FULL REFLECTION OF STATIONARY SETS BELOW $\aleph_\omega$

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Abstract

It is consistent that for every $n \geq 2$, every stationary subset of $\omega_n$ consisting of ordinals of cofinality $\omega_k$ where $k = 0$ or $k \leq n - 3$ reflects fully in the set of ordinals of cofinality $\omega_{n-1}$. We also show that this result is best possible.
1. Introduction.

A stationary subset $S$ of a regular uncountable cardinal $\kappa$ reflects at $\gamma < \kappa$ if $S \cap \gamma$ is a stationary subset of $\gamma$. For stationary sets $S, A \subseteq \kappa$ let

$$S < A \text{ if } S \text{ reflects at almost all } \alpha \in A$$

where “almost all” means modulo the closed unbounded filter on $\kappa$, i.e. with the exception of a nonstationary set of $\alpha$’s. If $S < A$ we say that $S$ reflects fully in $A$.

The trace of $S$, $\text{Tr}(S)$, is the set of all $\gamma < \kappa$ at which $S$ reflects. The relation $<$ is well-founded [1], and $o(S)$, the order of $S$, is the rank of $S$ in this well-founded relation.

In this paper we investigate the question which stationary subsets of $\omega_n$ reflect fully in which stationary sets; in other words the structure of the well founded relation $<$. Clearly, $o(S) < o(A)$ is a necessary condition for $S < A$, and moreover, a set $S \subseteq \omega_n$ has order $k$ just in case it has a stationary intersection with the set

$$S^n_k = \{ \alpha < \omega_n : cf\alpha = \omega_k \}.$$

Thus the problem reduces to the investigation of full reflection of stationary subsets of $S^n_k$ in stationary subsets of $S^n_m$ for $k < m < n$.

The problem for $n = 2$ is solved completely in Magidor’s paper [2]: It is consistent that every stationary $S \subseteq S^n_0$ reflects fully in $S^n_1$. The problem for $n > 2$ is more complicated. It is tempting to try the obvious generalization, namely $S < A$ whenever $o(S) < o(A)$, but this is provably false:

**Proposition 1.1.** There exist stationary sets $S \subset S^n_0$ and $A \subset S^n_1$ such that $S$ does not reflect at any $\gamma \in A$.

**Proof.** Let $S_i, i < \omega_2$, be any family of pairwise disjoint subsets of $S^n_0$, and let $(C_\gamma : \gamma \in S^n_1)$ be such that each $C_\gamma$ is a closed unbounded subset of $\gamma$ of order type $\omega_1$. Clearly, at most $\aleph_1$ of the sets $S_i$ can meet each $C_\gamma$, and so for each $\gamma$ there is $i(\gamma) < \omega_2$ such that $C_\gamma \cap S_i = \emptyset$ for all $i \geq i(\gamma)$.

There is $i < \omega_2$ such that $i(\gamma) = i$ for a stationary set of $\gamma$’s. Let $A \subset S^n_1$ be this stationary set and let $S = S_i$. Then $S \cap C_\gamma = \emptyset$ for all $\gamma \in A$ and so $S \cap \gamma$ is nonstationary. Hence $S$ does not reflect at any $\gamma \in A$. 

There is of course nothing special in the proof about $\aleph_1$ (or about $\aleph_1$) and so we have the following generalization:
Proposition 1.2. Let \( k < m < n - 1 \). There exist stationary sets \( S \subseteq S^n_k \) and \( A \subseteq S^n_m \) such that \( S \) does not reflect at any \( \gamma \in A \). 

Consequently, if \( n > 2 \) then full reflection in \( S^n_m \) is possible only if \( m = n - 1 \). This motivates our Main Theorem.

1.3 Main Theorem. Let \( \kappa_2 < \kappa_3 < \cdots < \kappa_n < \cdots \) be a sequence of supercompact cardinals. There is a generic extension \( V[G] \) in which \( \kappa_n = \aleph_n \) for all \( n \geq 2 \), and such that

(a) every stationary subset of \( S^n_0 \) reflects fully in \( S^n_1 \), and

(b) for every \( n \geq 3 \), every stationary subset of \( S^n_k \) for all \( k = 0, \ldots, n - 3 \), reflects fully in \( S^n_{n-1} \).

We will show that the result of the Main Theorem is best possible. But first we prove a corollary:

1.4 Corollary. In the model of the Main Theorem we have for all \( n \geq 2 \) and all \( m, 0 < m < n \):

(a) Any \( \aleph_m \) stationary subsets of \( S^n_0 \) reflect simultaneously at some \( \gamma \in S^n_m \).

