

# On the number of non-isomorphic subgraphs

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## Abstract

Let  $\mathcal{K}$  be the family of graphs on  $\omega_1$  without cliques or independent subsets of size  $\omega_1$ . We prove that

- (a) it is consistent with CH that every  $G \in \mathcal{K}$  has  $2^{\omega_1}$  many pairwise non-isomorphic subgraphs,
- (b) the following proposition holds in L: *(\*) there is a  $G \in \mathcal{K}$  such that for each partition  $(A, B)$  of  $\omega_1$  either  $G \cong G[A]$  or  $G \cong G[B]$ ,*
- (c) the failure of *(\*)* is consistent with ZFC.

## 1 Introduction

We assume only basic knowledge of set theory — simple combinatorics for section 2, believing in  $L \models \diamond^+$  defined below for section 3, and finite support iterated forcing for section 4.

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Answering a question of R. Jamison, H. A. Kierstead and P. J. Nyikos [5] proved that if an  $n$ -uniform hypergraph  $G = \langle V, E \rangle$  is isomorphic to each of its induced subgraphs of cardinality  $|V|$ , then  $G$  must be either empty or complete. They raised several new problems. Some of them will be investigated in this paper. To present them we need to introduce some notions.

An infinite graph  $G = \langle V, E \rangle$  is called *non-trivial* iff  $G$  contains no clique or independent subset of size  $|V|$ . Denote the class of all non-trivial graphs on  $\omega_1$  by  $\mathcal{K}$ . Let  $I(G)$  be the set of all isomorphism classes of induced subgraphs of  $G = \langle V, E \rangle$  with size  $|V|$ .

H. A. Kierstead and P. J. Nyikos proved that  $|I(G)| \geq \omega$  for each  $G \in \mathcal{K}$  and asked whether  $|I(G)| \geq 2^\omega$  or  $|I(G)| \geq 2^{\omega_1}$  hold or not. In [3] it was shown that (i)  $|I(G)| \geq 2^\omega$  for each  $G \in \mathcal{K}$ , (ii) under  $\diamond^+$  there exists a  $G \in \mathcal{K}$  with  $|I(G)| = \omega_1$ . In section 2 we show that if ZFC is consistent, then so is ZFC + CH + “ $|I(G)| = 2^{\omega_1}$  for each  $G \in \mathcal{K}$ ”. Given any  $G \in \mathcal{K}$  we will investigate its partition tree. Applying the weak  $\diamond$  principle of Devlin and Shelah [2] we show that if this partition tree is a special Aronszajn tree, then  $|I(G)| > \omega_1$ . This result completes the investigation of problem 2 of [5] for  $\omega_1$ .

Consider a graph  $G = \langle V, E \rangle$ . We say that  $G$  is *almost smooth* if it is isomorphic to  $G[W]$  whenever  $W \subset V$  with  $|V \setminus W| < |V|$ . The graph  $G$  is called *quasi smooth* iff it is isomorphic either to  $G[W]$  or to  $G[V \setminus W]$  whenever  $W \subset V$ . H. A. Kierstead and P. J. Nyikos asked (problem 3) whether an almost smooth, non-trivial graph can exist. In [3] various models of ZFC was constructed which contain such graphs on  $\omega_1$ . It was also shown that the existence of a non-trivial, quasi smooth graph on  $\omega_1$  is consistent with ZFC. But in that model CH failed. In section 3 we prove that  $\diamond^+$ , and so V=L, too, implies the existence of such a graph.

In section 4 we construct a model of ZFC in which there is no quasi-smooth  $G \in \mathcal{K}$ . Our main idea is that given a  $G \in \mathcal{K}$  we try to construct a partition  $(A_0, A_1)$  of  $\omega_1$  which is so bad that not only  $G \not\cong G[A_i]$  in the ground model but certain simple generic extensions can not add such isomorphisms to the ground model. We divide the class  $\mathcal{K}$  into three subclasses and develop different methods to carry out our plan.

The question whether the existence of an almost-smooth  $G \in \mathcal{K}$  can be proved in ZFC is still open.

We use the standard set-theoretical notation throughout, cf [4]. Given a graph  $G = \langle V, E \rangle$  we write  $V(G) = V$  and  $E(G) = E$ . If  $H \subset V(G)$  we

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define  $G[H]$  to be  $\langle H, E(G) \cap [H]^2 \rangle$ . Given  $x \in V$  take  $G(x) = \{y \in V : \{x, y\} \in E\}$ . If  $G$  and  $H$  are graphs we write  $G \cong H$  to mean that  $G$  and  $H$  are isomorphic. If  $f : V(G) \rightarrow V(H)$  is a function we denote by  $f : G \cong H$  the fact that  $f$  is an isomorphism between  $G$  and  $H$ .

Given a set  $X$  let  $\text{Bij}_p(X)$  be the set of all bijections between subsets of  $X$ . If  $G = \langle V, E \rangle$  is a graph take

$$\text{Iso}_p(G) = \{f \in \text{Bij}_p(V) : f : G[\text{dom}(f)] \cong G[\text{ran}(f)]\}.$$

We denote by  $\text{Fin}(X, Y)$  the set of all functions mapping a finite subset of  $X$  to  $Y$ .

Given a poset  $P$  and  $p, q \in P$  we write  $p \parallel_P q$  to mean that  $p$  and  $q$  are compatible in  $P$ .

The axiom  $\diamond^+$  claims that *there is a sequence  $\langle S_\alpha : \alpha < \omega_1 \rangle$  of countable sets such that for each  $X \subset \omega_1$  we have a closed unbounded  $C \subset \omega_1$  satisfying  $X \cap \nu \in S_\nu$  and  $C \cap \nu \in S_\nu$  for each  $\nu \in C$ .*

We denote by  $\text{TC}(x)$  the transitive closure of a set  $x$ . If  $\kappa$  is a cardinal take  $H_\kappa = \{x : |\text{TC}(x)| < \kappa\}$  and  $\mathcal{H}_\kappa = \langle H_\kappa, \in \rangle$ .

Let us denote by  $\mathcal{D}_{\omega_1}$  the club filter on  $\omega_1$ .

## 2 $\text{I}(G)$ can be always large

**Theorem 2.1** *Assume that GCH holds and every Aronszajn-tree is special. Then  $|\text{I}(G)| = 2^{\omega_1}$  for each  $G \in \mathcal{K}$ .*

**Remark:** S.Shelah proved, [7, chapter V. §6,7], that the assumption of theorem 2.1 is consistent with ZFC.

During the proof we will apply the following definitions and lemmas.

**Lemma 2.2** *Assume that  $G \in \mathcal{K}$ ,  $A \in [\omega_1]^{\omega_1}$  and  $|\{G(x) \cap A : x \in \omega_1\}| = \omega_1$ . Then  $|\text{I}(G)| = 2^{\omega_1}$ .*

**Proof:** See [3, theorem 2.1 and lemma 2.13]. ■

**Definition 2.3** Consider a graph  $G = \langle \omega_1, E \rangle$ .

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1. For each  $\nu \in \omega_1$  let us define the ordinal  $\gamma_\nu \in \omega_1$  and the sequence  $\langle \xi_\gamma^\nu : \gamma \leq \gamma_\nu \rangle$  as follows: put  $\xi_0^\nu = 0$  and if  $\langle \xi_\alpha^\nu : \alpha < \gamma \rangle$  is defined, then take

$$\xi_\gamma^\nu = \min \{ \xi : \forall \alpha < \gamma \ \xi > \xi_\alpha^\nu \text{ and } (\{ \xi_\alpha^\nu, \xi \} \in E \text{ iff } \{ \xi_\alpha^\nu, \nu \} \in E) \}.$$

If  $\xi_\gamma^\nu = \nu$ , then we put  $\gamma_\nu = \gamma$ .

2. Given  $\nu, \mu \in \omega_1$  write  $\nu \prec^G \mu$  iff  $\xi_\gamma^\nu = \xi_\gamma^\mu$  for each  $\gamma \leq \gamma_\nu$ .
3. Take  $\mathcal{T}^G = \langle \omega_1, \prec^G \rangle$ .  $\mathcal{T}_G$  is called the *partition tree of G*.

**Lemma 2.4** *If  $G = \langle \omega_1, E \rangle \in \mathcal{K}$  with  $|I(G)| < 2^{\omega_1}$ , then  $\mathcal{T}^G$  is an Aronszajn tree.*

**Proof:** By the construction of  $\mathcal{T}^G$ , if  $\nu, \mu \in \omega_1$ ,  $\nu < \mu$  and  $G(\nu) \cap \nu = G(\mu) \cap \nu$ , then  $\nu \prec^G \mu$ . So the levels of  $\mathcal{T}^G$  are countable by lemma 2.2. On the other hand,  $\mathcal{T}^G$  does not contain  $\omega_1$ -branches, because the branches are prehomogeneous subsets and  $G$  is non-trivial. ■

**Definition 2.5** 1. Let  $F : (2^\omega)^{<\omega_1} \rightarrow 2$  and  $A \subset \omega_1$ . We say that a function  $g : \omega_1 \rightarrow 2$  is an *A-diamond for F* iff, for any  $h \in (2^\omega)^{\omega_1}$ ,  $\{ \alpha \in A : F(h \upharpoonright \alpha) = g(\alpha) \}$  is a stationary subset of  $\omega_1$ .

2.  $A \subset \omega_1$  is called a *small subset of  $\omega_1$*  iff for some  $F : (2^\omega)^{<\omega_1} \rightarrow 2$  no function is an *A-diamond for F*.

3.  $\mathcal{J} = \{ A \subset \omega_1 : A \text{ is a small subset of } \omega_1 \}$ .

In [2] the following was proved:

**Theorem 2.6** *If  $2^\omega < 2^{\omega_1}$ , then  $\mathcal{J}$  is a countably complete, proper, normal ideal on  $\omega_1$ .*

After this preparation we are ready to prove theorem 2.1.

**Proof:** Assume that  $G = \langle \omega_1, E \rangle \in \mathcal{K}$ .

$|I(G)| < 2^{\omega_1}$  and a contradiction will be derived.

Since  $2^{\omega_1} = \omega_2$ , we can fix a sequence  $\{ G_\nu : \nu < \omega_1 \}$  of graphs on  $\omega_1$  such that for each  $Y \in [\omega_1]^{\omega_1}$  there is a  $\nu < \omega_1$  with  $G[Y] \cong G_\nu$ . Write  $G_\nu = \langle \omega_1, E_\nu \rangle$ .

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Consider the Aronszajn-tree  $\mathcal{T}^G = \langle \omega_1, \prec^G \rangle$ . Since every Aronszajn-tree is special and  $\mathcal{I}$  is a countably complete ideal on  $\omega_1$ , there is an antichain  $S$  in  $\mathcal{T}^G$  with  $S \notin \mathcal{J}$ . Take

$$A = \left\{ \alpha \in \omega_1 : \exists \sigma \in S (\alpha \prec^G \sigma) \right\}.$$

Now property (\*) below holds:

$$(*) \quad \forall \sigma \in S \forall \rho \in (S \cup A) \setminus \sigma + 1 \\ \exists \alpha \in A \cap \sigma (\{\sigma, \alpha\} \in E \text{ iff } \{\rho, \alpha\} \notin E).$$

Indeed, if for each  $\alpha \in A \cap \sigma$  we had  $\{\sigma, \alpha\} \in E$  iff  $\{\rho, \alpha\} \in E$ , then  $\sigma \prec^G \rho$  would hold by the construction of  $\mathcal{T}^G$ .

