On the number of automorphisms of uncountable models

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Abstract
Let $\sigma(\mathcal{A})$ denote the number of automorphisms of a model $\mathcal{A}$ of power ω_1. We derive a necessary and sufficient condition in terms of trees for the existence of an $\mathcal{A}$ with $\omega_1 < \sigma(\mathcal{A}) < 2^{\omega_1}$. We study the sufficiency of some conditions for $\sigma(\mathcal{A}) = 2^{\omega_1}$. These conditions are analogous to conditions studied by D.Kueker in connection with countable models.

The starting point of this paper was an attempt to generalize some results of D.Kueker [8] to models of power ω_1. For example, Kueker shows that for countable $\mathcal{A}$ the number $\sigma(\mathcal{A})$ of automorphisms of $\mathcal{A}$ is either $\leq \omega$ or 2^ω. In Corollary 13 we prove the analogue of this result under the set-theoretical assumption $I(\omega)$: if $I(\omega)$ holds and the cardinality of $\mathcal{A}$ is ω_1, then $\sigma(\mathcal{A}) \leq \omega_1$ or $\sigma(\mathcal{A}) = 2^{\omega_1}$. In Theorem 16 we show that the consistency strength of this statement + $2^{\omega_1} > \omega_2$ is that of an inaccessible cardinal. We use $||\mathcal{A}||$ to denote the universe of a model $\mathcal{A}$ and $|\mathcal{A}|$ to denote the cardinality of $||\mathcal{A}||$. Kueker proves also that if $|\mathcal{A}| \leq \omega$, $|\mathcal{B}| > \omega$ and $\mathcal{A} \equiv \mathcal{B}$ (in $L_{\infty\omega}$), then $\sigma(\mathcal{A}) = 2^\omega$. Theorem 1 below generalizes this to power ω_1. If $\mathcal{A}$ and $\mathcal{B}$ are countable, $\mathcal{A} \neq \mathcal{B}$ and $\mathcal{A} \prec \mathcal{B}$ (in $L_{\infty\omega}$), then we know that $\sigma(\mathcal{A}) = 2^\omega$. Theorem 7 shows that the natural analogue of this result fails for models of power ω_1. Theorem 14 links the existence of a model $\mathcal{A}$ such that $|\mathcal{A}| = \omega_1$, $\omega_1 < \sigma(\mathcal{A}) < 2^{\omega_1}$, to the existence of a tree T which is of power ω_1, of height ω_1 and has $\sigma(\mathcal{A})$ uncountable branches.

We use $\mathcal{A} \equiv_{\omega_1} \mathcal{B}$ to denote that $\exists$ has a winning strategy in the Ehrenfeucht-Fraïssé game $G(\mathcal{A},\mathcal{B})$ of length ω_1 between $\mathcal{A}$ and $\mathcal{B}$. During this game two players $\exists$ and $\forall$ extend a countable partial isomorphism π between $\mathcal{A}$ and $\mathcal{B}$. At the start of the game π is empty. Player $\forall$ begins the game by choosing an element a in either $\mathcal{A}$ or $\mathcal{B}$. Then $\exists$ has to pick an element b in either $\mathcal{A}$ or $\mathcal{B}$ so that a and b are in different models. Suppose that $a \in \mathcal{A}$. If the relation $\pi \cup \{(a,b)\}$ is not a partial isomorphism, then $\exists$ loses immediately, else the game continues in the same manner and the new value of π is the mapping $\pi \cup \{(a,b)\}$. The case $a \in \mathcal{B}$ is treated similarly, but we consider the relation $\pi \cup \{(b,a)\}$. The length of our game is ω_1 moves. Player $\exists$ wins, if he can move ω_1 times without losing. The only difference between this game and the ordinary game characterizing partial

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isomorphism is its length. M.Karttunen and T.Hyttinen have proved ([3,4,7]) that $\mathcal{A} \equiv_{\omega_1} \mathcal{B}$ is equivalent to elementary equivalence relative to the infinitely deep language $M_{\infty\omega_1}$. It may also be observed that $\mathcal{A} \equiv_{\omega_1} \mathcal{B}$ is equivalent to isomorphism in a forcing extension, where the set of forcing conditions is countably closed [9]. For the definition of $M_{\infty\omega_1}$ and other information of $\equiv_{\omega_1}$ the reader is referred to [3,4,7,9,10,11]. Our treatment is selfcontained, however. The definition of the language $M_{\infty\omega_1}$ is not needed in this paper.

One of the basic consequences of $\mathcal{A} \equiv_{\omega_1} \mathcal{B}$ is that if $\mathcal{A}$ and $\mathcal{B}$ both have power ω_1, then $\mathcal{A} \cong \mathcal{B}$ [7]. The proof of this is similar to the proof of the corresponding result for countable models.

We note in passing that there is a canonical infinitary game sentence $\varphi_{\mathcal{A}}$ (see [3], [4] or [7]), a kind of generalized Scott sentence, with the property that $\mathcal{B} \models \varphi_{\mathcal{A}}$ iff $\mathcal{A} \equiv_{\omega_1} \mathcal{B}$ for any $\mathcal{B}$. So, if $\mathcal{A} \equiv_{\omega_1} \mathcal{B}$ happens to imply that $\mathcal{B}$ has power $\leq \omega_1$, then $\varphi_{\mathcal{A}}$ characterizes $\mathcal{A}$ up to isomorphism.

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Theorem 1. *If a model of power ω_1 is $\equiv_{\omega_1}$-equivalent to a model of power $> \omega_1$, then it has 2^{ω_1} automorphisms.*

For the proof of this theorem we define the following game $G(\mathcal{A})$ where $\mathcal{A}$ is a model of power ω_1: There are ω_1 moves and two players $\exists$ and $\forall$. During the game a countable partial isomorphism π is extended. At each move $\forall$ first plays a point, to which $\exists$ then tries to extend π. $\forall$ can tell whether the point is to be on the image side or in the domain side. Moreover, $\exists$ has to come up with two contradictory extensions of π, from which $\forall$ chooses the one the game goes on with. $\exists$ wins, if he can play all ω_1 moves.

A model $\mathcal{A}$ is called *perfect*, if $\exists$ has a winning strategy in $G(\mathcal{A})$.

Proposition 2. *If $\mathcal{A} \equiv_{\omega_1} \mathcal{B}$ for some $\mathcal{B}$ of power $> \omega_1$, then $\mathcal{A}$ is perfect.*

Proof. Let S be a winning strategy of $\exists$ in the Ehrenfeucht-Fraïssé-game. An S*-mapping* is a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$ arising from S. We describe a winning strategy of $\exists$ in $G(\mathcal{A})$. During the game $\exists$ constructs S-mappings $\sigma : \mathcal{A} \to \mathcal{B}$ and $\rho : \mathcal{B} \to \mathcal{A}$ simultaneously with the required π. The idea is to keep $\pi = \rho \circ \sigma$.

Suppose now $\forall$ plays x and asks $\exists$ to extend the domain of π to x. If $x \notin \mathrm{dom}(\sigma)$ $(= \mathrm{dom}(\pi))$, $\exists$ uses S to extend σ to x. Likewise, if $\sigma(x) \notin \mathrm{dom}(\rho)$, $\exists$ uses S to extend ρ so that $\sigma(x) \in \mathrm{dom}(\rho)$. Let $\pi(x) = \rho(\sigma(x))$. This completes the first part of the move of $\exists$.

For the second part, $\exists$ has to come up with π' and π'', which are contradictory extensions of π. For any $b \in \mathcal{B}$ S gives some $s(b) \in \mathcal{A}$. If $b \notin \mathrm{ran}(\sigma)$, then $s(b) \notin \mathrm{dom}(\pi)$. As $|\mathcal{B} \backslash \mathrm{ran}(\sigma)| > |\mathcal{A}|$, there are $b \neq b' \in \mathcal{B} \backslash \mathrm{ran}(\sigma)$ with

Figure 1.

$s(b) = s(b')$. We extend ρ using S first to get an element a so that $\rho(b) = a$ and after that we extend ρ further to get $\rho(b') = a'$. Now, $a \neq a'$, since $b \neq b'$ (Figure 1). Now we can define π' and π''. In the first case we extend σ so that $\sigma(s(b)) = b$ and we let $\pi' = \rho \circ \sigma$. (Note here, that we do not extend σ to b'. It is not necessary to keep $\mathrm{ran}(\sigma) = \mathrm{dom}(\rho)$.) In the second case we extend σ so that $\sigma(s(b)) = b'$ and we define $\pi'' = \rho \circ \sigma$. Because $\pi'(s(b)) \neq \pi''(s(b))$, the two extensions are contradictory. $\square$

Proposition 3. *If $\mathcal{A}$ is perfect, then $\sigma(\mathcal{A}) = 2^{\omega_1}$.*

Proof. Suppose S is a winning strategy of $\exists$ in $G(\mathcal{A})$. Let us consider all games in which $\forall$ enumerates all of $\mathcal{A}$. Each such play determines an automorphism of $\mathcal{A}$. Since $\forall$ has a chance of splitting the game at each move, there are 2^{ω_1} different automorphisms. $\square$

This ends the proof of Theorem 1. $\square$

Now we define a game that characterizes the elementary submodel relation for the language $M_{\infty\omega_1}$. Suppose $\mathcal{A} \subseteq \mathcal{B}$. We describe the game $G_{\preceq}(\mathcal{A}, \mathcal{B})$. The game resembles very much the ordinary Ehrenfeucht-Fraïssé-game between $\mathcal{A}$ and $\mathcal{B}$. The difference is that at the start of the game $\forall$ can pick a countable set C of elements of $\mathcal{A}$ and set as the initial partial isomorphism $\pi = \{(a, a) \mid a \in C\}$. Then $\forall$ and $\exists$ continue the game like the usual Ehrenfeucht-Fraïssé-game extending π.

