THE CICHOŃ DIAGRAM

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Abstract. We conclude the discussion of additivity, Baire number, uniformity and covering for measure and category by constructing the remaining 5 models. Thus we complete the analysis of Cichoń's diagram.

1. Introduction

The goal of this paper is to describe the relationship between basic properties of measure and category.

Definition 1.1. Let $\mathcal{N}$ and $\mathcal{M}$ denote the ideals of null subsets of the real line and meager subsets of the real line respectively.

Define the following ten sentences:

$\mathbf{A}(m) \equiv$ unions of fewer than $2^{\aleph_0}$ null sets is null,

$\mathbf{B}(m) \equiv \Re$ is not the union of fewer than $2^{\aleph_0}$ null sets,

$\mathbf{U}(m) \equiv$ every subset of $\Re$ of size less than $2^{\aleph_0}$ is null,

$\mathbf{C}(m) \equiv$ ideal of null sets does not have a basis of size less than $2^{\aleph_0}$.

Sentences $\mathbf{A}(c)$, $\mathbf{B}(c)$, $\mathbf{U}(c)$ and $\mathbf{C}(c)$ are defined analogously by replacing word "null" by the word "meager" in the definitions above.

In addition define

$w\mathbf{D} \ \equiv \ \forall F \subset [\omega^\omega]^{<2^{\aleph_0}} \ \exists g \in \omega^\omega \ \forall f \in F \ \exists^\infty n \ f(n) < g(n)$

and

$\mathbf{D} \ \equiv \ \forall F \subset [\omega^\omega]^{<2^{\aleph_0}} \ \exists g \in \omega^\omega \ \forall f \in F \ \forall^\infty n \ f(n) < g(n)$.

The relationship between these sentences is described in the following diagram which is called Cichoń's diagram:

$$\begin{array}{ccccccc}
\mathbf{B}(m) & \rightarrow & \mathbf{U}(c) & \rightarrow & \mathbf{C}(c) & \rightarrow & \mathbf{C}(m) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \mathbf{D} & \rightarrow & w\mathbf{D} & & \\
 & & \uparrow & & \uparrow & & \\
\mathbf{A}(m) & \rightarrow & \mathbf{A}(c) & \rightarrow & \mathbf{B}(c) & \rightarrow & \mathbf{U}(m)
\end{array}$$

In addition

$$\mathbf{A}(c) \equiv \mathbf{B}(c) \ \& \ \mathbf{D}$$

and

$$\mathbf{C}(c) \equiv \mathbf{U}(c) \ \vee \ w\mathbf{D}.$$

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The proofs of these inequalities can be found in [1], [4] and [7].

In context of this diagram a natural question arises:

Are those the only implications between these sentences that are provable in ZFC?

It turns out that the answer to this question is positive. Every combination of those sentences which does not contradict the implications in the diagram is consistent with ZFC. This is proved in step-by-step fashion and this paper contains constructions of the last 5 models.

The tables below contain all known results on the subject. They are not symmetric but still one can recognize some patterns here. Let $\mathcal{L}$ be the set of sentences obtained from sentences $\mathbf{A}$, $\mathbf{B}$, $\mathbf{U}$, $\mathbf{C}$, $\mathbf{D}$ and $w\mathbf{D}$ using logical connectives. Define $\star : \mathcal{L} \longrightarrow \mathcal{L}$ as

$$\varphi^\star = \begin{cases} \neg\psi^\star & \text{if } \varphi = \neg\psi \\ \psi_1^\star \vee \psi_2^\star & \text{if } \varphi = \psi_1 \vee \psi_2 \\ \neg\mathbf{C} & \text{if } \varphi = \mathbf{A} \\ \neg\mathbf{U} & \text{if } \varphi = \mathbf{B} \\ \neg\mathbf{B} & \text{if } \varphi = \mathbf{U} \\ \neg\mathbf{A} & \text{if } \varphi = \mathbf{C} \\ \neg w\mathbf{D} & \text{if } \varphi = \mathbf{D} \\ \neg\mathbf{D} & \text{if } \varphi = w\mathbf{D} \end{cases} \quad \text{for } \varphi \in \mathcal{L}.$$

It turns out that if φ is consistent with ZFC then $\varphi^\star$ is consistent with ZFC. Moreover, in most cases one can find a notion of forcing $\mathbf{P}$ such that ω_2-iteration of $\mathbf{P}$ over a model for CH gives a model for φ while ω_1-iteration of $\mathbf{P}$ over a model for $\mathbf{MA}$ & $\neg$CH gives a model for $\varphi^\star$.

The first table known as, the Kunen-Miller chart, gives consistency results concerning sentences $\mathbf{A}$, $\mathbf{B}$, $\mathbf{U}$, $\mathbf{C}$ only. It was completed by H. Judah and S. Shelah in [5]. The remaining three tables give corresponding information including all 3 consistent combinations of $\mathbf{D}$ and $w\mathbf{D}$.

			Add	**T**	**F**	**F**	**F**	**F**	**F**
		Category	Baire	**T**	**T**	**T**	**F**	**F**	**F**
	Measure		Unif	**T**	**T**	**F**	**T**	**F**	**F**
Add	Baire	Unif	Cov	**T**	**T**	**T**	**T**	**T**	**F**
T	**T**	**T**	**T**	A					
F	**T**	**T**	**T**	B	C		D		
F	**T**	**F**	**T**				$E=E^\star$		
F	**F**	**T**	**T**	F	G	$H=H^\star$	$I=I^\star$	$G^\star$	$F^\star$
F	**F**	**F**	**T**				$D^\star$	$C^\star$	$B^\star$
F	**F**	**F**	**F**						$A^\star$

A ω_2-iteration with finite (countable) support of amoeba reals over a model for CH or any model for CH or **MA** works.

$A^\star$ ω_2-iteration with finite (countable) support of amoeba reals over a model for $\neg$CH or ω_2-iteration of Sacks or Silver reals over a model for CH.

B ω_2-iteration of random and dominating reals over a model for CH. [7]

$B^\star$ ω_1-iteration of random and dominating reals over a model for $\neg$CH & $\mathbf{B}(c)$.

C ω_2-iteration with finite support of random reals over a model for CH. [7]

$C^\star$ ω_1-iteration with finite support of random reals over a model for $\neg$CH& **D**. [7]

D Countable support ω_2-iteration of infinitely equal reals (see section 3) and random reals over a model for CH. [7]

$D^\star$ ω_2-iteration of Laver reals ([5]). We do not know if there exists a notion of forcing **P** such that ω_2-iteration of **P** over a model for CH gives D and ω_1-iteration of **P** over a model for **MA** & $\neg$CH gives $D^\star$.

$E=E^\star$ $\aleph_2$ random reals over a model for CH. This model is self-dual.

F ω_2-iteration with finite support of any σ-centered notion of forcing adding dominating reals over a model for CH . [7]

$F^\star$ ω_1-iteration with finite support of any σ-centered notion of forcing adding dominating reals over a model for **MA** & $\neg$CH . We can also get a model for this case by an ω_2-iteration of infinitely equal reals over a model for CH.

G ω_2-iteration with finite support of eventually different reals (see [7]) over a model for CH.

$G^\star$ ω_1-iteration with finite support of eventually different reals over a model for $\neg$CH & $\mathbf{B}(c)$.

$H = H^\star$ $\aleph_2$ Cohen reals over a model for CH. This model is self-dual.

$I = I^\star$ ω_2-iteration of Mathias forcing over a model for CH [7]. This model is self dual.

$w\mathbf{D}$ & $\neg\mathbf{D}$			Add	**T**	**F**	**F**	**F**	**F**	**F**
		Category	Baire	**T**	**T**	**T**	**F**	**F**	**F**
	Measure		Unif	**T**	**T**	**F**	**T**	**F**	**F**
Add	Baire	Unif	Cov	**T**	**T**	**T**	**T**	**T**	**F**
T	**T**	**T**	**T**						
F	**T**	**T**	**T**		A		B		
F	**T**	**F**	**T**				C		
F	**F**	**T**	**T**		D	$E = E^\star$	$F = F^\star$	$D^\star$	
F	**F**	**F**	**T**				$B^\star$	$A^\star$	
F	**F**	**F**	**F**						

A ω_2-iteration with finite support of random reals over a model for CH.

$A^\star$ ω_1-iteration with finite support of random reals over a model for $\neg$CH &$\mathbf{D}$.

B ω_2-iteration with countable support of forcing from [10] and random reals over a model for CH (see section 5).

$B^\star$ ω_2-iteration with countable support of rational perfect set forcing and forcing $\mathbf{Q}_{f,g}$ from [11] over a model for CH (see section 5).

C $\aleph_2$ Cohen and then $\aleph_2$ random reals over a model for CH. This model is self-dual.

D ω_2-iteration of eventually different reals over a model for CH. [7]
$D^\star$ ω_1-iteration of eventually different reals over a model for $\neg\mathsf{CH}$ & $\mathbf{B}(c)$.
$E{=}E^\star$ $\aleph_2$ Cohen reals over a model for CH. This model is self-dual.
$F{=}F^\star$ ω_2-iteration with countable support of forcing Q from [2] over a model for CH. This model is self-dual.

Models in the following two tables are dual to each other.

D			Add	**T**	**F**	**F**	**F**	**F**	**F**
		Category	Baire	**T**	**T**	**T**	**F**	**F**	**F**
	Measure		Unif	**T**	**T**	**F**	**T**	**F**	**F**
Add	Baire	Unif	Cov	**T**	**T**	**T**	**T**	**T**	**F**
T	**T**	**T**	**T**	A					
F	**T**	**T**	**T**	B			C		
F	**T**	**F**	**T**				F		
F	**F**	**T**	**T**	E			D		
F	**F**	**F**	**T**				G		
F	**F**	**F**	**F**						

$\neg w\mathbf{D}$			Add	**T**	**F**	**F**	**F**	**F**	**F**
		Category	Baire	**T**	**T**	**T**	**F**	**F**	**F**
	Measure		Unif	**T**	**T**	**F**	**T**	**F**	**F**
Add	Baire	Unif	Cov	**T**	**T**	**T**	**T**	**T**	**F**
T	**T**	**T**	**T**						
F	**T**	**T**	**T**				$G^\star$		
F	**T**	**F**	**T**				$F^\star$		
F	**F**	**T**	**T**				$D^\star$		$E^\star$
F	**F**	**F**	**T**				$C^\star$		$B^\star$
F	**F**	**F**	**F**						$A^\star$

A ω_2-iteration of amoeba reals over a model for CH or any model for $\mathbf{MA}$.
$A^\star$ ω_2-iteration of amoeba reals over a model for $\neg\mathsf{CH}$.
B ω_2-iteration of dominating and random reals over a model for CH. [7]
$B^\star$ ω_2-iteration of dominating and random reals over a model for $\neg\mathsf{CH}$ $\&\mathbf{B}(c)$.
C ω_2-iteration with countable support of Mathias and random reals (see section 5).
$C^\star$ ω_2-iteration with countable support of forcing $\mathbf{Q}_{f,g}$ from [11] (see section 2 and 3).
D ω_2-iteration with countable support of Mathias reals over a model for CH.
$D^\star$ ω_2-iteration with countable support of $\mathbf{Q}_{f,g}$ and infinitely equal reals over a model for CH. (section 2)
E ω_2-iteration of dominating reals over a model for CH. [7]
$E^\star$ ω_2-iteration of dominating reals over a model for $\neg\mathsf{CH}$ & $\mathbf{MA}$ or ω_2-iteration with countable support of eventually equal reals.
F $\aleph_2$ random reals over a model for $\mathbf{MA}$ & $2^{\aleph_0} = \aleph_2$.
$F^\star$ $\aleph_2$ random reals over a model for CH.
G ω_2-iteration with countable support of Laver reals over a model for CH.

