CATEGORICITY FOR ABSTRACT CLASSES WITH AMALGAMATION

SH394

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Abstract. Let $\mathfrak{K}$ be an abstract elementary class with amalgamation, and Lowenheim Skolem number $\text{LS}(\mathfrak{K})$. We prove that for a suitable Hanf number χ_0 if $\chi_0 < \lambda_0 \leq \lambda_1$, and $\mathfrak{K}$ is categorical in λ_1^+ then it is categorical in λ_0.

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Annotated Content

§0 Introduction

We try to find something on

$$\mathrm{Cat}_K = \{\lambda : K \text{ categorical in } \lambda\}$$

for $\mathfrak{K}$ an abstract elementary class with amalgamation (see 0.1 below).
The Łos conjecture = Morley theorem deals with the case where K is the class of models of a countable first order theory T. See [Sh:c] for more on first order theories. What for T a theory in an infinitary language? (For a theory T, K is the class $K_T = \{M : M \models T\}$ we may write Cat_T instead of $\mathrm{Cat}_{K_T} = \mathrm{Cat}_K$). Keisler gets what can be gotten from Morley's proof on $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}$. Then see [Sh 48] on categoricity in $\aleph_1$ for $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}$ and even $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}(\mathbf{Q})$, and [Sh 87a], [Sh 87b] on the behaviour in the $\aleph_n$'s. Makkai Shelah [MaSh 285] proved: if $T \subseteq \mathbb{L}_{\kappa,\aleph_0}, \kappa$ a compact cardinal then $\mathrm{Cat}_T \cap \{\mu^+ : \mu \ge \beth_{(2^\kappa+|T|)^+}\}$ is empty or is $\{\mu^+ : \mu \ge \beth_{(2^\kappa+|T|)^+}\}$ (it relies on some developments from [Sh 300] but is self-contained).

It was then reasonable to deal with weakening the requirement on κ to measurability. Kolman Shelah [KlSh 362] proved that if $\lambda \in \mathrm{Cat}_T$ where $T \subseteq \mathbb{L}_{\kappa,\omega}(\tau), \lambda \ge \beth_{(2^\chi)^+}$ where $\chi = |\tau| + \kappa, \kappa$ measurable, <u>then</u> (after cosmetic changes), for the right $\le_T$ the class $\{M : M \models T, \|M\| < \lambda\}$ has amalgamation and joint embedding property. This is continued in [Sh 472] which gets results on categoricity parallel to the one in [MaSh 285] for the "downward" implication.

In [Sh 88] we deal with abstract elementary classes (they include models of $T \subseteq \mathbb{L}_{\kappa,\aleph_0}$, see 0.1), prove a representation theorem (see 0.5 below), and investigate categoricity in $\aleph_1$ (and having models in $\aleph_2$, limit models, realizing and materializing types). Unfortunately, we do not have anything interesting to say here on this context. So we add amalgamation and the joint embedding properties thus getting to the framework of Jonsson [J] (they are the ones needed to construct homogeneous universal models). So this context is more narrow than the ones discussed above, but we do not use large cardinals. We concentrate here, for categoricity on λ, on the case "λ is successor $> \beth_{(2^{\mathrm{LS}(\mathfrak{K})})^+}$". See for later works [Sh 576], [Sh 600], [ShVi 635] and [Va02].
We quote the basics from [Sh 88] (or [Sh 576]).

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We thank John Baldwin for complaining repeatedly during 2003/2004 on §8,§9 and Alex Usvyatsov for help in proofreading their revisions, so 8.9 (and beginning of §9) were changed; elsewhere the changes are small.

0.1 Definition. $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an abstract elementary class <u>if</u> for some vocabulary $\tau = \tau(K) = \tau(\mathfrak{K}), K$ is a class of $\tau(K)$-models, and the following axioms hold.

$Ax0$: The holding of $M \in K, N \leq_{\mathfrak{K}} M$ depends on N, M only up to isomorphism i.e. $[M \in K, M \cong N \Rightarrow N \in K]$, and [if $N \leq_{\mathfrak{K}} M$ and f is an isomorphism from M onto the τ-model M' mapping N onto N' <u>then</u> $N' \leq_{\mathfrak{K}} M'$].

AxI: If $M \leq_{\mathfrak{K}} N$ then $M \subseteq N$ (i.e. M is a submodel of N).

$AxII$: $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2$ implies $M_0 \leq_{\mathfrak{K}} M_2$ and $M \leq_{\mathfrak{K}} M$ for $M \in K$.

$AxIII$: If λ is a regular cardinal, M_i (for $i < \lambda$) is a $\leq_{\mathfrak{K}}$-increasing (i.e. $i < j < \lambda$ implies $M_i \leq_{\mathfrak{K}} M_j$) and continuous (i.e. for limit ordinal $\delta < \lambda$ we have $M_\delta = \bigcup\limits_{i<\delta} M_i$) <u>then</u> $M_0 \leq_{\mathfrak{K}} \bigcup\limits_{i<\lambda} M_i \in \mathfrak{K}$.

$AxIV$: If λ is a regular cardinal, $M_i (i < \lambda)$ is $\leq_{\mathfrak{K}}$-increasing continuous and $M_i \leq_{\mathfrak{K}} N$ <u>then</u> $\bigcup\limits_{i<\lambda} M_i \leq_{\mathfrak{K}} N$.

AxV: If $M_0 \subseteq M_1$ and $M_\ell \leq_{\mathfrak{K}} N$ for $\ell = 0, 1$, <u>then</u> $M_0 \leq_{\mathfrak{K}} M_1$.

$AxVI$: LS($\mathfrak{K}$) exists[1]; see below Definition 0.3.

0.2 Definition. 1) $K_\mu =: \{M \in K : \|M\| = \mu\}$.
2) We say h is a $\leq_{\mathfrak{K}}$-embedding of M into N is for some $M' \leq_{\mathfrak{K}} N, h$ is an isomorphism from M onto M'.
3) We say that $\mathfrak{K}$ has amalgamation (or the amalgamation property) <u>when</u> if for any models $M_\ell \in \mathfrak{K}$ for $\ell = 0, 1, 2$ and $\leq_{\mathfrak{K}}$-embeddings h_ℓ of M_0 into M_ℓ for $\ell = 1, 2$ there are M_3, g_1, g_2 such that $M_3 \in \mathfrak{K}$ and g_ℓ is a $\leq_{\mathfrak{K}}$-embedding of M_ℓ into M_3 for $\ell = 1, 2$ and $g_1 \circ h_1 = g_2 \circ h_2$.
4) $\mathfrak{K}$ has the λ-amalgamation means that above $M_\ell \in \mathfrak{K}_\lambda$ for $\ell = 0, 1, 2$ and $\ell = 3$.
5) $\mathfrak{K}$ has the point embedding property, JEP means that for any $M_1, M_2 \in \mathfrak{K}$ there is $M_3 \in \mathfrak{K}$ then $\leq_{\mathfrak{K}}$-embedding g_1, g_2 of M_1, M_2 into M respectively. The λ-joint embedding property, JEP_λ means that above we assume $M_1, M_2 \in \mathfrak{K}_\lambda$.
6) Let $M <_{\mathfrak{K}} N$ mean $M \leq_{\mathfrak{K}} N \ \& \ N \neq N$.