We write $\mathcal{A} \preceq_{\omega_1} \mathcal{B}$, if $\exists$ has a winning strategy in the game $G_{\preceq}(\mathcal{A}, \mathcal{B})$. If $\mathcal{A} \preceq_{\omega_1} \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$, then we write $\mathcal{A} \prec_{\omega_1} \mathcal{B}$. It can be proved that the relation $\mathcal{A} \preceq_{\omega_1} \mathcal{B}$ holds if and only if $\mathcal{A}$ is an elementary submodel of $\mathcal{B}$ relative to the language $M_{\infty\omega_1}$. In this definition the formulas of $M_{\infty\omega_1}$ may contain only a countable number of free variables. The proof is very similar to the proof of the fact that $\mathcal{A} \equiv_{\omega_1} \mathcal{B}$ is equivalent to elementary equivalence of $\mathcal{A}$ and $\mathcal{B}$ ([7], [3], [4]).

We describe the game $G_{\leq}(\mathcal{A},\mathcal{B})$, which is more difficult for $\exists$ to win than $G_{\preceq}(\mathcal{A},\mathcal{B})$. The length of the game is ω_1 and it resembles the Ehrenfeucht-Fraïssé game. During it $\exists$ must extend a countable partial isomorphism $\pi:\mathcal{A}\to\mathcal{B}$ and at each move the rules are the following:

(i) if $a\in\mathcal{A}$, $a\notin \mathrm{dom}(\pi)$ and $a\notin \mathrm{ran}(\pi)$, then $\forall$ can move $a\in\mathcal{A}$ and demand $\exists$ to extend π to $\pi\cup\{(a,a)\}$;

(ii) if $a\in\mathcal{A}$ $(a\in\mathcal{B})$ then $\forall$ can move $a\in\mathcal{A}$ $(a\in\mathcal{B})$ and demand $\exists$ to extend π so that $a\in \mathrm{dom}(\pi)$ $(a\in\mathrm{ran}(\pi))$.

We write $\mathcal{A}\leq_{\omega_1}\mathcal{B}$, if $\mathcal{A}\subseteq\mathcal{B}$ and $\exists$ has a winning strategy in the game $G_{\leq}(\mathcal{A},\mathcal{B})$. If $\mathcal{A}\leq_{\omega_1}\mathcal{B}$ and $\mathcal{A}\neq\mathcal{B}$, then we write $\mathcal{A}<_{\omega_1}\mathcal{B}$.

Our aim is next to prove that if $\mathcal{A}<_{\omega_1}\mathcal{B}$ for some $\mathcal{B}$, then there are 2^{ω_1} automorphisms of $\mathcal{A}$.

Lemma 4. *Let $(\mathcal{A}_\alpha)_{\alpha<\delta}$ (δ limit) be uncountable models such that:*

(i) $\mathcal{A}_\alpha\subseteq\mathcal{A}_\beta$ *if* $\alpha<\beta$;

(ii) $\mathcal{A}_\gamma=\bigcup_{\alpha<\gamma}\mathcal{A}_\alpha$ *if* γ *is a limit;*

(iii) $\mathcal{A}_\alpha\leq_{\omega_1}\mathcal{A}_{\alpha+1}$ *if* $\alpha<\delta$.

Let $\mathcal{A}_\delta=\bigcup_{\alpha<\delta}\mathcal{A}_\alpha$. *Then* $\mathcal{A}_0\preceq_{\omega_1}\mathcal{A}_\delta$. *(The arity of relations and functions must be finite.)*

Proof. For simplicity of notation, we assume that in the games $G_{\preceq}(\mathcal{A},\mathcal{B})$ and $G_{\leq}(\mathcal{A},\mathcal{B})$ at each round α, $\exists$ extends the partial isomorphism π by just a single ordered pair (a_α,b_α), where $a_\alpha\in\mathcal{A}$ and $b_\alpha\in\mathcal{B}$.

For each $\alpha<\delta$, let σ_α be $\exists$'s fixed winning strategy in $G_{\leq}(\mathcal{A}_\alpha,\mathcal{A}_{\alpha+1})$.

We describe a winning strategy for $\exists$ in $G_{\preceq}(\mathcal{A}_0,\mathcal{A}_\delta)$. We modify the game $G_{\preceq}(\mathcal{A}_0,\mathcal{A}_\delta)$ so that $\forall$ and $\exists$ only move at infinite limit ordinal rounds, which is clearly equivalent to the original game. At each round $\gamma<\omega_1$, $\exists$ also constructs a sequence s_γ of length $\delta+1$, such that $s_\gamma(\alpha)\in\mathcal{A}_\alpha$ for all $\alpha\leq\delta$. At limit rounds γ, $\exists$ first constructs s_γ and then extends the partial isomorphism π in the game $G_{\preceq}(\mathcal{A}_0,\mathcal{A}_\delta)$ by (a,b), where $a=s_\gamma(0)$ and $b=s_\gamma(\delta)$.

Before round $\gamma\geq\omega$, we assume that the following conditions are true:

(1) For all $\alpha<\delta$, the sequence $((s_\epsilon(\alpha),s_\epsilon(\alpha+1)))_{\epsilon<\gamma}$ is a play in $G_{\leq}(\mathcal{A}_\alpha,\mathcal{A}_{\alpha+1})$ according to $\exists$'s winning strategy σ_α.

(2) For all $\epsilon<\gamma$, s_ϵ is continuous, that is, if ξ is a limit ordinal, and $s_\epsilon(\xi)=a$, there is $\zeta<\xi$, such that for all $\zeta<\alpha\leq\xi$, $s_\epsilon(\alpha)=a$.

(3) Suppose a is in the range of some sequence s_ϵ, $\epsilon<\gamma$, and α is the least ordinal such that $a\in A_\alpha$. Then there is an ordinal β such that $[\alpha,\beta]=\{\xi\mid$ for some $\epsilon<\gamma,\ s_\epsilon(\xi)=a\}$. If γ is a successor, then β is a successor ordinal or δ. If γ is a limit, then $\beta=\delta$.

$\forall$ starts the game $G_{\preceq}(\mathcal{A}_0,\mathcal{A}_\delta)$ by choosing the countable set C of elements of $\mathcal{A}_0$. $\exists$ chooses as the first sequences s_n, $n<\omega$, constant sequences whose values enumerate C. Let us consider round γ in the game, where γ is an infinite limit. In general there are two cases.

First the case where $\forall$ picks $a \in \mathcal{A}_0$ as his γth move. If there is some s_ϵ such that $s_\epsilon(0) = a$, then $\exists$ responds by $s_\epsilon(\delta) \in \mathcal{A}_\delta$ and defines $s_\gamma = s_\epsilon$. Else, by (3), $\exists$ can move $a \in \mathcal{A}_\delta$ and choose the appropriate constant sequence as s_γ. The inductive hypotheses are met and we can let $s_{\gamma+n} = s_\gamma$, for $n < \omega$.

Suppose then $\forall$ picks $b \in \mathcal{A}_\delta$ as his γth move. Again, if for some $\epsilon < \gamma$, $s_\epsilon(\delta) = b$, we are done. Else, let us construct the required sequence s_γ. Let α_0 be the least ordinal such that $b \in \mathcal{A}_{\alpha_0}$ and $s_\epsilon(\alpha_0) \neq b$ for all $\epsilon < \gamma$. Note that by hypothesis (3) and condition (ii) of the lemma, $\alpha_0 = \beta_0 + 1$, for some β_0 (or $\alpha_0 = 0$). We define $s_\gamma(\beta) = b$ for all $\beta > \beta_0$. Let c be the response of $\exists$ according to σ_{α_0} if $\forall$ continues $G_{\leq}(\mathcal{A}_{\beta_0}, \mathcal{A}_{\alpha_0})$ by moving $b \in \mathcal{A}_{\alpha_0}$. Let $s_\gamma(\beta_0) = c$. Then we continue the construction of s_γ by downward induction. $\exists$ then moves $s_\gamma(0) \in \mathcal{A}_0$ in the game $G_{\preceq}(\mathcal{A}_0, \mathcal{A}_\delta)$. Similarly, by a closing procedure, $\exists$ can construct $s_{\gamma+n}$, $n < \omega$, so that clause (3) is satisfied at $\gamma + \omega$. □

Proposition 5. *If $\mathcal{A}$ is of cardinality ω_1 and $\mathcal{A} <_{\omega_1} \mathcal{B}$ for some $\mathcal{B}$, then $\mathcal{A} \equiv_{\omega_1} \mathcal{B}$ for some $\mathcal{B}$ of power ω_2, whence $\mathcal{A}$ is perfect.*

Proof. We may assume $\mathcal{A}$ and $\mathcal{B}$ have both power ω_1. Thus, by remarks preceding Theorem 1, $\mathcal{A} \cong \mathcal{B}$. We construct a sequence $(\mathcal{A}_\alpha)_{\alpha<\omega_2}$ of models so that each is isomorphic to $\mathcal{A}$, $\mathcal{A}_\alpha \subset \mathcal{A}_\beta$, if $\alpha < \beta$, and $\mathcal{A}_\alpha <_{\omega_1} \mathcal{A}_{\alpha+1}$ for all $\alpha < \omega_2$. We handle the successor step by identifying $\mathcal{A}_\alpha$ with $\mathcal{A}$ via the isomorphism. Then from $\mathcal{B}$ we get $\mathcal{A}_{\alpha+1}$. At limits we take the union of models. Lemma 4 makes sure that the union is isomorphic to $\mathcal{A}$, if it is not of power ω_2. □

So, if $\mathcal{A}$ fulfills the condition of Proposition 5, then it has 2^{ω_1} automorphisms. The proof of the following result shows that $\mathcal{A} \leq_{\omega_1} \mathcal{B}$ is a much stricter condition than $\mathcal{A} \preceq_{\omega_1} \mathcal{B}$.