$G^\star$ ω_2-iteration with countable support of infinitely equal and random reals over a model for CH. [5]

2. Not adding unbounded reals

Our first goal is to construct a model for ZFC & $\neg w\mathbf{D}$ & $\mathbf{U}(c)$ & $\neg\mathbf{B}(m)$ & $\mathbf{U}(m)$. We start with the definition of the forcing which will be used in this construction. This family of forcing notions was defined in [11].

Definition 2.1. Let $f \in \omega^\omega$ and $g \in \omega^{\omega\times\omega}$ be two functions such that

(1) $f(n) > \prod_{j<n} f(j)$ for $n \in \omega$,
(2) $g(n, j+1) > f(n)^2 \cdot g(n, j)$ for $n, j \in \omega$,
(3) $\min\{j \in \omega : g(n, j) > f(n+1)\} \overset{n\to\infty}{\longrightarrow} \infty$.

Let
$$\mathsf{Seq}^f = \bigcup_{n\in\omega} \prod_{j<n} f(j).$$
For a tree T define $T^{[s]} = \{t \in T : s \subset t \text{ or } t \supset s\}$, $\mathsf{succ}_T(s) = \{t \in T : t \supset s,\ \mathsf{lh}(t) = \mathsf{lh}(s)+1\}$. If $T = T^{[s]}$ for some $s \in T$ then s is called a *stem* of T.

Let $\mathbf{Q}_{f,g}$ be the following notion of forcing: $T \in \mathbf{Q}_{f,g}$ iff

(1) T is a perfect subtree of Seq^f,
(2) there exists a function $h \in \omega^\omega$ diverging to infinity such that
$$\exists n\ \forall m \geq n\ \forall s \in T \cap \omega^m\ |\mathsf{succ}_T(s)| \geq g(m, h(m)).$$

Elements of $\mathbf{Q}_{f,g}$ are ordered by $\subseteq$.

Let $\mathbf{Q}'_{f,g} \subset \mathbf{Q}_{f,g}$ be the set defined as follows: $T \in \mathbf{Q}'_{f,g}$ iff there exists $s_0 \in \mathsf{Seq}^f$ such that $T = T^{[s_0]}$ and there exists an increasing function $h \in \omega^\omega$ such that
$$\forall m \geq \mathsf{lh}(s_0)\ \forall s \in T \cap \omega^{m-1}\ |\mathsf{succ}_T(s)| \geq g(m, h(m)).$$

Clearly $\mathbf{Q}'_{f,g}$ is dense in $\mathbf{Q}_{f,g}$ and therefore from now on we will work with conditions in this form. Notice that

Lemma 2.2. $\mathbf{V}^{\mathbf{Q}_{f,g}} \models$ *"* $\mathbf{V} \cap \omega^\omega$ *is meager in* ω^ω*".*

Proof Notice that if r is a $\mathbf{Q}_{f,g}$-generic real then by an easy density argument we show that
$$\forall h \in \mathbf{V} \cap \omega^\omega\ \forall^\infty n\ h(n) \neq r(n).$$
Therefore $\mathbf{V} \cap \omega^\omega \subset \{h \in \omega^\omega : \forall^\infty n\ h(n) \neq r(n)\}$ which is a meager set. $\square$

Definition 2.3. We say that notion of forcing $\mathbf{P}$ is ω^ω-bounding if
$$\forall \sigma \in \mathbf{V}^{\mathbf{P}} \cap \omega^\omega \quad \Vdash \exists r \in \mathbf{V} \cap \omega^\omega \forall n\ \sigma(n) \leq r(n).$$

The following theorem was proved in [11], we prove it here for completeness;

Theorem 2.4. $\mathbf{Q}_{f,g}$ *is* ω^ω*-bounding.*

Proof We will need the following

Definition 2.5. For $T, T' \in \mathbf{Q}_{f,g}$ and $\widehat{k} \in \omega$ define $T \geq_{\hat{k}} T'$ if

(1) $T \geq T'$,
(2) $\forall s \in T\ \mathsf{succ}_T(s) \neq \mathsf{succ}_{T'}(s) \to |\mathsf{succ}_T(s)| \geq g(\mathsf{lh}(s), \widehat{k})$.

Claim 2.6. *Suppose that $\{T^n : n \in \omega\}$ is a sequence of elements of $\mathbf{Q}_{f,g}$ such that $T^{n+1} \geq_{k_n} T^n$ for $n \in \omega$ where $\{k_n : n \in \omega\}$ is an increasing sequence of natural numbers. Then there exists $T \in \mathbf{Q}_{f,g}$ such that $T \geq_{k_n} T^n$ for $n \in \omega$.*

Proof For $n \in \omega$ define

$$u_n = \min\{j \in \omega : \forall k \geq j \ \forall s \in T^n \cap \omega^k \ |\mathsf{succ}_{T_n}(s)| \geq g(k, k_n)\}.$$

Let $T = \bigcup_{n\in\omega} T^n \restriction u_n$. Function $h(m) = k_{n-1}$ for $m \in [u_{n-1}, u_n)$ witnesses that $T \in \mathbf{Q}_{f,g}$. $\square$

Lemma 2.7. *Let $T \in \mathbf{Q}_{f,g}$ and τ be such that $T \Vdash \tau \in \omega$. Suppose that $\widehat{k} \in \omega$. Then there exists $\widehat{T} \geq_{\hat{k}} T$ and $n \in \omega$ such that*

$$\forall s \in \widehat{T} \cap \omega^n \ \exists a_s \in \omega \ \widehat{T}^{[s]} \Vdash \tau = a_s.$$

Proof Let $S \subseteq T$ be the set of all $t \in T$ such that $T^{[t]}$ satisfies the lemma. In other words

$$S = \{t \in T : \exists n_t \in \omega \ \exists \widehat{T} \geq_{\hat{k}} T^{[t]} \ \forall s \in \widehat{T} \cap \omega^{n_t} \ \exists a_s \in \omega \ \widehat{T}^{[s]} \Vdash \tau = a_s\}.$$

We want to show that stem of T belongs to S. Notice that if $s \notin S$ then

$$|\mathsf{succ}_T(s) \cap S| \leq g(\mathsf{lh}(s), \widehat{k}).$$

Suppose that stem of T does not belong to S and by induction on levels build a tree $\widehat{S} \geq_{\hat{k}} T$ such that for $s \in \widehat{S}$,

$$\mathsf{succ}_{\widehat{S}}(s) = \begin{cases} \mathsf{succ}_T(s) & \text{if } |\mathsf{succ}_T(s) \cap S| \leq g(\mathsf{lh}(s), \widehat{k}) \\ \mathsf{succ}_T(s) - \mathsf{succ}_S(s) & \text{otherwise} \end{cases}.$$

Clearly $\widehat{S} \in \mathbf{Q}_{f,g}$ since $g(\mathsf{lh}(s), m) - g(\mathsf{lh}(s), \widehat{k}) \geq g(\mathsf{lh}(s), m - \widehat{k})$ for all s and $m > \widehat{k}$.

Find $\widehat{S}_1 \geq \widehat{S}$ and $\widehat{n} \in \omega$ such that $\widehat{S}_1 \Vdash \tau = \widehat{n}$. Now get $t \in T$ and $\widehat{S}_2 \geq \widehat{S}_1$ such that $\widehat{S}_2 \geq_{\hat{k}} T^{[t]}$. But that contradicts the definition of the condition $\widehat{S}$. $\square$

We finish the proof of the theorem. Suppose that $T \Vdash \sigma \in \omega^\omega$. Build by induction sequences $\{T_n : n \in \omega\}$ and $\{k_n : n \in \omega\}$ such that for $n \in \omega$,

(1) $T_{n+1} \geq_{k_n} T_n$,
(2) $\forall s \in T_{n+1} \cap \omega^{k_n} \ \exists a_s \in \omega \ T_{n+1}^{[s]} \Vdash \sigma(n) = a_s$.

Let $T = \lim_{n\to\infty} T_n$ and let $r(n) = \max\{a_s : s \in T \cap \omega^{k_n}\}$ for $n \in \omega$. Then

$$T \Vdash \forall n \in \omega \ \sigma(n) \leq r(n)$$

which finishes the proof. $\square$

Notice that in fact we proved that

Lemma 2.8. *If $T \Vdash \sigma \in \omega^\omega$ then there exists a sequence $\{k_n : n \in \omega\}$ and a tree $\widehat{T} \geq T$ such that*

$$\forall s \in \widehat{T} \cap \omega^{k_n} \ \exists a_s \in \omega \ \widehat{T}^{[s]} \Vdash \sigma(n) = a_s. \quad \square$$

Our next goal is to show that forcing with $\mathbf{Q}_{f,g}$ does not add random reals. We will need the following

Definition 2.9. Let $f \in \omega^\omega$ and let $X_f = \prod_{n=0}^{\infty} f(n)$. Define S_f as follows: $T \in S_f$ if T is a perfect subtree of Seq^f and

$$\lim_{n\to\infty} \frac{|T \cap \omega^n|}{\prod_{m=1}^{n-1} f(m)} = 0.$$

Notion of forcing $\mathbf{Q}$ is called f-bounding if

$$\forall \sigma \in X_f \cap \mathbf{V}^{\mathbf{Q}}\ \exists T \in S_f \cap \mathbf{V}\ \forall n\ \sigma{\restriction}n \in T.$$

Theorem 2.10. *Let* $\mathbf{P}$ *be a notion of forcing. We have the following*

(1) *If* $\mathbf{P}$ *is an* f*-bounding notion of forcing then* $\mathbf{P}$ *does not add random reals.*
(2) *If* $\mathbf{P}$ *is* ω^ω*-bounding and* $\mathbf{P}$ *does not random reals then* $\mathbf{P}$ *is* f*-bounding for every* $f \in \omega^\omega$*.*

Proof Define a measure μ on X_f as a product of equally distributed, normalized measures on $f(n)$.