(b) For every \( k \leq m - 2 \), any \( \aleph_m \) stationary subsets of \( S^n_k \) reflect simultaneously at some \( \gamma \in S^n_m \).

Proof. Let us prove (a) as (b) is similar. Let \( m < n \) and let \( S_\xi, \xi < \omega_m \), be stationary subsets of \( S^n_0 \). First, each \( S_\xi \) reflects fully in \( S^n_{n-1} \) and so there exist club sets \( C_\xi, \xi < \omega_m \), such that each \( S_\xi \) reflects at all \( \alpha \in C_\xi \cap S^n_{n-1} \). As the club filter is \( \omega_n \)-complete, there exists an \( \alpha \in S^n_{n-1} \) such that \( S_\xi \cap \alpha \) is stationary, for all \( \xi < \omega_m \). Next we apply full reflection of subsets of \( S^n_{n-1} \) in \( S^n_{n-2} \) (to the ordinal \( \alpha \) of cofinality \( \omega_{n-1} \) rather than to \( \omega_{n-1} \) itself) and the \( \omega_{n-1} \)-completeness of the club filter on \( \omega_{n-1} \), to find \( \beta \in S^n_{n-2} \) such that \( S_\xi \cap \beta \) is stationary for all \( \xi < \omega_m \). This way we continue until we find a \( \gamma \in S^n_m \) such that every \( S_\xi \cap \gamma \) is stationary. □

Note that the amount of simultaneous reflection in 1.4 is best possible:

1.5 Proposition. If \( \text{cf} \gamma = \aleph_m \) and if \( S_\xi, \xi < \omega_{m+1} \), are disjoint stationary sets then some \( S_\xi \) does not reflect at \( \gamma \).

Proof. \( \gamma \) has a club subset of size \( \aleph_m \), and it can only meet \( \aleph_m \) of the sets \( S_\xi \cap \gamma \). □
By Corollary 1.4, the model of the Main Theorem has the property that whenever $2 \leq m < n$, every stationary subset of $S^n_k$ reflects quite strongly in $S^n_m$, provided $k \leq m - 2$. This cannot be improved to include the case of $k = m - 1$, as the following proposition shows:

1.6 Proposition. Let $m \geq 2$. Either

(a) for all $k < m - 1$ there exists a stationary set $S \subseteq S^m_k$ that does not reflect fully in $S^m_m$, or

(b) for all $n > m$ there exists a stationary set $A \subseteq S^m_n$ that does not reflect at any $\delta \in S^m_n$.

We shall give a proof of 1.6 in Section 3. In our model we have, for every $m \geq 2$, full reflection of subsets of $S^m_0$ in $S^m_m$ (and of subsets of $S^m_k$ for $k \leq m - 3$) and therefore 1.6 (a) fails in the model. Thus the model necessarily satisfies 1.6 (b), which shows that the consistency result is best possible.

2. Proof of Main Theorem

Let $\kappa_2 < \kappa_3 < \cdots < \kappa_n < \cdots$ be a sequence of cardinals with the property that for each $n \geq 2$, $\kappa_n$ is a $\kappa < \kappa_{n+1}$ - supercompact cardinal, i.e. for every $\gamma < \kappa_{n+1}$ there exists an elementary embedding $j : V \to M$ with critical point $\kappa_n$ such that $j(\kappa_n) > \gamma$ and $M^\gamma \subset M$.\footnote{We note in passing that the condition about the $\kappa_n$ is equivalent to “every $\kappa_n$ is $< \kappa_\omega$ - supercompact” where $\kappa_\omega = \sup_{m < \omega} \kappa_m$.}

We construct the generic extension by iterated forcing, an iteration of length $\omega$ with full support. The first stage of the iteration $P_1$ makes $\kappa_2 = \aleph_2$, and for each $n$, the $n$th stage $P_n$ (a forcing notion in $V(P_1 * \cdots * P_{n-1})$) makes $\kappa_{n+1} = \aleph_{n+1}$. In the iteration, we repeatedly use three standard notions of forcing: $\text{Col}(\kappa, \alpha)$, $C(\kappa)$ and $CU(\kappa, T)$.

Definition. Let $\kappa$ be a regular uncountable cardinal.

(a) $\text{Col}(\kappa, \alpha)$ is the forcing that collapses $\alpha \geq \kappa$ with conditions of size $< \kappa$:

A condition is a function $p$ from a subset of $\kappa$ of size $< \kappa$ into $\alpha$; a condition $q$ is stronger than $p$ if $q \supseteq p$.