Proposition 6.

$$\mathcal{A} \leq_{\omega_1} \mathcal{B} \Rightarrow \mathcal{A} \preceq_{\omega_1} \mathcal{B}$$

but

$$\mathcal{A} \preceq_{\omega_1} \mathcal{B} \not\Rightarrow \mathcal{A} \leq_{\omega_1} \mathcal{B}.$$

Proof. The first claim is trivial. For the second consider the following models. There is one equivalence relation R in the vocabulary. The model $\mathcal{A}$ contains simply ω_1 equivalence classes of size ω_1. The model $\mathcal{B} \supset \mathcal{A}$ contains one additional equivalence class of size ω_1. Then it is very easy to see that $\exists$ wins $G_{\preceq}(\mathcal{A}, \mathcal{B})$. But $\forall$ can win $G_{\leq}(\mathcal{A}, \mathcal{B})$ in two moves. First $\forall$ chooses some $b \in \mathcal{B}$, $b \notin \mathcal{A}$. Let π be $\forall$'s response. Let $a \in \mathcal{A}$, $\mathcal{A} \models R(a, \pi^{-1}(b))$, $a \notin \operatorname{ran}(\pi) \cup \operatorname{dom}(\pi)$. Then $\forall$ demands $\exists$ to map a identically. □

If $\mathcal{A}$ and $\mathcal{B}$ are countable, $\mathcal{A} \neq \mathcal{B}$ and $\mathcal{A} \prec \mathcal{B}$ (relative to $L_{\infty\omega}$), then $\sigma(\mathcal{A}) = 2^\omega$. This would suggest the analogous conjecture for uncountable models: if $|\mathcal{A}| = |\mathcal{B}| = \omega_1$ and $\mathcal{A} \prec_{\omega_1} \mathcal{B}$, then $\sigma(\mathcal{A}) = 2^{\omega_1}$. But this conjecture is false, as the following counterexample constructed by S.Shelah shows.

Theorem 7. *Let $\kappa > \omega$ be regular. There are models $\mathcal{M}_1 \subseteq \mathcal{M}_2$, $\mathcal{M}_1 \neq \mathcal{M}_2$, $|\mathcal{M}_1| = |\mathcal{M}_2| = \kappa$, such that*

(i) for every $A \subset ||\mathcal{M}_1||$, $|A| < \kappa$, there is an isomorphism from $\mathcal{M}_2$ onto $\mathcal{M}_1$ which is the identity on A;

(ii) $\sigma(\mathcal{M}_1) \leq \kappa$.

Remark. Hence $\mathcal{M}_1 \prec_\kappa \mathcal{M}_2$ but there is no $\mathcal{M}_3$ such that $\mathcal{M}_1 \equiv_\kappa \mathcal{M}_3$ and $|\mathcal{M}_3| > \kappa$, as then $\sigma(\mathcal{M}_1) = 2^\kappa$.

Proof. We first define such $\mathcal{M}_1$ and $\mathcal{M}_2$ with the vocabulary $L = \{R_\delta \mid 0 < \delta < \kappa, \delta \text{ limit}\}$, where R_δ has δ places and $|R_\delta^{\mathcal{M}_1}| = |R_\delta^{\mathcal{M}_2}| = \kappa$. We can then replace these models (in Proposition 8) by models with a vocabulary consisting of just one binary relation.

We define A, A_α, f^α and γ_α, $\alpha < \kappa$, such that:

(1) $\omega \leq \gamma_\alpha < \kappa$ for all $\alpha < \kappa$ and $\langle \gamma_\alpha \mid \alpha < \kappa \rangle$ is increasing and continuous;

(2) $\gamma_0 = \omega$, if $\alpha > 0$ is a limit, then $\gamma_\alpha = \bigcup_{\beta < \alpha} \gamma_\alpha$, and if $\alpha = \beta + 1$, then $\gamma_\alpha = \gamma_\beta + \gamma_\beta$;

(3) $A_\alpha = \{i < \gamma_\alpha \mid i \text{ even}\}$, $A = \{i < \kappa \mid i \text{ even}\}$;

(4) f^α is a 1–1 function from κ onto A mapping $\gamma_{\alpha+1}$ onto $A_{\alpha+1}$;

(5) f^α maps the interval $[\gamma_\beta, \gamma_{\beta+1})$ onto $[\gamma_\beta, \gamma_{\beta+1}) \cap A$ for $\beta > \alpha$;

(6) $f^\alpha \restriction A_\alpha$ is the identity function on A_α;

(7) f^α, $\alpha < \kappa$, are defined using free groups (see the construction of f^α below).

The definition of γ_α and A_α is clear from (1)–(3). We now describe the construction of f^α, $\alpha < \kappa$. If $\beta < \kappa$, let $T^\beta_{\text{at}} = \{s^\beta_\alpha \mid \alpha \leq \beta\}$ and $T^\beta_{\text{nat}} = \{(s^\beta_\alpha)^{-1} \mid \alpha \leq \beta\}$ be sets of arbitrary symbols. Let T_β be the set of all such sequences $\tau = \sigma_1 \ldots \sigma_n$ that:

(T1) $0 \leq n < \omega$;

(T2) $\sigma_k \in T^\beta_{\text{at}} \cup T^\beta_{\text{nat}}$ for all $1 \leq k \leq n$;

(T3) if $n > 0$ then $\sigma_n = s^\beta_\beta$;

(T4) $\sigma_k \in T^\beta_{\text{nat}} \Rightarrow \sigma_{k+1} \in T^\beta_{\text{at}}$ for all $1 \leq k < n$;

(T5) $\neg(\exists k, \alpha)(\{\sigma_k, \sigma_{k+1}\} = \{s^\beta_\alpha, (s^\beta_\alpha)^{-1}\})$.

Thus we see that T_β is a subset of the normal forms of the free group generated by $\{s^\beta_\alpha \mid \alpha \leq \beta\}$. If $\tau = \sigma_1 \ldots \sigma_n \in T_\beta$ and $s^\beta_\alpha \in T^\beta_{\text{at}}$, then we define the operation $s^\beta_\alpha \cdot \tau$ in the following way:

(a) if $\sigma_1 \neq (s^\beta_\alpha)^{-1}$ or $\tau = \emptyset$, then $s^\beta_\alpha \cdot \tau = s^\beta_\alpha \sigma_1 \ldots \sigma_n$ (i.e. just concatenate);

(b) if $\sigma_1 = (s^\beta_\alpha)^{-1}$, then $s^\beta_\alpha \cdot \tau = \sigma_2 \ldots \sigma_n$.

It is easy to check that $s^\beta_\alpha \cdot \tau \in T_\beta$. Thus $\cdot$ is defined like the multiplicative operation for the free group.

Lemma A. *Let $\tau, \tau' \in T_\beta$ and $\alpha \le \beta$. If $\tau \ne \tau'$, then $s_\alpha^\beta \cdot \tau \ne s_\alpha^\beta \cdot \tau'$.*

Proof. Straightforward. □ Lemma A.

For each $\alpha < \kappa$ let

$$\{(\tau_\xi, j_\xi) \mid \gamma_\alpha \le \xi < \gamma_{\alpha+1}\}$$

list the set

$$P_\alpha = \{(\tau, j) \mid \tau \in T_\alpha, \tau \ne \emptyset, j < \gamma_\alpha, j \notin A_\alpha\}$$

without repetitions in such a way that

$$\xi \text{ is even if and only if } \sigma_1^{\tau_\xi} \in T_{\text{at}}^\alpha,$$

where we denote $\tau_\xi = \sigma_1^{\tau_\xi} \ldots \sigma_{n_{\tau_\xi}}^{\tau_\xi}$.

If $(\tau, j) \in P_\alpha$ for some $\alpha < \kappa$, let $\xi(\tau, j)$ be the unique ξ such that $(\tau, j) = (\tau_\xi, j_\xi)$. Now we define f^α, $\alpha < \kappa$ (see Figure 3). For $\epsilon < \kappa$ let

$$f^\alpha(\epsilon) = \begin{cases} \epsilon & \text{if } \epsilon < \gamma_\alpha \text{ and } \epsilon \in A_\alpha, \\ \xi(s_\alpha^\alpha, \epsilon) & \text{if } \epsilon < \gamma_\alpha \text{ and } \epsilon \notin A_\alpha, \\ \xi(s_\alpha^\alpha \cdot \tau, j) & \text{if } \gamma_\alpha \le \epsilon < \gamma_{\alpha+1} \text{ and } \epsilon = \xi(\tau, j), \\ \xi(s_\alpha^\beta \cdot \tau, j) & \text{if } \gamma_\beta \le \epsilon < \gamma_{\beta+1},\ \beta > \alpha \text{ and } \epsilon = \xi(\tau, j). \end{cases}$$

Figure 3.

We have to check that f^α is well-defined, that is, $\xi(s_\alpha^\alpha \cdot \tau, j)$ and $\xi(s_\alpha^\beta \cdot \tau, j)$ must be defined above in appropriate conditions and their values must be even. We check only $\xi(s_\alpha^\alpha \cdot \tau, j)$, the other case is similar. Suppose $\gamma_\alpha \le \epsilon < \gamma_{\alpha+1}$ and $\epsilon = \xi(\tau, j)$. Then $\tau \in T_\alpha$, $\tau \ne \emptyset$. Let $\tau = \sigma_1 \ldots \sigma_n$. If $\sigma_1 \ne (s_\alpha^\alpha)^{-1}$, then $s_\alpha^\alpha \cdot \tau = s_\alpha^\alpha \sigma_1 \ldots \sigma_n \ne \emptyset$. Thus $\xi(s_\alpha^\alpha \cdot \tau, j)$ is defined and it is even, since $s_\alpha^\alpha \in T_{\text{at}}^\alpha$. Suppose $\sigma_1 = (s_\alpha^\alpha)^{-1}$. Then $s_\alpha^\alpha \cdot \tau = \sigma_2 \ldots \sigma_n$. Now $n \ge 2$ by (T3) and $\sigma_2 \in T_{\text{at}}^\alpha$ by (T4). Thus $\sigma_2 \ldots \sigma_n \ne \emptyset$ and $\xi(s_\alpha^\alpha \cdot \tau, j)$ is defined and even.