0.3 Definition. 1) We say that μ is a Lowenheim Skolem number of $\mathfrak{K}$ <u>if</u> $\mu \geq \aleph_0$ and:

$(*)^\mu_K$ for every $M \in K, A \subseteq M, |A| \leq \mu$ there is $M', A \subseteq M' \leq_{\mathfrak{K}} M$ and $\|M'\| \leq \mu$.

[1] We normally assume $M \in \mathfrak{K} \Rightarrow \|M\| \geq LS(\mathfrak{K})$, here there is no loss in it. It is also natural to assume $|\tau(\mathfrak{K})| \leq LS(\mathfrak{K})$ which just means increasing $LS(\mathfrak{K})$.

2) $\mathrm{LS}'(\mathfrak{K}) = \mathrm{Min}\{\mu : \mu$ is a Skolem Lowenheim number of $\mathfrak{K}\}$.
3) $\mathrm{LS}(\mathfrak{K}) = \mathrm{LS}'(\mathfrak{K}) + |\tau(K)|$.

0.4 Claim. *1) If I is a directed partial order, $M_t \in K$ for $t \in I$ and $s <_I t \Rightarrow M_s \leq_{\mathfrak{K}} M_t$ then*

(a) $M_s \leq_{\mathfrak{K}} \bigcup_{t \in I} M_t \in K$ *for every* $s \in I$

(b) *if* $(\forall t \in I)[M_t \leq_{\mathfrak{K}} N]$ *then* $\bigcup_{t \in I} M_t \leq_{\mathfrak{K}} N$.

2) If $A \subseteq M \in K, |A| + \mathrm{LS}'(\mathfrak{K}) \leq \mu \leq \|M\|$, then there is $M_1 \leq_{\mathfrak{K}} M$ such that $\|M_1\| = \mu$ and $A \subseteq M_1$.
3) If I is a directed partial order, $M_t \leq N_t \in K$ for $t \in I$ and $s \leq_I t \Rightarrow M_s \leq_{\mathfrak{K}} M_t$ & $N_s \leq_{\mathfrak{K}} N_t$ then $\bigcup_t M_t \leq_{\mathfrak{K}} \bigcup_t N_t$.

0.5 Claim. *Let $\mathfrak{K}$ be an abstract elementary class. Then there are τ^+, Γ such that:*

(a) *τ^+ is a vocabulary extending $\tau(K)$ of cardinality* $\mathrm{LS}(\mathfrak{K})$
(b) *Γ is a set of quantifier free types in τ^+ (each is an m-type for some $m < \omega$)*
(c) *$M \in K$ iff for some τ^+-model M^+ omitting every $p \in \Gamma$ we have $M = M^+ \restriction \tau$*
(d) *$M \leq_{\mathfrak{K}} N$ iff there are τ^+-models M^+, N^+ omitting every $p \in \Gamma$ such that $M^+ \subseteq N^+, M = M^+ \restriction \tau(K)$ and $N = N^+ \restriction \tau(K)$.*
(e) *if $M \leq_{\mathfrak{K}} N$ and M^+ is an expansion of M to a τ^+-model omitting every $p \in \Gamma$ then we can find a τ^+-expansion of N omitting every $p \in \Gamma$ such that $M^+ \subseteq N^+$.*

0.6 Claim. *Assume $\mathfrak{K}$ has a member of cardinality $\geq \beth_{(2^{\mathrm{LS}(\mathfrak{K})})^+}$ (here and elsewhere we can weaken this to: has a model of cardinality $\geq \beth_\alpha$ for every $\alpha < (2^{\mathrm{LS}(\mathfrak{K})})^+$). Then there is Φ proper for linear orders (see [Sh:c, Ch.VII,§2]) such that:*

(a) $|\tau(\Phi)| = \mathrm{LS}(\mathfrak{K})$
(b) *for linear orders $I \subseteq J$ we have* $\mathrm{EM}_\tau(I, \Phi) \leq_{\mathfrak{K}} \mathrm{EM}(J, \Phi) (\in K)$.
(c) $\mathrm{EM}_\tau(I, \Phi)$ *has cardinality* $|I| + \mathrm{LS}(\mathfrak{K})$ *(so $\mathfrak{K}$ has a model in every cardinality $\geq \mathrm{LS}(\mathfrak{K})$).*

Proof. Here see 8.6.

PART 1
§1 The Framework

1.1 Hypothesis.

(a) $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ an abstract elementary class (0.1) so $K_\lambda = \{M \in K : \|M\| = \lambda\}$

(b) $\mathfrak{K}$ has amalgamation and the joint embedding property

(c) K has members of arbitrarily large cardinality, equivalently: K has a member of cardinality at least $\beth_{(2^{\mathrm{LS}(\mathfrak{K})})^+}$.

1.2 Convention. 1) So there is a monster $\mathfrak{C}$ (see [Sh:a, Ch.I,§1] = [Sh:c, Ch.I,§1]).

1.3 Definition. 1) We say K (or $\mathfrak{K}$) is categorical in λ <u>if</u> it has one and only one model of cardinality λ, up to isomorphism.
2) $I(\lambda, K)$ is the number of models in K_λ (i.e., in K of cardinality λ) up to isomorphism.

1.4 Definition. 1) We can define $\mathrm{tp}(\bar a, M, N)$ (when $M \leq_{\mathfrak{K}} N$ and $\bar a \subseteq N$), as $(\bar a, M, N)/E$ where E is the following equivalence relation: $(\bar a^1, M^1, N^1)\, E\, (\bar a^2, M^2, N^2)$ <u>iff</u> $M^\ell \leq_{\mathfrak{K}} N^\ell, \bar a^\ell \in {}^\alpha(N^\ell)$ (for some α but the same for $\ell = 1,2$) and $M^1 = M^2$ and there is $N \in K$ satisfying $M^1 = M^2 \leq_{\mathfrak{K}} N$ and $\leq_{\mathfrak{K}}$-embedding $f^\ell : N^\ell \to N$ over M^ℓ (i.e. $f \restriction M^\ell$ is the identity) for $\ell = 1,2$ satisfying $f^1(\bar a^1) = f^2(\bar a^2)$. We can define $\mathrm{tp}(\bar a, \emptyset, N)$ similarly.[2]
2) We may omit N when $N = \mathfrak{C}$ (see 1.2) and may then write $\frac{\bar a}{M} = \bar a/M = \mathrm{tp}(\bar a, M, \mathfrak{C})$. We define "$N$ is κ-saturated" (when $\kappa > \mathrm{LS}(\mathfrak{K})$) by: if $M \leq_{\mathfrak{K}} N, \|M\| < \kappa$ and $p \in \mathscr{S}^{<\omega}(M)$ (see below) <u>then</u> p is realized in M, i.e. for some $\bar a \subseteq N, p = \mathrm{tp}(\bar a, M, N)$.
3) $\mathscr{S}^\alpha(M) = \{\mathrm{tp}(\bar a, M, N) : \bar a \in {}^\alpha N, M \leq_{\mathfrak{K}} N\}$.
4) $\mathscr{S}(M) = \mathscr{S}^1(M)$ (we could have just as well used $\mathscr{S}^{<\omega}(M) = \bigcup\limits_{n<\omega} \mathscr{S}^n(M)$).
5) If $M_0 \leq_{\mathfrak{K}} M_1$ and $p_\ell \in \mathscr{S}^\alpha(M_\ell)$ for $\ell = 1,2$, <u>then</u> $p_0 = p_1 \restriction M_0$ means that for some $\bar a, N$ we have $M_1 \leq_{\mathfrak{K}} N$ and $\bar a \in {}^\alpha N$ and $p_\ell = \mathrm{tp}(\bar a, M_\ell, N)$ for $\ell = 1,2$. See [Sh 300, Ch.II] or [Sh 576, §0] and see 1.10 below.