(b) $C(\kappa)$ is the forcing that adds a Cohen subset of $\kappa$: A condition is an 0-1-function $p$ on a subset of $\kappa$ of size $< \kappa$; a condition $q$ is stronger than $p$ if $q \supseteq p$.
(c) $CU(\kappa, T)$ is the forcing that shoots a club through a stationary set $T \subseteq \kappa$:
A condition is a closed bounded subset of $T$; a condition $q$ is stronger than $p$ if $q$ end-extends $p$.

The first stage $P_1$ of the iteration $P = (P_n : n = 1, 2, \cdots)$ is a forcing of size $\kappa_2$ that is $\omega$-closed$^2$, satisfies the $\kappa_2$-chain condition and collapses each cardinal between $\aleph_1$ and $\kappa_2$ (it is essentially the Levy forcing with countable conditions.)

For each $n \geq 2$, we construct (in $V(P|n)$) the $n^{th}$ stage $P_n$ such that

\begin{enumerate}
\item $|P_n| = \kappa_{n+1}$
\item $P_n$ is $\aleph_{n-2}$ closed
\item $P_n$ satisfies the $\kappa_{n+1}$-chain condition
\item $P_n$ collapses each cardinal between $\aleph_n = \kappa_n$ and $\kappa_{n+1}$
\item $P_n$ does not add any $\omega_{n-1}$-sequences of ordinals
\end{enumerate}

and such that $P_n$ guarantees the reflection of stationary subsets of $\aleph_n$ stated in the theorem.

It follows, by induction, that each $\kappa_n$ becomes $\aleph_n$: Assuming that $\kappa_n = \aleph_n$ in $V(P|n)$, the $n^{th}$ stage $P_n$ preserves $\aleph_n$ by (e), and the rest of the iteration $(P_{n+1}, P_{n+2}, \cdots)$ also preserves $\aleph_n$ because it is $\aleph_{n-1}$-closed by (b); $P_n$ makes $\kappa_{n+1}$ the successor of $\kappa_n$ by (c) and (d).

We first define the forcing $P_1$:

$P_1$ is an iteration, with countable support, $(Q_\alpha : \alpha < \kappa_2)$ where for each $\alpha$,

\[ Q_\alpha = \text{Col}(\aleph_1, \aleph_1 + \alpha) \times C(\aleph_1). \]

It follows easily from well known facts that $P_1$ is an $\omega$-closed forcing of size $\kappa_2$, satisfies the $\kappa_2$-chain condition and makes $\kappa_2 = \aleph_2$.

Next we define the forcing $P_2$. (It is a modification of Magidor’s forcing from [2], but the added collapsing of cardinals requires a stronger assumption on $\kappa_2$ than weak compactness. The iteration is padded up by the addition of Cohen forcing which will make the main argument of the proof work more smoothly). The definition of $P_2$ is inside the model $V(P_1)$, and so $\kappa_2 = \aleph_2$:

$P_2$ is an iteration, with $\aleph_1$-support, $(Q_\alpha : \alpha < \kappa_3)$ where for each $\alpha$,

\[ Q_\alpha = \text{Col}(\aleph_2, \aleph_2 + \alpha) \times C(\aleph_2) \times CU(T_\alpha) \]

$^2$A forcing notion is $\lambda$-closed if every descending sequence of length $\leq \lambda$ has a lower bound.
where $T_\alpha$ is, in $V(P_1 * P_2|\alpha)$, some stationary subset of $\omega_2$. We choose the $T_\alpha$’s so that each $T_\alpha$ contains all limit ordinals of cofinality $\omega$. It follows easily that for each $\alpha < \kappa_3$, $P_2|\alpha\parallel Q_\alpha$ is $\omega$-closed.

The crucial property of the forcing $P_2$ will be the following:

Lemma 2.2. $P_2$ does not add new $\omega_1$ - sequences of ordinals.

One consequence of Lemma 2.2 is that the conditions $(p,q,s) \in Q_\alpha$ can be taken to be sets in $V(P_1)$ (rather than in $V(P_1 * P_2|\alpha)$). Once we have Lemma 2.2, the properties (2.1) (a) - (e) follow easily.

It remains to specify the choice of the $T_\alpha$’s. By a standard argument using the $\kappa_3$ - chain condition, we can enumerate all potential subsets of $\omega_2$ by a sequence $\langle S_\alpha : \alpha < \kappa_3 \rangle$ in such a way that each $S_\alpha$ is in $V(P_1 * P_2|\alpha)$. At the stage $\alpha$ of the iteration, we let $T_\alpha = \omega_2$, unless $S_\alpha$ is, in $V(P_1 * P_2|\alpha)$, a stationary set of ordinals of cofinality $\omega$. If that is the case, we let $\mathcal{T}_\alpha = (\mathcal{Tr}(S_\alpha) \cap S_2^1) \cup S_2^0$.