Lemma B. *Conditions (4), (5) and (6) above are met.*

Proof. From the definition of f^α we see easily that f^α maps $\gamma_{\alpha+1}$ to $A_{\alpha+1}$ and $[\gamma_\beta, \gamma_{\beta+1})$ to $[\gamma_\beta, \gamma_{\beta+1}) \cap A$, if $\beta > \alpha$. We show first that f^α is a 1–1 function $\kappa \to A$. Suppose $\epsilon_1 \neq \epsilon_2$. We prove $f^\alpha(\epsilon_1) \neq f^\alpha(\epsilon_2)$. There are several cases, of which we treat the two most interesting. The proof in other cases is similar or trivial.

(a) Suppose $\epsilon_1 < \gamma_\alpha$, $\epsilon_1 \notin A_\alpha$ and $\epsilon_2 \in [\gamma_\alpha, \gamma_{\alpha+1})$. Let $\epsilon_2 = \xi(\tau, j)$. Since $\tau \neq \emptyset$, by Lemma A $s^\alpha_\alpha \cdot \tau \neq s^\alpha_\alpha$. Thus $f^\alpha(\epsilon_1) = \xi(s^\alpha_\alpha, \epsilon_1) \neq \xi(s^\alpha_\alpha \cdot \tau, j) = f^\alpha(\epsilon_2)$.

(b) Suppose $\epsilon_1, \epsilon_2 \in [\gamma_\alpha, \gamma_{\alpha+1})$. Let $\epsilon_1 = \xi(\tau_1, j_1)$ and $\epsilon_2 = \xi(\tau_2, j_2)$. If $j_1 \neq j_2$, then the claim is clear. If $j_1 = j_2$, then $\tau_1 \neq \tau_2$ and by Lemma A $s^\alpha_\alpha \cdot \tau_1 \neq s^\alpha_\alpha \cdot \tau_2$ and again the claim holds.

Next we prove that f^α is onto. Let $\delta \in A$. We try to find $\epsilon < \kappa$, for which $\delta = f^\alpha(\epsilon)$. If $\delta \in A_\alpha$, then we set $\epsilon = \delta$. Suppose then $\delta \in [\gamma_\alpha, \gamma_{\alpha+1}) \cap A$. Denote $\delta = \xi(\tau, j)$, where $\tau = \sigma_1 \ldots \sigma_n$, $\tau \neq \emptyset$. We know $\sigma_1 \in T^\alpha_{\mathrm{at}}$, since δ is even.

(a) If $n = 1$, then $\tau = s^\alpha_\alpha$ by (T3) and we set $\epsilon = j$.
(b) If $n > 1$ and $\sigma_1 = s^\alpha_\alpha$, then we set $\epsilon = \xi(\sigma_2 \ldots \sigma_n, j)$.
(c) If $n > 1$ and $\sigma_1 \neq s^\alpha_\alpha$, then $\epsilon = \xi((s^\alpha_\alpha)^{-1}\sigma_1 \ldots \sigma_n, j)$. Here ξ is defined and (T4) fulfilled because $\sigma_1 \in T^\alpha_{\mathrm{at}}$.

Suppose then $\delta \in [\gamma_\beta, \gamma_{\beta+1}) \cap A$, $\beta > \alpha$.

(a) If $\sigma_1 = s^\beta_\alpha$, then $n > 1$ by (T3) and $\epsilon = \xi(\sigma_2 \ldots \sigma_n, j)$.
(b) If $\sigma_1 \neq s^\beta_\alpha$, then $\epsilon = \xi((s^\beta_\alpha)^{-1}\sigma_1 \ldots \sigma_n, j)$.

Thus we have proved that $f^\alpha : \kappa \to A$ is 1–1 and onto. Now (4), (5) and (6) are clear. $\square$ Lemma B.

If $\alpha < \kappa$, let $\gamma(\alpha)$ denote the unique β for which $\gamma_\beta \leq \alpha < \gamma_{\beta+1}$. Let G_1 be the group of permutations of A generated by $\{f^\beta(f^\alpha)^{-1} \mid \alpha, \beta < \kappa\}$. Let G_2 be the group of permutations of κ generated by $\{(f^\beta)^{-1}f^\alpha \mid \alpha, \beta < \kappa\}$.

We are ready to define the models. We define $\mathcal{M}_1$ and $\mathcal{M}_2$ as follows:

(i) $||\mathcal{M}_1|| = A$;
(ii) $||\mathcal{M}_2|| = \kappa$;
(iii) $R^{\mathcal{M}_k}_\alpha = \{\langle i_0 i_2 \ldots i_\epsilon \ldots\rangle_{\epsilon<\alpha, \epsilon \text{ even}} \mid \exists g \in G_k(\bigwedge_{\epsilon<\alpha \text{ even}} g(i_\epsilon) = \epsilon)\}$, $k = 1, 2$, $0 < \alpha < \kappa$, α limit.

Lemma C. $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

Proof. Suppose $\langle i_\epsilon \mid \epsilon < \alpha \text{ even}\rangle \in R^{\mathcal{M}_1}_\alpha$. Thus there are $k < \omega$, $\alpha_r, \beta_r < \kappa$, for $1 \leq r \leq k$ such that (using $(f^\beta(f^\alpha)^{-1})^{-1} = f^\alpha(f^\beta)^{-1}$)

$$\bigwedge_{\epsilon<\alpha \text{ even}} f^{\beta_1}(f^{\alpha_1})^{-1}f^{\beta_2}(f^{\alpha_2})^{-1}\ldots f^{\beta_k}(f^{\alpha_k})^{-1}(i_\epsilon) = \epsilon.$$

If $\gamma < \kappa$ is chosen large enough, then by (6) $f^\gamma(i_\epsilon) = i_\epsilon$ and $f^\gamma(\epsilon) = \epsilon$ for all $\epsilon < \alpha$, ϵ even, and thus

$$\bigwedge_{\epsilon<\alpha \text{ even}} ((f^\gamma)^{-1}f^{\beta_1})((f^{\alpha_1})^{-1}f^{\beta_2})\ldots((f^{\alpha_{k-1}})^{-1}f^{\beta_k})((f^{\alpha_k})^{-1}f^\gamma)(i_\epsilon) = \epsilon.$$

But this means $\langle i_\epsilon \mid \epsilon < \alpha \text{ even}\rangle \in R_\alpha^{\mathcal{M}_2}$. The other direction is similar. $\square$ Lemma C.

Lemma D. *For each α, f^α is an isomorphism from $\mathcal{M}_2$ onto $\mathcal{M}_1$ which is the identity on A_α. (Hence G_k is a group of automorphisms of $\mathcal{M}_k$.)*

Proof. Suppose $\langle i_\epsilon \mid \epsilon < \alpha \text{ even}\rangle \in R_\alpha^{\mathcal{M}_2}$. Then

$$\bigwedge_{\epsilon<\alpha \text{ even}} (f^{\beta_1})^{-1} f^{\alpha_1} \dots (f^{\beta_k})^{-1} f^{\alpha_k}(i_\epsilon) = \epsilon.$$

If γ is chosen large enough, then

$$\bigwedge_{\epsilon<\alpha \text{ even}} f^\gamma (f^{\beta_1})^{-1} f^{\alpha_1} \dots (f^{\beta_k})^{-1} f^{\alpha_k} (f^\alpha)^{-1}(f^\alpha(i_\epsilon)) = \epsilon,$$

which means $\langle f^\alpha(i_\epsilon) \mid \epsilon < \alpha \text{ even}\rangle \in R_\alpha^{\mathcal{M}_1}$. The other direction is similar. $\square$ Lemma D.

Since κ is regular, Lemma D proves part (i) of the theorem. To show (ii) it is enough to prove the following lemma, because $|G_1| \le \kappa$.

Lemma E. *G_1 is the group of all automorphisms of $\mathcal{M}_1$.*

Proof. Let $g^* \in \mathrm{AUT}(\mathcal{M}_1)$, $g^* \notin G_1$. Let G_1^δ be the group generated by $\{f^\beta (f^\alpha)^{-1} \mid \alpha, \beta < \delta\}$. As κ is regular, by taking successive closures we can find a limit ordinal $\delta < \kappa$ such that:

($\delta 1$) g^* maps A_δ onto A_δ;
($\delta 2$) for every $g \in G_1^\delta$, $g^* \restriction A_\delta \ne g \restriction A_\delta$.

(In fact the set of such δ is a closed unbounded subset of κ.)

Let $i_\epsilon = g^*(\epsilon)$ for $\epsilon < \alpha$, $\alpha = \gamma_\delta$, ϵ even. As $g^* \in \mathrm{AUT}(\mathcal{M}_1)$ and $\langle \epsilon \mid \epsilon < \alpha \text{ even}\rangle \in R_\alpha^{\mathcal{M}_1}$, there is some $g_1 \in G_1$ with $\bigwedge_{\epsilon<\alpha \text{ even}} g_1(i_\epsilon) = \epsilon$. Let $g = g_1^{-1} \in G_1$. Then

$$\bigwedge_{\epsilon<\alpha \text{ even}} g(\epsilon) = i_\epsilon.$$

Thus $g^* \restriction A_\delta = g \restriction A_\delta$. By ($\delta 2$) $g \restriction A_\delta \notin \{h \restriction A_\delta \mid h \in G_1^\delta\}$ and by ($\delta 1$) g maps A_δ onto itself. To get a contradiction it is enough to prove:

(Γ) If $g \in G_1$ and $g \restriction A_\delta \notin \{h \restriction A_\delta \mid h \in G_1^\delta\}$, then g does not map A_δ onto itself.