(1) Every element of S_f corresponds to a closed, measure zero subset of X_f. This finishes the proof as X_f is isomorphic to the Cantor space with standard measure.

(2) Suppose that $\Vdash \sigma \in X_f$. Since we assume that $\mathbf{P}$ does not add random reals we can find a null G_δ subset $H \in \mathbf{V}$ of X_f such that $\Vdash \sigma \in H$.

Claim 2.11. *Suppose that* $H \subseteq X_f$*. Then* $\mu(H) = 0$ *iff there exists a sequence* $\{J_n \subseteq \mathsf{Seq}^f \cap \omega^n : n \in \omega\}$ *such that*

(1) $H \subseteq \{x \in X_f : \exists^\infty n\ x{\restriction}n \in J_n\}$,
(2) $\sum_{n=0}^{\infty} \mu(\{x \in X_f : x{\restriction}n \in J_n\}) < \infty$.

Proof ($\leftarrow$) This implication is an immediate consequence of Borel-Cantelli lemma.

($\rightarrow$) Since $\mu(H) = 0$ there are open sets $\{G_n : n \in \omega\}$ covering H such that $\mu(G_n) < \frac{1}{2^n}$ for $n \in \omega$. Write each G_n as a union of disjoint basic sets i.e.

$$G_n = \bigcup_{m\in\omega} [s^n_m] \ for\ n \in \omega.$$

Let $J_n = \{s \in \mathsf{Seq}^f \cap \omega^n : s = s^l_k$ for some $k, l \in \omega\}$ for $n \in \omega$. Verification of (1) and (2) is straightforward. □

Let $\{J_n : n \in \omega\}$ be a sequence obtained by applying the above to the set H. In particular $\{n \in \omega : \sigma{\restriction}n \in J_n\}$ is infinite. Using the fact that forcing $\mathbf{P}$ is ω^ω-bounding find a function $h \in \omega^\omega$ such that $\forall n\ \exists m \in [h(n), h(n+1))\ \sigma{\restriction}m \in J_m$. Let

$$C = \bigcap_{n\in\omega} \bigcup_{m=h(n)}^{h(n+1)} \bigcup_{s\in J_m} [s].$$

It is easy to see that C is a closed set and that $\Vdash \sigma \in C$. As C is a closed set C is a set of branches of some tree T. This tree has required properties. □

The following theorem was proved in [11], we prove it here for completness.

Theorem 2.12. *Forcing* $\mathbf{Q}_{f,g}$ *is* f*-bounding.*

Proof We start with the following

Lemma 2.13. *If $\widetilde{T} \Vdash \forall n\ \sigma(n) \leq f(n)$ then there exists tree $\widehat{T} \geq \widetilde{T}$ such that*

$$\forall s \in \widehat{T} \cap \omega^n\ \exists a_s \leq f(n)\ \widehat{T}^{[s]} \Vdash \sigma(n) = a_s.$$

PROOF By applying 2.8 we get a tree $T \geq \widetilde{T}$ and a sequence $\{k_n : n \in \omega\}$ such that

$$\forall s \in T \cap \omega^{k_n}\ \exists a_s \in \omega\ T^{[s]} \Vdash \sigma(n) = a_s.$$

Without loss of generality we can assume that $k_n \geq n$ for all $n \in \omega$. Suppose that function $h \in \omega^\omega$ witnesses that $T \in \mathbf{Q}_{f,g}$. In other words $|\mathsf{succ}_T(s)| \geq g(\mathsf{lh}(s), h(\mathsf{lh}(s)))$ for $s \in T$.

Build by induction a family of trees $\{T_{n,l} : n \in \omega, n \leq l \leq k_n\}$ such that

(1) $T_{n,l} \geq T_{n,l'}$ for $l \leq l',\ n \in \omega$,
(2) $T_{n,l}{\restriction}l = T_{n,l'}{\restriction}l$ for $l \leq l',\ n \in \omega$,
(3) $T_{n,l} \geq T_{m,l'}$ for $n < m$ and all $l, l' \in \omega$,
(4) $T_{n,l}{\restriction}n = T_{m,l'}{\restriction}n$ for $n < m$ and all l, l',
(5) $\forall n\ \forall s \in T_{n,l} \cap \omega^l\ \exists a_s \leq f(n)\ T_{n,l}^{[s]} \Vdash \sigma(n) = a_s$,
(6) $\forall n\ \forall s \in T_{n,n} \cap \omega^{\leq n}\ |\mathsf{succ}_{T_{n,l}}(s)| \geq g(\mathsf{lh}(s), h(\mathsf{lh}(s)) - 1)$.

It is clear that

$$\widehat{T} = \lim_{n \to \infty} T_{n,n}$$

has the required properties and the function $h'(n) = h(n) - 1$ witnesses that $\widehat{T} \in \mathbf{Q}_{f,g}$.

Suppose that the tree $T_{n,n}$ is given for some $n \in \omega$. Trees $T_{n+1,k_n} \geq T_{n+1,k_n-1} \geq \ldots \geq T_{n+1,n+1}$ are constucted by induction as follows:

Let $T_{n+1,k_n} = T_{n,n}$ and suppose that $T_{n+1,l}$ is given. Tree $T_{n,l-1}$ will be defined in the following way: $T_{n,l-1}{\restriction}l - 1 = T_{n,l}{\restriction}l - 1$ and for each $t \in T_{n,l} \cap \omega^{l-1}$ we will specify which of the immediate successors of t belong to $T_{n,l-1}$.

Take $t \in T_{n+1,l} \cap \omega^{l-1}$ and let $s \in \mathsf{succ}_{T_{n+1,l}}(t)$. By (5) there exists $a_s \leq f(n)$ such that $T_{n+1,l}^{[s]} \Vdash \sigma(n) = a_s$. That defines a partition of the set $\mathsf{succ}_{T_{n+1,l}}(t)$ into $f(n)$ many pieces. Let the set of immediate successors of t in $T_{n+1,l-1}$ be the largest piece in this partition.

Notice that for $t \in T \cap \omega^n$ the set $\mathsf{succ}_T(t)$ will be altered at most n times and each time its size will decrease by a factor $f(i)$ for $i \leq n$. Therefore

$$|\mathsf{succ}_{T_{n,n}}(t)| > \frac{g(n, h(n))}{\prod_{i \leq n} f(i)} \geq g(n, h(n) - 1).$$

This verifies (6) and finishes the proof of the lemma. □

Now we can prove the theorem. Let σ be a $\mathbf{Q}_{f,g}$-name such that $\widetilde{T} \Vdash \forall n\ \sigma(n) \leq f(n)$ for some $\widetilde{T} \in \mathbf{Q}_{f,g}$.

Let $\widehat{T} \geq \widetilde{T}$ be the condition as in the lemma above. The tree T' we are looking for will be defined as follows:

$$s \in T' \text{ iff } \exists t \in \widehat{T}\ \widehat{T}^{[t]} \Vdash \sigma{\restriction}\mathsf{lh}(s) = s.$$

By trimming $\widehat{T}$ some more we can see that

$$\frac{|T' \cap \omega^n|}{\prod_{m=1}^n f(m)} \leq \frac{|\widehat{T} \cap \omega^n|}{\prod_{m=1}^n f(m)} \overset{n\to\infty}{\longrightarrow} 0\ . \quad \square$$

To conclude this section we need some preservation theorems. We have to show that a countable support iteration of ω^ω-bounding forcings is ω^ω-bounding. This has been proved for proper forcings (see [9]). Here we present a much easier proof that works for a more limited class of partial orderings. Similarly we need to know that the iterations we use do not add random reals. Unfortunately f-boundedness is not preserved by a countable support iteration. We will prove it only for certain partial orderings. For a general preservation theorem of a slightly stronger property called (f, g)-boundedness see [12].

Definition 2.14. Let $\mathbf{P}$ be a notion of forcing satisfying axiom A (see [3]). We say that $\mathbf{P}$ has property $(\star)$ if for every $p \in \mathbf{P}, \widehat{n} \in \omega$ and a $\mathbf{P}$-name τ for a natural number there exists $N \in \omega$ and $q \geq_{\hat{n}} p$ such that $q \Vdash \tau < N$.

It is easy to see that partial orderings having property $(\star)$ are ω^ω-bounding.

Theorem 2.15. *Let $\{\mathbf{P}_\xi, \dot{\mathbf{Q}}_\xi : \xi < \alpha\}$ be a countable support iteration of forcings that have the property $(\star)$. Then $\mathbf{P}_\alpha = \lim_{\xi<\alpha} \mathbf{P}_\xi$ is ω^ω-bounding.*

Proof For $p, q \in \mathbf{P}_\alpha,\ F \in [\alpha]^{<\omega}$ and $\widehat{n} \in \omega$ write $p \geq_{F,\hat{n}} q$ if

(1) $p \geq q$,
(2) $\forall \xi \in F\ p{\restriction}\xi \Vdash p(\xi) \geq_{\hat{n}} q(\xi)$.

The proof of the theorem is based on the following general fact:

Lemma 2.16. *Suppose that $p \in \mathbf{P}_\alpha,\ F \in [\alpha]^{<\omega}$ and $\widehat{n} \in \omega$ are given. Let τ be a $\mathbf{P}$-name for a natural number. Then there exists $q \geq_{F,\hat{n}} p$ and $N \in \omega$ such that $q \Vdash \tau < N$.*

Proof It will be proved by induction on $(|F|, \min F)$ over all possible models. Suppose that $|F| = n+1$ and $\min F = \alpha_0 < \alpha$. By induction hypothesis in $\mathbf{V}^{P_{\alpha_0+1}}$ the lemma is true for $F' = F - \{\alpha_0\}$. Therefore there exists a $\mathbf{Q}_{\alpha_0}$ name $\sigma \in \mathbf{V}^{P_{\alpha_0}}$ such that

$$\mathbf{V}^{P_{\alpha_0}} \models \text{“}p(\alpha_0) \Vdash \exists q\text{”} \geq_{F',\hat{n}} p{\restriction}(\alpha_0, \alpha)\ q\text{”} \Vdash \tau < \sigma\text{”}.$$

Since $\mathbf{Q}_{\alpha_0}$ has property $(\star)$ in $\mathbf{V}^{P_{\alpha_0}}$ we can find $q' \geq_{\hat{n}} p(\alpha_0)$ and N such that

$$\mathbf{V}^{P_{\alpha_0}} \models \text{“}q' \Vdash \sigma < N\text{”}.$$

The last statement is forced by a condition $q_0 \in \mathbf{P}_{\alpha_0}$. Let $q = q_0{}^\frown q'{}^\frown q''$. It is the condition we were looking for. $\square$

Let p_0 be any element of $\mathbf{P}_\alpha$. Suppose that $p_0 \Vdash \sigma \in \omega^\omega$. Using 2.16 define by induction sequences $\{p_n : n \in \omega\}, \{F_n : n \in \omega\}$ and a function $r \in \omega^\omega$ such that

(1) $p_{n+1} \geq_{F_n,n} p_n$ for $n \in \omega$,
(2) $\forall \xi \in \mathsf{supp}(p_n)\ \exists j \in \omega\ \xi \in F_j$,
(3) $F_n \subset F_{n+1}$ for $n \in \omega$,
(4) $p_{n+1} \Vdash \sigma(n) < r(n)$.