Assuming that Lemma 2.2 holds, we now show that in $V(P_1 * P_2)$, every stationary $S \subseteq S_2^0$ reflects fully in $S_2^1$:

The set $S$ appears as $S_\alpha$ at some stage $\alpha$, and because it is stationary in $V(P_1 * P_2)$, it is stationary in the smaller model $V(P_1 * P_2|\alpha)$. The forcing $Q_\alpha$ creates a closed unbounded set $C$ such that $C \cap S_2^1 \subseteq \mathcal{Tr}(S)$ (note that because $P_2$ does not add $\omega_1$ - sequences, the meaning of $\mathcal{Tr}(S)$ or of $S_2^1$ does not change).

Thus in $V(P_1 * P_2)$ we have full reflection of subsets of $S_2^0$ in $S_2^1$. The later stages of the iteration do not add new subsets of $\omega_2$ and so this full reflection remains true in $V(P)$.

We postpone the proof of Lemma 2.2 until after the definition of the rest of the iteration.

We now define $P_n$ for $n \geq 3$. We work in $V(P_1 * \cdots * P_{n-1})$. By the induction hypothesis we have $\kappa_n = \aleph_n$.

$P_n$ is an iteration with $\aleph_n$ - support, $\langle Q_\alpha : \alpha < \kappa_{n+1} \rangle$, where for each $\alpha$,

$$Q_\alpha = \text{Col}(\aleph_n, \aleph_n + \alpha) \times C(\aleph_n) \times CU(T_\alpha)$$

where $T_\alpha$ is a $P_n|\alpha$ - name for a subset of $\omega_n$. To specify the $T_\alpha$’s, let $\langle S_\alpha : \alpha < \kappa_{n+1} \rangle$ be an enumeration of all potential subsets of $\omega_n$ such that each $S_\alpha$ is a $P_n|\alpha$ - name.
At the stage \( \alpha \), let \( T_\alpha = \omega_n \) unless \( S_\alpha \) a stationary set of ordinals and \( S_\alpha \subseteq S_k^n \) for some \( k = 0, \ldots, n - 3 \), in which case let

\[
T_\alpha = (Tr(S_\alpha) \cap S_{n-1}^n) \cup (S_0^n \cup \cdots \cup S_{n-2}^n)
\]

\[
= \{ \gamma < \omega_n : cf \gamma \leq \omega_{n-2} \text{ or } S_\alpha \cap \gamma \text{ is stationary} \}
\]

Due to the selection of the \( T_\alpha \)'s, \( Q_\alpha \) is \( \omega_{n-2} \)-closed, and so is \( P_n \). The crucial property of the forcing is the analog of Lemma 2.2:

**Lemma 2.3.** \( P_n \) does not add new \( \omega_{n-1} \)-sequences of ordinals.

Given this lemma, properties (2.1) (a) - (e) follow easily. The same argument as given above for \( P_2 \) shows that in \( V(P_1 \ast \cdots \ast P_n) \), and therefore in \( V(P) \) as well, every stationary subset of \( S_k^n, k = 0, \ldots, n - 3 \), reflects fully in \( S_{n-1}^n \).

It remains to prove Lemmas 2.2 and 2.3. We prove Lemma 2.3, as 2.2 is an easy modification.

**Proof of Lemma 2.3.**

Let \( n \geq 3 \), and let us give the argument for a specific \( n \), say \( n = 4 \). We want to show that \( P_4 \) does not add \( \omega_3 \)-sequences of ordinals.

We will work in \( V(P_1 \ast P_2) \) (and so consider the forcing \( P_3 \ast P_4 \)). As \( P_1 \ast P_2 \) has size \( \kappa_3, \kappa_4 \) is a \( < \kappa_5 \)-supercompact cardinal in \( V(P_1 \ast P_2) \), and \( \kappa_3 = \aleph_3 \). The forcing \( P_3 \) is an iteration of length \( \kappa_4 \) that makes \( \kappa_4 = \aleph_4 \) and is \( \aleph_1 \)-closed; then \( P_4 \) is an iteration of length \( \kappa_5 \). By induction on \( \alpha < \kappa_5 \) we show

\[
(2.4) \quad P_4|\alpha \text{ does not add } \omega_3 \text{-sequences of ordinals.}
\]

As \( P_4 \) has the \( \aleph_5 \)-chain condition, (2.4) is certainly enough for Lemma 2.3. Let \( \alpha < \kappa_5 \).