Let  $\nu \in \omega_1$ ,  $\sigma \in S$ ,  $T \subset S \cap \sigma$  and  $f : G[(A \cap \sigma) \cup T] \rightarrow G_\nu$  be an embedding. Define  $F(\nu, \sigma, T, f) \in 2$  as follows:

$$F(\nu, \sigma, T, f) = 1 \text{ iff } \exists x \in G_\nu (\forall \alpha \in A \cap \sigma) (\{x, f(\alpha)\} \in E_\nu \text{ iff } \{\sigma, \alpha\} \in E).$$

In case  $\omega\sigma = \sigma$ , under suitable encoding,  $F$  can be viewed as a function from  $(2^\omega)^{<\omega_1}$  to 2.

Since  $S \notin \mathcal{J}$ , there is a  $g \in 2^{\omega_1}$  such that for every  $\nu \in \omega_1 = 2^\omega$ ,  $T \subset S$  and  $f : G[A \cup T] \cong G_\nu$ , the set

$$S_T = \{ \sigma \in S : g(\sigma) = F(\nu, \sigma, T \cap \sigma, f[\sigma]) \}$$

is stationary. Take  $T = \{ \sigma \in S : g(\sigma) = 0 \}$ . Choose an ordinal  $\nu < \omega_1$  and a function  $f$  with  $f : G[A \cup T] \cong G_\nu$ . For each  $\sigma < \omega_1$  with  $\sigma = \omega\sigma$  it follows, by (\*), that

$$\sigma \in T \text{ iff } \exists x \in \omega_1 \forall \alpha \in S \cap \sigma (\{x, f(\alpha)\} \in E_\nu \text{ iff } \{\sigma, \alpha\} \in E).$$

Thus  $g(\sigma) = 0$  iff  $F(\nu, \sigma, T \cap \sigma, f[\sigma]) = 1$ , for each  $\sigma \in S$ , that is,  $S_T = \emptyset$ , which is a contradiction. ■

### 3 A quasi-smooth graph under $\diamond^+$

**Theorem 3.1** *If  $\diamond^+$  holds, then there exists a non-trivial, quasi-smooth graph on  $\omega_1$ .*

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**Proof:** Given a set  $X$ ,  $\mathcal{A} \subset P(X)$  and  $\mathcal{F} \subset \text{Bij}_p(X)$  take

$$\text{Cl}(\mathcal{A}, \mathcal{F}) = \bigcap \{ \mathcal{B} : \mathcal{B} \supset \mathcal{A} \text{ and } \forall B_0, B_1 \in \mathcal{B} \forall f \in \mathcal{F} \forall Y \in [X]^{<\omega} \\ \{B_0 \cup B_1, f''B_0, B_0 \Delta Y\} \subset \mathcal{B} \}.$$

We say that  $\mathcal{A}$  is  $\mathcal{F}$ -closed if  $\mathcal{A} = \text{Cl}(\mathcal{A}, \mathcal{F})$ . Given  $\mathcal{A}, \mathcal{D} \subset P(X)$ , we say that  $\mathcal{D}$  is *uncovered by*  $\mathcal{A}$  if  $|D \setminus A| = \omega$  for each  $A \in \mathcal{A}$  and  $D \in \mathcal{D}$ .

**Lemma 3.2** *Assume that  $\mathcal{F} \subset \text{Bij}_p(X)$  is a countable set,  $\mathcal{A}^0, \mathcal{A}^1 \subset P(X)$  are countable,  $\mathcal{F}$ -closed families. If  $\mathcal{D} \subset P(X)$  is a countable family which is uncovered by  $\mathcal{A}^0 \cup \mathcal{A}^1$ , then there is a partition  $(B_0, B_1)$  of  $X$  such that  $\mathcal{D}$  is uncovered by  $\text{Cl}(\mathcal{A}^i \cup \{B_i\}, \mathcal{F})$  for  $i < 2$ .*

**Proof:** We can assume that  $\mathcal{F}$  is closed under composition. Fix an enumeration  $\{ \langle D_n, k_n, F_n, i_n, A_n \rangle : n \in \omega \}$  of  $\mathcal{D} \times \omega \times \mathcal{F}^{<\omega} \times \{ \langle i, A \rangle : i \in 2, A \in \mathcal{A}^i \}$ . By induction on  $n$ , we will pick points  $x_n \in X$  and will define finite sets,  $B_n^0$  and  $B_n^1$ , such that  $B_n^0 \cap B_n^1 = \emptyset$  and  $B_n^i \subset B_{n+1}^i$ .

Assume that we have done it for  $n - 1$ . Write  $F_n = \langle f_0, \dots, f_{k-1} \rangle$ . Take  $B_{n-1} = B_{n-1}^0 \cup B_{n-1}^1$  and

$$B_n^- = B_{n-1} \cup \bigcup \{ f_j'' B_{n-1} : j < k \}.$$

Pick an arbitrary point  $x_n \in D_n \setminus (A_n \cup B_n^-)$ . Put

$$B_n^{i_n} = B_{n-1}^{i_n}$$

and

$$B_n^{1-i_n} = B_{n-1}^{1-i_n} \cup \{x_n\} \cup \{ f_j^{-1}(x_n) : j < k \}.$$

Next choose a partition  $(B^0, B^1)$  of  $X$  with  $B^i \supset \cup \{ B_n^i : n < \omega \}$  for  $i < 2$ . We claim that it works. Indeed, a typical element of  $\text{Cl}(\mathcal{A}^i \cup \{B^i\}, \mathcal{F})$  has the form

$$C = A \cup \bigcup \{ f_j'' B^i : j < k \},$$

where  $A \in \mathcal{A}$ ,  $k < \omega$  and  $f_0, \dots, f_{k-1} \in \mathcal{F}$ . So, if  $D \in \mathcal{D}$ , then

$$D \setminus C \supset \{ x_n : D_n = D, A_n = A, i_n = i \text{ and } F_n = \langle f_0, \dots, f_{k-1} \rangle \}$$

because  $x_n \notin A$  and  $f_j^{-1}(x_n) \in B^{1-i}$  by the constuction. ■

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Consider a sequence  $F = \langle f_0, \dots, f_{n-1} \rangle$ . Given a family  $\mathcal{F} \subset \text{Bij}_p(X)$  we say that  $F$  is an  $\mathcal{F}$ -term provided  $f_i = f$  or  $f_i = f^{-1}$  for some  $f \in \mathcal{F}$ , for each  $i < n$ . We denote the function  $f_0 \circ \dots \circ f_{n-1}$  by  $F$  as well. We will assume that the empty term denotes the identity function on  $X$ . If  $l \leq n$  take  $(l)F = \langle f_0, \dots, f_{l-1} \rangle$  and  $F_{(l)} = \langle f_l, \dots, f_{n-1} \rangle$ . Let

$$\text{Sub}(F) = \left\{ \langle f_{i_0}, \dots, f_{i_{l-1}} \rangle : l \leq n, i_0 < \dots < i_{l-1} < n \right\}.$$

Given  $f \in \mathcal{F}$  and  $x, y \in X$  with  $x \notin \text{dom}(f)$  and  $y \notin \text{ran}(f)$  let  $F^{f,x,y}$  be the term that we obtain replacing each occurrence of  $f$  and of  $f^{-1}$  in  $F$  with  $f \cup \{\langle x, y \rangle\}$  and with  $f^{-1} \cup \{\langle y, x \rangle\}$ , respectively.

**Lemma 3.3** *Assume that  $\mathcal{F} \subset \text{Bij}_p(X)$ ,  $\mathcal{A} \subset P(X)$  is  $\mathcal{F}$ -closed,  $F_0, \dots, F_{n-1}$  are  $\mathcal{F}$ -terms,  $z_0, \dots, z_{n-1} \in X$ ,  $A_0, \dots, A_{n-1} \in \mathcal{A}$  such that for each  $i < n$*

$$(*) \quad z_i \notin \bigcup \{F''A_i : F \in \text{Sub}(F_i)\}.$$

*If  $f \in \mathcal{F}$ ,  $x \in X \setminus \text{dom}(f)$ ,  $Y \in [X \setminus \text{ran}(f)]^\omega$  with  $|A \cap Y| < \omega$  for each  $A \in \mathcal{A}$ , then there are infinitely many  $y \in Y$  such that  $(*)$  remains true when replacing  $f$  with  $f \cup \{\langle x, y \rangle\}$ , that is,*

$$(**) \quad z_i \notin \bigcup \{F''A_i : F \in \text{Sub}(F_i^{f,x,y})\}$$

*for each  $i < n$ .*

**Proof:** It is enough to prove it for  $n = 1$ . Write  $F = \langle f_0 \dots, f_{k-1} \rangle$ ,  $A = A_0$ ,  $z = z_0$ . Take

$$Y_{F,A} = \{y \in Y : (**) \text{ holds for } y\}.$$

Now we prove the lemma by induction on  $k$ .

If  $k = 0$ , then  $Y_{F,A} = Y \setminus A$ . Suppose we know the lemma for  $k - 1$ . Using the induction hypothesis we can assume that  $(\dagger)$  below holds:

$$(\dagger) \quad Y = \bigcap \left\{ Y_{G, F''_{(l)}A} : l \leq n, G \in \text{Sub}((l)F^{f,x,y}), G \neq F^{f,x,y} \right\}.$$

Assume that  $|Y_{F,A}| < \omega$  and a contradiction will be derived.

First let us remark that either  $f_{k-1} = f$  or  $f_{k-1} = f^{-1}$  by  $(\dagger)$ .

**Case 1:**  $f_{k-1} = f^{-1}$ .

Then  $Y_{F,A} \supset Y \setminus A$  by  $(\dagger)$ , so we are done.

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**Case 2:**  $f_{k-1} = f$ .

In this case  $x \in A$  and for all but finitely many  $y \in Y$  we have  $z = F^{f,x,y}(x)$ . Then for each  $y, y' \in Y$  take

$$l(y, y') = \max \left\{ l \leq n : \forall i < l \ F_{(i)}^{f,x,y}(x) = F_{(i)}^{f,x,y'}(x) \right\}.$$

By Ramsey's theorem, we can assume that  $l(y, y') = l$  whenever  $y, y' \in Y$ . Clearly  $l < n$ . Then  $F_{(l)}^{f,x,y}(x) \neq F_{(l)}^{f,x,y'}(x)$  but  $F_{(l-1)}^{f,x,y}(x) = F_{(l-1)}^{f,x,y'}(x)$ , so  $f_l = f^{-1}$  and  $F_{(l-1)}^{f,x,y}(x) = x$  for each  $y \in Y$ . Thus  $z = (l-1)F^{f,x,y}(x)$  for each  $y \in Y$ , which contradicts  $(\dagger)$  because  $x \in A$ .

The lemma is proved. ■

We are ready to construct our desired graph.

First fix a sequence  $\langle M_\alpha : \alpha < \omega_1 \rangle$  of countable, elementary submodels of some  $\mathcal{H}_\lambda$  with  $\langle M_\gamma : \gamma < \alpha \rangle \in M_\alpha$  for each  $\alpha < \omega_1$ , where  $\lambda$  is a large enough regular cardinal.