Let q be the limit of $\{p_n : n \in \omega\}$. Then $q \Vdash \forall n \in \omega\ \sigma(n) < r(n)$. $\square$

Finally we can prove:

Theorem 2.17. $\mathsf{Con(ZFC)} \to \mathsf{Con}(\mathsf{ZFC}\ \&\ \neg w\mathbf{D}\ \&\ \mathbf{U}(c)\ \&\ \neg\mathbf{B}(m)\ \&\ \mathbf{U}(m))$.

Proof The following notion of forcing was introduced in [7]: let $f \in \omega^\omega$. Define

$p \in \mathbf{Q}_f$ iff

(1) $p : \mathsf{dom}(p) \longrightarrow \omega$,
(2) $\mathsf{dom}(p) \subset \omega$ and $\omega - \mathsf{dom}(p)$ is infinite,
(3) $\forall n\ p(n) \leq f(n)$.

For $p, q \in \mathbf{Q}_f$ $p \geq q$ if $p \supseteq q$ and for $n \in \omega$ $p \geq_n q$ iff $p \geq q$ and the first n elements of $\omega - \mathsf{dom}(p)$ and $\omega - \mathsf{dom}(q)$ are the same.

The following fact is well known:

Lemma 2.18. *Let $\mathbf{P}$ be a notion of forcing. If $\mathbf{P}$ has the Laver property then $\mathbf{P}$ is f-bounding for all functions $f \in \omega^\omega$.* $\square$

Lemma 2.19. *Let $f \in \omega^\omega$ be a strictly increasing function such that $f(n) > 2^n$ for $n \in \omega$. Then*

(1) *$\mathbf{V} \cap 2^\omega$ has measure zero in $\mathbf{V}^{\mathbf{Q}_f}$,*
(2) *$\mathbf{Q}_f$ is f-bounding.*

Proof (1) It is enough to show that $X_f \cap \mathbf{V}$ has measure zero in $\mathbf{V}^{\mathbf{Q}_f}$. Notice that for $h \in X_f$ the set

$$H_h = \{x \in X_f : \exists^\infty n\ x(n) = h(n)\}$$

has measure zero. It is easy to see that

$$X_f \cap \mathbf{V} \subset H_{h_G}$$

where h_G is a generic real.

(2) Let p_0 be any element of $\mathbf{Q}_f$. Suppose that $p_0 \Vdash \sigma \in X_f$. Define by induction sequences $\{p_n : n \in \omega\}$, $\{k_n : n \in \omega\}$ and $\{J_n : n \in \omega\}$ such that

(1) $J_n \subset \mathsf{Seq}^f \cap \omega^{k_n}$ for $n \in \omega$,
(2) $p_{n+1} \geq_n p_n$ for $n \in \omega$,
(3) $p_{n+1} \Vdash \sigma{\restriction}k_n \in J_n$ for $n \in \omega$,
(4) $\dfrac{|J_n|}{\prod_{m=1}^{k_n} f(m)} \leq \dfrac{1}{n}$ for $n \in \omega$.

Let $q \geq p_0$ be the limit of $\{p_n : n \in \omega\}$ and $T = \bigcup_{n\in\omega} J_n$. By removing all nodes whose ancestors are missing we can make sure that T is a tree. Then q forces that σ is a branch through T and by (4) T has measure zero. $\square$

Let $\{\mathbf{P}_\xi, \dot{\mathbf{Q}}_\xi : \xi < \aleph_2\}$ be a countable support iteration such that
$\Vdash_\xi$ "$\dot{\mathbf{Q}}_\xi \cong \mathbf{Q}_{f,g}$" if ξ is even
$\Vdash_\xi$ "$\dot{\mathbf{Q}}_\xi \cong \mathbf{Q}_f$" if ξ is odd.

Let $\mathbf{P} = \mathbf{P}_{\aleph_2}$. Then $\mathbf{V}^{\mathbf{P}} \models \neg w\mathbf{D}$ since $\mathbf{P}$ is ω^ω-bounding, and $\mathbf{V}^{\mathbf{P}} \models \mathbf{U}(c)$ & $\mathbf{U}(m)$ by the properties of forcings $\mathbf{Q}_{f,g}$ and $\mathbf{Q}_f$ (note that $\mathbf{Q}_{f,g}$ has property $(\star)$). To finish the proof we need

Lemma 2.20. *$\mathbf{P}$ is f-bounding.*

Proof For $p, q \in \mathbf{P}$, $F \in [\aleph_2]^{<\omega}$ and $\widehat{n} \in \omega$ denote $p \geq_{F,\hat{n}} q$ if

(1) $p \geq q$,
(2) $\forall \xi \in F\ p{\restriction}\xi \Vdash p(\xi) \geq_{\hat{n}} q(\xi)$.

Let p_0 be any element of $\mathbf{P}$. Suppose that $p_0 \Vdash \sigma \in X_f$. Using the fact that both $\mathbf{Q}_{f,g}$ and $\mathbf{Q}_f$ are f-bounding and arguing as in the proofs of 2.13 and 2.19, define by induction sequences $\{p_n : n \in \omega\}, \{F_n : n \in \omega\}$, $\{k_n : n \in \omega\}$ and $\{J_n : n \in \omega\}$ such that

(1) $J_n \subset \mathsf{Seq}^f \cap \omega^{k_n}$ for $n \in \omega$,
(2) $p_{n+1} \geq_{F_n,n} p_n$ for $n \in \omega$,
(3) $\forall \xi \in \mathsf{supp}(p_n)\ \exists j \in \omega\ \xi \in F_j$,
(4) $F_n \subset F_{n+1}$ for $n \in \omega$,
(5) $p_{n+1} \Vdash \sigma \restriction k_n \in J_n$ for $n \in \omega$,
(6) $\frac{|J_n|}{\prod_{m=1}^{k_n} f(m)} \leq \frac{1}{n}$ for $n \in \omega$.

Let $q \geq p_0$ be the limit of $\{p_n : n \in \omega\}$ and $T = \bigcup_{n\in\omega} J_n$. As before, by removing non-splitting nodes we can assume that T is a tree. Then q forces that σ is a branch through T and by (6) T has measure zero. $\square$

Notice that 2.20 can be proved in the same way for many other forcings including perfect set forcing from section 5.

3. Preserving "old reals have outer measure 1"

In this section we construct a model for ZFC & $\neg w\mathbf{D}$ & $\mathbf{U}(c)$ & $\neg\mathbf{U}(m)$ & $\neg\mathbf{B}(m)$. It is obtained by ω_2-iteration with countable support of $\mathbf{Q}_{f,g}$.

The main problem is to verify that $\neg\mathbf{U}(m)$ holds in that model.

We will use the following technique from [5].

Definition 3.1. Let $\mathbf{P}$ be a notion of forcing. Define

$\star_1[\mathbf{P}]$ iff for every sufficiently large cardinal κ, and for every countable elementary submodel $N \prec H(\kappa, \in)$, if $\mathbf{P} \in N$ and $\{\dot{I}_n : n \in \omega\} \in N$ is a $\mathbf{P}$-name for a sequence of rational intervals and $\{p_n : n \in \omega\} \in N$ is a sequence of elements of $\mathbf{P}$ such that $p_0 \Vdash \sum_{n=1}^{\infty} \mu(\dot{I}_n) < \infty$ and $p_n \Vdash \dot{I}_n = I_n$ for $n \in \omega$ then for every random real x over N, if $x \notin \bigcup_{n\in\omega} I_n$ then there exists $q \geq p_0$ such that

(1) q is $(N, \mathbf{P})$-generic,
(2) $q \Vdash x$ is random over $N[G]$ for every $\mathbf{P}$-generic filter over N containing p_0,
(3) $q \Vdash x \notin \bigcup_{n\in\omega} \dot{I}_n$.

$\star_2[\mathbf{P}]$ iff for every $\mathbf{P}$-name $\dot{A}$ for a subset of 2^ω and every $p \in \mathbf{P}$, if $p \Vdash \mu(\dot{A}) \leq \varepsilon$ then

$$\mu^\star(\{x \in 2^\omega : \exists q \geq p\ q \Vdash x \notin \dot{A}\}) \geq 1 - \varepsilon.$$

$\star_3[\mathbf{P}]$ iff for every $A \subset \mathbf{V} \cap 2^\omega$ of positive measure $\mathbf{V}^{\mathbf{P}} \models \mu^\star(A) > 0$.

$\star_4[\mathbf{P}]$ iff for every sufficiently large cardinal κ, and for every countable elementary submodel $N \prec H(\kappa, \in)$, if $\mathbf{P} \in N$ and $\{p_n : n \in \omega\} \in N$ is a sequence of $\mathbf{P}$ and $\{\dot{A}_n : n \in \omega\} \in N$ is a sequence of elements of $\mathbf{P}$-names such that for $n \in \omega$ $p_n \Vdash \dot{A}_n$ is a Borel set of measure $\leq \varepsilon_n$, and $\lim_{n\to\infty} \varepsilon_n = 0$ then for every random real x over N there exists a condition $q \in \mathbf{P}$ such that

(1) q is $(N, \mathbf{P})$-generic,
(2) $q \Vdash x$ is random over $N[G]$ for every $\mathbf{P}$-generic filter over N containing p_0,
(3) there exists $n \in \omega$ such that $q \geq p_n$ and $q \Vdash x \notin \dot{A}_n$.

In [5] it is proved that

Lemma 3.2. *For every notion of forcing* $\mathbf{P}$*,*

(1) *If* $\mathbf{P}$ *is weakly homogenous then* $\star_2[\mathbf{P}] \leftrightarrow \star_3[\mathbf{P}]$*,*
(2) $\star_1[\mathbf{P}] \leftrightarrow \star_4[\mathbf{P}]$. $\square$

Lemma 3.3. *Suppose that* $\mathbf{P}$ *has property* $\star_1$. *Then* $\mathbf{V}^{\mathbf{P}} \models$ "$\mathbf{V} \cap 2^\omega$ *is not measurable".*

PROOF It is enough to show that $\mathbf{V} \cap 2^\omega$ has positive outer measure. Let $\{\dot{I}_n : n \in \omega\}$ be a $\mathbf{P}$-name for a sequence of rational intervals such that $p_0 \Vdash \sum_{n\in\omega} \mu(\dot{I}_n) \leq \varepsilon < 1$. Find sequences $\{p_n : n \in \omega\}$, $\{j_n : n \in \omega\}$, and $\{I_n : n \in \omega\}$ such that for $n \in \omega$

(1) $p_{n+1} \geq p_n$,
(2) $p_{n+1} \Vdash \dot{I}_j = I_j$ for $j \leq j_n$,
(3) $p_{n+1} \Vdash \sum_{j=j_n}^{\infty} \mu(\dot{I}_j) \leq \varepsilon - \frac{1}{n}$.