Let \( j \) be an elementary embedding \( j : V \to M \) (as we work in \( V(P_1 \ast P_2) \), \( V \) means \( V(P_1 \ast P_2) \)) such that \( j(\kappa_4) > \beta \) and \( M^\beta \subset M \) for some inaccessible cardinal \( \beta > \alpha \). Consider the forcing \( j(P_3) \) in \( M \). It is an iteration of which \( P_3 \) is an initial segment. By a standard argument, the elementary embedding \( j : V \to M \) can be extended to an elementary embedding \( j : V(P_3) \to M(j(P_3)) \). We claim that every \( \beta \)-sequence of ordinals in \( V(P_3) \) belongs to \( M(j(P_3)) \): the name for such a set has size \( \leq \beta \) and so it belongs to \( M \), and since \( P_3 \in M \) and \( M(P_3) \subseteq M(j(P_3)) \), the claim follows. In particular, \( P_4|\alpha \in M(j(P_3)) \).
Let $p, \dot{F} \in V(P_3)$ be such that $p \in P_4[\alpha]$ and $\dot{F}$ is a $(P_4[\alpha])$-name for an $\omega_3$-sequence of ordinals. We shall find a stronger condition that decides all the values of $\dot{F}$. By the elementarity of $j$, it suffices to prove that

\[(2.5) \exists \bar{p} \leq j(p) \text{ in } j(P_4[\alpha]) \text{ that decides } j(\dot{F}).\]

The rest of the proof is devoted to the proof of (2.5).

Let $G$ be an $M$-generic filter on $j(P_3)$.

**Lemma 2.6.** In $M[G]$ there is a generic filter $H$ on $P_4[\alpha]$ over $M[G \cap P_3]$ such that $M[G]$ is a generic extension of $M[G \cap P_3][H]$ by an $\aleph_1$-closed forcing, and such that $p \in H$.

**Proof.** There is an $\eta < j(\kappa_4)$ such that $P_4[\alpha]$ has size $\aleph_3$ in $M_\eta = M[G \cap (j(P_3)[\eta])]$. Since $P_4[\alpha]$ is $\aleph_2$-closed, it is isomorphic in $M_\eta$ to the Cohen forcing $C(\aleph_3)$. But $Q_\eta = (j(P_3))(\eta) = \text{Col}(\aleph_3, \aleph_3+\eta) \times C(\aleph_3) \times CU(T_\eta)$, so $G|Q_\eta = G_{\text{Col}} \times G_C \times G_{CU}$, and using $G_C$ and the isomorphism between $P_4[\alpha]$ and $C(\aleph_3)$ we obtain $H$. Since the quotient forcing $j(P_3)/(P_3 \times C(\aleph_3))$ is an iteration of $\aleph_1$-closed forcings, it is $\aleph_1$-closed. \(\square\)

**Lemma 2.7.** In $M[G]$ there is a condition $\bar{p} \in j(P_4[\alpha])$ that extends $p$, and extends every member of $j''H$.

Lemma 2.7 will complete the proof of (2.5): since every value of $\dot{F}$ is decided by some condition in $H$, every value of $j(\dot{F})$ is decided by some condition in $j''H$, and therefore by $\bar{p}$.

**Proof of Lemma 2.7.** Working in $M[G]$, we construct $\bar{p} \in j(P_4[\alpha])$, a sequence $(p_\xi : \xi < j(\alpha))$ of length $j(\alpha)$, by induction. When $\xi$ is not in the range of $j$, we let $p_\xi$ be the trivial condition; that guarantees that the support of $\bar{p}$ has size $|\alpha|$ which is $\aleph_3$ (because $\alpha < j(\kappa_4) = \aleph_4$ in $M[G]$). So let $\xi < \alpha$ be such that $\bar{p}|j(\xi)$ has been defined, and construct $p_{j(\xi)}$.

The condition $p_{j(\xi)}$ has three parts $u, v, s$ where $u \in \text{Col}(j(\kappa_4), j(\kappa_4)+j(\xi)), v \in C((\kappa_4))$ and $s \in CU(T_{j(\xi)})$. It is easy to construct the $u$-part and the $v$-part, as follows: The filter $H|P_4(\xi)$ has three parts; a collapsing function $f$ of $\kappa_4$ onto $\kappa_4+\xi$, a 0-1-function $g$ on $\kappa_4$, and a club subset $C$ of $T_\xi$. We let $u = j''f$ and $v = j''g$, and these are functions of size $\aleph_3$ and therefore members of $\text{Col}$ and $C$ respectively. For
the s - part, let s = j\"C ∪ {κ₄}. In order that this set be a condition in CU(T[j(ξ)]), we have to verify that κ₄ ∈ T[j(ξ)].