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[2] what about $\mathrm{tp}(\bar a, A, N)$? The cumbersomeness is that we end up defining essentially $\mathrm{tp}_*(\bar a \cup A, \emptyset, N)$

1.5 Definition. Let $\mathfrak{K}$ stable in λ mean: $\|M\| \le \lambda \Rightarrow |\mathscr{S}(M)| \le \lambda$ and $\lambda \ge$ LS$(\mathfrak{K})$.

1.6 Convention. If not said otherwise, Φ is as in 0.6.

1.7 Claim. *If K is categorical in λ and $\lambda \ge \text{LS}(\mathfrak{K})$, then*

- (a) *$\mathfrak{K}$ is stable in every μ which satisfies $\text{LS}(\mathfrak{K}) \le \mu < \lambda$, hence*
- (b) *the model $M \in K_\lambda$ is $\text{cf}(\lambda)$-saturated (if $\text{cf}(\lambda) > \text{LS}(\mathfrak{K})$).*

Remark. The first proof below gives more and uses more.

Proof. Like [KlSh 362] but this is immersed with ultrapowers.
First Proof: So[3] let Φ be as in 0.6. Now let I be such that:

- (a) I is a linear order of cardinality λ
- (b) for every $J \subseteq I, |J| = \mu$ there is J_1 satisfying
  - (α) $J \subseteq J_1 \subseteq I$
  - (β) $|J_1| = \mu$
  - (γ) if $I' \subseteq I$ is finite then for some automorphism g of I we have
    $g \restriction J = \text{id}_J$
    $g(I') \subseteq J_1$.

(see [Sh 220, AP]).

Now suppose toward contradiction that $M_0 \in K_\mu, |\mathscr{S}(M_0)| > \mu$, then we can find $M_1 \in K_{\mu^+}$ and $a_i \in M_1$ for $i < \mu^+$ such that $M \le_{\mathfrak{K}} M_1$ and $i < j < \mu^+ \Rightarrow \text{tp}(a_i, M, M_1) \neq \text{tp}(a_j, M, M_1)$. By 1.10(1) below we can find $M_2 \in K_\lambda$ such that $M_1 \le_{\mathfrak{K}} M_2$. Let Φ be as in 0.6. Now M_2 and $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ are both models in K_λ hence are isomorphic, so by renaming equal. So let $a_i = \sigma_i(\bar a_{\bar t_i})$ with $\bar t_i \in {}^{n_i}I$. By the pigeon-hole principle without loss of generality $\sigma_i = \sigma^*, n_i = n^*$ and let $J \subseteq I, |J| \le \mu$ be such that $M \subseteq \text{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ and let J_1 satisfy clauses $(b)(\alpha)(\beta), (\gamma)$ above. For each $i < \mu^+$ there is an automorphism f_i of the linear order I such that $f_i \restriction J = \text{id}_J, f_i(\bar t_i) \subseteq J_1$. So without loss of generality $f_i(\bar t_i) = \bar s$. Now each f_i induces naturally an automorphism $\hat f_i$ of $\text{EM}_{\bar u(\Phi)}(I, \Phi)$, which in particular is an automorphism of $\text{EM}_{\tau(\mathfrak{K})}(I, \Phi) = M_2$. This automorphism is the identity

[3] the proof was written because of a requisition of Rami Grossberg!

on $\mathrm{EM}_{\tau(\Phi)}(J,\Phi)$ hence on M. Also $\hat{f}_i(a_i) = \hat{f}_i(\sigma^*(\bar{t}_1),\dots,\sigma^*(f_i(\bar{t}_i)) = \sigma^*(\bar{s})$, so is the same. Clearly $\mathrm{tp}(a_i,M,M_1) = \mathrm{tp}(a_i,M,M_2) = \mathrm{tp}(\hat{f}_i(a_i),\hat{f}_1(M),M_2) = \mathrm{tp}(\sigma^*(\bar{s}),M,M_2)$, as this holds for every $i<\mu^+$ we have gotten a contradiction.

Alternative proof: Let Φ be as in 0.6. Assume $N = \mathrm{EM}_\tau(\lambda,\Phi)$ and $M \le_{\mathfrak{K}} N, M \in K_\mu$. We can find $u \subseteq \lambda, |u| = \mu$ such that $M \le_{\mathfrak{K}} \mathrm{EM}_\tau(u,\Phi)$. Clearly it is enough to show

(∗) if $\bar{b}_\ell = \sigma(\dots,\bar{a}_{\alpha(\ell,k)},\dots)_{k<k(*)} \in N$ for $\ell = 1,2$ as $\alpha(\ell,0) < \dots < \alpha(\ell,k(*))-1$ and

$$(\forall \beta \in u)(\forall k < k(*)))[(\alpha(\ell,k) < \beta \equiv \alpha(2,k) < \beta) \wedge (\alpha(1,k)) > \beta \equiv \alpha(2,k) > \beta)]$$

then $\mathrm{tp}(b_1,M,N) = \mathrm{tp}(b_2,M,N)$.

But (∗) is immediate: let $N_i = \mathrm{EM}_\tau(u_\ell,\Phi)$ where $u_i = u \vee \{\alpha(\ell,k) : k < k(*)\}$ for $\ell = 1,2$, so there is an isomorphism f from $\mathrm{EM}(u_1,\Phi)$ onto $\mathrm{EM}(u_2,\Phi)$ which is the identity on $\{\bar{a}_\beta : \beta \in u\}$ and maps $\bar{a}_{\alpha(1,k)}$ to $\bar{a}_{\alpha(2,k)}$ for $k < k(*)$. So f can be extended to an automorphism f^+ of $\mathfrak{C}, f^+ \restriction M = \mathrm{id}_\mu, f^+(b_1) = b_2$.

So if $M \in K_\mu, |\mathscr{S}(M)| > \mu$, there is M^+ such that $M \le_{\mathfrak{K}} M^+ \in K_{\mu^+}$ and $|\{\mathrm{tp}(b,M,M^+) : b \in M^+\}| = \mu^+$. Let $N^+ \in K_\lambda$ be such that $M^+ \le_{\mathfrak{K}} N^+ \in K_\lambda$. So there is an isomorphism g from N^+ onto N. Now $g(M), g(M^+)$ contradicts what we have proved above.