Then choose a  $\diamond$ -sequence  $\langle S_\alpha : \alpha < \omega_1 \rangle \in M_0$  for the uncountable subsets of  $\omega_1$ , that is,  $\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\} \notin NS(\omega_1)$  whenever  $X \in [\omega_1]^{\omega_1}$ . We can also assume that  $S_\alpha$  is cofinal in  $\alpha$  for each limit  $\alpha$ .

We will define, by induction on  $\alpha$ ,

1. graphs  $G_\alpha = \langle \omega\alpha, E_\alpha \rangle$  with  $G_\beta = G_\alpha[\omega\beta]$  for  $\beta < \alpha$ ,
2. countable sets  $\mathcal{F}_\alpha \in \text{Iso}_p(G_\alpha)$ ,

satisfying the induction hypotheses (I)–(II) below:

(I)  $\{S_{\omega\gamma} : \gamma \leq \alpha\}$  is uncovered by  $I_\alpha \cup J_\alpha$  where

$$I_\alpha = \text{Cl}(\{G(\nu) \cap \nu : \nu \in \omega\alpha\}, \bigcup_{\beta \leq \alpha} \mathcal{F}_\beta)$$

and

$$J_\alpha = \text{Cl}(\{\nu \setminus G(\nu) : \nu \in \omega\alpha\}, \bigcup_{\beta \leq \alpha} \mathcal{F}_\beta).$$

To formulate (II) we need the following definition.

**Definition 3.4** Assume that  $\alpha = \beta + 1$  and  $Y \subset \omega\alpha$ . We say that  $Y$  is *large* if  $\forall n \in \omega, \forall \langle \langle f_i, x_i \rangle : i < n \rangle, \forall h$   
**if**



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1.  $\forall i < n \exists \alpha_i < \beta f_i \in \mathcal{F}_{\alpha_i}$ ,
2.  $\forall i < n \omega \alpha_i \leq x_i < \omega \beta$ ,
3.  $\forall i < n \text{ran}(f_i) \subset Y$ ,
4.  $\forall i \neq j < n \text{ran}(f_i) \cap \text{ran}(f_j) = \emptyset$
5.  $h \in \text{Fin}(Y \cap \omega \beta, 2)$  and  $\text{dom}(h) \cap \bigcup \{\text{ran}(f_i) : i < n\} = \emptyset$ ,

**then**

$\exists y \in Y \cap [\omega \beta, \omega \alpha)$  such that

6.  $\forall i < n \forall x \in \text{dom}(f_i) (\{y, f_i(x)\} \in E_\alpha \text{ iff } \{x_i, x\} \in E_\alpha)$ ,
7.  $\forall z \in \text{dom}(h) \{y, z\} \in E_\alpha \text{ iff } h(z) = 1$ .

Take

(II) If  $\alpha = \beta + 1$ , then  $\omega \alpha$  is large.

The construction will be carried out in such a way that  $\langle G_\beta : \beta \leq \alpha \rangle \in M_\alpha$  and  $\langle \mathcal{F}_\beta : \beta < \alpha \rangle \in M_\alpha$ .

To start with take  $G_0 = \langle \emptyset, \emptyset \rangle$  and  $\mathcal{F} = \{\emptyset\}$ . Assume that the construction is done for  $\beta < \alpha$ .

**Case 1:**  $\alpha$  is limit.

We must take  $G_\alpha = \bigcup \{G_\beta : \beta < \alpha\}$ . We will define sets  $\mathcal{F}_\alpha^0, \mathcal{F}_\alpha^1 \subset \text{Iso}_p(G_\alpha)$  and will take  $\mathcal{F}_\alpha = \mathcal{F}_\alpha^0 \cup \mathcal{F}_\alpha^1$ .

Let

$$\mathcal{F}_\alpha^0 = \{f \in \text{Iso}_p(G_\alpha) \cap M_\alpha : \exists \langle \alpha_n : n < \omega \rangle \subset \alpha \text{ sup } \{\alpha_n : n < \omega\} = \alpha, \\ f[\omega \alpha_n \in \mathcal{F}_{\alpha_n} \text{ and } f[\omega \alpha_n : G_{\alpha_n} \cong G_{\alpha_n}[\text{ran}(f)]] \text{ for each } n \in \omega\}.$$

Take  $\mathcal{F}^- = \bigcup_{\beta < \alpha} \mathcal{F}^\beta \cup \mathcal{F}_\alpha^0$ ,  $I_\alpha^- = \bigcup_{\beta < \alpha} I_\beta$  and  $J_\alpha^- = \bigcup_{\beta < \alpha} J_\beta$ . Clearly  $\mathcal{F}^- \subset M_\alpha$  with  $\mathcal{F}^- \in M_{\alpha+1}$ , so  $M_{\alpha+1} \models “|\mathcal{F}^-| = \omega”$ . Obviously both  $I_\alpha^-$  and  $J_\alpha^-$  are  $\mathcal{F}^-$ -closed and  $\mathcal{S} = \{S_{\omega \beta} : \beta \leq \alpha\}$  is uncovered by them.

From now on we work in  $M_{\alpha+1}$  to construct  $\mathcal{F}_\alpha^1$ . For  $W \subset \omega \alpha$  write  $L_W = \{\nu < \alpha : W \cap (\omega \nu + \omega) \text{ is large}\}$ .

Take

$$\mathcal{W}_\alpha = \{\langle W, f \rangle \in (P(\omega \alpha) \cap \mathcal{M}_\alpha) \times (\bigcup_{\beta < \alpha} \mathcal{F}_\beta) : L_W \text{ is cofinal in } \alpha$$

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and  $f : G_{\gamma_f} \cong G_{\gamma_f}[W \cap \omega\gamma_f]$  for some  $\gamma_f < \alpha$  }.

We want to find functions  $g^{W,f} \supset f$  for  $\langle W, f \rangle \in \mathcal{W}_\alpha$  such that

- (A)  $g^{W,f} : G_\alpha \cong G_\alpha[W]$
- (B) taking  $\mathcal{F}_\alpha^1 = \{g^{W,f} : \langle W, f \rangle \in \mathcal{W}_\alpha\}$  the induction hypothesis (I) remains true.

First we prove a lemma:

**Lemma 3.5** *If  $\langle W, f \rangle \in \mathcal{W}_\alpha$ ,  $g \in \text{Iso}_p(G_\alpha, G_\alpha[W])$ ,  $g \supset f$ ,  $|g \setminus f| < \omega$ , then*

- (i) *for each  $x \in W \setminus \text{dom}(f)$  the set*

$$\{y \in W : g \cup \{\langle x, y \rangle\} \in \text{Iso}_p(G_\alpha, G_\alpha[W])\}$$

*is cofinal in  $\omega\alpha$ .*

- (ii) *for each  $y \in W \setminus \text{ran}(f)$  the set*

$$\{x \in W : g \cup \{\langle x, y \rangle\} \in \text{Iso}_p(G_\alpha, G_\alpha[W])\}$$

*is cofinal in  $\omega\alpha$ .*

**Proof:** (i): Define the function  $h : \text{ran}(g) \setminus \text{ran}(f) \rightarrow 2$  with  $h(g(z)) = 1$  iff  $\{z, x\} \in E_\alpha$ . Choose  $\beta \in L_W$  with  $\text{ran}(h) \subset \omega\beta$  and  $\gamma_f \leq \beta$ . Since  $W \cap (\omega\beta + \omega)$  is large, we have a  $y \in W \cap [\omega\beta, \omega\beta + \omega)$  such that

1.  $\{y, f(z)\} \in E_\alpha$  iff  $\{x, z\} \in E_\alpha$  for each  $z \in \text{dom}(f)$
2.  $\{y, g(z)\} \in E_\alpha$  iff  $h(g(z)) = 1$  for each  $z \in \text{dom}(g) \setminus \text{dom}(f)$ .

But this means that  $g \cup \{\langle x, y \rangle\} \in \text{Iso}_p(G_\alpha, G_\alpha[W])$ .

- (ii) The same proof works using that  $\omega\beta + \omega$  is large for each  $\beta < \alpha$ . ■

By induction on  $n$ , we will pick points  $z_n \in \omega\alpha$  and will construct families of partial automorphisms,  $\{g_n^{W,f} : \langle W, f \rangle \in \mathcal{W}_\alpha\}$  such that  $g^{W,f} = \bigcup \{g_n^{W,f} : n < \omega\}$  will work.

During the inductive construction we will speak about  $\mathcal{F}_\alpha$ -terms and about functions which are represented by them in the  $n^{\text{th}}$  step.

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If  $F = \langle h_0, \dots, h_{k-1} \rangle$  is an  $\mathcal{F}_\alpha$ -term and  $n \in \omega$  take  $F_{[n]} = j_0 \circ \dots \circ j_{k-1}$  where

$$j_i = \begin{cases} g_n^{W,f} & \text{if } h_i = g^{W,f}, \\ (g_n^{W,f})^{-1} & \text{if } h_i = (g^{W,f})^{-1}, \\ h_i & \text{otherwise.} \end{cases}$$

First fix an enumeration  $\{\langle \langle W_n, f_n \rangle, u_n, i_n \rangle : 1 \leq n < \omega\}$  of  $\mathcal{W}_\alpha \times \omega\alpha \times 2$  and an enumeration  $\langle \langle \langle F_{n,i} : i < l_n \rangle, j_n, \langle A_{n,i} : i < l_n \rangle, D_n \rangle : n < \omega \rangle$  of the quadruples  $\langle \langle F_0, \dots, F_{k-1} \rangle, j, \langle A_0, \dots, A_{k-1} \rangle, D \rangle$  where  $k < \omega$ ,  $F_0, \dots, F_{k-1}$  are  $\mathcal{F}_\alpha$ -terms,  $j \in 2$ ,  $D \in \mathcal{S}$  and either  $j = 0$  and  $A_0, \dots, A_{k-1} \in I_\alpha^-$  or  $j = 1$  and  $A_0, \dots, A_{k-1} \in J_\alpha^-$ .

During the inductive construction conditions (i)–(v) below will be satisfied:

- (i)  $g_0^{W,f} = f$
- (ii)  $g_n^{W,f} \in \text{Iso}_p(G_\alpha, G_\alpha[W])$
- (iii)  $g_n^{W,f} \supset g_{n-1}^{W,f}$ ,  $|g_n^{W,f} \setminus f| < \omega$
- (iv)  $z_k \notin \bigcup \{F''_{[n]} A_{k,i} : F \in \text{Sub}(F_{k,i})\}$  for each  $i < l_k$  and  $k < n$
- (v) if  $i_n = 1$ , then  $u_n \in \text{dom}(g_n^{W_n, f_n})$ ,  
if  $i_n = 0$ , then either  $u_n \notin W_n$  or  $u_n \in \text{ran}(g_n^{W_n, f_n})$ .

If  $n = 0$ , then take  $g_0^{W,f} = f$ .