Proof of (Γ). So let

$$g = f^{\beta_k}(f^{\alpha_k})^{-1} \dots f^{\beta_1}(f^{\alpha_1})^{-1}$$

be a counterexample with k minimal. Clearly $\alpha_i \ne \beta_i$ and $\alpha_{i+1} \ne \beta_i$ by the minimality of k.

As $g \notin G_1^\delta$, for some $1 \le r \le k$ holds $\alpha_r \ge \delta$ or $\beta_r \ge \delta$. If $\alpha_r \ge \delta$, then we can consider $g^{-1} = f^{\alpha_1}(f^{\beta_1})^{-1} \dots f^{\alpha_k}(f^{\beta_k})^{-1}$, which is also a counterexample with k minimal. Thus we may assume without loss of generality that $\beta_r \ge \delta$ for some r. Let

$$\mu = \max(\{\alpha_r \mid r \in \{1,\dots,k\}, \alpha_r < \delta\} \cup \{\beta_r \mid r \in \{1,\dots,k\}, \beta_r < \delta\}) + 1.$$

Let $\xi_0 \in A_\delta$ be arbitrary. We denote

$$\begin{aligned}
\eta_1 &= (f^{\alpha_1})^{-1}(\xi_0),\\
\xi_1 &= f^{\beta_1}(\eta_1),\\
&\vdots\\
\eta_k &= (f^{\alpha_k})^{-1}(\xi_{k-1}),\\
\xi_k &= f^{\beta_k}(\eta_k).
\end{aligned}$$

Thus $\xi_k = g(\xi_0)$. For $i = 0,\dots,k$ let

$$b_{\le i} = \max\{\mu, \beta_1, \dots, \beta_i\}.$$

Lemma F. *Suppose $\xi_0 \in A_\delta$. Then $\gamma(\xi_i) < \max\{b_{\le i}+1, \delta\}$ for $i = 0,\dots,k$.*

Proof. By induction. First, $\gamma(\xi_0) < \delta$. Suppose $\gamma(\xi_i) < \max\{b_{\le i}+1,\delta\}$. From the definition of f^α we see $\gamma((f^\alpha)^{-1}(\epsilon)) \le \gamma(\epsilon)$ for all α, ϵ. Thus $\gamma(\eta_{i+1}) \le \gamma(\xi_i)$. We see also that if $\gamma(f^\alpha(\epsilon)) > \gamma(\epsilon)$, then $\gamma(f^\alpha(\epsilon)) = \alpha$. Thus $\gamma(\xi_{i+1}) \le \gamma(\eta_{i+1})$ or $\gamma(\xi_{i+1}) = \beta_{i+1}$. In both cases $\gamma(\xi_{i+1}) < \max\{b_{\le i+1}+1, \delta\}$. ▫ Lemma F.

Lemma G. *For all $1 \le i \le k$ either $\beta_i \le b_{\le i-1}$ or $\alpha_i \le b_{\le i-1}$.*

Proof. Suppose $\beta_i > b_{\le i-1}$ and $\alpha_i > b_{\le i-1}$. Since $b_{\le i-1} \ge \mu$, this implies $\alpha_i, \beta_i \ge \delta$. Thus $\alpha_i, \beta_i \ge \max\{b_{\le i-1}+1, \delta\}$. Suppose $\xi_0 \in A_\delta$ is arbitrary. By Lemma F $\gamma(\xi_{i-1}) < \max\{b_{\le i-1}+1,\delta\}$ and by (6) $f^{\beta_i}(f^{\alpha_i})^{-1}(\xi_{i-1}) = \xi_{i-1}$. But now we see

$$\begin{aligned}
&f^{\beta_k}(f^{\alpha_k})^{-1} \dots f^{\beta_1}(f^{\alpha_1})^{-1} \restriction A_\delta\\
&\quad = f^{\beta_k}(f^{\alpha_k})^{-1} \dots f^{\beta_{i+1}}(f^{\alpha_{i+1}})^{-1} f^{\beta_{i-1}}(f^{\alpha_{i-1}})^{-1} \dots f^{\beta_1}(f^{\alpha_1})^{-1} \restriction A_\delta,
\end{aligned}$$

a contradiction with the minimality of k. ▫ Lemma G.

The following lemma shows that g maps $\xi(s^\mu_\mu, 1)$ outside A_δ, which contradicts our assumption and proves (Γ).

Lemma H. *Let $\xi_0 = \xi(s^\mu_\mu, 1)$. Then for all $1 \le i \le k$ ξ_i is of the form $\xi(s^{b_{\le i}}_{\beta_i}\sigma^i_2 \dots \sigma^i_{n_i}, j_i)$, where $s^{b_{\le i}}_{\beta_i}\sigma^i_2 \dots \sigma^i_{n_i} \in T_{b_{\le i}}$ and $n_i \ge 1$. Hence $\gamma(\xi_i) = b_{\le i}$.*

Proof. Suppose first the claim holds for ξ_i, $i \geq 1$. We prove it holds for ξ_{i+1}.

(a) Suppose $\alpha_{i+1} > b_{\leq i} = \gamma(\xi_i)$. Then $\eta_{i+1} = (f^{\alpha_{i+1}})^{-1}(\xi_i) = \xi_i$. By Lemma G $\beta_{i+1} \leq b_{\leq i}$. Now

$$\xi_{i+1} = f^{\beta_{i+1}}(\xi_i) = \xi(s^{\beta_{\leq i}}_{\beta_{i+1}} s^{\beta_{\leq i}}_{\beta_i} \sigma^i_2 \ldots \sigma^i_{n_i}, j_i).$$

Hence the claim holds for $i+1$.

(b) Suppose $\alpha_{i+1} \leq b_{\leq i} = \gamma(\xi_i)$. Then

$$\eta_{i+1} = (f^{\alpha_{i+1}})^{-1}(\xi_i) = \xi((s^{\beta_{\leq i}}_{\alpha_{i+1}})^{-1} s^{\beta_{\leq i}}_{\beta_i} \sigma^i_2 \ldots \sigma^i_{n_i}, j_i),$$

where $\alpha_{i+1} \neq \beta_i$ by the minimality of k. Note that η_{i+1} is odd. If $\beta_{i+1} > b_{\leq i}$, then

$$\xi_{i+1} = f^{\beta_{i+1}}(\eta_{i+1}) = \xi(s^{\beta_{i+1}}_{\beta_{i+1}}, \eta_{i+1})$$

and the claim holds for $i+1$. If $\beta_{i+1} \leq b_{\leq i}$, then

$$\xi_{i+1} = f^{\beta_{i+1}}(\eta_{i+1}) = \xi(s^{\beta_{\leq i}}_{\beta_{i+1}} (s^{\beta_{\leq i}}_{\alpha_{i+1}})^{-1} s^{\beta_{\leq i}}_{\beta_i} \sigma^i_2 \ldots \sigma^i_{n_i}, j_i),$$

where $\beta_{i+1} \neq \alpha_{i+1}$ by the minimality of k and the claim holds.

Next we prove that the claim is true for $i = 1$.

(a) Suppose $\alpha_1 > b_{\leq 0} = \mu$. Then $\eta_1 = \xi_0 = \xi(s^\mu_\mu, 1)$ and $\beta_1 \leq b_{\leq 0} = \mu$. As above we get $\xi_1 = \xi(s^\mu_{\beta_1} s^\mu_\mu, 1)$.

(b) Suppose $\alpha_1 \leq b_{\leq 0} = \mu$. Then $\eta_1 = \xi((s^\mu_{\alpha_1})^{-1} s^\mu_\mu, 1)$, where $\alpha_1 \neq \mu$ by the definition of μ. If $\beta_1 > \mu$, then $\xi_1 = \xi(s^{\beta_1}_{\beta_1}, \eta_1)$. If $\beta_1 \leq \mu$, then $\xi_2 = (s^\mu_{\beta_1} (s^\mu_{\alpha_1})^{-1} s^\mu_\mu, 1)$. $\square$ Lemma H.

Let $\xi_0 = \xi(s^\mu_\mu, 1)$. By Lemma H $\gamma(\xi_k) = b_{\leq k} \geq \delta$, since $\beta_i \geq \delta$ for some i. Thus $\xi_k \notin A_\delta$, which proves (Γ). This ends the proof of Lemma E and the whole theorem. $\square$

Proposition 8. *We can find models $\mathcal{M}_1$ and $\mathcal{M}_2$ which satisfy Theorem 7 and have a vocabulary of one binary relation.*

Proof. Suppose $\mathcal{M}$ is a model of the vocabulary $\{R_\delta \mid 0 < \delta < \kappa, \delta \text{ limit}\}$, such that $|\mathcal{M}| = \kappa$, $|R^{\mathcal{M}}_\delta| \leq \kappa$ and R_δ has δ places. We define a model $\mathcal{A} = F(\mathcal{M})$ of one binary relation R. Let

$$||\mathcal{A}|| = ||\mathcal{M}|| \cup \bigcup_\delta \{((a_\alpha)_{\alpha<\delta}, \beta) \mid \mathcal{M} \models R_\delta(a_0, \ldots, a_{\alpha<\delta}, \ldots), \beta < \delta\}.$$

The relation R holds in $\mathcal{A}$ exactly in the following two cases:

(i) if $b_1, b_2 \in ||\mathcal{A}||$, $b_1 = ((a_\alpha)_{\alpha<\delta}, \beta_1)$ and $b_2 = ((a_\alpha)_{\alpha<\delta}, \beta_2)$, where $\beta_1 < \beta_2$, then $\mathcal{A} \models R(b_1, b_2)$;

(ii) if $b \in ||\mathcal{A}||$ and $b = ((a_\alpha)_{\alpha<\delta}, \beta)$, then $\mathcal{A} \models R(a_\beta, b)$.

In other words, for each tuple $(a_\alpha)_{\alpha<\delta}$, such that $\mathcal{M} \models R_\delta(a_0, \ldots, a_{\alpha<\delta}, \ldots)$ we add δ new elements to $||\mathcal{A}||$. The new δ elements are wellordered by R and for all $\beta < \delta$ a_β is in relation R with the βth added element.