Similarly $\mathrm{LS}(\mathfrak{K}) \le \mu = \mu < \lambda, M \in K_\mu \Rightarrow |\mathscr{S}^\theta(M)| \le \mu$. So we have proved clause (a).

Now for proving clause (b); it just follows from clause (a). $\square_{1.7}$

1.8 Definition. 1) For $\mu \ge \mathrm{LS}(\mathfrak{K}), E_\mu = E^1_\mu[\mathfrak{K}], E_\mu$ is the following relation,

$p\ E_\mu\ q$ iff for some $M \in K, m < \omega$ we have

$$p,q \in \mathscr{S}^m(M) \text{ and } [N \le_{\mathfrak{K}} M \ \& \ \|N\| \le \mu \Rightarrow p \restriction N = q \restriction N].$$

2) We say $p \in \mathscr{S}^m(M)$ is μ-local if p/E_μ is a singleton.
3) We say $\mathfrak{K}$ is μ-local if every $p \in \mathscr{S}^{<\omega}(M)$ is μ-local.
4) We say "c realizes p/E_μ in M^*" if $M \le_{\mathfrak{K}} M^*, \bar{c} \in M^*$ and $[N \le_{\mathfrak{K}} M \ \& \ \|N\| \le \mu \Rightarrow \mathrm{tp}(c,N,M^*) = p \restriction N]$.

1.9 Remark. 0) Obviously E_μ is an equivalence relation.
1) In previous contexts $E_{\mathrm{LS}(\mathfrak{K})}$ is equality, e.g. the axioms of NF in

[Sh 300, Ch.II,§1] implies it; but here we do not know — this is the main difficulty. We may look at this as our bad luck, or inversely, a place to encounter some of the difficulty of dealing with $\mathbb{L}_{\mu,\omega}$ (in which our context is included).
2) Note that the μ-local does not imply μ-compactness which means: if $m < \omega, M \in \mathfrak{K}$ and $\bar{p} = \langle p_N : N \leq_{\mathfrak{K}} M, \|N\| \leq \mu \rangle, p_N \in \mathscr{S}^m(N)$ and $[N_1 \leq_{\mathfrak{K}} N_2 \leq_{\mathfrak{K}} M \;\&\; \|N_2\| \leq \mu \Rightarrow p_{N_1} = p_{N_2} \restriction N_1]$ then there is $p \in \mathscr{S}^m(M)$ such that $N \leq_{\mathfrak{K}} M \;\&\; \|N\| \leq \mu \Rightarrow p \restriction N = p_N$.

1.10 Claim. *1) There is no maximal member in K, in fact for every $M \in K$ there is $N, M <_{\mathfrak{K}} N \in K, \|N\| \leq \|M\| + \mathrm{LS}(\mathfrak{K})$, hence for every $\lambda \geq \|M\| + \mathrm{LS}(\mathfrak{K})$ there is $N \in K_\lambda$ such that $M <_{\mathfrak{K}} N \in K_\lambda$.*
2) If $p_2 \in \mathscr{S}^\alpha(M_2)$ and $M_1 \leq_{\mathfrak{K}} M_2 \in K$ then for one and only one $p_1 \in \mathscr{S}^\alpha(M_1)$ we have $p_1 = p_2 \restriction M_1$.
3) If $p_1 \in \mathscr{S}^\alpha(M_1)$ and $M_1 \leq_{\mathfrak{K}} M_2 \in K$ then for some $p_2 \in \mathscr{S}^\alpha(M_2)$ we have $p_1 = p_2 \restriction M_1$.
4) If $M_1 \leq_{\mathfrak{K}} M_2 \leq_{\mathfrak{K}} M_3$ and $p_\ell \in \mathscr{S}^\alpha(M_\ell)$ for $\ell = 1,2,3$ then $p_3 \restriction M_2 = p_2 \;\&\; p_2 \restriction M_1 = p_1 \Rightarrow p_3 \restriction M_1 = p_1$.

Proof. 1) Immediate by clause (c) of the hypothesis 1.1 and claim 0.6.
2) Straightforward.
3) By amalgamation.
4) Check. $\square_{1.10}$

1.11 Claim. *If $\langle M_i : i \leq \omega \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous and $p_n \in \mathscr{S}^\alpha(M_n)$ and $p_n = p_{n+1} \restriction M_n$ for $n < \omega$, then there is $p_\omega \in \mathscr{S}^\alpha(M_\omega)$ such that $n < \omega \Rightarrow p_\omega \restriction M_n = p_n$.*

Proof. Let N_0 be such that $M_0 \leq_{\mathfrak{K}} N_0$ and $p_0 = \mathrm{tp}(a, M_0, N_0)$. We now choose (N_n, h_n) by induction on n such that:

⊛ (a) $N_n \in \mathfrak{K}$ is $\leq_{\mathfrak{K}}$-increasing
  (b) h_n is a $\leq_{\mathfrak{K}}$-embedding of M_n into M_n
  (c) h_n increases with n
  (d) $\mathrm{tp}(a, h_n(M_n), N_n)$ is $h_n(p_n)$.

For $n = 0, (N_n, \mathrm{id}_{M_0})$ are as required.
For $n+1$, use $p_{n+1} \restriction M_n = p_n$ and straight chasing diagrams. $\square_{1.11}$

1.12 Remark. In 1.11 we do not claim uniqueness and do not claim existence replacing ω for δ of uncountable cofinality. In general not true [Saharon add]. Compare with 8.5, 9.2.

§2 Variant of Saturated

2.1 Definition. Assuming $\mathfrak{K}$ stable in μ and α is an ordinal $< \mu^+$, $\mu \times \alpha$ means ordinal product.
1) $M <^\circ_{\mu,\alpha} N$ if: $M \in K_\mu, N \in K_\mu, M \le_{\mathfrak{K}} N$ and there is a $\le_{\mathfrak{K}}$-increasing sequence $\bar M = \langle M_i : i \le \mu \times \alpha \rangle$ which is continuous, $M_0 = M, M_{\mu \times \alpha} \le_{\mathfrak{K}} N$ and every $p \in \mathscr{S}^1(M_i)$ is realized in M_{i+1}.
2) We say $M <^1_{\mu,\alpha} N$ iff $M \in K_\mu, N \in K_\mu, M \le_{\mathfrak{K}} N$ and there is a $\le_{\mathfrak{K}}$-increasing sequence $\bar M = \langle M_i : i \le \mu \times \alpha \rangle, M_0 = M, M_{\mu \times \alpha} = N$ and every $p \in \mathscr{S}^1(M_i)$ is realized in M_{i+1}.
3) If $\alpha = 1$, we may omit it.