This is a nontrivial requirement if S_j(ξ) is in M((j(P₃))* (j(P₄)|(j(ξ)))) a stationary subset of j(κ₄) and is a subset of either S₃₀ or of S₃₁ (of S₃ₙ for n = 4 and k ≤ n - 3). Then κ₄ has to be reflecting point of S_j(ξ), i.e. we have to show that S_j(ξ) ∩ κ₄ is stationary, in M((j(P₃))* (j(P₄)|j(ξ))).

By the assumption and by elementarity of j, S_ξ is a stationary subset of κ₄ in V(P₃ * P₄|ξ), and S_ξ ⊆ S₃₀ or S_ξ ⊆ S₃₁, i.e. consists of ordinals of cofinality < λ₁. Since S_j(ξ) ∩ κ₄ = j(S_ξ) ∩ κ₄ = S_ξ, it suffices to show that S_ξ is stationary not only in V(P₃ * P₄|ξ) but also in M((j(P₃))* (j(P₄)|j(ξ))).

Firstly M(P₃ * P₄|ξ) ⊆ V(P₃ * P₄|ξ), and so S_ξ is stationary in M(P₃ * P₄|ξ). Secondly, j(P₄) is N₄ - closed, and by Lemma 2.6, M(j(P₃)) is an N₁ - closed forcing extension of M(P₃ * P₄|ξ), and so the proof is completed by application of the following lemma (taking κ = N₀ or N₁, λ = N₄).

**Lemma 2.8** Let κ < λ be regular cardinals and assume that for all α < λ and all β < κ, αβ < λ. Let Q be a κ - closed forcing and S a stationary subset of λ of ordinals of cofinality κ. Then Q|| S is stationary.

This lemma is due to Baumgartner and we include the proof for lack of reference.

**Proof of Lemma 2.8.** Let q be a condition and let Ĉ be a Q - name for a closed unbounded subset of λ. We shall find ȶ ≤ q and γ ∈ S such that ȶ|| γ ∈ Ĉ. Let M be a transitive set such that M is a model of enough set theory, is closed under < κ - sequences and such that M ⊇ λ, q ∈ M, Q ∈ M, Ĉ ∈ M. Let ⟨N_γ : γ < λ⟩ be an elementary chain of submodels of M such that each N_γ has size < λ, contains q, Q and Ĉ, N_γ ∩ λ is an ordinal, and N_γ⁺ contains all < κ - sequences in N_γ. Since S is stationary, there exists a γ ∈ S such that N_γ ∩ λ = γ. As cfγ = κ, N = N_γ is closed under < κ - sequences.

Let {γ_ξ : ξ < κ} be an increasing sequence with limit γ. We construct a descending sequence {q_ξ : ξ < κ} of conditions such that q₀ = q, such that for all ξ < κ, q_ξ ∈ N and for some β_ξ ∈ N greater than γ_ξ, q_ξ⁺|| β_ξ ∈ Ĉ. At successor stages, q_ξ⁺ exists because in N, q_ξ forces that Ĉ is unbounded. At limit stages η < κ, the η - sequence ⟨q_ξ : ξ < η⟩ is in N and has a lower bound in N because N |= Q is κ - closed.
Since $Q$ is $\kappa$-closed, the sequence $\langle q_\xi : \xi < \kappa \rangle$ has a lower bound $\bar{q}$, and because of the $\beta$'s, $\bar{q}$ forces that $\bar{C}$ is unbounded in $\gamma$. Therefore $\bar{q} \parallel \gamma \in \bar{C}$. \hfill \qed

3. Negative results.

We shall now present several negative results on the structure of the relation $S < T$ below $\aleph_\omega$. With the exception of the proof of Proposition 1.6, we state the results for the particular case of reflection of subsets of $S^3_0$ in $S^3_1$, but the results generalize easily to other cardinalities and other cofinalities.

The first result uses a simple calculation (as in Proposition 1.1):

**Proposition 3.1.** For any $\aleph_3$ stationary sets $A_\alpha \subseteq S^3_1$, $\alpha < \omega_3$, there exists a stationary set $S \subseteq S^3_0$ such that $S \not< A_\alpha$ for all $\alpha$.