If  $n > 0$ , then let  $g_n^{W,f} = g_{n-1}^{W,f}$  whenever  $\langle W, f \rangle \neq \langle W_n, f_n \rangle$ . Assume that  $i_n = 0$ ,  $\langle W, f \rangle = \langle W_n, f_n \rangle$  and  $u_n \notin \text{dom}(g_{n-1}^{W_n, f_n})$ . Then, by lemma 3.5, the set  $Y = \{y \in W : g_{n-1}^{W,f} \cup \{\langle u_n, y \rangle\} \in \text{Iso}_p(G_\alpha, G_\alpha[W])\}$  is unbounded in  $\omega\alpha$ . Since the members of  $I_\alpha^- \cup J_\alpha^-$  are bounded in  $\omega\alpha$ , we can apply lemma 3.3 to pick a point  $y \in Y$  such that taking  $g_n^{W_n, f_n} = g_{n-1}^{W_n, f_n} \cup \{\langle u_n, y \rangle\}$  condition (iv) holds.

If  $i_n = 1$  and  $\langle W, f \rangle = \langle W_n, f_n \rangle$ , then the same argument works.

Finally pick a point

$$z_n \notin D_n \setminus \bigcup \{F''_{[n]} A_{n,i} : F \in \text{Sub}(F_{n,i}), i < l_n\}.$$

The inductive construction is done.

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Take  $g^{W,f} = \cup \{g_n^{W,f} : n < \omega\}$ . By (v),

$$g_n^{W,f} : G_{\omega\alpha} \cong G_{\omega\alpha}[W].$$

By (iv), we have

$$z_k \in D_k \setminus \bigcup \{F''_{k,i} A_{k,i} : i < l_k\}$$

and so it follows that  $\{S_{\omega\beta} : \beta \leq \alpha\}$  is uncovered by  $I_\alpha \cup J_\alpha$ .

**Case 2:**  $\alpha = \beta + 1$ .

To start with we fix an enumeration  $\{\langle \langle f_i^k, x_i^k \rangle : i < n_k \rangle, h_k \rangle : k \in \omega\}$  of pairs  $\langle \langle f_i, x_i \rangle : i < n \rangle, h \rangle$  satisfying 3.4.1–5.

If  $k \in \omega$  take

$$B_k^0 = h_k^{-1} \{0\} \cup \{f_i^k(\nu) : i < n_k, \nu \in \text{dom}(f_i^k) \text{ and } \{\nu, x_i^k\} \notin E_\beta\}$$

and

$$B_k^1 = h_k^{-1} \{1\} \cup \{f_i^k(\nu) : i < n_k, \nu \in \text{dom}(f_i^k) \text{ and } \{\nu, x_i^k\} \in E_\beta\}.$$

Applying lemma 3.2  $\omega$ -many times we can find partitions  $(C_k^0, C_k^1)$ ,  $k < \omega$ , of  $\omega\beta$  such that taking

$$I_\beta^+ = \text{Cl}((I_\beta \cup \{C_k^1 : k \in \omega\}), \bigcup_{\gamma \leq \beta} \mathcal{F}_\gamma)$$

and

$$J_\beta^+ = \text{Cl}((I_\beta \cup \{C_k^0 : k \in \omega\}), \bigcup_{\gamma \leq \beta} \mathcal{F}_\gamma)$$

the set  $\{S_{\omega\gamma} : \gamma \leq \beta\}$  is uncovered by  $I_\beta^+ \cup J_\beta^+$ .

We can assume that  $B_k^i \subset C_k^i$  for  $i < 2$  and  $k < \omega$  because  $B_k^0 \in J_\beta$  and  $B_k^1 \in I_\beta$ . Take

$$E_\alpha = E_\beta \cup \{\{\nu, \omega\beta + n\} : \nu < \omega\beta, n \in \omega \text{ and } \nu \in B_n^1\}$$

and

$$\mathcal{F}_\alpha = \emptyset.$$

By the construction of  $G_\alpha = \langle \omega\alpha, E_\alpha \rangle$ , it follows that  $\omega\alpha$  is large, so (II) holds. On the other hand

$$I_\alpha = \{X \cup Y : X \in I_\beta^+, Y \in [\omega\alpha]^{<\omega}\}$$

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and

$$J_\alpha = \{X \cup Y : X \in J_\beta^+, Y \in [\omega\alpha]^{<\omega}\}.$$

so  $\{S_{\omega\gamma} : \gamma < \alpha\}$  is uncovered by  $I_\alpha \cup J_\alpha$ . Finally  $S_{\omega\alpha}$  is cofinal in  $\omega\alpha$  but the elements of  $I_\alpha \cup J_\alpha$  are all bounded, so the induction hypothesis (I) also holds.

The construction is done. Take  $E = \cup \{E_\alpha : \alpha < \omega_1\}$  and  $G = \langle \omega_1, E \rangle$ . By (I),  $G$  is non-trivial. Finally, we must prove that  $G$  is quasy smooth. Consider a set  $Y \subset \omega_1$ . The following lemma is almost trivial.

**Lemma 3.6** *For each  $\alpha < \omega_1$  either  $Y \cap (\omega\alpha + \omega)$  or  $(\omega\alpha + \omega) \setminus Y$  is large.*

**Proof:** Assume on the contrary that there are pairs  $\langle \langle \langle f_i, x_i \rangle : i < n \rangle, h \rangle$  and  $\langle \langle \langle f_i, x_i \rangle : n \leq i < n+k \rangle, h' \rangle$  showing that neither  $Y \cap (\omega\alpha + \omega)$  nor  $(\omega\alpha + \omega) \setminus Y$  is large. Then  $\langle \langle \langle f_i, x_i \rangle : i < n+k \rangle, h \cup h' \rangle$  shows that  $\omega\alpha + \omega$  is not large. ■

So we can assume that the set

$$L = \{\alpha < \omega_1 : Y \cap (\omega\alpha + \omega) \text{ is large}\}$$

is uncountable and to complete the proof of theorem 3.1 it is enough to show that in this case  $G \cong G[Y]$ . By  $\diamond^+$ , we can find a club subset  $C \subset L$  such that  $Y \cap \omega\gamma \in M_\gamma$ ,  $C \cap \omega\gamma \in M_\gamma$  and  $\omega\gamma = \gamma$  whenever  $\gamma \in C$ . We can assume that  $0 \in C$ .

Write  $C = \{\gamma_\nu : \nu < \omega_1\}$ . By induction on  $\nu < \omega_1$ , we will construct functions  $f_\nu$  such that

- (a)  $f_\nu : G_{\gamma_\nu} \cong G_{\gamma_\nu}[Y]$ ,  $f_\nu \in \mathcal{F}_{\gamma_\nu}$ ,
- (b)  $\langle f_\mu : \mu < \nu \rangle \in M_{\sup\{\gamma_\mu+1 : \mu < \nu\}}$ .

Take  $f_0 = \emptyset$ . If  $\nu = \mu + 1$ , then let  $f_\nu = g^{Y \cap \omega\gamma_\nu, f_\mu}$ . If  $\nu$  is limit, then put  $f_\nu = \cup \{f_\mu : \mu < \nu\}$ . Clearly (a) and (b) remains valid. Finally put  $f = \cup \{f_\nu : \nu < \omega_1\}$ . Then  $f : G \cong G[Y]$ , so the theorem is proved. ■

## 4 A model without quasi-smooth graphs

Given an Aronszajn-tree  $T = \langle \omega_1, < \rangle$  define the poset  $\mathcal{Q}_T$  as follows: the underlying set of  $\mathcal{Q}_T$  consists of all functions  $f$  mapping a finite subset of

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$\omega_1$  to  $\omega$  such that  $f^{-1}\{n\}$  is antichain in  $T$  for each  $n \in \omega$ . The ordering on  $\mathcal{Q}_T$  is as expected:  $f \leq_{\mathcal{Q}_T} g$  iff  $f \supset g$ . For  $\gamma < \omega_1$  denote by  $T_\gamma$  the set of elements of  $T$  with height  $\gamma$ . Take  $T_{<\delta} = \bigcup_{\gamma < \delta} T_\gamma$ . If  $x \in T_\delta$  and  $\gamma < \delta$ , let  $x \upharpoonright \gamma$  be the unique element of  $T_\gamma$  which is comparable with  $x$ . We write  $\mathcal{C}$  for the poset  $\langle \text{Fin}(\omega_1, 2), \supset \rangle$ , that is, forcing with  $\mathcal{C}$  adds  $\omega_1$ -many Cohen reals to the ground model.

**Theorem 4.1** *If ZF is consistent, then so is ZFC + “there are no non-trivial quasi-smooth graphs on  $\omega_1$ ”.*

**Proof:** Assume that GCH holds in the ground model. Consider a finite support iteration  $\langle P_i, Q_j : i \leq \omega_2, j < \omega_2 \rangle$  satisfying (a)–(c) below:

- (a) If  $j < \omega_2$  is even, then  $Q_j = \mathcal{C}$ .
- (b) If  $j < \omega_2$  is odd, then  $V^{P_j} \models “Q_j = \mathcal{Q}_{T_j} \text{ for some Aronszajn-tree } T_j”$ .
- (c)  $V^{P_{\omega_2}} \models “\text{every Aronszajn tree is special}”$ .

We will show that  $V^{P_{\omega_2}}$  does not contain non-trivial, quasi-smooth graphs on  $\omega_1$ .

To start with we introduce some notation. Consider a graph  $G = \langle V, E \rangle$ . For  $x \in V$  define the function  $\text{tp}_G(x) : V \setminus \{x\} \rightarrow 2$  by the equation  $G(x) = \text{tp}_G(x)^{-1}\{1\}$ . Given  $A \subset V$  write  $\text{tp}_G(x, A) = \text{tp}_G(x) \upharpoonright A$ .

If  $A \subset V$  and  $t \in 2^A$ , take  $\text{rl}_G(t) = \{x \in V \setminus A : \text{tp}_G(x, A) = t\}$  and  $\text{rl}_G^*(t) = \{x \in V \setminus A : |\text{tp}_G(x, A) \Delta t| < \omega\}$ . For  $x \in V$  and  $A \subset V$  put  $\text{twin}_G(x, A) = \text{rl}_G(\text{tp}_G(x, A))$ .

For  $A \subset V$  define the equivalence relation  $\equiv_{G,A}$  on  $V \setminus A$  as follows:

$$x \equiv_{G,A} y \text{ iff } |\text{tp}_G(x, A) \Delta \text{tp}_G(y, A)| < \omega.$$

For  $x \in V \setminus A$  denote by  $[x]_{G,A}$  the equivalence class of  $x$  in  $\equiv_{G,A}$ . Clearly  $[x]_{G,A} = \text{rl}_G^*(\text{tp}_G(x, A))$ . Write  $G / \equiv_{G,A}$  for the family of equivalence classes of  $\equiv_{G,A}$ .

We divide  $\mathcal{K}$  into three subclasses,  $\mathcal{K}_0$ ,  $\mathcal{K}_1$  and  $\mathcal{K}_2$ , and investigate them separately to show that  $V^{P_{\omega_2}} \models “(\forall G \in \mathcal{K}_i) G \text{ is not quasi-smooth}”$  for  $i < 3$ . Take

$$\begin{aligned} \mathcal{K}_0 &= \{G \in \mathcal{K} : \exists A \in [\omega_1]^\omega \ |G / \equiv_{G,A}| = \omega_1\}, \\ \mathcal{K}_1 &= \{G \in \mathcal{K} : \forall A \in [\omega_1]^\omega \ \exists x \ |\omega_1 \setminus [x]_{G,A}| < \omega_1\} \end{aligned}$$

and

$$\mathcal{K}_2 = \mathcal{K} \setminus (\mathcal{K}_0 \cup \mathcal{K}_1).$$

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#### 4.1 $G \in \mathcal{K}_0$

First we recall a definition of [1].