Obviously $|F(\mathcal{M})| = \kappa$. It is a routine task to check that there is a 1–1 correspondence between $\mathrm{AUT}(\mathcal{M})$ and $\mathrm{AUT}(F(\mathcal{M}))$. (Note that $\mathcal{A} \models \neg\exists x R(x, a)$ iff $a \in ||\mathcal{M}||$.) Thus $\sigma(\mathcal{M}) = \sigma(F(\mathcal{M}))$. It also easy to see that if $\mathcal{M} \prec_\kappa \mathcal{M}'$, then $F(\mathcal{M}) \prec_\kappa F(\mathcal{M}')$. Let $\mathcal{M}$ and $\mathcal{M}'$ be the models constructed in Theorem 7. Let $\mathcal{M}_1 = F(\mathcal{M})$ and $\mathcal{M}_2 = F(\mathcal{M}')$. □

We say that a chain of models $(\mathcal{A}_\alpha)_{\alpha<\kappa}$ is *continuous*, if $\mathcal{A}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{A}_\alpha$ for γ a limit. A chain is an *elementary* chain, if $\mathcal{A}_\alpha \preceq_{\omega_1} \mathcal{A}_\beta$ for all $\alpha < \beta$. If the relation $\preceq_{\omega_1}$ were preserved under unions of continuous chains of models, then we could replace $<_{\omega_1}$ by $\prec_{\omega_1}$ in Proposition 5, as is easy to see. This raised the question, whether $\preceq_{\omega_1}$ is preserved under unions of continuous chains. Since Theorem 7 shows that $<_{\omega_1}$ cannot be replaced by $\prec_{\omega_1}$, it also proves that $\preceq_{\omega_1}$ is not always preserved. Below we present also two other counterexamples. They are continuous elementary chains of length ω and ω_1. The problem, whether $\preceq_{\omega_1}$ is preserved under unions of continuous chains of length ω_2 or greater, is open to the authors.

We define the linear order η, which we shall use in the proofs below. The linear order η consists of functions $f : \omega \to \omega_1$, for which the set $\{n \in \omega \mid f(n) \neq 0\}$ is finite. If $f, g \in \eta$, then $f < g$ iff $f(n) < g(n)$, where n is the least number, where f and g differ. By $\eta^{<\alpha}$ we mean the restriction of η to those functions f for which $f(0) < \alpha$. Similarly we define $\eta^{\geq\alpha}$.

Let ξ and θ be arbitrary linear orders. By $\xi \times \theta$ we mean a linear order where we have a copy of ξ for every $x \in \theta$. The order between the copies is determined by θ. By $\theta + \xi$ we mean a linear order, where ξ is on top of θ. If α is an ordinal, then α^* denotes α in a reversed order.

We first prove a lemma about η.

Lemma 9.
(i) $\eta^{\geq\alpha} \cong \eta$ *for all* α,
(ii) $\eta \times n \cong \eta$ *for all* $n \in \omega$,
(iii) $\eta \times \alpha^* \cong \eta$ *for all* $\alpha < \omega_1$.

Proof. (i) Let $f \in \eta^{\geq\alpha}$. Simply map f to $g \in \eta$, where $g(0) = f(0) - \alpha$ and $g(n) = f(n)$, if $n \neq 0$.

(ii) We prove the claim by induction on n. Suppose $\eta \times n \cong \eta$. Clearly $\eta^{<1} \cong \eta$, thus $\eta \times n \cong \eta^{<1}$. By (i) $\eta \cong \eta^{\geq 1}$. So $\eta \times (n+1) \cong \eta^{<1} + \eta^{\geq 1} \cong \eta$.

(iii) We prove this by induction on α. The successor step is easy, because $\eta + \eta \cong \eta$. Suppose then that α is a limit ordinal. Let $(\alpha_n)_{n<\omega}$ be an increasing sequence cofinal in α. Then $\alpha = \sum_{n<\omega} \alpha_{n+1} - \alpha_n$. All the differences in the sum are $< \alpha$, so we can use our induction assumption and we get $\eta\times\alpha^* \cong \eta\times\omega^*$. Thus the limit case is reduced to showing that $\eta\times\omega^* \cong \eta$. We describe the isomorphism. First we map the topmost copy of η in $\omega^*\times\eta$ to $\{f \in \eta \mid f(0) > 0\}$. This mapping goes as in (i). Then we map the next copy of η to $\{f \in \eta \mid f(0) = 0, f(1) > 0\}$, and continuing this way we get an isomorphism. □

Proposition 10. *There exists an elementary chain $(\mathcal{A}_n)_{n<\omega}$ of models of cardinality ω_1 such that*

$$\mathcal{A}_n \not\preceq_{\omega_1} \bigcup_{n<\omega} \mathcal{A}_n$$

for all n.

Proof. We let $\mathcal{A}_n = \eta \times n$. Then the union of the chain is $\mathcal{A} = \eta \times \omega$. We can choose an increasing sequence of points in $\mathcal{A}$ so that the length of the sequence is ω and the sequence has no upper bound in $\mathcal{A}$. It is not possible to find such a sequence in any $\mathcal{A}_n$. Thus it is clear that no $\mathcal{A}_n$ is an elementary submodel of $\mathcal{A}$.

It remains to prove that our chain is really an elementary chain. We start to play the game $G_\preceq(\mathcal{A}_n, \mathcal{A}_m)$, $m > n$. First $\forall$ chooses a countable set C in $\mathcal{A}_n$, which is mapped identically to $\mathcal{A}_m$. Some of the points of C are in the topmost copy of η in $\mathcal{A}_n$. Let $\alpha < \omega_1$ be so big that none of these points f has $f(0) \geq \alpha$. We form an isomorphism between $\mathcal{A}_n$ and $\mathcal{A}_m$ so that it maps the points in C identically. We map the part $\eta \times (n-1) + \eta^{<\alpha}$ in $\mathcal{A}_n$ identically to $\mathcal{A}_m$. The remaining part of $\mathcal{A}_n$ is $\eta^{\geq\alpha}$ and thus isomorphic to η. The remaining part of $\mathcal{A}_m$ is isomorphic to $\eta + \eta \times (m-n)$ and thus isomorphic to η. So we get an isomorphism between the remaining parts. Now $\exists$ can win the game simply by playing according to our isomorphism. □

Proposition 11. *There exists an elementary chain $(\mathcal{A}_\alpha)_{\alpha<\omega_1}$ of models of cardinality ω_1 such that*

$$\mathcal{A}_\alpha \not\preceq_{\omega_1} \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$$

for all α. In this chain $\mathcal{A}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{A}_\alpha$, if γ is a limit ordinal.

Proof. We let $\mathcal{A}_\alpha = \eta + \eta \times \alpha^*$. Then there is a descending ω_1-sequence in $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$, but no descending ω_1-sequence in any $\mathcal{A}_\alpha$. This shows that $\mathcal{A}_\alpha \not\preceq_{\omega_1} \mathcal{A}$.

We have to prove that our chain is elementary. We start to play the game $G_\preceq(\mathcal{A}_\alpha, \mathcal{A}_\beta)$, where $\alpha < \beta$. First $\forall$ chooses a countable set C of points in $\mathcal{A}_\alpha$. Let $\delta < \omega_1$ be so big that for no $f \in C$ $f(0) \geq \delta$. We form an isomorphism between our models so that it maps the points in C identically. First we map the part $\eta\times\alpha^*$ in $\mathcal{A}_\alpha$ identically to $\mathcal{A}_\beta$. We map the part $\eta^{<\delta}$ in the bottom copy of η in $\mathcal{A}_\alpha$ again identically to $\mathcal{A}_\beta$. Now it remains to map $\eta^{\geq\delta}$ to $\eta^{\geq\delta} + \eta\times\gamma^*$, where

$\gamma = \beta - \alpha$. But, according to Lemma 7 (i) and (ii), these both are isomorphic to η, so we get the isomorphism between $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$. Then $\exists$ wins the game by playing according to this isomorphism. $\square$

We shall now consider a totally different kind of condition which also guarantees perfectness. Let $I(\omega)$ denote the assumption (taken from [2]) that

> "there is an ideal I on ω_2 which is ω_2-complete, normal, contains all singletons $\{\alpha\}$, $\alpha < \omega_2$, and
>
> $$I^+ = \{X \subseteq \omega_2 \mid X \notin I\}$$
>
> has a dense subset K such that every descending chain of length $< \omega_1$ of elements of K has a lower bound in K. "

Remark. $I(\omega)$ implies that I is precipitous and hence that ω_2 is measurable in an inner model. On the other hand, if a measurable cardinal is Levy-collapsed to ω_2, $I(\omega)$ becomes true [1].

We prove that $I(\omega)$ implies CH. Suppose $2^\omega \geq \omega_2$. Let T be a full binary tree of height $\omega + 1$. Let $A \subseteq \{t \in T \mid \text{height}(t) = \omega\}$, $|A| = \omega_2$. Let I be the ideal on A given by $I(\omega)$. Now it is very easy to construct $t_0 < \cdots < t_n < \cdots$ and $X_0 \supseteq \cdots \supseteq X_n \supseteq \cdots$, $n < \omega$, such that $\text{height}(t_n) = n$, $X_n \in K$, and for all $a \in X_n$ holds $a > t_n$. Now $\bigcap_{n<\omega} X_n$ contains at most one element, a contradiction.

Theorem 12. *Assume $I(\omega)$. If a model $\mathcal{A}$ of power ω_1 satisfies $\sigma(\mathcal{A}) > \omega_1$, then $\mathcal{A}$ is perfect.*

Proof. (Inspired by [2].) Let I satisfy $I(\omega)$. We may assume I is an ideal on a set AUT of automorphisms of power ω_2. We describe a winning strategy of $\exists$ in $G(\mathcal{A})$. Let $X \subseteq \text{AUT}$ and $f \in X$. We say that f is an I-*point* of X, if for all countable $\pi \subseteq f$, it holds that $[\pi] \cap X \in I^+$, where $[\pi] =$ the set of all extensions of π.