**Proof.** Let $A_\alpha$, $\alpha < \omega_3$, be stationary subsets of $S^3_1$. By [3], there exist $\aleph_4$ almost disjoint stationary subsets of $S^3_0$; let $S_i$, $i < \omega_4$, be such sets. Assuming that each $S_i$ reflects fully in some $A_\alpha(i)$, we can find $\aleph_4$ of them that reflect fully in the same $A_\alpha$. Take any $\aleph_2$ of them and reduce each by a nonstationary set to get $\aleph_2$ pairwise disjoint stationary subsets $\{T_\xi : \xi < \omega_2\}$ of $S^3_0$, such that each of them reflects fully in $A_\alpha$. Hence there are clubs $C_\xi$, $\xi < \omega_2$, such that $Tr(T_\xi) \supseteq A_\alpha \cap C_\xi$ for every $\xi$. Let $\gamma \in \bigcap_{\xi < \omega_2} C_\xi \cap A_\alpha$. Then every $T_\xi$ reflects at $\gamma$, and so $\gamma$ has $\aleph_2$ pairwise disjoint stationary subsets $\{T_\xi \cap \gamma : \xi < \omega_2\}$. This is a contradiction because $\gamma$ has a closed unbounded subset of size $\text{cf} \gamma = \aleph_1$. \hfill \qed

The next result uses the fact that under GCH there exists a $\lozenge$-sequence for $S^3_1$.

**Proposition 3.2.** (GCH) There exists a stationary set $A \subseteq S^3_1$ that is not the trace of any $S \in S^3_0$; precisely: for every $S \subseteq S^3_0$ the set $A \Delta (Tr(S) \cap S^3_1)$ is stationary.

**Proof.** Let $\langle S_\gamma : \gamma \in S^3_1 \rangle$ be a $\lozenge$-sequence for $S^3_1$; it has the property that for every set $S \subseteq \omega_3$, the set $D(S) = \{ \gamma \in S^3_1 : S \cap \gamma = S_\gamma \}$ is stationary. Let

$$A = \{ \gamma \in S^3_1 : S_\gamma \text{ is nonstationary} \}.$$

The set $A$ is stationary because $A \supseteq D(\emptyset)$. If $S$ is any stationary subset of $S^3_0$, then for every $\gamma$ in the stationary set $D(S)$, $\gamma \in A$ if and only if $\gamma \notin Tr(S)$, and so $D(S) \subseteq A \Delta Tr(S)$. \hfill \qed

The remaining negative results use the following theorem of Shelah which proves the existence of sets with the “square property”.

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Theorem ([4], Lemma 4.2). Let $1 \leq k \leq n - 2$. The set $S^n_k$ is the union of $\aleph_{n-1}$ stationary sets $A$, each having the following property. There exists a collection $\{C_\gamma : \gamma \in A\}$ (a “square sequence for $A$”) such that for each $\gamma \in A$, $C_\gamma$ is a club subset of $\gamma$ of order type $\omega_k$, consisting of limit ordinals of cofinality $< \omega_k$, and such that for all $\gamma_1, \gamma_2 \in A$ and all $\alpha$, if $\alpha \in C_{\gamma_1} \cap C_{\gamma_2}$ then $C_{\gamma_1} \cap \alpha = C_{\gamma_2} \cap \alpha$.

Square sequences can be used to construct a number of counterexamples. For instance, if $S_n, n < \omega$, are $\aleph_0$ stationary subsets of $S_3^0$, then $\text{Tr}(\bigcup_{n=0}^{\infty} S_n) = \bigcup_{n=0}^{\infty} S_n$.

Proposition 3.3. There is a stationary set $A \subseteq S_3^1$ and stationary subsets $S_i, i < \omega_1$, of $S_3^0$ such that $\text{Tr}(S_i) \cap A = \emptyset$ for each $i$ but $\text{Tr}(\bigcup_{i<\omega_1} S_i) \supseteq A$.

Proof. Let $A$ be a stationary subset of $S_3^1$ with a square sequence $\{C_\gamma : \gamma \in A\}$, and let $S = \bigcup_{\gamma \in A} C_\gamma$. Clearly, $S \subseteq S_3^0$ is stationary, and $\text{Tr}(S) \supseteq A$. For each $\xi < \omega_1$, let

$$S_\xi = \{\alpha \in S : \text{order type } (C_\gamma \cap \alpha) = \xi\}$$

(this is independent of the choice of $\gamma \in A$). For every $\gamma \in S$ and every $\xi < \omega_1$, the set $S_\xi \cap C_\gamma$ has exactly one element, and so $S_\xi$ does not reflect at $\gamma$. It is easy to see that $\aleph_1$ of the sets $S_\xi$ are stationary. [The definition of $S_\xi$ is a well known trick]

The argument used in the above proof establishes the following:

Proposition 3.4. If a stationary set $A \subseteq S^n_m$ has a square sequence and if $k < m$ then there exists a stationary $S \subseteq S^n_k$ that does not reflect at any $\gamma \in A$.

Proof of Proposition 1.6. Let $2 \leq m < n$ and let us assume that (b) fails, i.e. that every stationary set $A \subseteq S^n_{m-1}$ reflects at some $\delta$ of cofinality $\aleph_m$. We shall prove that (a) holds. For each $k < m - 1$ we want a stationary set $S \subseteq S^n_k$ that does not reflect fully in $S^n_{m-1}$. Let $k < m - 1$.