**Definition 4.2** A poset  $P$  is *stable* if

$$\forall B \in [P]^\omega \exists B^* \in [P]^\omega \forall p \in P \exists p' \leq p \exists p^* \in B^* \forall b \in B (p' \parallel_P b \text{ iff } p^* \parallel_P b).$$

We will say that  $p'$  and  $p^*$  are *twins for  $B$*  and that  $B^*$  *shows the stability of  $P$  for  $B$* .

**Lemma 4.3**  $P_{\omega_2}$  is stable.

**Proof:** First let us remark that it is enough to prove that both  $\mathcal{C}$  and  $\mathcal{Q}_T$  are stable for any Aronszajn-tree for in [1] it was proved that any finite support iteration of stable, c.c.c. posets is stable.

It is clear that  $\mathcal{C}$  is stable. Assume that  $T$  is an Aronszajn tree and  $B \subset [\mathcal{Q}_T]^\omega$ . Fix a countable ordinal  $\delta$  with  $\{\text{dom}(p) : p \in B\} \subset T_{<\delta}$  and take  $B^* = \{p \in \mathcal{Q}_T : \text{dom}(p) \subset T_{<\delta+\omega}\}$ . It is not hard to see that  $B^*$  shows the stability of  $P$  for  $B$ . ■

For  $G \in \mathcal{K}$  take  $G \in \mathcal{K}'_0$  iff there is an  $A \in [\omega_1]^\omega$  such that the set  $\{x : |[x]_{G,A}| \leq \omega\}$  is uncountable.

Given  $G \in \mathcal{K}_0$  we will write  $G \in \mathcal{K}'_0$  iff there are disjoint sets  $A_0, A_1 \in [\omega_1]^\omega$  such that

- (1)  $x \equiv_{G,A_0} y$  iff  $x \equiv_{G,A_1} y$  for each  $x, y \in \omega_1 \setminus A_0 \cup A_1$ ,
- (2) the set  $\{x : |[x]_{G,A_0}| \leq \omega\}$  is uncountable.

**Lemma 4.4** Assume CH. If  $G \in \mathcal{K}'_0$ , then there is a partition  $(V_0, V_1)$  of  $\omega_1$  so that for each stable c.c.c. poset  $P$  we have

$$V^P \models "G \text{ is not isomorphic to } G[V_i] \text{ for } i \in 2".$$

**Proof:** Pick  $A_0, A_1 \in [\omega_1]^\omega$  witnessing  $G \in \mathcal{K}'_0$ . Write  $A = A_0 \cup A_1$ . Take  $E = E(G)$ .

Let  $\kappa$  be a large enough regular cardinal and fix an increasing sequence  $\langle N_\nu : \nu < \omega_1 \rangle$  of countable, elementary submodels of  $\mathcal{H}_\kappa$  such that

- (i)  $G, A, A_0, A_1 \in N_0$ ,

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(ii)  $\langle N_\nu : \nu < \mu \rangle \in N_\mu$  for  $\mu < \omega_1$ .

For  $x \in \omega_1 \setminus A$  take

$$\text{rank}(x) = \min\{\nu : x \in N_\nu\}.$$

Fix a partition  $(S_0, S_1)$  of  $\omega_1$  with  $|S_0| = |S_1| = \omega_1$ . Take

$$V_i = A_i \cup \text{rank}^{-1}S_i$$

for  $i \in 2$ .

We show that the partition  $(V_0, V_1)$  works.

Assume on the contrary that  $P$  is a stable c.c.c. poset,  $\dot{f}$  is a  $P$ -name of a function,  $p_0 \in P$  and

$$p_0 \Vdash \dot{f} : G \cong G[V_0].$$

Without loss of generality we can assume that  $p_0 = 1_P$ . Now for each  $c \in A_0$  choose a maximal antichain  $J_c \subset P$  and a function  $h_c : J_c \rightarrow V$  such that  $q \Vdash \dot{f}^{-1}(\hat{c}) = \widehat{h_c(r)}$  for each  $q \in J_c$ .

Take  $B = \bigcup\{J_c : c \in A_0\}$  and pick a countable  $B^* \subset P$  showing the stability of  $P$  for  $B$ .

For  $b \in P$  define the partial function  $\text{dt}_b : \omega_1 \rightarrow 2^{A_0}$  as follows. Let  $x \in \omega_1$ . If there is a function  $t \in 2^{A_0}$  so that

- (a)  $t(c) = 1 \iff$  for each  $q \in I_c$  if  $q$  and  $b$  are compatible conditions, then  $\{x, h_c(q)\} \in E$ ,
- (b)  $t(c) = 0 \iff$  for each  $q \in I_c$  if  $q$  and  $b$  are compatible conditions, then  $\{x, h_c(q)\} \notin E$ ,

then take  $\text{dt}_b(x) = t$ . Otherwise  $x \notin \text{dom dt}_b$ .

**Sublemma 4.4.1** *If  $p \Vdash \dot{f}(x) = y$ , then there are  $p' \leq p$  and  $b \in B^*$  such that  $b$  and  $p'$  are twins for  $B$  and  $\text{dt}_b(x) = \text{tp}_G(y, A_0)$ .*

**Proof:** By the choice of  $B^*$ , we can find a  $p' \leq p$  and a  $b \in B^*$  so that  $p'$  and  $b$  are twins for  $B$ . Let  $c \in A_0$ . For each  $q \in J_c$ , if  $q$  and  $p'$  are compatible in  $P$ , then  $\{y, c\} \in E$  iff  $\{x, h_c(q)\} \in E$ , because, taking  $r$  as a common extension of  $q$  and  $p'$ , we have  $r \Vdash \dot{f}(\hat{x}) = \hat{y}$  and  $\dot{f}(\widehat{h_q(c)}) = \hat{c}$ . So  $\{y, c\} \in E$  iff for each  $q \in I_c$  if  $q$  and  $p'$  are compatible, then  $\{x, h_c(q)\} \in E$ . But  $p'$  and  $b$  are twins for  $\bigcup\{J_c : c \in A_0\}$ , so  $\text{dt}_b(x) = \text{dt}_{p'}(x) = \text{tp}_G(y, A_0)$ . ■



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**Sublemma 4.4.2** *There is a  $b \in B^*$  such that*

$$(*) \quad |\{t \in \text{ran dt}_b : |\text{rl}_G^*(t)| \leq \omega\}| = \omega_1.$$

**Proof:** Let  $\mathcal{G}$  be a  $P$ -generic filter over  $V$ . Put

$$\mathcal{F} = \{\text{tp}_G(y, A_0) : y \in V_0 \setminus A_0, |[y]_{G, A_0}| \leq \omega\}.$$

Then  $|\mathcal{F}| = \omega_1$ , so we can write  $\mathcal{F} = \{t_\nu : \nu < \omega_1\}$ . Fix sequences  $\langle p_\nu : \nu < \omega_1 \rangle \subset \mathcal{G}$ ,  $\langle x_\nu : \nu < \omega_1 \rangle \subset \omega_1$  and  $\langle y_\nu : \nu < \omega_1 \rangle \subset \omega_1$  such that  $p_\nu \Vdash \dot{f}(x_\nu) = y_\nu$  and  $\text{tp}_G(y_\nu, A_0) = t_\nu$ . By sublemma 4.4.1,

$$\bigcup_{b \in B^*} \text{ran dt}_b \supseteq \mathcal{F}.$$

But  $B^*$  is countable, so we can find a  $b \in B^*$  satisfying  $(*)$  above. ■

Fix  $b \in B^*$  with property  $(*)$ . Consider the structure

$$\mathcal{N} = \langle P[(B \cup B^*), B, B^*, \langle J_c, h_c : c \in A_0 \rangle] \rangle.$$

By CH, there is a  $\nu < \omega_1$  with  $\mathcal{N} \in N_\nu$ . Pick  $\mu \in S_1 \setminus \nu$ . Since  $G, \mathcal{N}, b \in N_\nu$ , it follows that  $\text{dt}_b \in N_\nu \subset N_\mu$ . By  $(*)$  and (ii), there is a

$$t \in \text{ran dt}_b \cap (N_\mu \setminus \bigcup_{\xi < \mu} N_\xi)$$

with  $|\text{rl}_G^*(t)| \leq \omega$ . Then

$$(\dagger) \quad \text{rl}_G^*(t) \subset N_\mu \setminus \bigcup_{\xi < \mu} N_\xi.$$

Pick  $x \in \omega_1$  with  $\text{dt}_b(x) = t$ . Find  $p \in \mathcal{G}$  and  $y \in V_0$  such that  $p \leq b$  and  $p \Vdash \dot{f}(x) = y$ . By sublemma 4.4.1, there are  $p' \leq p$  and  $b' \in B^*$  such that  $p'$  and  $b'$  are twins for  $B$  and  $\text{dt}_{b'}(x) = \text{tp}_G(y, A_0)$ . But  $p \leq b$ , so  $\text{dt}_b(x) = \text{dt}_{b'}(x)$ . Indeed, let  $c \in A_0$  and assume that  $\text{dt}_{b'}(x)(c) = 1$ . Pick  $q \in J_q$  which is compatible with  $b'$ . By the definition of  $\text{dt}_{b'}$ , it follows that  $\{h_c(q), x\} \in E$ . Since  $p'$  and  $b'$  are twins for  $B$ , so  $p'$  and  $q$  also have a common extension  $q'$  in  $P$ . But  $p' \leq p \leq b$ , so  $q'$  witnesses that  $b$  and  $q$  are compatible. Thus, by the definition of  $\text{dt}_b$ , we have  $\text{dt}_b(x)(c) = 1$ .

