\}$. Furthermore B^*/A^* is almost free but essentially non-free. It remains to prove that if $k \leq m+1$ is not zero, then B^* is free over the free product of $\{F_i^* : i \neq k\}$. After we have established this fact we can use that λ implies $\mathrm{CP}_{n,m+1}$, to deduce that CP_{n+m+1} holds in the variety.

Assume first that $k \leq m$. We know that B is free over the free product of $\{F_i : i \neq k\}$. So $B \cap N$ is free over the algebra generated by $(B \cap M) \cup \cup\{F_i \cap N : i \leq m, i \neq k\}$. But the algebra generated by $(B \cap M) \cup *_{i \leq m, i \neq k} F_i$ is the same as $A_0 * F_{m+1}^1 * *_{0 < i \leq m, i \neq k} F_i^1$. Next assume that $k = m+1$. We must show that $B \cap N$ is free over $A_0 * F_1^* * \cdots * F_m^*$. As above $B \cap N$ is free over A and so as before $B \cap N$ is essentially free over $A \cap N$. Since $A \cap N = A_0 * F_1^* * \cdots * F_m^* * (F_0^1 \cap N)$, $B \cap N$ is essentially free over $A_0 * F_1^* * \cdots * F_m^*$. So replacing, if necessary, $B \cap N$ by its product with a free algebra we are done. $\square$

Discussion: In order to get a universe where the existence of an $\aleph_n$-freeessentially non-free algebra implies CP_n, we will use various reflection principles. We will consider sentences of the form $Q_1 X_1 Q_2 X_2 \ldots Q_n X_n \psi(X_1, \ldots X_n)$, where $Q_1, \ldots Q_n$ are either *aa* or *stat* and $\psi(X_1, \ldots, X_n)$ is any (second-order) sentence about $X_1, \ldots, X_n$ (i.e., ψ is just a first order sentence where $X_1, \ldots, X_n$ are extra predicates). We call this language $L_2(aa)$. To specify the semantics of this language we first fix a cardinal λ, and say in the λ-interpretation, a model A satisfies $aa\, X\, \psi(X)$ if there is a closed unbounded set, $\mathcal{C}$ in $\mathcal{P}_{<\lambda}(A)$ so that $\psi(X)$ for all $X \in \mathcal{C}$. Similarly *stat* means "for a stationary set". To denote the λ interpretation we will write $L_2(aa^\lambda)$ Notice that the Lévy collapse to λ^+ preserves any statement in the λ-interpretation.

Definition 4.3 For regular cardinals κ, λ, let $\Delta_{\kappa\lambda}$ denote the following principle:

> Suppose A is a structure of whose underlying set is κ and φ is any $L_2(aa^\lambda)$-sentence. Let C be any club subset of κ. If A satisfies φ then there is a substructure in C of cardinality λ which satisfies φ.

This principle is adapted from the one with the same name in [8]. They use it to show that that $\aleph_{\omega^2+1}$ may be outside the incompactness spectrum of abelian groups.

THEOREM 4.9 ([8]) *(1) If κ is weakly compact and κ is collapsed (by the Lévy collapse) to $\aleph_2$, then $\Delta_{\aleph_2\aleph_1}$ holds.*

(2) Suppose it is consistent that that there are $\aleph_0$ supercompact cardinals. Then it is consistent that for every $m > n$, $\Delta_{\aleph_m,\aleph_n}$ holds.

THEOREM 4.10 *(1) Suppose $\Delta_{\aleph_2\aleph_1}$ holds. If in a variety there is an $\aleph_2$-free essentially non-free algebra of cardinality $\aleph_2$, then* CP$_2$ *holds.*

(2) Suppose $m > 1$ and for every $m \geq n > k$, $\Delta_{\aleph_n,\aleph_k}$ holds. If in a variety there is an $\aleph_m$-free essentially non-free algebra of cardinality $\aleph_m$, then CP$_m$ *holds.*

PROOF. (1) Suppose A is an $\aleph_2$-free essentially non-free algebra of cardinality $\aleph_2$. Without loss of generality we can assume that $A \cong A * F$ where F is a free algebra of cardinality $\aleph_2$. Hence if $A^* \subseteq A$, $|A^*| < |A|$, $\mu \leq \aleph_2$ then A/A^* is essentially μ-free if and only if A/A^* is μ-free. Consider $C = \{X: |X| = \aleph_0$ and A/X is $\aleph_1$-free$\}$. We claim that C must contain a club. Otherwise we would be able to reflect to a free subalgebra of cardinality $\aleph_1$ which satisfies (in the $\aleph_0$ interpretation)

$$stat X\, stat Y (Y/X \text{ is essentially non-free})$$

which contradicts the freeness of the subalgebra.