Let $A$ be a stationary subset of $S^n_{m-1}$ that have a square sequence $\{C_\gamma : \gamma \in A\}$. The set $A$ reflects at some $\delta$ of cofinality $\omega_m$. Let $C$ be a club subset of $\delta$ of order type $\omega_m$. Using the isomorphism between $C$ and $\omega_m$, the sequence $\{C_\gamma \cap C : \gamma \in A\}$ becomes a square sequence for a stationary subset $B$ of $S^n_{m-1}$. It follows that there is a stationary subset of $S^n_k$ that does not reflect at any $\gamma \in B$. 

\[
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\]
The last counterexample also uses a square sequence.

**Proposition 3.5.** (GCH) There is a stationary set $A \subseteq S^1_1$ and $\aleph_4$ stationary sets $S_i \subseteq S^1_0$ such that the sets $\{Tr(S_i) \cap A : i < \omega_4\}$ are stationary and pairwise almost disjoint.

**Proof.** Let $A$ be a stationary subset of $S^1_1$ with a square sequence $(C_\gamma : \gamma \in A)$, and let $S = \bigcup \{ C_\gamma : \gamma \in A \}$. Let $\{f_i : i < \omega_1\}$ be regressive functions on $S^1_0 \cup S^1_1$ with the property that for any two $f_i, f_j$, the set $\{ \alpha : f_i(\alpha) = f_j(\alpha) \}$ is nonstationary (such a family exists by [3]). For each $i$ and each $\gamma \in A$, the function $f_i$ is regressive on $C_\gamma$ and so there is some $\eta = \eta(i, \gamma) < \gamma$ such that $\{ \alpha \in C_\gamma : f_i(\alpha) < \eta \}$ is stationary. Let $T_\gamma \subseteq \omega_1$ be the stationary set $\{ o.t.(C_\gamma \cap \alpha) : f_i(\alpha) < \eta \}$ and let $H_{i, \gamma}$ be the function on $T_\gamma$ (with values $< \omega_1$) defined by $H(\xi) = f_i(\xi^{th} \text{ element of } C_\gamma)$. For each $i$, the function on $A$ that to each $\gamma$ assigns $(T_{i, \gamma}, H_{i, \gamma})$ is regressive, and so constant = $(T_i, H_i)$ on a stationary set. By a counting argument, $(T_i, H_i)$ is the same for $\aleph_4$'s; so w.l.o.g. we assume that they are the same $(T, H)$ for all $i$.

Now we let, for each $i$,

$$A_i = \{ \gamma \in A : (\forall \alpha \in C_\gamma) \text{ if } \xi = o.t.(C_\gamma \cap \alpha) \in T \text{ then } f_i(\alpha) = H(\xi) \}$$

$$S_i = \{ \alpha \in S : o.t.(C_\gamma \cap \alpha) \in T \text{ and } (\forall \beta \leq \alpha, \beta \in C_\gamma) \text{ if } \xi = o.t.(C_\gamma \cap \beta) \in T \text{ then } f_i(\beta) = H(\xi) \}$$

By the definition of $T$ and $H$, each $A_i$ is a stationary set, and each $S_i$ reflects at every point of $A_i$. We claim that if $\gamma \in A$ and $S_i \cap \gamma$ is stationary then $\gamma \in A_i$. So let $\gamma \in A$ be such that $S_i \cap \gamma$ is stationary. Let $\xi \in T$ and let $\alpha$ be the $\xi^{th}$ element of $C_\gamma$; we need to show that $f_i(\alpha) = H(\xi)$. As $S_i \cap \gamma$ is stationary, there exists a $\beta \in S_i \cap C_\gamma$ greater than $\alpha$. By the definition of $S_i$, $f_i(\alpha) = H(\xi)$. Thus $\gamma \in A_i$, and $A_i = A \cap Tr(S_i)$.

Finally, we show that the sets $A_i$ are pairwise almost disjoint. Let $C$ be a club disjoint from the set $\{ \alpha : f_i(\alpha) = f_j(\alpha) \}$. We claim that the set $C'$ of all limit points of $C$ is disjoint from $A_i \cap A_j$. If $\gamma \in C'$ then $C \cap C'$ is a club in $\gamma$, and so is $C \cap C_\gamma$. Since $T$ is stationary in $\omega_1$, there is a $\xi \in T$ such that the $\xi^{th}$ element $\alpha$ of $C_\gamma$ is in $C$, and therefore $f_i(\alpha) \neq f_j(\alpha)$; it follows that $\gamma$ cannot be both in $A_i$ and in $A_j$. □


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