Thus  $\text{tp}_G(y, A_0) = \text{dt}_{b'}(x) = \text{dt}_b(x) = t$ . By  $(\dagger)$ , this implies that  $\text{rank}(y) = \mu$ . But, by the construction of the partition  $(V_0, V_1)$ , there are no  $y \in V_0$  with  $\text{rank}(y) = \mu$ . Contradiction, the lemma is proved. ■

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**Lemma 4.5** *Assume CH. If  $G \in \mathcal{K}_0^*$ , then*

$V^{\mathcal{C}} \models$  “there is a partition  $(V_0, V_1)$  of  $\omega_1$  so that

for each stable c.c.c. poset  $P$  we have:

$$V^{\mathcal{C}*P} \models \text{“}G \text{ is not isomorphic to } G[V_i] \text{ for } i \in 2\text{”}.$$

**Proof:** . Fix a set  $A \in [\omega_1]^\omega$  witnessing  $G \in \mathcal{K}_0^*$  and a bijection  $f : A \rightarrow \omega$  in  $V$ . Let  $r : \omega \rightarrow 2$  be the characteristic function of a Cohen real from  $V^{\mathcal{C}}$ . Take  $A_i = (f \circ r)^{-1}\{i\}$  for  $i < 2$ . Then  $(A_0, A_1)$  is a partition of  $A$ . Using a trivial density argument we can see that  $x \not\equiv_{G,A} y$  implies  $x \not\equiv_{G,A_i} y$  for  $i < 2$  and for  $x, y \in \omega_1 \setminus A$ . Thus  $V^{\mathcal{C}} \models$  “ $A_0$  and  $A_1$  witness  $G \in \mathcal{K}_0^*$ ”. Applying lemma 4.4 in  $V^{\mathcal{C}}$  we get the desired partition of  $\omega_1$ . ■

**Lemma 4.6** *In  $V^{P_{\omega_2}}$ , if  $G \in \mathcal{K}_0$  is quasi-smooth, then  $G \in \mathcal{K}_0^*$ .*

**Proof:** Choose a set  $A \in [\omega_1]^\omega$  witnessing  $G \in \mathcal{K}_0$  and a bijection  $f : A \rightarrow \omega$ . Pick  $\alpha < \omega_2$ ,  $\alpha$  is even, with  $A, f, G \in V^{P_\alpha}$ . From now on we work in  $V^{P_\alpha}$ . Let  $\{[x_\nu]_{G,A} : \nu < \omega_1\}$  be an enumeration of the equivalence classes of  $\equiv_{G,A}$ . Fix a partition  $(I_0, I_1)$  of  $\omega_1$  into uncountable pieces. Let  $r : \omega \rightarrow 2$  be the characteristic function of a Cohen real from  $V^{P_\alpha * \mathcal{C}}$ . Take  $A_i = (f \circ r)^{-1}\{i\}$  for  $i < 2$ . Then  $(A_0, A_1)$  is a partition of  $A$ . Using a trivial density argument we can see that  $x \not\equiv_{G,A} y$  implies  $x \not\equiv_{G,A_i} y$  for  $i < 2$  and for  $x, y \in \omega_1 \setminus A$ . For  $i \in 2$  put

$$B_i = A_i \cup \{x_\nu : \nu \in I_i\} \cup \{[x_\nu]_{G,A} \setminus \{x_\nu\} : \nu \in I_{1-i}\}.$$

Clearly  $(B_0, B_1)$  is a partition of  $\omega_1$  and

$$B_i \cap [x_\nu]_{G,A_i} = B_i \cap [x_\nu]_{G,A} = \{x_\nu\}.$$

So  $G[B_i] \in \mathcal{K}^*$ . But  $G$  is quasi-smooth, so  $G \cong G[B_i]$  for some  $i \in 2$  in  $V^{P_{\omega_2}}$ . Thus  $G \in \mathcal{K}_0^*$  is proved. ■

## 4.2 $G \in \mathcal{K}_1$

We say that a poset  $P$  has property  $Pr$  iff for each sequence  $\langle p_\nu : \nu < \omega_1 \rangle \subset P$  there exist disjoint sets  $U_0, U_1 \in [\omega_1]^{\omega_1}$  such that whenever  $\alpha \in U_0$  and  $\beta \in U_1$  we have  $p_\alpha \parallel_P p_\beta$ .

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**Lemma 4.7**  $\mathcal{C}$  has property Pr.

Indeed,  $\mathcal{C}$  has property K.

**Lemma 4.8** If  $T$  is an Aronszajn-tree, then  $\mathcal{Q}_T$  has property Pr.

**Proof:** Let  $\langle p_\alpha : \alpha < \omega_1 \rangle \subset P$  be given. We can assume that there are a stationary set  $S \subset \omega_1$ ,  $p^* \in \mathcal{Q}_T$ ,  $\gamma^* < \omega_1$ ,  $n \in \omega$  and  $\{z_i : i < n\} \subset T$  such that for each  $\alpha \in S$

- (a)  $x \upharpoonright \alpha \in \text{dom}(p_\alpha)$  for each  $x \in \text{dom}(p_\alpha)$  with  $\text{height}_T(x) \geq \alpha$ ,
- (b)  $p_\alpha \upharpoonright T_{<\alpha} = p^*$ ,
- (c)  $|\text{dom}(p_\alpha) \cap T_\alpha| = n$ ,
- (d) writing  $\text{dom}(p_\alpha) \cap T_\alpha = \{x_0^\alpha, \dots, x_{n-1}^\alpha\}$ ,  $x_0^\alpha <_{\text{on}} \dots <_{\text{on}} x_{n-1}^\alpha$ , the sequence  $\langle p_\alpha(x_0^\alpha), \dots, p_\alpha(x_{n-1}^\alpha) \rangle$  is independent from  $\alpha$ ,
- (e)  $\gamma^* < \alpha$  and the elements  $x_0^\alpha \upharpoonright \gamma^*, \dots, x_{n-1}^\alpha \upharpoonright \gamma^*$  are pairwise distinct,
- (f)  $x_i^\alpha \upharpoonright \gamma^* = z_i$  for  $i < n$ .

For each  $\beta < \omega_1$  and  $\bar{y} = \langle y_0, \dots, y_{n-1} \rangle \in (T_\beta)^n$  take

$$S_{\bar{y}} = \{\alpha \in S \setminus \beta : x_i^\alpha \upharpoonright \beta = y_i \text{ for each } i < n\}.$$

Let

$$C^* = \{\delta < \omega_1 \setminus \gamma^* : \forall \beta < \delta \forall \bar{y} \in (T_\beta)^n (|S_{\bar{y}}| \leq \omega \rightarrow S_{\bar{y}} \subset \delta)\}.$$

Now  $\{p_\alpha : \alpha \in S \cap C^*\}$  are  $\omega_1$  members of  $P$ , so for some  $\alpha < \beta \in S \cap C^*$  the conditions  $p_\alpha$  and  $p_\beta$  are compatible. Since  $p_\alpha(x_l^\alpha) = p_\beta(x_l^\beta)$ ,  $x_l^\alpha$  and  $x_l^\beta$  are incomparable in  $T$  for  $l < n$ . So for some  $\nu < \alpha$ ,  $x_l^\alpha \upharpoonright \nu \neq x_l^\beta \upharpoonright \nu$  whenever  $l < n$ . On the other hand, for  $l \neq m < n$  we have  $x_l^\alpha \upharpoonright \nu \neq x_m^\beta \upharpoonright \nu$  because  $x_l^\alpha \upharpoonright \gamma^* = z_l \neq z_m = x_m^\beta \upharpoonright \gamma^*$ . Take  $y_l^a = x_l^\alpha \upharpoonright \nu$  and  $y_l^b = x_l^\beta \upharpoonright \nu$  for  $l < n$  and write  $\bar{a} = \langle y_0^a, \dots, y_{n-1}^a \rangle$ ,  $\bar{b} = \langle y_0^b, \dots, y_{n-1}^b \rangle$ . The elements  $\{y_i^a, y_i^b : i < n\}$  are pairwise different, so for each  $\alpha' \in S_{\bar{a}}$  and  $\beta' \in S_{\bar{b}}$  the conditions  $p_{\alpha'}$  and  $p_{\beta'}$  are compatible. But  $|S_{\bar{a}}| = |S_{\bar{b}}| = \omega_1$ , because  $\alpha \in S_{\bar{a}}$ ,  $\beta \in S_{\bar{b}}$ ,  $\nu < \alpha$  and  $\alpha \in C^*$ . ■

A poset  $P$  is called *well-met* if any two compatible elements  $p_0$  and  $p_1$  of  $P$  have a greatest lower bound denoted by  $p_0 \wedge p_1$ .

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**Lemma 4.9** *Assume that the poset  $P$  has property  $\text{Pr}$  and  $V^P \models$  “the poset  $Q$  has property  $\text{Pr}$ ”. Let  $\langle \langle p_\alpha, q_\alpha \rangle : \alpha < \omega_1 \rangle \subset P * Q$ . Then there are disjoint sets  $U_0, U_1 \in [\omega_1]^{\omega_1}$  such that for each  $\gamma \in U_0$  and  $\delta \in U_1$  the conditions  $\langle p_\gamma, q_\gamma \rangle$  and  $\langle p_\delta, q_\delta \rangle$  are compatible, in other words,  $p_\gamma$  and  $p_\delta$  have a common extension  $p_{\gamma,\delta}$  in  $P$  with  $p_{\gamma,\delta} \Vdash$  “ $q_\gamma \parallel_Q q_\delta$ ”. If  $P$  is well-met, then we can find conditions  $\{p'_\alpha : \alpha \in U_0 \cup U_1\}$  in  $P$  with  $p'_\alpha \leq p_\alpha$  such that  $p'_\gamma \wedge p'_\delta \Vdash$  “ $q_\gamma \parallel_Q q_\delta$ ” for each  $\gamma \in U_0$  and  $\delta \in U_1$ .*

**Proof:** Let  $\dot{U}$  be a  $P$ -name for the set  $U = \{\alpha : p_\alpha \in \mathcal{G}_P\}$ , where  $\mathcal{G}_P$  is the  $P$ -generic filter. Since  $P$  satisfies c.c.c., there is a  $p^* \in P$  with  $p^* \Vdash$  “ $|\dot{U}| = \omega_1$ ”. Since  $V^P \models$  “ $Q$  has property  $\text{Pr}$ ”, there is a condition  $p \leq p^*$  and there are  $P$ -names such that  $p \Vdash$  “ $\forall_i = \{\dot{\alpha}_\gamma^i : \gamma < \omega_1\} \in [U]^{\omega_1}$ , for  $i \in 2$ , and  $q_{\dot{\alpha}_\gamma^0}$  and  $q_{\dot{\alpha}_\delta^1}$  are compatible whenever  $\gamma, \delta \in \omega_1$ ”. Choose conditions  $p_\gamma^* \leq p$  and ordinals  $\beta_\gamma^0, \beta_\gamma^1$ , with  $p_\gamma^* \Vdash$  “ $\dot{\alpha}_\gamma^i = \dot{\beta}_\gamma^i$ ” for  $i < 2$ .

Now consider the sequence  $A = \{p_\gamma^* : \gamma < \omega_1\}$ . Since  $P$  has property  $\text{Pr}$ , there are disjoint, uncountable sets  $C_0, C_1 \subset A$  such that  $p_\gamma^*$  and  $p_\delta^*$  are compatible whenever  $\gamma \in C_0$  and  $\delta \in C_1$ . Take  $U_i = \{\beta_\gamma^i : \gamma \in C_i\}$  for  $i \in 2$ . We can assume that  $U_0 \cap U_1 = \emptyset$ . Let  $\gamma \in C_0$  and  $\delta \in C_1$  and let  $p''$  be a common extension of  $p_\gamma^*$  and  $p_\delta^*$ . Then  $p'' \Vdash$  “ $\beta_\gamma^0, \beta_\delta^1 \in \dot{U}$ , that is,  $p_{\beta_\gamma^0}$  and  $p_{\beta_\delta^1}$  are in  $\mathcal{G}_P$ ”, so  $p''$ , must be a common extension of  $p_{\beta_\gamma^0}$  and  $p_{\beta_\delta^1}$ . So  $p'' \Vdash$  “ $\beta_\gamma^0 \in V_0$  and  $\beta_\delta^1 \in V_1$ ”, thus  $p'' \Vdash$  “ $q_{\beta_\gamma^0}$  and  $q_{\beta_\delta^1}$  are compatible in  $Q$ ”, so  $\langle p_{\beta_\gamma^0}, q_{\beta_\gamma^0} \rangle \parallel_{P*Q} \langle p_{\beta_\delta^1}, q_{\beta_\delta^1} \rangle$ .