Claim: Every $X \in I^+$ has an I-point.

Otherwise every $f \in X$ has a $\pi_f \subseteq f$ with $X \cap [\pi_f] \in I$. Because CH holds, there are only ω_1 countable π. This implies $X \subseteq \bigcup_{f \in X} X \cap [\pi_f] \in I$, a contradiction.

The idea of $\exists$ is to construct a descending sequence $(X_\alpha)_{\alpha<\omega_1}$ of elements of K. We denote by π_α the countable partial isomorphism at stage α. The descending sequence is chosen so that for all $f \in X_\alpha$ holds $\pi_\alpha \subset f$.

Suppose the players have played α moves. Then $\forall$ demands $\exists$ to extend π_α to a point x and give two contradictory extensions. For example, $\forall$ demands x to be on the domain side. Because functions f can have only ω_1 different values at x and I is ω_2-closed, we can find $Y \in I^+$, $Y \subseteq X_\alpha$, such that all the functions in Y agree at x. Now let f be an I-point of Y and let f' be an I-point of $Y\setminus\{f\}$. Because f and f' are two different mappings, we can choose countable $\pi \subset f$ and $\pi' \subset f'$ so that π and π' are contradictory extensions of π_α and they are defined at x. Now we can choose $X \in K$ and $X' \in K$, $(X, X' \subseteq Y)$, so that for all $g \in X$ $\pi \subset g$ and for all $g \in X'$ $\pi' \subset g$. The extensions π and π' are the demanded contradictory extensions. For example, if $\forall$ picks π, then we set $X_{\alpha+1} = X$ and $\pi_{\alpha+1} = \pi$.

Limit steps in the game do not cause trouble, because countable descending chains in K have a lower bound in K. □

Corollary 13. *Assume $I(\omega)$. Then the following condition $(*)$ holds:*
$(*)$ *If $\mathcal{A}$ is a model of power ω_1, then the conditions*

(i) $\sigma(\mathcal{A}) > \omega_1$,
(ii) $\sigma(\mathcal{A}) = 2^{\omega_1}$,
(iii) *$\mathcal{A}$ is perfect,*

are equivalent.

Remark. T. Jech has proved [5] it consistent that $2^\omega = \omega_1$, $2^{\omega_1} > \omega_2$ and there is a tree of power ω_1 with ω_2 automorphisms. Hence $(*)$ cannot hold without some set-theoretical assumption. We shall later show that the consistency strength of $(*)$ is that of an inaccessible cardinal. Note that $(*)$ implies CH.

The following result of S.Shelah shows a dependence between trees and the number of automorphisms of an uncountable model.

Theorem 14. *Suppose that there exists a tree T of height ω_1 such that:*
(i) T has λ uncountable branches, where $\omega_1 < \lambda < 2^{\omega_1}$;
(ii) each level in the tree has $\leq \omega_1$ nodes.
Then we can build a structure $\mathcal{M}$ of cardinality ω_1 with exactly λ automorphisms.

Proof. Let $T_\alpha = \{t \in T \mid \text{height}(t) = \alpha\}$ and

$$G_\alpha = \{X \subset T_\alpha \mid |X| < \omega\}$$

for each $\alpha < \omega_1$. If $X, Y \in G_\alpha$, we define

$$X + Y = (X\setminus Y) \cup (Y\setminus X),$$

i.e. $X + Y$ is the symmetric difference of X and Y. Clearly, $+$ makes G_α into an Abelian group. Actually, G_α is a linear vector space over the field $Z_2 = \{0, 1\}$, but below we need only to know that G_α is Abelian.

Let G be the Abelian group, which consists of all functions (ω_1-sequences) $s : \omega_1 \to \bigcup_{\alpha<\omega_1} G_\alpha$, where $s(\alpha) \in G_\alpha$, and addition is defined coordinatewise:

$(s_1+s_2)(\alpha) = s_1(\alpha)+s_2(\alpha)$. If $B = (t_\alpha)_{\alpha<\omega_1}$ is an ω_1-branch in T, then B determines naturally a sequence $b \in G$, where $b(\alpha) = \{t_\alpha\}$. Let $G' \subseteq G$ be the Abelian group generated by all sequences b corresponding to ω_1-branches. (Equivalently, G' is the vector subspace spanned by such sequences.)

Suppose $s \in G'$ is arbitrary. Then $s = b_1 + \cdots + b_n$ for some ω_1-branches $b_1, \ldots, b_n$. Clearly, if $t \in T_\alpha$, then $t \in s(\alpha)$ iff an odd number of branches $b_1, \ldots, b_n$ passes through t. From this we see that if $\alpha < \beta$ and $t \in T_\alpha$, then

(∗) $t \in s(\alpha)$ iff t has an odd number of successors in $s(\beta)$.

Let $\mathcal{M}'$ be a model of vocabulary $\{R_s \mid s \in G'\}$ such that

(i) $||\mathcal{M}'|| = \{s \mid s \in G'\}$;
(ii) $\mathcal{M}' \models R_s(s_1, s_2)$ iff $s_2 = s_1 + s$.

The model $\mathcal{M}'$ is like an affine space, where the set of points is $||\mathcal{M}'||$ and the space of differences G' is kept rigid. Obviously, $|\mathcal{M}'| = \lambda$ and $\mathrm{AUT}(\mathcal{M}')$ consists of all mappings π'_s, $s \in ||\mathcal{M}'||$, where $\pi'_s(x) = x + s$. Thus $\mathcal{M}'$ has exactly λ automorphisms.

Let $\mathcal{M}$ be a model such that:

(i) $||\mathcal{M}|| = \{s \mid \alpha \mid s \in ||\mathcal{M}'||, \alpha < \omega_1\}$;
(ii) the vocabulary of $\mathcal{M}'$ is $\{F\} \cup \{R_s \mid s \in ||\mathcal{M}||\}$;
(iii) $\mathcal{M} \models R_s(s_1, s_2)$ iff the domains of s, s_1, s_2 are equal and $s_2 = s_1 + s$ (where the sum is defined coordinatewise);
(iv) $\mathcal{M} \models F(s_1, s_2)$ iff s_1 is an initial segment of s_2.

Since $|T| = \omega_1$, there are only ω_1 countable initial segments of ω_1-branches, and $|\mathcal{M}| = \omega_1$. We show that there is a 1–1 correspondence between $\mathrm{AUT}(\mathcal{M}')$ and $\mathrm{AUT}(\mathcal{M})$. Let $s \in ||\mathcal{M}'||$ be arbitrary. Then $\pi'_s \in \mathrm{AUT}(\mathcal{M}')$. We define from π'_s an automorphism π_s of $\mathcal{M}$: if $r \in ||\mathcal{M}||$ and $\mathrm{dom}(r) = \alpha$, then $\pi_s(r) = r + s \mid \alpha$. Obviously, if $s \neq s'$, then $\pi_s \neq \pi_{s'}$.

Suppose then π is an automorphism of $\mathcal{M}$. We denote by $s^\beta_\emptyset$ a function, such that $\mathrm{dom}(s^\beta_\emptyset) = \beta$ and $s^\beta_\emptyset(\alpha) = \emptyset$ for all $\alpha < \beta$. We define $s \in G$ in the following way: $s \mid \beta = \pi(s^\beta_\emptyset)$ for all $\beta < \omega_1$. We show that $s \in ||\mathcal{M}'||$. By (∗) $|s(\alpha)| \geq |s(\beta)|$ if $\alpha \geq \beta$. Since $|s(\alpha)|$ is finite for all α, there must be n and β such that $|s(\alpha)| = n$ for all $\alpha \geq \beta$. Thus from (∗) we see that from β up s determines some ω_1-branches $b_1, \ldots, b_n$, such that $s \mid (\omega_1 \backslash \beta) = b \mid (\omega_1 \backslash \beta)$, where $b = b_1 + \cdots + b_n$. It remains to show that $s \mid (\beta+1) = b \mid (\beta+1)$. We know $s \mid (\beta+1) = \pi(s^{\beta+1}_\emptyset) = s' \mid (\beta+1)$ for some $s' \in ||\mathcal{M}'||$. Since $s'(\beta) = b(\beta)$, (∗) implies that $s' \mid (\beta+1) = b \mid (\beta+1)$, and thus $s = b \in ||\mathcal{M}'||$.

Now it is very easy to show that $\pi = \pi_s$. Thus there is a 1–1 correspondence and $\mathcal{M}$ has exactly λ automorphisms. □

Remark. If the tree T above is a Kurepa tree, then the resulting model $\mathcal{M}$ is clearly not perfect.

We can modify the preceding proof to get a suitable model with a finite vocabulary. We add to the model $\mathcal{M}$ the set $\{a_s \mid s \in ||\mathcal{M}||\}$ of new elements and wellorder them with a new relation $<$. Then we can use these new elements to code the relations R_s into a single relation and we get a finite vocabulary. This modification does not affect the number of automorphisms.

Theorem 14 is of use only, if the conditions in it are consistent with ZFC. We show that this is indeed the case.

A tree T is a *Kurepa tree* if:

(i) $\mathrm{height}(T) = \omega_1$;
(ii) each level of T is at most countable;
(iii) T has at least ω_2 uncountable branches.