Consider now a filtration $\{A_\alpha: \alpha < \omega_2\}$ (such that each A_α is free). If

$$\{\alpha: A/A_\alpha \text{ is } \aleph_1\text{-free and essentially non-free}\}$$

contains a club, hence $\{\alpha : A/A_\alpha$ is $\aleph_1$-free and not almost free$\}$ is stationary and then we are done since in this case by Theorem 4.8, $\aleph_1$ implies CP$_1$ and we have have an instance of $(*)_{\aleph_1,1}$. Hence the variety satisfies CP$_2$.

Assume the set does not contain a club, i.e.,

$$S = \{\alpha\colon A/A_\alpha \text{ is essentially not } \aleph_1\text{-free}\}$$

is stationary. Let $\mathbb{P}$ be the Lévy collapse of $\aleph_2$ to $\aleph_1$. Then, by the previous paragraph,

$$\Vdash_{\mathbb{P}} A \text{ is free.}$$

Let $\dot{X}$ be the name for a free basis of A. Choose some cardinal χ which is large enough for $\mathbb{P}$. Let $M \prec (\mathrm{H}(\chi), \in, <)$ where $<$ is a well-ordering, everything relevant is an element of M, $\omega_1 \subseteq M$, and $M \cap \omega_2 = \delta \in S$. Next choose a countable $N \prec (\mathrm{H}(\chi), \in, <)$ so that $M \in N$. Hence $(N \cap A)/A_\delta$ is essentially non-free. We will contradict this statement and so finish the proof. By Proposition 4.7, $(N \cap A)/(N \cap A_\delta)$ is essentially non-free.

Since $N \cap A \in C$, we have $A/(N \cap A)$ is $\aleph_1$-free. We shall show that $A/(N \cap A_\delta)$ is $\aleph_1$-free. Let us see why this finishes the proof. By the two facts we can find a countable subalgebra B so that $B/(N \cap A)$ and $B/(N \cap A_\delta)$ are both free. But since $B = (N \cap A) * F$ for some free algebra F, we would contradict the fact $(N \cap A)/(N \cap A_\delta)$ is essentially non-free.

Let p be an $N \cap M$-generic condition. Then

$$p \Vdash N \cap M \cap A \ (= N \cap A_\delta) \text{ is generated by } N \cap A_\delta \cap \tilde{X}.$$

So p forces that $A/(N \cap A_\delta)$ is $\aleph_1$-free. But being $\aleph_1$-free is absolute for Lévy forcing.

The proof of part (2) is similar. $\Box$

The situation in part (2) of the theorem above is perhaps the most satisfying. On the other hand we need very strong large cardinal assumptions to make it true. (It is not only our proof which required the large cardinals but the result itself, since if the conclusion of (2) is satisfied then we have for all m, any stationary subset of $\aleph_{m+1}$ consisting of ordinals of cofinality less than $\aleph_m$ reflects.) It is of interest to know if the classes can be separated via a large cardinal notion which is consistent with $\mathrm{V} = \mathrm{L}$. Rather than stating a large cardinal hypothesis we will state the consequence which we will use.

Definition 4.4 Say a cardinal μ is an *ω-limit of weakly compacts* if there are disjoint subsets S, T_n $(n < \omega)$ of μ consisting of inaccessible cardinals so that

1. for every n and $\kappa \in T_n$ and $X \subseteq \kappa$, there is $\lambda \in S$ so that $(V_\lambda, X \cap \lambda, \in) \prec_{\Sigma^1_1} (V_\kappa, X, \in)$
2. for every n and $\kappa \in T_{n+1}$, $T_n \cap \kappa$ is stationary in κ.

Notice that (1) of the definition above implies that every element of T_n is weakly compact.

THEOREM 4.11 *Suppose that μ is a ω-limit of weakly compacts and that GCH holds. Let S, T_n $(n < \omega)$ be as in the definition. Then there is a forcing extension of the universe satisfying: for all n and $\kappa \in T_n$ there is a κ-free essentially non-free algebra of cardinality κ if and only if* CP_{n+1} *holds.*