It is easy to see that $\sum_{n\in\omega} \mu(I_n) \leq \varepsilon$.

Choose a countable, elementary submodel N of $H(\kappa)$ containing $\mathbf{P}$ and $\{p_n, j_n, \dot{I}_n, I_n : n \in \omega\}$. Since N is countable there exists $x \in \mathbf{V} \cap 2^\omega$ such that x is a random real over N and $x \notin \bigcup_{n\in\omega} I_n$. Using $\star_1[\mathbf{P}]$ we get $q \geq p$ such that $q \Vdash x \notin \bigcup_{n\in\omega} \dot{I}_n$.

Since $\{\dot{I}_n : n \in \omega\}$ was arbitrary it shows that

$$\mathbf{V}^{\mathbf{P}} \models \mu^\star(\mathbf{V} \cap 2^\omega) = 1$$

which finishes the proof. □

The lemma above would be even easier to prove if we assume $\star_3[\mathbf{P}]$. The reason for using property $\star_1[\mathbf{P}]$ is in the following:

Theorem 3.4 ([5])**.** *Suppose that* $\{\mathbf{P}_\xi, \dot{\mathbf{Q}}_\xi : \xi < \alpha\}$ *is a countable support iteration such that* $\Vdash_\xi$ "$\dot{\mathbf{Q}}_\xi$ *has property* $\star_1$ *for* $\xi < \alpha$. *Let* $\mathbf{P} = \mathbf{P}_\alpha$. *Then* $\mathbf{P}$ *has property* $\star_1$. □

To construct the model satisfying ZFC & $\neg w\mathbf{D}$ & $\mathbf{U}(c)$ & $\neg\mathbf{U}(m)$ & $\neg\mathbf{B}(m)$ we show that forcing $\mathbf{Q}_{f,g}$ has property $\star_1$. At the first step we show that it has property $\star_3$ i.e.

Theorem 3.5. *Let* $A \subset 2^\omega$ *be such that* $\mu(A) = \varepsilon_0 > 0$. *Then* $\mathbf{V}^{\mathbf{Q}_{f,g}} \models \mu^\star(A) > 0$.

PROOF Suppose that this theorem is not true. Then there exists a set $A \subset 2^\omega$ such that $\mu^\star(A) = \varepsilon_0 > 0$, a condition $T \in \mathbf{Q}_{f,g}$ and a sequence $\{\dot{I}_n : n \in \omega\}$ of $\mathbf{Q}_{f,g}$-names for rational intervals such that

(1) $T \Vdash \sum_{n=1}^{\infty} \mu(\dot{I}_n) = 1$,
(2) $T \Vdash A \subset \bigcap_{m\in\omega} \bigcup_{n>m} \dot{I}_n$.

Let s_0 be the stem of T. By 2.8 without losing generality we can assume that there exists an increasing sequence of natural numbers $\{k_n : n \in \omega\}$ such that

(1) For every $s \in T \cap \omega^{k_n}$ $T^{[s]}$ forces a value to $\{\dot{I}_j : j \leq n\}$,
(2) $T \Vdash \sum_{n \geq \mathsf{lh}(s_0)} \mu(\dot{I}_n) < \frac{1}{2} \cdot \varepsilon_0$,
(3) $\prod_{n=\mathsf{lh}(s_0)}^{\infty} (1 - \frac{1}{f(n)}) > \frac{1}{2}$.

For $s \in T$ and $j \in \omega$ define

$$I_j^s = \begin{cases} I & \text{if } T^{[s]} \Vdash \dot{I}_j = I \\ \emptyset & \text{otherwise} \end{cases}.$$

Suppose that a function $h \in \omega^\omega$ witnesses that $T \in \mathbf{Q}_{f,g}$ and consider a function $h' \in \omega^\omega$ such that $h'(n) \leq h(n)$ for $n \in \omega$.

Claim 3.6. *For* $x \in 2^\omega$ *the following condition are equivalent:*

(1) *There exists* $T' \geq T$ *such that* h' *witnesses that* $T' \in \mathbf{Q}_{f,g}$ *and* $T' \Vdash x \notin \bigcup_{n\in\omega} \dot{I}_n$,
(2) *For every* $k \geq \mathsf{lh}(s_0)$ *there exists a finite tree* t *of height* k *such that*
  (a) $t \subset T \cap \omega^{\leq k}$,
  (b) $|\mathsf{succ}_t(s)| \geq g(\mathsf{lh}(s), h'(\mathsf{lh}(s)))$ *for* $s \in t \cap \omega^{\geq \mathsf{lh}(s_0)}$,
  (c) *If* $s \in t \cap \omega^k$ *then* $x \notin \bigcup_{j\in\omega} I^s_j$.

Proof (1) $\to$ (2) If T' satisfies (1) then $T'\restriction k$ satisfies (2)

(2) $\to$ (1) Build a sequence $\{t_k : k \in \omega\}$ satisfying (2) and apply the compactness theorem to construct T'. □

Define a set $D \subset 2^\omega$ as follows:
$y \in D$ iff there exists $T' \in \mathbf{Q}_{f,g}$ such that
(1) $T' \geq T$ has the same stem as T ($=s_0$),
(2) $T' \Vdash y \notin \bigcup_{n\geq \mathsf{lh}(s_0)} \dot{I}_n$,
(3) $\forall n \geq \mathsf{lh}(s_0)\ \forall s \in T' \cap \omega^n\ |\mathsf{succ}_{T'}(s)| \geq g(n, h(n)-1)$.

Notice that the set D is defined in $\mathbf{V}$ and since $T \Vdash A \subset \bigcup_{n\geq \mathsf{lh}(s_0)} \dot{I}_n$ we have $\mu(2^\omega - D) > \varepsilon_0$.

For $k \geq \mathsf{lh}(s_0)$ define sets D_k as follows:
$y \in D_k$ iff there exists a finite tree t such that
(1) $t \subset T \cap \omega^{\leq k}$,
(2) $\forall n \geq \mathsf{lh}(s_0)\ \forall s \in t \cap \omega^n\ |\mathsf{succ}_t(s)| \geq g(n, h(n)-1)$,
(3) $\forall s \in t \cap \omega^k\ y \notin \bigcup_{n\geq \mathsf{lh}(s_0)} I^s_n$.

By the above claim $D = \bigcap_{k\in\omega} D_k$. Since sets D_k form a decreasing family we can find $k \in \omega$ such that $\mu(2^\omega - D_k) > \varepsilon_0$.

For every $s \in T$ such that $\mathsf{lh}(s_0) \leq \mathsf{lh}(s) \leq k$ define set $D_{k,s}$ as follows:
$y \in D_{k,s}$ iff there exists a finite tree t such that
(1) $t \subset T \cap \omega^{\leq k}$ and $t = t^{[s]}$,
(2) $\forall n \geq \mathsf{lh}(s)\ \forall s' \in t \cap \omega^n\ |\mathsf{succ}_t(s')| \geq g(n, h(n)-1)$,
(3) $\forall s' \in t \cap \omega^k\ y \notin \bigcup_{n\geq \mathsf{lh}(s_0)} I^{s'}_n$.

Notice that $D_k = D_{k,s_0}$. Observe also that for $s \in T \cap \omega^k$

$$\mu(2^\omega - D_{k,s}) \leq \sum_{n\geq \mathsf{lh}(s_0)} \mu(I^s_n) < \frac{\varepsilon_0}{2}.$$

Claim 3.7. *Suppose that for some* $m \in [\mathsf{lh}(s_0), k-1]$ *and* $s \in T \cap \omega^m$,

$$\mu(2^\omega - D_{k,t}) \ \leq\ a \text{ for } t \in \mathsf{succ}_T(s).$$

Then

$$\mu(2^\omega - D_{k,s}) \leq \frac{a}{1 - \dfrac{g(m, h(m)-1)}{g(m, h(m))}}.$$

Proof Notice that $y \notin D_{k,s}$ iff $|\{t \in \mathsf{succ}_T(s) : y \notin D_{k,t}\}| > g(m, h(m)) - g(m, h(m)-1)$.

Claim 3.8. *Let* $N_1 > N_2$ *be two natural numbers. Suppose that* $\{A_j : j \leq N_1\}$ *is a family of subsets of* 2^ω *of measure* $\leq a$. *Let* $U = \{x \in 2^\omega : x$ *belongs to at least* N_2 *sets* $A_j\}$. *Then*

$$\mu(U) \leq a \cdot \frac{N_1}{N_2}.$$

PROOF Let χ_{A_i} be the characteristic function of the set A_i for $i \leq N_1$. It follows that $\int \sum_{i \leq N_1} \chi_{A_i} \leq N_1 \cdot a$ and therefore

$$\mu\left(\left\{x \in 2^\omega : \sum_{i \leq N_1} \chi_{A_i}(x) \geq N_2\right\}\right) \leq \frac{N_1}{N_2} \cdot a. \quad \square$$

By applying the claim above we get

$$\mu(2^\omega - D_{k,s}) \leq a \cdot \frac{g(m,h(m))}{g(m,h(m)) - g(m,h(m)-1)} = \frac{a}{1 - \dfrac{g(m,h(m)-1)}{g(m,h(m))}}. \quad \square$$

Finally by induction we have

$$\mu(2^\omega - D_k) = \mu(2^\omega - D_{k,s_0}) \leq \frac{\varepsilon_0}{2} \cdot \frac{1}{M}$$

where

$$M = \prod_{\mathsf{lh}(s_0)}^{m=k} \left(1 - \frac{g(m,h(m)-1)}{g(m,h(m))}\right) \geq \prod_{\mathsf{lh}(s_0)}^{m=k} \left(1 - \frac{1}{f(m)}\right) > \frac{1}{2}.$$

Therefore $\mu(2^\omega - D_k) < \varepsilon_0$ which gives a contradiction. $\square$

Now we can prove

Theorem 3.9. $\mathbf{Q}_{f,g}$ *has property* $\star_1$.