2.2 Lemma. *Assume $\mathfrak{K}$ stable in μ and $\alpha < \mu^+$.*
0) If $\ell \in \{0,1\}$ and $\alpha_1 < \alpha_2 < \mu^+$ and there is $b \subseteq \alpha_2$ such that $\mathrm{otp}(b) = \alpha_1$ and $[\ell = 1 \Rightarrow b$ unbounded in $\alpha_2]$ then $<^\ell_{\mu,\alpha_2} \subseteq <^\ell_{\mu,\alpha_1}$.
1) If $M \in K_\mu$, then for some N we have $M <^\circ_{\mu,\alpha} N$ and for some $N, M <^1_{\mu,\alpha} N$.
2) (a) If $M \in K_\mu, M \le_{\mathfrak{K}} M' \le^\ell_{\mu,\alpha} N$ then $M \le^\ell_{\mu,\alpha} N$.
(b) If $M \in K_\mu, M \le_{\mathfrak{K}} M' \le^\ell_{\mu,\alpha} N' \le_{\mathfrak{K}} N \in K_\mu$ then $M \le^\circ_{\mu,\alpha} N$ (so $\le^1_{\mu,\alpha} \subseteq \le^\circ_{\mu,\alpha}$).
3) If $\langle M_i : i < \alpha \rangle$ is $\le_{\mathfrak{K}}$-increasing sequence in $K_\mu, M_i \le^\circ_\mu M_{i+1}$ and $\alpha < \mu^+$ is a limit ordinal, then $M_0 \le^1_{\mu,\alpha} \bigcup\limits_{i<\alpha} M_i$.
4) If $M \le^\circ_\mu N$ then:

(a) any $M' \in K_\mu$ can be $\le_{\mathfrak{K}}$-embedded into N (here we can weaken $\|M'\| = \mu$ to $\|M'\| \le \mu$)
(b) If $M' \le_{\mathfrak{K}} N' \in K_{\le \mu}$, h is a $\le_{\mathfrak{K}}$-embedding of M' into M then h can be extended to a $\le_{\mathfrak{K}}$-embedding of N' into N.

5) If $M^\ell \le^1_{\mu,\kappa} N^\ell$ for $\ell = 1,2$, h an isomorphism from M^1 into [onto] M^2 then h can be extended to an isomorphism from N^1 into [onto] N^2.
6) If $M \le^1_{\mu,\kappa} N^\ell$ for $\ell = 1,2$ then $N^1 \cong N^2$ (even over M).
7) If $M \le^\circ_{\mu,\kappa} N$, $M \le_{\mathfrak{K}} M' \in K_\mu$ then M' can be $\le_{\mathfrak{K}}$-embedded into N over M.
8) If $\mu \ge \kappa > \mathrm{LS}(\mathfrak{K})$ and $M <^1_{\mu,\kappa} N$ then N is $\mathrm{cf}(\kappa)$-saturated.

Proof. See [Sh 300, Ch.II,3.10,p.319] and around, we shall explain and prove part (8) below.

2.3 Discussion: There (in [Sh 300, Ch.II,3.6]) the main point was that for $\kappa > \mathrm{LS}(\mathfrak{K})$, the notions "$\kappa$-homogeneous universal" and κ-saturation (i.e., every "small" 1-type is realized) are equivalent.

Not hard, still [Sh 300, Ch.II,3.6] was a surprise to some (including myself). In first order the equivalence saturated $\equiv$ homogeneous universal for $\prec$ seemed, with a posteriori wisdom, natural as the homogeneity used was anyhow for sequences of elements realizing the same first order formulas so (forgetting about the models) to some extent this seemed natural; i.e. asking this for any type of 1-element was very natural.

But here, types of 1-element are really meaningful only over a model. So it seems that if over any small submodel every type of 1-element is realized (say in $\mathfrak{A}$) and we would like to embed $N \geq_{\mathfrak{K}} N_0, N_0 \leq_{\mathfrak{K}} \mathfrak{A}$ into $\mathfrak{A}$ over N_0, we encounter the following problem: we cannot continue this as after ω stages, as we get a set which is not a model (if $\mathrm{LS}(\mathfrak{K}) > \aleph_0$ this absolutely necessarily fails; and if $\mathrm{LS}(\mathfrak{K}) = \aleph_0$ at best the situation is as in [Sh 87a]).

This explains a natural preconception making you not believe; i.e. psychological barrier to prove. It does not mean that the proof is hard.
Note that in [Sh 48], [Sh 87a], [Sh 87b] and even [Sh 88] the types are a still set of formulas and essentially (after cleaning) first order.

2.4 Remark. Note that $\leq^1_{\mu,\kappa}, \kappa$ regular are the interesting ones as $\leq^1_{\mu,\delta} = \leq^1_{\mu,\mathrm{cf}(\delta)}$.
[Why? For limit ordinal $\delta < \mu^+$, $\leq^1_{\mu,\delta} \subseteq \leq^1_{\mu,\mathrm{cf}(\delta)}$ by 2.2(0) and equality holds by the uniqueness 2.2(6).]
Still $\leq^0_{\mu,\kappa}$ is enough for universality (2.2(4)) and is natural, $\leq^1_{\mu,\kappa}$ is natural for uniqueness. BUT $<^1_{\mu,\aleph_0} = <^1_{\mu,\aleph_1}$ can be proved only under categoricity (or something like superstability assumptions). For understanding this we may consider a first order T stable in μ. Then, $M <^1_{\mu,\kappa} N$ is equivalent to:

$\circledast$ (i) $\|M\| = \|N\| = \mu, M, N \models T$

(ii) and there is $\langle M_i : i \leq \kappa \rangle$ which is $\prec$-increasing continuous such that

(α) $M_0 = M$ $\quad$ $M_\kappa = N$

(β) $(M_{i+1}, c)_{c \in M_i}$ is saturated.

Question: Now, is N saturated when $M <^1_{\mu,\kappa} N$?

Answer: It is saturated iff $\mathrm{cf}(\kappa) \geq \kappa_r(T)$. See [Sh:c, Ch.III,§3].

This is similar to S-limit models for S a stationary subset of μ^+, see [Sh 88]. That is, if $M <^1_{\mu,\kappa} N$ then N is a $\{\delta < \lambda^+ : \mathrm{cf}(\delta) = \kappa\}$-limit model. If T is first order stable in μ, then there are such models for every $\kappa \leq \mu$; for $\alpha = 1$ we get the saturated model. Of course, N is a superlimit model (see [Sh 88]) if $\kappa = \text{cf}(\kappa) \leq \mu \Rightarrow (\exists M)(M \leq^1_{\mu,\kappa} N)$.

Before we prove 2.2(8), recall

2.5 Definition. $M \in K$ is κ-saturated if $\kappa > \text{LS}(\mathfrak{K})$ and:
$N \leq_{\mathfrak{K}} M, \|N\| < \kappa, p \in \mathscr{S}^1(N) \Rightarrow p$ realized in M.

Proof of 2.2(8).
Statement: If $M <^1_{\mu,\kappa} N$ (κ regular) then N is κ-saturated.
Note: if $\kappa \leq \text{LS}(\mathfrak{K})$ the conclusion is essentially empty, but there is no need for the assumption "$\kappa > \text{LS}(\mathfrak{K})$".

Proof. Let $\bar{M} = \langle M_i : i \leq \mu \times \kappa \rangle$ witness $M \leq^1_{\mu,\kappa} N$ so $M_0 = M, M_{\mu\times\kappa} = N, M_i \leq_{\mathfrak{K}}$-increasing continuous and every $p \in \mathscr{S}(M_i)$ is realized in M_{i+1}.