It is well-known (see e.g. [6]) that Kurepa trees exist in the constructible universe. Let $\mathcal{M}$ be a countable standard model of ZFC + $V = L$. Let T be a Kurepa tree in $\mathcal{M}$. Let λ be the number of uncountable branches in T. Now we use forcing to get a model where $2^{\omega_1} > \lambda$. We utilize Lemma 19.7 of [6]. In $\mathcal{M}$ the equation $2^{<\omega_1} = \omega_1$ holds. Let $\kappa > \lambda$ be such that $\kappa^{\omega_1} = \kappa$. Let P be the set of all functions p such that:

(i) $\mathrm{dom}(p) \subseteq \kappa \times \omega_1$ and $|\mathrm{dom}(p)| < \omega_1$,
(ii) $\mathrm{ran}(p) \subseteq \{0, 1\}$,

and let p be stronger than q iff $p \supset q$. The generic extension $\mathcal{M}[G]$ has the same cardinals as $\mathcal{M}$ and $\mathcal{M}[G] \models 2^{\omega_1} = \kappa$. P is a countably closed notion of forcing. Hence Lemma 24.5 of [6] says that the Kurepa tree T contains in $\mathcal{M}[G]$ just those branches that are in the ground model. Thus there are exactly λ uncountable branches in T also in the extended model $\mathcal{M}[G]$. CH is true in L, therefore $\mathcal{M}[G] \models 2^{\omega} = \omega_1$ by the countable closure of P. We have obtained a model $\mathcal{M}[G]$ of ZFC + CH with a tree T, which has the properties (i)–(ii) of Theorem 14.

From Theorem 14 and the above remarks we obtain a new proof of Jech's result [5]:

> If ZF is consistent, then ZFC + $2^{\omega} = \omega_1$ + "there exists a model of cardinality ω_1 with λ automorphisms, $\omega_1 < \lambda < 2^{\omega_1}$ " is consistent.

If we assume CH, we can prove the other direction in Theorem 14.

Proposition 15. *Assume CH. Suppose that we have a model $\mathcal{M}$ of cardinality ω_1 and $\mathcal{M}$ has λ automorphisms, $\omega_1 < \lambda < 2^{\omega_1}$. Then there exists a tree T of height ω_1 such that the conditions (i)–(ii) in Theorem 14 hold.*

Proof. To avoid some complications, we assume that $\mathcal{M}$ has a relational vocabulary. If not, we can transform the vocabulary to relational and that does not affect the number of automorphisms. The tree T will consist of partial automorphisms of $\mathcal{M}$. Let $(a_\alpha)_{\alpha<\omega_1}$ enumerate $\mathcal{M}$. Let $\mathcal{M}_\alpha = \mathcal{M}\restriction \{a_\beta \mid \beta < \alpha\}$. We let $T = \{f \mid f$ is an automorphism of some $\mathcal{M}_\alpha\}$. If $f, g \in T$, then $f \leq g$ iff g extends f.

Suppose f is an automorphism of $\mathcal{M}$. Let $\alpha < \omega_1$ be arbitrary. It may be that the restriction of f to $\mathcal{M}_\alpha$ is not a bijection from $\mathcal{M}_\alpha$ to $\mathcal{M}_\alpha$, but by taking successively closures we find $\beta > \alpha$, for which f gives an automorphism of $\mathcal{M}_\beta$. Thus f determines an uncountable branch in T.

For the other direction, if we have an uncountable branch in T, it is clear that it determines an automorphism of $\mathcal{M}$. Thus T has λ uncountable branches.

The tree T may contain at most $\omega_1 \times \omega^\omega$ nodes. Since we assumed CH, this is equal to ω_1. So, each level of T contains $\leq \omega_1$ nodes. □

Theorem 16. *CH + $(*)$ is equiconsistent with the existence of an inaccessible cardinal. Also CH + $2^{\omega_1} > \omega_2$ + "for all $\mathcal{A}$ of power ω_1, $\sigma(\mathcal{A}) > \omega_1$ implies $\sigma(\mathcal{A}) = 2^{\omega_1}$" is equiconsistent with the existence of an inaccessible cardinal.*

Proof. Let λ be a strongly inaccessible cardinal and $\mu \geq \lambda$ so that $\mu = \mu^{\aleph_1}$. Let $P = Q \times R$, where Q is the Levy collapse of λ to $\aleph_2$ (see [6], p. 191) and R is the set of Cohen conditions for adding μ subsets to $\aleph_1$. We show that $V^P \models (*)$. Suppose $p \models \sigma(\mathcal{A}) > \omega_1$. We may assume, without loss of generality, that $\mathcal{A} \in V$. Hence there is a P-name $\tilde{f}$ and $p \in P$ so that $p \models$ "$\tilde{f}$ is an automorphism of $\mathcal{A}$ and $\tilde{f} \notin V$." For any extension q of p let

$$f^q = \{(\alpha, \beta) \mid q \models \tilde{f}(\alpha) = \beta\}.$$

Now for each extension q of p and for all countable sets $A, B \subseteq \omega_1$ there are extensions q^0 and q^1 of q in P and an element a of ω_1 so that

(i) $A \cup \{a\} \subseteq \mathrm{dom}(f^{q^0}) \cap \mathrm{dom}(f^{q^1})$,
(ii) $B \subseteq \mathrm{ran}(f^{q^0}) \cap \mathrm{ran}(f^{q^1})$,
(iii) $f^{q^0}(a) \neq f^{q^1}(a)$.

Using this fact it is easy to see that $p \models$ "$\exists$ wins $G(\mathcal{A})$". This ends the proof of one half of the claims.

For the other half of the first claim we assume that CH + $(*)$ holds. If $\aleph_2$ is not inaccessible in L, then there is a Kurepa tree with $\geq \aleph_2$ branches, and hence by the remark after Theorem 14, a non-perfect model of cardinality ω_1 with $> \omega_1$ automorphisms.

For the other half of the second claim we show that under our assumption $\aleph_2$ has to be inaccessible in L. For this end, suppose $\aleph_2$ is not inaccessible in L. Then there is $A \subseteq \omega_1$ so that $\aleph_2^{L[A]} = \aleph_2$, $\aleph_1^{L[A]} = \aleph_1$ and GCH holds in $L[A]$ (see, e.g., Jech [6], p.252). We shall construct a tree with $\aleph_1$ nodes and exactly $\aleph_2$ branches. Let C be the set of δ with $\omega_1 < \delta < \omega_2$, and $L_\delta[A] \models$ ZFC- + "there is cardinal ω_1 and there are no cardinals $> \omega_1$". Note that $C \in L[A]$.

If $\gamma < \beta$, we denote by $(L_\beta[B], \gamma)$ a model of vocabulary $(\in, U_1, U_2)$, where U_1 and U_2 are unary relations, the interpretation of U_1 is B and the interpretation of U_2 is the single element $\gamma \in L_\beta[B]$.

We form the Skolem hulls in this proof by choosing as a witness the element which is the smallest possible in the canonical well-ordering of the corresponding model.

Fact A. An easy argument shows that if $\delta \in C$ and $\gamma < \delta$, then there cannot be any gaps between ordinals which are included in the Skolem hull of $\omega_1 \cup \{\gamma\}$ (or ω_1, as γ is definable in the model) in $(L_\delta[A], \gamma)$.

Let $\mathcal{B}$ be the class of pairs $(\alpha, (L_\beta[B], \gamma)) \in L[A]$, where $L_\beta[B] \models$ ZFC- + "there is cardinal ω_1 and there are no cardinals $> \omega_1$", $B = A \cap \omega_1^{L_\beta[B]}$, $\alpha < \omega_1^{L_\beta[B]}$, $\gamma < \beta$ and $\gamma > \omega_1^{L_\beta[B]}$.

We define a partial ordering of these pairs as follows:

$$(\alpha, (L_\beta[B], \gamma)) < (\alpha', (L_{\beta'}[B'], \gamma'))$$

if $\alpha < \alpha'$, $\beta \leq \beta'$ and $(L_\beta[B], \gamma)$ is the transitive collapse of the Skolem hull of $\alpha \cup \{\gamma'\}$ in $(L_{\beta'}[B'], \gamma')$. We define a tree T as follows: Nodes of the tree are pairs $(\alpha, (L_\beta[B], \gamma)) \in \mathcal{B}$ with $\alpha < \beta < \omega_1$. The ordering of T is the same as that of $\mathcal{B}$. The cardinality of T is $\aleph_1$.

If $G = (\alpha_\xi, (L_{\beta_\xi}[B_\xi], \gamma_\xi))$, $\xi < \omega_1$, is an uncountable branch in T, then the direct limit of $(L_{\beta_\xi}[B_\xi], \gamma_\xi)$, $\xi < \omega_1$, is isomorphic to some $(L_\delta[A], \gamma)$, where $\delta \in C$. If we denote by H_α the transitive collapse of the Skolem hull of $\alpha \cup \{\gamma\}$, $\alpha < \omega_1$, in $(L_\delta[A], \gamma)$, then (α, H_α), $\alpha < \omega_1$, is a branch H in T. A straightforward argument shows that G and H coincide. So the original branch G is in fact in $L[A]$. Since T has at most $\aleph_2$ uncountable branches in $L[A]$, it has at most $\aleph_2$ uncountable branches altogether. On the other hand, by Fact A above, T clearly has at least $\aleph_2$ uncountable branches. We have shown that T has $\aleph_1$ nodes and exactly $\aleph_2$ uncountable branches. □

In this paper we have considered models of cardinality ω_1 and games of length ω_1. When we generalize the model theory of countable models to uncountable cardinalities, many problems arise. We chose to concentrate our attention on ω_1, because it offers the simplest example of an uncountable cardinal, and even this simple case seems to present enough problems. Naturally, the results in this paper can be generalized to many other cardinalities κ, i.e. we can consider models of power κ and games of length κ. The results 1–6 above are valid for any uncountable cardinal κ. Proposition 10 can be generalized for any regular uncountable cardinal κ, thus we get an elementary chain of length ω, for which $\preceq_\kappa$ is not preserved under the union. From the ideas of Proposition 11 we obtain the following result: if κ is a regular uncountable cardinal, λ is a successor cardinal and $\lambda \leq \kappa$, then there is an elementary chain of length λ, for which $\preceq_\kappa$ is not preserved under the union. Theorem 14, which shows a dependence between trees and automorphisms, holds for any uncountable κ. Proposition 15 has a counterpart for any regular uncountable κ.

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