PROOF We will need several definitions:

Definition 3.10. Let $\{\dot{I}_n : n \in \omega\}$ be a $\mathbf{Q}_{f,g}$-name for a sequence of rational intervals. We say that $T \in \mathbf{Q}_{f,g}$ interprets $\{\dot{I}_n : n \in \omega\}$ if there exists an increasing sequence $\{k_n : n \in \omega\}$ such that for every $j \leq n \in \omega$ and $s \in T \cap \omega^{k_n}$ $T^{[s]}$ decides a value of $\dot{I}_j$ i.e. $T^{[s]} \Vdash \dot{I}_j = I^s_j$ for some rational interval I^s_j.

By 2.8 we know that

$$\{T \in \mathbf{Q}_{f,g} : T \text{ interprets } \{\dot{I}_n : n \in \omega\}\}$$

is dense in $\mathbf{Q}_{f,g}$. Suppose that $T \in \mathbf{Q}_{f,g}$. Subset $S \subseteq T$ is called *front* if for every branch b through T there exists $n \in \omega$ such that $b{\restriction}n \in S$.

Suppose that $D \subseteq \mathbf{Q}_{f,g}$ is an open set. Define

$$cl(D) = \{T \in \mathbf{Q}_{f,g} : \{s \in T : T^{[s]} \in D\} \textit{ is a front in } T\}.$$

Let $\{\dot{I}_n : n \in \omega\}$ be a $\mathbf{Q}_{f,g}$-name for a sequence of rational intervals such that for some $T_0 \in \mathbf{Q}_{f,g}$ $T_0 \Vdash \sum_{n=1}^{\infty} \mu(\dot{I}_n) < \varepsilon < 1$ and T_0 interprets $\{\dot{I}_n : n \in \omega\}$.

Let $N \prec H(\kappa)$ be a countable model containing $\mathbf{Q}_{f,g}$, T_0, $\{\dot{I}_n : n \in \omega\}$.

Define a set $Y \subseteq 2^\omega$ as follows:

$x \in Y$ iff there exists $\widehat{T} \in \mathbf{Q}_{f,g}$ such that

(1) $\widehat{T} \leq T_0$,
(2) If $D \in N$ is an open, dense subset of $\mathbf{Q}_{f,g}$ then there exists $T' \in cl(D) \cap N$ such that $\widehat{T} \leq T'$,
(3) $\widehat{T} \Vdash x \notin \bigcup_{n \in \omega} \dot{I}_n$,

(4) Suppose that $J = \{\ddot{I}_n : n \in \omega\} \in N$ is a $\mathbf{Q}_{f,g}$-name for a sequence of rational intervals such that $\Vdash \sum_{n=1}^{\infty} \mu(\ddot{I}_n) < \infty$ and let $D_J = \{T \in \mathbf{Q}_{f,g} :$ T interprets $\{\ddot{I}_n : n \in \omega\}$ (with sequence $\{k_n^T : n \in \omega\}$). Then there exists $T \in D_J \cap N$ and $k \in \omega$ such that
$\forall m \geq k \ \forall s \in \widehat{T} \cap \omega^{k_m^T} \ x \notin I_m^{T,s}$.

Notice that (2) guarantees that $\widehat{T}$ is $(N, \mathbf{Q}_{f,g})$-generic while (4) guarantees that x is random over $N[G]$.

Lemma 3.11.
(1) *Y is a Σ^1_1 set of reals (in $\mathbf{V}$),*
(2) $\mu(Y) \geq 1 - \varepsilon$.

PROOF (1) It is easy to see that conditions (1)-(4) in the definition of Y are Borel provided that we have an enumeration (we can code as a real number) of the objects appearing in (2) and (4).
(2) easy computation using the fact that $\mathbf{Q}_{f,g}$ has property $\star_3$ and $\star_2$. $\square$

Work in N. Let $G \subset \mathsf{Coll}(\aleph_0, 2^{\aleph_0})$ be generic over N and let x be a random real over $N[G]$. Let $\mathbf{B}$ denotes the measure algebra. Since parameters of the definition of Y are in $N[G]$ we can ask whether $N[G][x] \models x \in Y$.

Since in $N[G]$, Y is a measurable set we can find two disjoint, Borel sets A and B such that $\mu(A \cup B) = 1$ and $A \Vdash_{\mathbf{B}} x \in Y$ and $B \Vdash_{\mathbf{B}} x \notin Y$. Morover $\mu(A) \geq 1 - \varepsilon$. In other words $A \subseteq Y$ *a.e.* and $B \subseteq 2^\omega - Y$ *a.e.*

Since x is a random real over N as well we have

$$\mathsf{Coll}(\aleph_0, 2^{\aleph_0}) \star \mathbf{B} \cong Q_x \star \dot{\mathbf{R}} \cong \mathbf{B} \star \dot{\mathbf{R}}$$

where Q_x is the smallest subalgebra which adds x.

Find a Borel set of positive measure $A^\star$ such that

$$N \models A^\star \Vdash_{\mathbf{B}} \text{“}\exists p \in \dot{\mathbf{R}} \ p \Vdash x \in A''$$

and

$$N \models 2^\omega - A^\star \Vdash_{\mathbf{B}} \text{“} \Vdash x \in B''.$$

It is clear that $A^\star - A$ has measure zero and therefore $\mu(A^\star) \geq 1 - \varepsilon$.

Notice that the definitions above do not depend on the choice of random real x as long as $x \in A^\star$. Thus if x is *any* random real over N such that $x \in A^\star$ then we can find an N-generic filter $G \subset \mathsf{Coll}(\aleph_0, 2^{\aleph_0})$ such that (G, x) is $\mathsf{Coll}(\aleph_0, 2^{\aleph_0}) \star \mathbf{B}$-generic over N and $N[G][x] \models x \in Y$. Since Y is a Σ^1_1 set it means that $\mathbf{V} \models x \in Y$. In other words there exists a Borel set $A^\star$ of measure $\geq 1 - \varepsilon$ such that if $x \in \mathbf{V} \cap A^\star$ is a random real over N then $x \in Y$.

Now we finish the proof of the theorem. Let N , $\{p_n : n \in \omega\}$, $\{\dot{I}_n : n \in \omega\}$ and x be such that

(1) $p_{n+1} \geq p_n$ for $n \in \omega$,
(2) $p_n \Vdash \dot{I}_n = I_n$ for $n \in \omega$,
(3) $x \notin \bigcup_{n \in \omega} I_n$,
(4) $\sum_{n=1}^{\infty} \mu(I_n) = \varepsilon$.

Define for $n \in \omega$, Y_n = set Y defined for model N, condition p_n and set $\{\dot{I}_{m+n} : m \in \omega\}$.

By the above remarks we can find Borel sets $\{A_n^\star : n \in \omega\} \in N$ such that for $n \in \omega$ $\mu(A_n^\star) \geq 1 - (\varepsilon - \sum_{j \leq n} \mu(I_j))$ and for every $x \in \mathbf{V} \cap A_n^\star$ if x is random over

N then $x \in Y_n$. Since $\mu(\bigcup_{n\in\omega} A^\star_n) = 1$ if x is random over N then $x \in A^\star_n$ for some $n \in \omega$. Therefore $x \in Y_n$ and this finishes the proof as $Y_n \subset Y_0$ for all $n \in \omega$. From the fact that $x \in Y_0$ follows the existence of the condition witnessing $\star_1$. $\square$

Theorem 3.12. $\mathsf{Con}(\mathsf{ZFC}) \to \mathsf{Con}(\mathsf{ZFC}$ & $\neg w\mathbf{D}$ & $\mathbf{U}(c)$ & $\neg\mathbf{U}(m)$ & $\neg\mathbf{B}(m))$.

Proof Let $\{\mathbf{P}_\xi, \dot{\mathbf{Q}}_\xi : \xi < \aleph_2\}$ be a countable support iteration such that $\Vdash_\xi$ "$\dot{\mathbf{Q}}_\xi = \mathbf{Q}_{f,g}$ for $\xi < \aleph_2$. Let $\mathbf{P} = \mathbf{P}_{\aleph_2}$. Then $\mathbf{V}^{\mathbf{P}} \models \neg\mathbf{U}(m)$ because $\mathbf{P}$ has property $\star_1$ and $\mathbf{V}^{\mathbf{P}} \models \neg\mathbf{B}(m)$ and $\neg w\mathbf{D}$ since $\mathbf{P}$ is f-bounding and ω^ω-bounding by 2.20 and 2.15. Finally $\mathbf{V}^{\mathbf{P}} \models \mathbf{U}(c)$ by 2.2. $\square$

4. Rational perfect set forcing

Our next goal is to construct a model for

$$\mathsf{ZFC} \ \&\ w\mathbf{D} \ \&\ \neg\mathbf{D} \ \&\ \mathbf{U}(c) \ \&\ \neg\mathbf{U}(m) \ \&\ \neg\mathbf{B}(m).$$

We will do it in the next section. H This model is obtained as a ω_2-iteration with countable support of $\mathbf{Q}_{f,g}$ and rational perfect set forcing. In this section we will prove several facts about rational perfect set forcing which we will need later.

Recall that rational perfect set forcing is defined as follows:

$T \in \mathbf{R}$ iff T is a perfect subtree of $\omega^{<\omega}$ and for every $s \in T$ there exists $s \subseteq t \in T$ such that $\mathsf{succ}_T(t)$ is infinite.

Elements of $\mathbf{R}$ are ordered by $\subseteq$.

Without loss of generality we can assume that for every $T \in \mathbf{R}$ and $s \in T$ the set $\mathsf{succ}_T(s)$ is either infinite or contains exactly one element since elements of this form are dense in $\mathbf{R}$.

For $T \in \mathbf{R}$ define

$$\mathsf{split}(T) = \{s \in T : \mathsf{succ}_T(s) \text{ is infinite }\}.$$

For $T, T' \in \mathbf{R}$ let

$T \geq_0 T'$ if $T \geq T'$ and T and T' have the same stem.

$T' \geq_n T$ if $T' \geq T$ and for every $s \in \mathsf{split}(T)$ if exactly n proper segments of s belong to $\mathsf{split}(T)$ then $s \in \mathsf{split}(T')$.

First we have to show that forcing $\mathbf{R}$ preserves outer measure.

Definition 4.1. Let $\{\dot{I}_n : n \in \omega\}$ be an $\mathbf{R}$-name for sequence of rational intervals such that $\Vdash \sum_{n=1}^\infty \mu(\dot{I}_n) = \frac{1}{2}$.