Suppose that  $P$  is well-met. Take  $p'_{\beta_\gamma^0} = p_\gamma^* \wedge p_{\beta_\gamma^0}$  and  $p'_{\beta_\delta^1} = p_\delta^* \wedge p_{\beta_\delta^1}$ . It works because we can use  $p_\gamma^* \wedge p_\delta^*$  as  $p''$  in the argument of the previous paragraph. ■

**Lemma 4.10** *If  $\langle R_\alpha : \alpha \leq \mu, S_\beta : \beta < \mu \rangle$  is a finite support iteration such that  $V^{R_\alpha} \models$  “ $S_\alpha$  has property  $\text{Pr}$ ” for  $\alpha < \mu$ , then  $R_\mu$  has property  $\text{Pr}$ , as well.*

**Proof:** We prove this lemma by induction on  $\mu$ . The successor case is covered by lemma 4.9. Assume that  $\mu$  is limit. Let  $\langle p_\xi : \xi < \omega_1 \rangle \subset R_\mu$ . Without loss of generality we can assume that  $\langle \text{supp}(p_\xi) : \xi < \omega_1 \rangle$  forms a  $\Delta$ -system with kernel  $d$ . Fix  $\nu < \mu$  with  $d \subset \nu$ . By the induction hypothesis, the poset  $R_\nu$  has property  $\text{Pr}$ , so there exist disjoint sets  $U_0, U_1 \in [\omega_1]^{\omega_1}$  such that whenever  $\xi \in U_0$  and  $\eta \in U_1$  we have  $p_\xi \parallel_{R_\nu} p_\eta$ . But  $p_\xi \parallel_{R_\nu} p_\eta$  implies  $p_\xi \parallel_{R_\mu} p_\eta$  because  $\text{supp}(p_\xi) \cap \text{supp}(p_\eta) \subset \nu$ , so  $R_\mu$  has property  $\text{Pr}$ , as well. ■

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The previous lemmas yield the following corollary.

**Lemma 4.11**  $P_{\omega_2}$  has property Pr.

Given  $G = \langle \omega_1, E \rangle \in \mathcal{K}_1$  and  $\xi, \alpha, \beta \in \omega_1$  with  $\xi \in \alpha \cap \beta$  take

$$D_\xi^G(\alpha, \beta) = \{\nu \in \xi : \{\alpha, \nu\} \in E \text{ iff } \{\beta, \nu\} \notin E\}.$$

**Lemma 4.12** If  $G \in \mathcal{K}_1$ , then

$$(*) \quad \forall \xi \in \omega_1 \exists \epsilon_G(\xi) \in \omega_1 \forall \alpha, \beta \in \omega_1 \setminus \epsilon_G(\xi) |D_\xi^G(\alpha, \beta)| < \omega.$$

**Proof:** Since  $G \in \mathcal{K}_1$ , we have an  $x \in \omega_1$  with  $|\omega_1 \setminus [x]_\xi| < \omega_1$ . Choose  $\epsilon_G(\xi) \in \omega_1 \setminus \xi$  with  $\omega_1 \setminus [x]_\xi \subset \epsilon_G(\xi)$ . It works because  $\alpha, \beta > \epsilon_G(\xi)$  implies  $\alpha, \beta \in [x]_\xi$ . ■

The bipartite graph  $\langle \omega_1 \times 2, \{\{\langle \nu, 0 \rangle, \langle \mu, 1 \rangle\} : \nu < \mu < \omega_1\} \rangle$  will be denoted by  $[\omega_1; \omega_1]$ .

**Lemma 4.13** If  $G \in \mathcal{K}_1$ , then neither  $G$  nor its complement may have a — not necessarily spanned — subgraph isomorphic to  $[\omega_1; \omega_1]$ .

**Proof:** Let  $G = \langle \omega_1, E \rangle$ . Write  $E(\alpha) = \{\xi \in \omega_1 : \{\xi, \alpha\} \in E\}$ . Assume on the contrary that  $A, B \in [\omega_1]^{\omega_1}$  are disjoint sets such that  $\{\alpha, \beta\} \in E$  whenever  $\alpha \in A$  and  $\beta \in B$  with  $\alpha < \beta$ . Without loss of generality we can assume that  $(A \setminus \alpha + 1) \cap E(\alpha) = \emptyset$  for each  $\alpha \in A$ . Write  $A = \{\alpha_\xi : \xi < \omega_1\}$ . Then for  $\xi \in \omega_1$  the set  $F(\xi) = (A \cap \alpha_\xi) \setminus E(\alpha_{\xi+1})$  is finite because  $\alpha_{\xi+1} > \epsilon(\alpha_\xi)$  and  $(A \cap \alpha_\xi) \setminus E(\beta) = \emptyset$  for all but countable many  $\beta \in B$ . By Fodor's lemma, we can assume that  $F(\xi) = F$  for each  $\xi \in S$ , where  $S$  is a stationary subset of  $\omega_1$  containing limit ordinals only. Let  $T = \{\xi \in S : F \subset \alpha_\xi\}$  and take  $W = \{\alpha_{\xi+1} : \xi \in T\}$ . Then  $G[W]$  is an uncountable complete subgraph of  $G$ . Contradiction. ■

**Lemma 4.14** If  $G \in \mathcal{K}_1$  and  $V^C \models$  “Q has property Pr”, then

$$V^{C*Q} \models “G \not\cong G[f^{-1}\{i\}] \text{ for } i < 2”,$$

where  $f : \omega_1 \rightarrow 2$  is the  $\mathcal{C}$ -generic function over  $V$ .

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**Proof:** Assume on the contrary that

$$\langle p, q \rangle \Vdash \text{“}\dot{h} : G \cong G[f^{-1}\{0\}]\text{”}.$$

To simplify our notations, we will write  $E$  for  $E(G)$ ,  $D_\xi(\alpha, \beta)$  for  $D_\xi^G(\alpha, \beta)$  and  $\epsilon(\xi)$  for  $\epsilon_G(\xi)$ .

Let  $C_0 = \{\delta < \omega_1 : \xi < \delta \text{ implies } \epsilon(\xi) < \delta\}$ . Clearly  $C_0$  is club. Take  $C_1 = \{\delta < \omega_1 : \langle p, q \rangle \Vdash \text{“}\dot{h}''\hat{\delta} = f^{-1}\{0\} \cap \hat{\delta}\text{”}\}$ . Since  $\mathcal{C} * Q$  satisfies c.c.c, the set  $C_1$  is club. Put  $C_2 = C_0 \cap C_1$ .

Now for each  $\alpha < \omega_1$  let  $\delta_\alpha = \min(C_2 \setminus \alpha + 1)$  and choose a condition  $\langle p_\alpha, q_\alpha \rangle \leq \langle p, q \rangle$  and a countable ordinal  $\gamma_\alpha$  such that

$$\langle p_\alpha, q_\alpha \rangle \Vdash \text{“}\dot{h}(\hat{\delta}_\alpha) = \hat{\gamma}_\alpha\text{”}.$$

Since  $\gamma_\alpha \geq \delta_\alpha > \epsilon(\alpha)$  for each  $\alpha \in \omega_1$ , we can fix a stationary set  $S \subset \omega_1$  and a finite set  $D$  such that  $D_\alpha(\delta_\alpha, \gamma_\alpha) = D$  for each  $\alpha \in S$ . Since  $\mathcal{C}$  is well-met, applying lemma 4.9 we can find disjoint uncountable subsets  $S_0, S_1 \subset S$  and a sequence  $\langle p'_\alpha : \alpha \in S_0 \cup S_1 \rangle \subset \mathcal{C}$  with  $p'_\alpha \leq p_\alpha$  such that  $p'_\alpha \wedge p'_\beta \Vdash \text{“}q_\alpha \parallel_Q q_\beta\text{”}$  for each  $\alpha \in S_0$  and  $\beta \in S_1$ .

We can assume that the sets  $\{\text{dom}(p'_\alpha) : \alpha \in S_0\}$  and  $\{\text{dom}(p'_\beta) : \beta \in S_1\}$  form  $\Delta$ -systems with kernels  $d_0$  and  $d_1$ , respectively.

Take  $Y_\xi^0 = \{\alpha \in S_0 : \{\xi, \delta_\alpha\} \in E\}$  and  $Y_\xi^1 = \{\alpha \in S_1 : \{\xi, \delta_\alpha\} \notin E\}$  for  $\xi < \omega_1$ . Write  $Y_i = \{\xi < \omega_1 : |Y_\xi^i| = \omega_1\}$  and  $Z_i = \omega_1 \setminus Y_i$  for  $i < 2$ .

By 4.13, the sets  $Z_i$  are countable. Pick  $\xi \in C_2$  with  $D \cup d_0 \cup d_1 \cup Z_0 \cup Z_1 \subset \xi$ . Let  $\xi' = \min(C_2 \setminus \xi + 1)$  and  $\xi'' = \min(C_2 \setminus \xi' + 1)$ . Since  $d_0 \cup d_1 \subset \xi$  and  $|Y_\xi^0| = |Y_\xi^1| = \omega_1$ , we can choose  $\alpha_i \in Y_\xi^i \setminus \xi''$  with  $\text{dom}(p'_{\alpha_i}) \cap [\xi, \xi'] = \emptyset$  for  $i = 0, 1$ . The set  $W = D_{\xi'}(\delta_{\alpha_0}, \delta_{\alpha_1}) \cap [\xi, \xi']$  is finite because  $\delta_{\alpha_i} \geq \alpha_i \geq \xi'' > \epsilon(\xi')$  for  $i < 2$ . Choose a  $\mathcal{C}$ -name  $q$  such that  $p'_{\alpha_0} \wedge p'_{\alpha_1} \Vdash \text{“}q \text{ is a common extension of } q_{\alpha_0} \text{ and } q_{\alpha_1} \text{ in } Q\text{”}$  and take

$$r = \langle p'_{\alpha_0} \cup p'_{\alpha_1} \cup \{(\nu, 1) : \nu \in W\}, q \rangle.$$

Since  $W \cap (\text{dom}(p'_{\alpha_0}) \cup \text{dom}(p'_{\alpha_1})) = \emptyset$ ,  $r$  is a condition.

Pick a condition  $r' \leq r$  from  $\mathcal{C} * Q$  and an ordinal  $\eta$  such that  $r' \Vdash \text{“}\dot{h}(\hat{\xi}) = \hat{\eta}\text{”}$ . Now  $\eta \in [\xi, \xi')$  because  $\xi, \xi' \in C_1$ . Since

$$r' \Vdash \text{“}\dot{h}(\delta_{\alpha_i}) = \hat{\gamma}_{\alpha_i}, \dot{h}(\hat{\xi}) = \hat{\eta} \text{ and } \dot{h} \text{ is an isomorphism”},$$

so  $\{\delta_{\alpha_0}, \xi\} \in E$  and  $\{\delta_{\alpha_1}, \xi\} \notin E$  imply that  $\{\gamma_{\alpha_0}, \eta\} \in E$  and  $\{\gamma_{\alpha_1}, \eta\} \notin E$ . But  $D_{\alpha_i}(\delta_{\alpha_i}, \gamma_{\alpha_i}) = D$  and  $D \subset \xi$  so  $\{\delta_{\alpha_0}, \eta\} \in E$  and  $\{\delta_{\alpha_1}, \eta\} \notin E$ , that is,  $\eta \in W$ . But  $r \Vdash \text{“}\text{ran}(\dot{h}) = f^{-1}\{0\} \text{ and } f^{-1}\{0\} \cap \hat{W} = \emptyset\text{”}$ , contradiction. ■

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### 4.3 $G \in \mathcal{K}_2$

Given a non-trivial graph  $G = \langle V, E \rangle$  with  $V \in [\omega_1]^{\omega_1}$  define

$$\Gamma(G) = \{\delta \in \omega_1 : \exists \alpha \in V \ \alpha \geq \delta \text{ and } |\text{twin}_G(\alpha, V \cap \delta)| \leq \omega\}.$$

The following lemma obviously holds.