PROOF. The forcing notion will be a reverse Easton forcing of length μ. That is we will do an iterated forcing with Easton support to get our poset $\mathbb{P}$. The iterated forcing up to stage α will be denoted $\mathbb{P}_\alpha$ and the iterate at α will be $\dot{Q}_\alpha$. For α outside of $S \cup \cup_{n<\omega} T_n$, let $\dot{Q}_\alpha$ be the $\mathbb{P}_\alpha$-name for the trivial poset. For $\alpha \in S$, let $\dot{Q}_\alpha$ be the $\mathbb{P}_\alpha$-name for the poset which adds a Cohen generic subset of α. For $\alpha \in T_0$, let $\dot{Q}_\alpha$ be the $\mathbb{P}_\alpha$-name for the poset which adds a stationary non-reflecting subset of α consisting of ordinals of cofinality ω. Finally for $\alpha \in T_{n+1}$, let $\dot{Q}_\alpha$ be the $\mathbb{P}_\alpha$-name for the poset which adds a stationary non-reflecting subset of α consisting of ordinals in T_n. We will refer to this set as E_α

The first fact that we will need is essentially due to Silver and Kunen (see [4]).

Fact. *1)* For all n and $\kappa \in T_n$, if $\dot{Q}$ is the $\mathbb{P}_\kappa$-name for the forcing which adds a Cohen subset of κ, then

$$\Vdash_{\mathbb{P}_\kappa * \dot{Q}} \kappa \text{ is weakly compact.}$$

2) If $\dot{R}$ is the $\mathbb{P}_\alpha * \dot{Q}_\kappa$-name for the forcing which shoots a club through the complement of E_κ, then

$$\Vdash_{\mathbb{P}_\kappa} \dot{Q}_\kappa * \dot{R} \text{ is equivalent to } \dot{Q}.$$

We now want to work in the universe $V^{\mathbb{P}}$. It is easy to to see that each E_α is a stationary non-reflecting set (since the stages of the iteration after $\mathbb{P}_{\alpha+1}$ add no subsets of α). We claim that for all n and $\alpha \in T_n$, if $D \subseteq \alpha$ is a stationary set and D is disjoint from E_α then D reflects in a regular cardinal. This is easy based on the fact. It is enough to work in $V^{\mathbb{P}_{\alpha+1}}$. Let R be the poset which shoots a club through the complement of E_α. After forcing with R, D remains stationary. On the other hand, α becomes weakly compact. So in the extension D reflects and so it must reflect before we force with R.

It is standard (cf. Theorem 1.1) to show that for all n and $\kappa \in T_n$ if CP_{n+1} holds then there is a essentially non-free algebra of cardinality κ which is κ-free. In fact we can construct such an algebra to have E_κ as its Γ-invariant. To complete the proof we will show by induction on n that if $\kappa \in T_n$ then κ implies CP_n. For $n = 0$, there is nothing to prove. Suppose the result is true for n and that B/A is κ-free for some $\kappa \in T_{n+1}$. By Theorem 3.2, $\Gamma(B/A) \subseteq \dot{E}_\kappa$. By the proof of Theorem 3.2, we can write B as $\cup_{\alpha<\kappa} B_\alpha$ a continuous union of free algebras, so that for all α, $B_{\alpha+1} + A$ is κ-pure and if λ is a regular uncountable limit cardinal, then $B_{\lambda+1} + A/B_\lambda + A$ is λ-free of cardinality λ and essentially non-free if and only if it is not free. By Theorem 4.8 we are done. □

That some large cardinal assumption is needed in the previous theorem is clear. For example, if there is no Mahlo cardinal in L, then every uncountable regular cardinal has a stationary subset consisting of ordinals of cofinality ω which does not reflect. So if there is no Mahlo cardinal in L, then the essentially non-free incompactness spectrum of any variety which satisfies CP_1 is the class of regular uncountable cardinals. As we shall see in the next theorem, the existence of a Mahlo cardinal is equiconsistent with the existence of a cardinal κ which is in the essentially non-free incompactness spectrum of a variety if and only if the variety satisfies CP_2. However the situation with CP_3 and higher principles seems different. It seems that the existence of a cardinal which is in the essentially non-free spectrum of a variety if and only if it satisfies CP_3 implies the consistency of the existence of weakly compact cardinals.

THEOREM 4.12 *The existence of a Mahlo cardinal is equiconsistent with the existence of a cardinal κ which is in the essentially non-free incompactness spectrum of a variety if and only if the variety satisfies* CP_2.

PROOF. We can work in L and suppose that κ is the first Mahlo cardinal. Fix $E \subseteq \kappa$ a set of inaccessible cardinals which does not reflect. By Theorem 12 of [7], there is a forcing notion which leaves E stationary so that every stationary set disjoint to E reflects in a regular cardinal. Now we can complete the proof as above to show that κ is as demanded by the theorem. $\square$

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