We say that $T \in \mathbf{R}$ *interprets* $\{\dot{I}_n : n \in \omega\}$ if for every $s \in \mathsf{split}(T)$ there exist rational intervals $\{I^s_1, \dots, I^s_{n_s}\}$such that

(1) $T^{[s]} \Vdash \forall j \leq n_s\ \dot{I}_j = I^s_j$,
(2) for every $\varepsilon > 0$ and every branch y through T there exists $m \in \omega$ such that for $k \geq m$

$$\mu\left(\bigcup_{j \leq n_{y\restriction k}} I^{y\restriction k}_j\right) \geq \frac{1}{2} - \varepsilon.$$

Lemma 4.2. *Suppose that $\{\dot{I}_n : n \in \omega\}$ is an $\mathbf{R}$-name for sequence of rational intervals. Assume that $T \Vdash \sum_{n=1}^\infty \mu(\dot{I}_n) = \frac{1}{2}$. Then there exists $\widehat{T} \geq T$ such that $\widehat{T}$ interprets $\{\dot{I}_n : n \in \omega\}$.*

PROOF Construct a sequence $\{T_n : n \in \omega\} \subset \mathbf{R}$ such that $T_{n+1} \geq_n T_n$ for $n \in \omega$ as follows:

$T_0 = T$ and suppose that T_n is already constructed.

For every $s \in \mathsf{split}(T_n)$ such that exactly n proper segments of s belong to $\mathsf{split}(T_n)$ and every $m \in \omega$ such that $s^\frown\{m\} \in \mathsf{succ}_{T_n}(s)$ extend $T^{[s^\frown\{m\}]}$ to decide a sufficiently long part of $\{\dot{I}_n : n \in \omega\}$. Paste all extensions together to get T_{n+1}.

Clearly $\widehat{T} = \bigcap_{n\in\omega} T_n$ has required property. $\square$

Now we are ready to show:

Theorem 4.3. *If $A \subseteq 2^\omega$ and $\mu(A) = 1$ then $\Vdash_{\mathbf{R}} \mu^\star(A) > 0$.*

PROOF Suppose not. Then there exists a measure one set $A \subseteq 2^\omega$, a $\mathbf{R}$-name for sequence of rational intervals $\{\dot{I}_n : n \in \omega\}$ and a condition $T \in \mathbf{R}$ such that

(1) $T \Vdash \sum_{n=1}^\infty \mu(\dot{I}_n) = \frac{1}{2}$.
(2) $T \Vdash A \subset \bigcap_{n\in\omega} \bigcup_{m\geq n} \dot{I}_m$.

By the above lemma we can assume that T interprets $\{\dot{I}_n : n \in \omega\}$.

For $s \in \mathsf{split}(T)$ and $\varepsilon > 0$ define

$$h^\varepsilon(s) = \min\left\{ j \in \omega : \sum_{i\leq j} \mu(I_i^s) \geq \frac{1}{2} - \varepsilon \right\}$$

and

$$A_s^\varepsilon = \bigcup_{i\geq h^\varepsilon(s)} I_i^s.$$

Note that $h^\varepsilon(s)$ may be undefined for some ε and s.

Let N be a countable, elementary submodel of $H(\kappa)$ for sufficiently big κ.

Let $x \in A$ be a random real over N . The following holds in $N[x]$.

Lemma 4.4. *For every $\varepsilon > 0$ there exists a tree $T_\varepsilon \subset T$ such that*

(1) *T_ε has no infinite branches.*
(2) *for every $s \in T_\varepsilon$ either $x \in A_s^\varepsilon$ or $\{n \in \omega : s^\frown\{n\} \in \mathsf{succ}_T(s) - \mathsf{succ}_{T_\varepsilon}(s)\}$ is finite.*

PROOF Fix $\varepsilon > 0$. For $s \in \mathsf{split}(T)$ define an ordinal $r_\varepsilon(s)$ as follows:

$r_\varepsilon(s) = 0$ iff $x \in A_s^\varepsilon$,

$r_\varepsilon(s) = \limsup\{r_\varepsilon(t) + 1 : t \in \mathsf{succ}(s)$ and $r_\varepsilon(t)$ is defined$\}$.

In other words $r_\varepsilon(s) \geq \alpha$ iff for all $\beta < \alpha$ there exists infinitely many $t \in \mathsf{succ}_T(s)$ such that $r_\varepsilon(t) \geq \beta$.

Claim 4.5. *For every $s \in \mathsf{split}(T)$ ordinal $r_\varepsilon(s)$ is well defined.*

PROOF If not we inductively build a condition $T' \geq T^{[s]}$ such that $r_\varepsilon(t)$ is not defined for all $t \in \mathsf{split}(T')$. But then $T' \Vdash x \notin \bigcup_{n\geq h^\varepsilon(s)} \dot{I}_n$. Contradiction. $\square$

Let s_0 be the stem of T. Define

$T_\varepsilon = \{s \in T : s_0 \subseteq s$ or for all $k < l$ if $r_\varepsilon(s{\restriction}k)$ and $r_\varepsilon(s{\restriction}l)$ are defined then $r_\varepsilon(s{\restriction}k) > r_\varepsilon(s{\restriction}l))\}$

It is easy to see that T_ε has no branches since for every branch y through T there exists $m \in \omega$ such that for $k \geq m$ $r_\varepsilon(y{\restriction}k) = 0$.

On the other hand if $x \notin A_s^\varepsilon$ then by the definition of rank the set $\{t \in \mathsf{succ}(s) : r_\varepsilon(t) \geq r_\varepsilon(s)\}$ is at most finite which verifies (2). $\square$

By the above lemma for every $\varepsilon > 0$ there exists a tree T_ε together with a function $r_\varepsilon : \mathsf{split}(T_\varepsilon) \longrightarrow \omega_1$ such that

$$\forall s, t \in \mathsf{split}(T_\varepsilon)\ s \subset t \rightarrow r_\varepsilon(s) > r_\varepsilon(t).$$

Since $N[x]$ is a generic extension of N there exists Borel set $B \subset 2^\omega$ of positive measure such that

$$N \models B \Vdash_{\mathbf{B}} \forall \varepsilon > 0 \text{ there exist } r_\varepsilon \text{ and } T_\varepsilon \text{ as in 4.4 }.$$

Fix $\varepsilon_0 = \mu(B)/2$ and let $\dot{r}$ and $\dot{T}$ be $\mathbf{B}$-names for r_{ε_0} and T_{ε_0}.

We can find Borel set $B' \subset B$ such that $\mu(B') > \frac{1}{2} \cdot \mu(B)$ and for $s \in \mathsf{split}(T)$

(1) $\{n \in \omega : \exists B'' \subset B'\ B'' \Vdash s \in \dot{T}\ \&\ s^\frown\{n\} \notin \dot{T}\ \&\ \mu(B'' \cap A_s^{\varepsilon_0}) = 0\}$ is finite,
(2) $\{\alpha \in \omega_1 : \exists B'' \subset B'\ B'' \Vdash \dot{r}(s) = \alpha\}$ is finite.

To show this we use the fact that the measure algebra $\mathbf{B}$ is ω^ω-bounding and $\dot{T}$ is forced to satisfy 4.4(2).

Now define in N
$\widehat{T} = \{s \in T : \exists B'' \subset B'\ B'' \Vdash_{\mathbf{B}} s \in \dot{T}\}$
and
$\widehat{r}(s) = \max(\{\alpha < \omega_1 : \exists B'' \subset B'\ B'' \Vdash_{\mathbf{B}} s \in \dot{T}\ \&\ \dot{r}(s) = \alpha\})$.

Notice that these definitions do not depend on the initial choice of random real x as long as $x \in B'$.

Lemma 4.6.

(1) *$\widehat{T}$ is a subtree of T,*
(2) *If $s \in \widehat{T}$ and $x \in B'$ is any random real over N such that $x \notin A_s^{\varepsilon_0}$ and $s \in \dot{T}[x]$ then $\{n \in \omega : s^\frown\{n\} \in \dot{T}[x] - \widehat{T}\}$ is finite,*
(3) *If $t \subset s \in \widehat{T}$ then $\widehat{r}(t) > \widehat{r}(s)$.*

PROOF (1) and (2) follow immediately from the definition of $\widehat{T}$ and the choice of the set B'.

(3) Suppose that $\widehat{r}(s) = \alpha$. It means that there exists a set $B'' \subset B'$ such that

$$B'' \Vdash \dot{r}(s) = \alpha.$$

Thus

$$B'' \Vdash \dot{r}(t) \text{ is well defined and } > \alpha$$

so $\alpha < \widehat{r}(t)$. $\square$

In particular it follows from (3) that the tree $\widehat{T}$ is well-founded, i.e. has no infinite branches, and that $\widehat{r} : \widehat{T} \longrightarrow \omega_1$ is a rank function such that

$$\forall s \subset t \in \widehat{T}\ \widehat{r}(s) > \widehat{r}(t).$$

By induction on rank define sets $X_s \subset 2^\omega$ for $s \in \mathsf{split}(\widehat{T})$ as follows:

If $\widehat{r}(s) = 0$ then $X_s = A_s^{\varepsilon_0}$. If $\widehat{r}(s) > 0$ then $X_s = \{z \in 2^\omega : z$ belongs to all but finitely many sets X_t where t is an immediate successor of t is $\mathsf{split}(\widehat{T})\}$.

It is easy to check that $\mu(X_s) \leq \varepsilon_0$ for $s \in \mathsf{split}(\widehat{T})$.

Choose $x \in A \cap (B' - X_{s_0})$ which is random over N. Since $x \notin X_{s_0}$ we can find infinitely many immediate successors s of s_0 in $\mathsf{split}(\widehat{T})$ such that $x \notin X_s$. Choose one of them, say $s_1 \supset s_0$

such that $x \notin X_{s_1}$ and $s_1 \in \dot{T}[x]$. By repeating this argument with s_1 instead of s_0 and so on we construct a branch through $\dot{T}[x]$. Contradiction since the tree $\dot{T}[x]$ is well-founded. $\square$.

By repeating the proof of 3.9 we get

Theorem 4.7. $\mathbf{R}$ *has property* $\star_1$. $\square$

5. Not adding dominating and Cohen reals

In this section we construct models for

1. ZFC & $\mathbf{D}$ & $\mathbf{B}(m)$ & $\neg\mathbf{B}(c)$ & $\mathbf{U}(m)$,
2. ZFC & $w\mathbf{D}$ & $\neg\mathbf{D}$ & $\neg\mathbf{B}(c)$ & $\mathbf{B}(m)$ & $\mathbf{U}(m)$,
3. ZFC & $w\mathbf{D}$ & $\neg\mathbf{D}$ & $\mathbf{U}(c)$ & $\neg\mathbf{U}(m)$ & $\neg\mathbf{B}(m)$).

We need the following definitions.