Assume

(∗) $N' \leq_{\mathfrak{K}} N, \|N'\| < \kappa, p \in \mathscr{S}(N')$.

We should prove that "p is realized in N". But $\langle M_i : i \leq \mu \times \kappa \rangle$ is increasing continuous

$$\text{cf}(\mu \times \kappa) = \kappa > \|N'\|$$

so $N' \leq_{\mathfrak{K}} M_{\mu\times\kappa} = \bigcup_{i<\mu\times\kappa} M_i$ implies there is $i(*) < \mu \times \kappa$, such that $N' \subseteq M_{i(*)}$ hence by Axiom V we have $N' \leq_{\mathfrak{K}} M_{i(*)}$. So p has (by amalgamation, i.e., 1.10(3)) an extension $p^* \in \mathscr{S}(M_{i(*)})$, i.e., $p^* \restriction N = p$. By the choice of $\langle M_i : i \leq \mu \times \kappa \rangle, p^*$ is realized in $M_{i(*)+1}$ so in $M_{\mu\times\kappa} = N$ and the same element realizes p by the definition of μ we are done. $\square_{2.2}$

Comment: Hence length μ (instead of $\mu \times \kappa$) suffices.

But for the uniqueness seemingly[4] it does not. See 2.2(4) + (5).

Comment: The definitions of $\leq^0_{\mu,\kappa}, \leq^1_{\mu,\kappa}$ are also essentially taken from [Sh 300, Ch.II,3.10]. We need the intermediate steps to construct models so we have to have μ of them in order to deal with all the elements.

[4]but see [Sh 600, §4] it suffices

2.6 Claim. *If K is categorical in $\lambda, M \in K_\lambda$ and $\mathrm{cf}(\lambda) > \mu$ then: if $N <_{\mathfrak{K}} M \in K_\lambda, N \in K_\mu, N' <_{\mathfrak{K}} M, h$ an isomorphism from N onto N', then h can be extended to an automorphism of M.*

Proof. By 1.4 we have $\mathrm{LS}(\mathfrak{K}) \le \mu < \lambda \Rightarrow \mathfrak{K}$ stable in μ. We can find $\langle M_i : i < \lambda \rangle$ which is $<_{\mathfrak{K}}$-increasing continuous, $\|M_i\| = |i| + \mathrm{LS}(\mathfrak{K})$ and $M_i <^1_{|i|+\mathrm{LS}(\mathfrak{K}),|i|+\mathrm{LS}(\mathfrak{K})} M_{i+1}$. By the categoricity assumption without loss of generality $M = \bigcup\limits_{i<\lambda} M_i$. As $\mathrm{cf}(\lambda) > \mu$ for some $i_0 < \lambda$ we have $N, N' \prec M_{i_0}$.

By 2.2(5) we can build an automorphism. $\square_{2.6}$

To restate in later names

2.7 Definition. For $\mu \ge \mathrm{LS}(\mathfrak{K})$, we say $N \in K_\mu$ is (μ,κ)-brimmed if for some M we have $M <^1_{\mu,\kappa} N$ (so κ is $\le \mu$, normally regular); we then say N is (μ,κ)-brimmed over M.

Restating the earlier statements

2.8 Claim. *1) The (μ,κ)-brimmed model is unique (even over M) if it exists at all.*
2) If M is (μ,κ)-brimmed, $\kappa = \mathrm{cf}(\kappa) > \mathrm{LS}(\mathfrak{K})$ then M is κ-saturated.
3) If M is (μ,κ)-brimmed for every $\kappa = \mathrm{cf}(\kappa) \le \mu$ and $\mu > \mathrm{LS}(\mathfrak{K})$ then M is μ-saturated.

2.9 Discussion: It is natural to define saturated as $\|M\|$-saturated. (It may cause confusions using the closely related notion of being (μ,κ)-brimmed for every regular $\kappa \le \mu$.) This is particularly reasonable when the cardinal is regular, e.g. if K categorical in $\lambda, \lambda = \mathrm{cf}(\lambda)$ the model in K_λ is λ-saturated.

Part of the program is to prove that all the definitions are equivalent in the "superstable" case.

For now in Definition 2.7 we have not said when such a model exists; stability in μ (for our classes which has amalgamation and JEP) is sufficient and necessary.

§3 Splitting

Whereas non-forking is very nice in [Sh:c], in more general contexts, non first order, it is not clear whether we have so good a notion, hence we go back to earlier notions from [Sh 3], like splitting. It still gives for many cases $p \in \mathscr{S}(M)$, a "definition" of p over some "small" $N \leq_{\mathfrak{K}} M$. We need μ-splitting because $E_{LS(\mathfrak{K})}$ is not known to be equality (see 1.8). We concentrate (in Definition 3.2 below) on the case N_1, N_2 are models not sequences as in this work this is the most useful case (though those sequences can be of length $< \|N\|$)

3.1 Context. Inside the monster model $\mathfrak{C}$.

3.2 Definition. $p \in \mathscr{S}(M)$ does μ-split over $N \leq_{\mathfrak{K}} M$ if:

> $\|N\| \leq \mu$, and there are N_1, N_2, h such that:
> $\|N_1\| = \|N_2\| \leq \mu$ and $N \leq_{\mathfrak{K}} N_\ell \leq_{\mathfrak{K}} M$, for $\ell = 1, 2$
> h an elementary mapping from N_1 onto N_2 over N such that
> the types $p \restriction N_2$ and $h(p \restriction N_1)$ are contradictory (equivalently distinct).

3.3 Claim. *1) Assume $\mathfrak{K}$ is stable in $\mu, \mu \geq \mathrm{LS}(\mathfrak{K})$. If $M \in \mathfrak{K}_{\geq \mu}$ and $p \in \mathscr{S}^1(M)$, then for some $N_0 \subseteq M, \|N_0\| = \mu, p$ does not μ-split over N_0 (see Definition 3.2).*
2) Moreover, if $2^\kappa > \mu, \langle M_i : i \leq \kappa + 1\rangle$ is $<_{\mathfrak{K}}$-increasing, $\bar a \in {}^m(M_{\kappa+1})$, $\mathrm{tp}(\bar a, M_{i+1}, M_{\kappa+1})$ does $(\leq \mu)$-split over M_i, then $\mathfrak{K}$ is not stable in μ.

Proof of 3.3. 1) If not, we can choose by induction on $i < \mu$ N_i, N_i^1, N_i^2, h_i such that:

- (a) $\langle N_i : i \leq \mu\rangle$ is increasing continuous, $N_i <_{\mathfrak{K}} M$, $\|N_i\| = \mu$
- (b) $N_i \leq_{\mathfrak{K}} N_i^\ell \leq_{\mathfrak{K}} N_{i+1}$
- (c) h_i is an elementary mapping from N_i^1 onto N_i^2 over N_i,
- (d) $p \restriction N_i^2, h_i(p \restriction N_i^1)$ are contradictory, equivalently distinct (we could have defined them for $i < \mu^+$).