**Lemma 4.15** *If  $G_0$  and  $G_1$  are graphs on uncountable subsets of  $\omega_1$ ,  $G_0 \cong G_1$ , then  $\Gamma(G_0) = \Gamma(G_1) \text{ mod } \text{NS}_{\omega_1}$ .*

**Lemma 4.16** *Given  $G \in \mathcal{K} \setminus \mathcal{K}_0$  and  $S \subset \omega_1$  there is a partition  $(V_0, V_1)$  of  $\omega_1$  such that  $\Gamma(G[V_0]) \subset S \text{ mod } \text{NS}_{\omega_1}$  and  $\Gamma(G[V_1]) \subset \omega_1 \setminus S \text{ mod } \text{NS}_{\omega_1}$ .*

**Proof:** Let  $\kappa$  be a large enough regular cardinal and fix an increasing, continuous sequence  $\langle N_\nu : \nu < \omega_1 \rangle$  of countable, elementary submodels of  $\mathcal{H}_\kappa = \langle H_\kappa, \in \rangle$  such that  $G, S \in N_0$  and  $\langle N_\nu : \nu \leq \mu \rangle \in N_{\mu+1}$  for  $\mu < \omega_1$ . Write  $\gamma_\nu = N_\nu \cap \omega_1$  and  $C = \{\gamma_\nu : \nu < \omega_1\}$ . Take  $V_0 = \bigcup_{\nu \in S} (\gamma_{\nu+1} \setminus \gamma_\nu)$  and  $V_1 = \omega_1 \setminus V_0 = \bigcup_{\nu \in \omega_1 \setminus S} (\gamma_{\nu+1} \setminus \gamma_\nu)$ .

It is enough to prove that  $\Gamma(G[V_0]) \subset S \text{ mod } \text{NS}_{\omega_1}$ . Assume that  $\gamma_\nu \in \Gamma(G[V_0])$ ,  $\gamma_\nu = \nu$ ,  $\alpha \geq \gamma_\nu$ ,  $\alpha \in V_0$  and  $|\text{twin}_{G[V_0]}(\alpha, \gamma_\nu \cap V_0)| = \omega$ . Since  $G, \nu, \gamma_\nu \cap V_0 \in N_{\nu+1}$  and  $|G / \equiv_{G, V_0 \cap \gamma_\nu}| \leq \omega$ , we have  $\text{tp}_{G[V_0]}(\alpha, \gamma_\nu \cap V_0) \in N_{\nu+1}$  and so  $\text{twin}_{G[V_0]}(\alpha, \gamma_\nu) \subset N_{\nu+1}$  as well. Thus  $\alpha \in \gamma_{\nu+1} \setminus \gamma_\nu$ . Hence  $\alpha \in V_0$  implies  $\gamma_n = \nu \in S$  which was to be proved. ■

**Lemma 4.17** *If  $G \in \mathcal{K} \setminus \mathcal{K}_0$  and  $\Gamma(G) \neq \emptyset \text{ mod } \text{NS}_{\omega_1}$ , then  $G$  is not quasi-smooth.*

**Proof:** Assume that  $S = \Gamma(G)$  is stationary and let  $(S_0, S_1)$  be a partition of  $S$  into stationary subsets. By lemma 4.16, there is a partition  $(V_0, V_1)$  of  $\omega_1$  with  $\Gamma(G[V_i]) \cap S \subset S_i$ . Then  $G[V_i]$  and  $G$  can not be isomorphic by lemma 4.15. ■

Let us remark that  $G \in \mathcal{K}_2$  iff  $G \in \mathcal{K} \setminus \mathcal{K}_0$  and there is an  $A \in [\omega_1]^\omega$  and  $x \in \omega_1 \setminus A$  such that  $|[x]_{G,A}| = |\omega_1 \setminus [x]_{G,A}| = \omega_1$ .

Given  $G \in \mathcal{K}_2$  we will write  $G \in \mathcal{K}'_2$  iff there are two disjoint, countable subsets of  $\omega_1$ ,  $A_0$  and  $A_1$ , and there is an  $x \in \omega_1$ , such that  $|[x]_{G,A_0}| = |\omega_1 \setminus [x]_{G,A_0}| = \omega_1$  and  $[x]_{G,A_0} \setminus A_1 = [x]_{G,A_1} \setminus A_0$ .

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**Lemma 4.18** *If  $G \in \mathcal{K}_2$ , then  $G \in (\mathcal{K}'_2)^{V^C}$ .*

**Proof:** Assume that  $A \in [\omega_1]^\omega$  and  $x \in \omega_1$  witness  $G \in \mathcal{K}_2$  in the ground model. Fix a bijection  $f : A \rightarrow \omega$  in  $V$ . Let  $r : \omega \rightarrow 2$  be the characteristic function of a Cohen real from  $V^C$ . Take  $A_i = (f \circ r)^{-1}\{i\}$ . By a simple density argument, we can see that  $[x]_{G,A_0} = [x]_{G,A} = [x]_{G,A_1}$ . Thus  $A_0, A_1$  and  $x$  show that  $g \in \mathcal{K}'_2$ . ■

**Lemma 4.19** *Assume that every Aronszajn tree is special. If  $G \in \mathcal{K}'_2$ , then there is a partition  $(V_0, V_1)$  of  $\omega_1$  such that  $\Gamma(G[V_i])$  is stationary for  $i < 2$ .*

**Proof:** Choose  $A_0, A_1$  and  $x$  witnessing  $G \in \mathcal{K}'_2$ . Let  $A = A_0 \cup A_1$ . Take  $C_0 = [x]_{G,A_0} \setminus A$ ,  $C_1 = (\omega_1 \setminus [x]_{G,A_0}) \setminus A$  and consider the partition trees  $\mathcal{T}_i$  of  $G[C_i]$  for  $i \in 2$  (see definition 2.3). These trees are Aronszajn-trees because  $G$  is non-trivial. Fix functions  $h_i : C_i \rightarrow \omega$  specializing  $\mathcal{T}_i$ . We can find natural numbers  $n_0$  and  $n_1$  such that the sets  $S_i = \{\nu : h_i^{-1}\{n_i\} \cap (\mathcal{T}_i)_\nu \neq \emptyset\}$  are stationary, that is,  $h_i^{-1}\{n_i\}$  meets stationary many level of  $\mathcal{T}_i$ . Take  $B_i = h_i^{-1}\{n_i\}$  and  $Y_i = \{c \in C_i : \exists b \in B_i \ c \preceq_{\mathcal{T}_i} b\}$ .

Pick any  $\delta \in S_i$ . Let  $b \in B_i \cap (\mathcal{T}_i)_\delta$ . If  $c \in Y_i \setminus (\mathcal{T}_i)_{<\delta}$ ,  $c \neq b$ , then  $c \upharpoonright \delta \neq b$  by the construction of  $Y_i$ . So  $\text{tp}_{GG[Y_i]}(c, (\mathcal{T}_i)_{<\delta}) = \text{tp}_{GG[Y_i]}(c \upharpoonright \delta, (\mathcal{T}_i)_{<\delta}) \neq \text{tp}_{GG[Y_i]}(b, (\mathcal{T}_i)_{<\delta})$  by the definition of the partition tree. This means that  $\text{twin}_{G[Y_i]}(b, (\mathcal{T}_i)_{<\delta}) = \{b\}$ . Thus  $\delta \in \Gamma(G[Y_i])$  provided  $(\mathcal{T}_i)_{<\delta} \subset \delta$  and  $b \geq \delta$ . But these requirements exclude only a non-stationary subset of  $S_i$ . So  $\Gamma(G[Y_i]) \supset S_i \text{ mod } \text{NS}_{\omega_1}$ .

Let  $V_i = Y_i \cup A_i \cup (C_{1-i} \setminus Y_{1-i})$  for  $i \in 2$  and consider the partition  $(V_0, V_1)$  of  $\omega_1$ . If  $z \in V_i \setminus (Y_i \cup A_i)$ , then  $\text{tp}_G(z, A_i) \neq \text{tp}_G(b, A_i)$  for any  $b \in B_i$  because  $C_0 \subset [x]_{G,A_i}$  and  $C_1 \subset \omega_1 \setminus [x]_{G,A_i}$ . So  $\Gamma(G[V_i]) \supset S_i \text{ mod } \text{NS}_{\omega_1}$  holds. ■

Now we are ready to conclude the proof of theorem 4.1. We will work in  $V^{P_{\omega_2}}$ . Assume that  $G \in \mathcal{K}$ . We must show that  $G$  is not quasi-smooth.

Pick a  $\nu < \omega_2$  with  $G \in (\mathcal{K})^{V^{P_\nu}}$  and  $Q_\nu = \mathcal{C}$ . Assume first that  $G \in (\mathcal{K}_0)^{V^{P_\nu}}$ . If  $G$  were quasi-smooth in  $V^{P_{\omega_2}}$ ,  $G \in (\mathcal{K}_0^*)^{P_{\omega_2}}$  would hold by lemma 4.6. So we can assume that  $G \in (\mathcal{K}_0^*)^{P_\nu}$ . Since  $P_{\omega_2}$  is a stable, c.c.c. poset, so is  $P_{\omega_2}/P_{\nu+1}$ . So, by lemma 4.5, there is a partition  $(V_0, V_1)$  of  $\omega_1$  in  $V^{P_{\nu+1}}$  such that  $V^{P_{\omega_2}} \models "G \text{ is not isomorphic to } G[V_i] \text{ for } i < 2"$ .

Assume that  $G \in (\mathcal{K}_1)^{V^{P_\nu}}$ . Since  $P_{\omega_2}$  has property  $Pr$ , so is  $P_{\omega_2}/P_{\nu+1}$ . Thus, by lemma 4.14, the partition  $(V_0, V_1)$  of  $\omega_1$  given by the  $Q_\nu$ -generic



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Cohen reals in  $V^{P_{\nu+1}}$  has the property that  $V^{P_{\omega_2}} \models$  “ $G$  is not isomorphic to  $G[V_i]$  for  $i < 2$ ”.

Finally assume that  $G \in (\mathcal{K}_2)^{V^{P_\nu}}$ . By lemma 4.18, we have  $G \in (\mathcal{K}'_2)^{V^{P_{\nu+1}}}$ . Since  $P_{\omega_2}$  satisfies c.c.c, it follows that  $G \in (\mathcal{K}'_2)^{V^{P_{\omega_2}}}$ . So applying lemma 4.19 we can find a partition  $(V_0, V_1)$  of  $\omega_1$  such that both  $\Gamma(G[V_0])$  and  $\Gamma(G[V_1])$  are stationary. Thus, by lemma 4.17, neither  $G[V_0]$  nor  $G[V_1]$  are quasi-smooth. So  $G$  itself can not be quasi-smooth. ■

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