Definition 5.1. Let $\mathbf{P}$ be a notion of forcing. We say that $\mathbf{P}$ is *almost* ω^ω*-bounding* if for every $\mathbf{P}$-name σ such that $p \Vdash \sigma \in \omega^\omega$ there exists a function $f \in \mathbf{V} \cap \omega^\omega$ such that for every subset $A \in \mathbf{V} \cap [\omega]^\omega$ there exists $q \geq p$ such that

$$q \Vdash \exists^\infty n \in A\ \sigma(n) \leq f(n).$$

We say that $\mathbf{P}$ is *weakly* ω^ω*-bounding* if for every $\mathbf{P}$-name σ such that $p \Vdash \sigma \in \omega^\omega$ there exists a function $f \in \mathbf{V} \cap \omega^\omega$ such that there exists $q \geq p$ such that

$$q \Vdash \exists^\infty n\ \sigma(n) \leq f(n).$$

We will use the following two preservation theorems.

Theorem 5.2 ([10]). *Let* $\{\mathbf{P}_\xi, \dot{\mathbf{Q}}_\xi : \xi < \alpha\}$ *be a countable support iteration such that for* $\xi < \alpha$

$\Vdash_\xi \dot{\mathbf{Q}}_\xi$ *is almost* ω^ω*-bounding.*

Then $\mathbf{P}_\alpha = \lim_{\xi<\alpha} \mathbf{P}_\xi$ *is weakly* ω^ω*-bounding.* $\square$

Definition 5.3. Let $\mathbf{P}$ be a notion of forcing satisfying axiom A. We say that $\mathbf{P}$ has *Laver property* if there exists a function $f_{\mathbf{P}} \in \omega^\omega$ such that for every finite set $A \subset \mathbf{V}$, $\mathbf{P}$-name $\dot{a}$, $p \in \mathbf{P}$ and $n \in \omega$ if $p \Vdash \dot{a} \in A$ then there is $q \geq_n p$ and a set $B \subset A$ of size $\leq f_{\mathbf{P}}(n)$ such that $q \Vdash \dot{a} \in B$.

Notice that this definition is actually stronger than standard definition of Laver property.

Theorem 5.4 ([6]). *Let* $S \subset \alpha$ *and suppose that* $\{\mathbf{P}_\xi, \dot{\mathbf{Q}}_\xi : \xi < \alpha\}$ *is a countable support iteration such that*

$\Vdash_\xi$ *"*$\dot{\mathbf{Q}}_\xi$ *is a random real forcing" if* $\xi \in S$

$\Vdash_\xi$ *"*$\dot{\mathbf{Q}}_\xi$ *has Laver property" if* $\xi \notin S$.

Let $\mathbf{P} = \mathbf{P}_\alpha$. *Then no real in* $\mathbf{V}^{\mathbf{P}}$ *is Cohen over* $\mathbf{V}$.

Now we can prove that:

Theorem 5.5.

1. Con(ZFC) $\to$ Con(ZFC & $\mathbf{D}$ & $\mathbf{B}(m)$ & $\neg\mathbf{B}(c)$ & $\mathbf{U}(m)$),
2. Con(ZFC) $\to$ Con(ZFC & $w\mathbf{D}$ & $\neg\mathbf{D}$ & $\neg\mathbf{B}(c)$ & $\mathbf{B}(m)$ & $\mathbf{U}(m)$),
3. Con(ZFC) $\to$ Con(ZFC & $w\mathbf{D}$ & $\neg\mathbf{D}$ & $\mathbf{U}(c)$ & $\neg\mathbf{U}(m)$ & $\neg\mathbf{B}(m)$).

Proof (1) Let $\{\mathbf{P}_\xi, \dot{\mathbf{Q}}_\xi : \xi < \aleph_2\}$ be a countable support iteration such that
$\Vdash_\xi$ "$\dot{\mathbf{Q}}_\xi$ is a random real forcing" if ξ is even
$\Vdash_\xi$ "$\dot{\mathbf{Q}}_\xi$ is Mathias forcing" if ξ is odd.
Let $\mathbf{P} = \mathbf{P}_{\aleph_2}$. Then
$\mathbf{V}^{\mathbf{P}} \models \mathbf{D}$ & $\mathbf{B}(m)$ & $\mathbf{U}(m)$ because Mathias and random reals are added cofinally in the iteration and
$\mathbf{V}^{\mathbf{P}} \models \neg\mathbf{B}(c)$ by 5.4.

(2) Let $\{\mathbf{P}_\xi, \dot{\mathbf{Q}}_\xi : \xi < \aleph_2\}$ be a countable support iteration such that
$\Vdash_\xi$ "$\dot{\mathbf{Q}}_\xi$ is a random real forcing" if ξ is even
$\Vdash_\xi$ "$\dot{\mathbf{Q}}_\xi$ is Shelah forcing from [2]" if ξ is odd.
Let $\mathbf{P} = \mathbf{P}_{\aleph_2}$. Then
$\mathbf{V}^{\mathbf{P}} \models w\mathbf{D}$ & $\mathbf{B}(m)$ & $\mathbf{U}(m)$ because of properties of Shelah forcing and random forcing. To show that $\mathbf{V}^{\mathbf{P}} \models \neg\mathbf{B}(c)$ we use 5.4 and the fact that Shelah forcing has the Laver property.

(3) Let $\{\mathbf{P}_\xi, \dot{\mathbf{Q}}_\xi : \xi < \aleph_2\}$ be a countable support iteration such that
$\Vdash_\xi$ "$\dot{\mathbf{Q}}_\xi \cong \mathbf{Q}_{f,g}$" if ξ is even
$\Vdash_\xi$ "$\dot{\mathbf{Q}}_\xi \cong \mathbf{R}$" if ξ is odd.
Let $\mathbf{P} = \mathbf{P}_{\aleph_2}$. Since $\mathbf{R}$ is has Laver property ([8]) exactly as in 2.20 we show that $\mathbf{P}$ is f-bounding. Therefore $\mathbf{V}^{\mathbf{P}} \models \neg\mathbf{B}(m)$. $\mathbf{V}^{\mathbf{P}} \models \neg\mathbf{U}(m)$ since $\mathbf{Q}_{f,g}$ and $\mathbf{R}$ have property $\star_1$. Also $\mathbf{V} \models w\mathbf{D}$ & $\mathbf{U}(c)$ since $\mathbf{R}$ adds unbounded reals and by 2.2.

To finish the proof of (2) and (3) we have to check that forcings used there do not add dominating reals. By 5.2 it is enough to verify that both Shelah forcing and rational perfect set forcing are almost ω^ω-bounding and this will be proved in the next theorem. □

Theorem 5.6. (1) *Rational perfect set forcing* $\mathbf{R}$ *is almost* ω^ω*-bounding,*
(2) *The Shelah forcing is almost* ω^ω*-bounding.*

Proof Let σ be an $\mathbf{R}$-name such that $T \Vdash \sigma \in \omega^\omega$ for some $T \in \mathbf{R}$. As in 4.2 we can assume that for every $s \in \mathsf{split}(T)$ and $t \in \mathsf{succ}_T(s)$, $T^{[t]}$ decides the value of $\sigma{\restriction}\mathsf{lh}(s)$. Notice that in this case every branch through T gives an interpretation to σ. Let N be a countable, elementary submodel of $H(\kappa)$ such that $\mathbf{R}$, T and σ belong to N. Let $g \in \mathbf{V} \cap \omega^\omega$ be a function which dominates all elements of $N \cap \omega^\omega$. Fix a set $A \in \mathbf{V} \cap [\omega]^\omega$. Since forcing $\mathbf{R}$ has absolute definition it is enough to show that for every $m \in \omega$ and every condition $T' \in N \cap \mathbf{R}$, $T \leq T'$ there exists a condition $T'' \in N \cap \mathbf{R}$, $T' \leq T''$ and $n \in A - [0, m]$ such that $N \models T'' \Vdash \sigma(n) \leq g(n)$. Choose $T' \geq T$ and let $b \in N$ be a branch through T'. Let $\sigma_b \in N \cap \omega^\omega$ be the interpretation of σ obtained using b. By the assumption there exists $n \in A$, $n \geq m$ such that $\sigma_b(n) \leq g(n)$. Choose $T'' = T'^{[t]}$ where $t = b{\restriction}n$.

(2) The proof presented here uses notation from [2]. Since the definition of Shelah's forcing and all the necessary lemmas can be found in [2] we give here only a skeleton of the proof.

Let $p = (w, T) \in \mathbf{S}$ and let τ be an $\mathbf{S}$-name for an element of ω^ω. Let q be a pure extension of p satisfying 2.4 of [2]. Suppose that $q = (w, t_0, t_1, \dots)$. We define by induction a sequence $\{q_l : l \in \omega\}$ satisfying the following conditions:

(1) $q_0 = q$,
(2) $q_{l+1} = (w, t_0^{l+1}, t_1^{l+1}, \dots)$ is an l-extension of q_l,

(3) if $k \leq l+1$ and $(w, w') \in t_0^{l+1} \dots t_k^{l+1}$ and $w' \cap [n(t_k^{l+1}), m(t_k^{l+1})) \neq \emptyset$ when $t_k^{l+1} \in K_{n(t_k^{l+1}), m(t_k^{l+1})}$ then $(w', t_{k+1}^{l+1}, t_{k+1}^{l+1}, \dots)$ forces value for $\tau \restriction k$,

(4) $Dp(t_{l+1}^{l+1}) > l$.

Before we construct this sequence let us see that this is enough to finish the proof.

Let $q^\star = (w, t_1^1, t_2^2, \dots)$. By (4), $q^\star \in \mathbf{S}$.

Let $g(n) = \max\{k : \exists w'\ (w, w') \in t_1^1 \dots t_n^n$ and $(w', t_{n+1}^{n+1}, t_{n+2}^{n+2}, \dots) \Vdash \tau(n) = k\}$ for $n \in \omega$.

Clearly $g \in \omega^\omega$. Suppose that $A \subset \omega$. Define

$$p_A = (w, (t_i^i : i \in A)).$$

It is easy to see that

$$p_A \Vdash \exists^\infty n \in A\ \tau(n) \leq g(n)$$

which finishes the proof.

We build the sequence $\{q_l : l \in \omega\}$ by induction on l. Suppose that q_l is already given. By the definition of $\mathbf{S}$ it is enough to build the condition for some fixed $w^\star = w \cap m(t_0^l, \dots t_l^l)$.

Define a function $C : \omega^{<\omega} \longrightarrow 2$ as follows:

$$C(v) = 1 \text{ iff } \exists k\ (w^\star, v) \in t_{l+1}^l, \dots, t_k^l \text{ and } (v, t_{k+1}^l, t_{k+2}^l, \dots) \text{ forces value for } \tau(l).$$

Using lemma 2.6 from [2] we get a condition where the function C is constantly 0 or 1. The first is impossible since the set of conditions forcing a value for $\tau(l)$ is dense. Therefore we get a condition $q = q_{l+1}$ on which C is constantly 1. Moreover we can assume that q_{l+1} is an l-extension of q_l.

This finishes the induction and the proof. □

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