Let $\chi = \mathrm{Min}\{\chi : 2^\chi > \mu\}$ so $2^{<\chi} \leq \mu$. Now contradict stability in μ as in part (2).
2) Similar to [Sh:a, Ch,I,§2] or [Sh:c, Ch.I,§2] (by using models), but we give details. Without loss of generality $M_i \in K_{\leq \mu}$ for $i \leq \kappa + 1$. For each $i < \kappa$ let $N_{i,1}, N_{i,2}$ be such that $M_i \leq_{\mathfrak{K}} N_{i,\ell} \leq_{\mathfrak{K}} M_{i+1}, g_i$ an isomorphism from $N_{i,1}$ onto $N_{i,2}$ over M_i and $\mathrm{tp}(\bar a, N_{i,2}, M_{i+1}) \neq g_i(\mathrm{tp}(\bar a, N_{i,1}, M_{i+1})$. Without loss of generality $2^{<\kappa} \leq \mu$.

We define by induction on $\alpha \le \kappa$ a model M^*_α and for each $\eta \in {}^\alpha 2$, a mapping h_η such that:

(a) $M^*_\alpha \in K_\mu$ is $\le_{\mathfrak{K}}$-increasing continuous

(b) for $\eta \in {}^\alpha 2, h_\eta$ is a $\le_{\mathfrak{K}}$-embedding of M_α into M^*_α

(c) if $\beta < \alpha, \eta \in {}^\alpha 2$, <u>then</u> $h_{\eta \restriction \beta} \subseteq h_\eta$

(d) if $\alpha = \beta + 1, \nu \in {}^\beta 2$, <u>then</u> $h_{\nu \hat{} <0>} \restriction N_{i,1} = h_{\nu \hat{} <1>} \restriction N_{i,2}$.

There is no problem to carry the definition (we are using amalgamation only in $K_{\le \mu}$ and if we start with $M_0 \in K_\mu$ only in K_μ). Now for each $\eta \in {}^\kappa 2$ we can find $M^*_\eta \in K_\mu, M^*_\kappa \le_{\mathfrak{K}} M^*_\eta$ and $\le_{\mathfrak{K}}$-embedding h^+_η of $M_{\kappa+1}$ into M^*_η extending $h_\eta = \bigcup\limits_{\alpha<\kappa} h_{\eta \restriction \alpha}$. Now $\{\text{tp}(h^+_\eta(\bar a), M^*_\kappa, M^*_\eta) : \eta \in {}^\kappa 2\}$ is a family of $2^\kappa > \mu$ distinct members of $\mathscr{S}^m(M^*_\kappa)$ and recall $M^*_\kappa \in K_\mu$ so we are done. $\square_{3.3}$

3.4 Conclusion. [Assume the conclusion of 3.2]. If $p \in \mathscr{S}^m(M), M$ is μ^+-saturated, $\kappa = \text{cf}(\kappa) \le \mu$, <u>then</u> for some $N_0 <^\circ_{\mu,\kappa} N_1 \le_{\mathfrak{K}} M$, (so $\|N_1\| = \mu$) we have:
p is the E_μ-unique extension of $p \restriction N_1$ which does not μ-split over N_0, which means: if $q \in \mathscr{S}(M), q \restriction N_1 = p \restriction N_1$ and p does not μ-split over N_0, <u>then</u> $pE_\mu q$.

§4 Indiscernibles and E.M. Models

4.1 Notation. We can below replace h_i by the sequence $\langle h_i(t) : t \in Y \rangle$.

4.2 Definition. Let $h_i : Y \to \mathfrak{C}$ for $i < i^*$.
1) $\langle h_i : i < i^* \rangle$ is an indiscernible sequence (of character $< \kappa$) (over A) if for every g, a partial one to one order preserving map from i^* to i^* (with domain of cardinality $< \kappa$) there is $f \in \text{AUT}(\mathfrak{C})$, such that

$$g(i) = j \Rightarrow h_j \circ h_i^{-1} \subseteq f$$

$$(\text{and } \text{id}_A \subseteq f).$$

So omitting κ means $\kappa > i^*$.
2) $\langle h_i : i < i^* \rangle$ is an indiscernible set (of character $< \kappa$) (over A) if: for every g, a partial one to one map from i^* to i^* (with $|\text{Dom}(g)| < \kappa$) there is $f \in \text{AUT}(\mathfrak{C})$, such that

$$g(i) = j \Rightarrow h_j \circ h_i^{-1} \subseteq f$$

$$(\text{and } \text{id}_A \subseteq f).$$

3) $\langle h_i : i < i^* \rangle$ is a strictly indiscernible sequence, if $i^* \geq \omega$ and for some Φ, proper for linear orders (see [Sh:a, Ch.VII] or [Sh:c, Ch.VII]) in vocabulary $\tau_1 = \tau(\Phi)$ extending $\tau(K)$, there are $M^1 = \text{EM}^1(i^*, \Phi)$ with skeleton $\langle x_i : i < i^* \rangle$ (so M^1 is the Skolem Hull of $\{x_i : i < i^*\}$ which is an indiscernible sequence for quantifier free formulas), and there is a sequence of unary terms $\langle \sigma_t : t \in Y \rangle$ such that:

$$\sigma_t(x_i) = h_i(t) \text{ for } i < i^*, t \in Y$$

$$M^1 \restriction \tau(K) <_{\mathfrak{K}} \mathfrak{C}.$$

4) Let $h_i : Y_i \to \mathfrak{C}$ for $i < i^*$ we say that $\langle h_i : i < i^* \rangle$ has localness θ if (θ is a cardinal and):

- (∗) if $h'_i : Y_i \to \mathfrak{C}$ for $i < i^*$ and for every $u \in [i^*]^{<\theta}$ there is an automorphism f_u of $\mathfrak{C}$ such that $f_u \restriction A = \text{id}_A$ and $i \in u \Rightarrow f_u \circ h_i = h'_i$, then there is an automorphism f of $\mathfrak{C}$ such that $f \restriction A = \text{id}_A$ and $i < i^* \to f \circ h_i = h'_i$.

4.3 Definition. 1) $\mathfrak{K}$ has the (κ,θ)-order property <u>if</u> for every α there are $A \subseteq \mathfrak{C}$ and $\langle \bar a_i : i < \alpha\rangle$, where $\bar a_i \in {}^\kappa\mathfrak{C}$ and $|A| \le \theta$ such that:

(∗) if $i_0 < j_0 < \alpha, i_1 < j_1 < \alpha$ then for no $f \in \mathrm{AUT}(\mathfrak{C})$ do we have $f \restriction A = \mathrm{id}_A, f(\bar a_{i_0}{}^\frown \bar a_{j_0}) = \bar a_{j_1}{}^\frown \bar a_{i_1}$.

If $A = \emptyset$ i.e. $\theta = 0$, we write "κ-order property".
2) $\mathfrak{K}$ has the $(\kappa_1,\kappa_2,\theta)$ order property <u>if</u> for every α there are $A \subseteq \mathfrak{C}$ satisfying $|A| \le \theta, \langle \bar a_i : i < \alpha\rangle$ where $\bar a_i \in {}^{\kappa_1}\mathfrak{C}$ and $\langle \bar b_i : i < \alpha\rangle$ where $\bar b_i \in {}^{\kappa_2}\mathfrak{C}$ such that

(∗) if $i_0 < j_0 < \alpha, i_1 < j_1 < \alpha$, <u>then</u> for no $f \in \mathrm{AUT}(\mathfrak{C})$ do we have $f \restriction A = \mathrm{id}_A, f(\bar a_{i_0}) = \bar a_{j_1}, f(\bar b_{j_0}) = \bar b_{i_1}$.

4.4 Observation. So we have obvious monotonicity properties and if $\theta \le \kappa$ we can let $A = \emptyset$; so the (κ,θ)-order property implies the $(\kappa+\theta)$-order property.

4.5 Claim. *1) Any strictly indiscernible sequence (over A) is an indiscernible sequence (over A).*
2) Any indiscernible set (over A) is an indiscernible sequence (over A); can add "of character $< \kappa$".

Proof. Obvious.

4.6 Claim. *1) If $\mu \ge \mathrm{LS}(\mathfrak{K}) + |Y|$ and for each $\theta < \beth_{(2^\mu)^+}$ we have $h^\theta_i : Y \to \mathfrak{C}$, for $i < \theta$ (e.g. $h^\theta_i = h_i$) <u>then</u> for any infinite i^*, we can find $\langle h'_j : j < i^*\rangle$, a strictly indiscernible sequence, with $h'_j : Y \to \mathfrak{C}$ such that:*

(∗) *for every $n < \omega, j_1 < \cdots < j_n < i^*$ for arbitrarily large $\theta < \beth_{(2^\mu)^+}$ we can find $i_1 < \cdots < i_n < \theta$ and $f \in \mathrm{AUT}(\mathfrak{C})$ such that $h'_{j_\ell} \circ (h^\theta_{i_\ell})^{-1} \subseteq f$.*

2) If in part (1) for each θ, the sequence $\langle h^\theta_j : j < \theta\rangle$ is an indiscernible sequence of character $\aleph_0$, in (∗) any $i_1 < \cdots < i_n < i^*$ will do.
3) In Definition 4.3 we can restrict α to $\alpha < \beth_{(2^{\kappa+\theta+\mathrm{LS}(\mathfrak{K})})^+}$ and get an equivalent version.
4) In Definition 4.3(1) we can demand $\langle \bar a{}^\frown \bar a_i : i < \alpha\rangle$ is strictly indiscernible (where $\bar a$ lists A) and get an equivalent version. Similarly in 4.7(2).
5) If $\mu \ge \mathrm{LS}(\mathfrak{K}) + |Y|, N \le_{\mathfrak{K}} \mathfrak{C}$ and for each $\theta < \beth_{(2^\mu)^+}$ we have $h^\theta_i : Y \to N$ for $i < \theta$ and N^1 is an expansion of N with $|\tau(N^1)| \le \mu$, <u>then</u> for some expansion N^2 of N^1 with $|\tau(N^2)| \le \mu$ and Ψ we have:

(a) $\tau(\Psi) = \tau(N^2)$

(b) *for linear orders* $I \subseteq J$ *we have*
$\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Psi) \le_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(J,\Psi) \in K$
and the skeleton of $\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Psi)$ *is* $\langle \bar a_t : t \in I\rangle, \bar a_t = \langle a_{t,y} : y \in Y\rangle$

(c) *for every* $n < \omega$ *for arbitrarily large* $\theta < \beth_{(2^\mu)^+}$ *for some* $i_0 < \dots i_{n-1} < \theta$, *for every linear order* I *and* $t_0 < \dots < t_{n-1}$ *in* I, *letting* $J = \{t_0,\dots,t_{n-1}\}$ *there is an isomorphism* g *from* $\mathrm{EM}(J,\Psi) \subseteq \mathrm{EM}(I,\Psi)$ *(those are* $\tau(N^2)$*-models) onto the submodel of* N^2 *generated by* $\bigcup_{\ell<n} \mathrm{Rang}(h^\theta_{i_\ell})$ *such that*
$h^\theta_{i_\ell}(y) = g(a_{t,y})$.

Proof. As in [Sh:c, Ch.VII,§5] and [Sh 88], see 8.7 for a similar somewhat more complicated proof. $\square_{4.6}$

As in the first order case:

4.7 Lemma. *1) If there is a strictly indiscernible sequence which is not an indiscernible set of character* $\aleph_0$ *called* $\langle \bar a^i : i < \omega\rangle$, *then* $\mathfrak{K}$ *has the* $|\ell g(a^i)|$*-order property.*

Remark. Permutation of infinite sets is a more complicated issue. That is, assume $\langle \bar a^i : i < i^*\rangle$ is a strictly indiscernible sequence over A of character θ^+ but is not an indiscernible set over A of character θ^+ and $i^* \ge \theta^+$. Does $\mathfrak{K}$ have the $(\ell g(\bar a^0), |A| + \theta \times \ell g(\bar a^0))$-order property.

4.8 Claim. *1) If* $\mathfrak{K}$ *has the* κ*-order property then:*

$$I(\chi,\mathfrak{K}) = 2^\chi \text{ for every } \chi > (\kappa + LS(\mathfrak{K}))^+$$

(and other strong non-structure properties).
2) If $\mathfrak{K}$ *has the* $(\kappa_1,\kappa_2,\theta)$*-order property and* $\chi \ge \kappa = \kappa_1+\kappa_2+\theta$ *then for some* $M \in K_\chi$, *we have* $|\mathscr{S}^{\kappa_2}(M)/E_\kappa| > \chi$.

Proof. 1) By [Sh:e, Ch.III,§3] (preliminary version appears in [Sh 300, Ch.III,§3]) (note the version on e.g., $\triangle(\mathbb{L}_{\lambda^+,\omega})$).
2) Straight. $\square_{4.8}$

4.9 Definition. 1) Suppose $M \leq_{\mathfrak{K}} N$ and $p \in \mathscr{S}^m(N)$. Then p divides over M if there are elementary maps $\langle h_i : i < \bar{\kappa}\rangle$, $\mathrm{Dom}(h_i) = N$, $h_i \restriction M = \mathrm{id}_M$, $\langle h_i : i < \bar{\kappa}\rangle$ is a strictly indiscernible sequence and $\{h_i(p) : i < \bar{\kappa}\}$ is contradictory i.e. no element (in some $\mathfrak{C}', \mathfrak{C} <_{\mathfrak{K}} \mathfrak{C}'$) realizing all of them; recall $\bar{\kappa}$ is the cardinality of $\mathfrak{C}$. Let μ-divides mean no elements realize $\geq \mu$ of them.
2) $\kappa_\mu(\mathfrak{K})$ [or $\kappa^*_\mu(\mathfrak{K})$] is the set of regular κ such that for some $\leq_{\mathfrak{K}}$-increasing continuous $\langle M_i : i \leq \kappa+1\rangle$ in K_μ and $b \in M_{\kappa+1}$ for every $i < \kappa$ we have: $\mathrm{tp}(b, M_\kappa, M_{\kappa+1})$ [or $\mathrm{tp}(b, M_{i+1}, M_{\kappa+1})$] divides over M_i; so $\kappa \leq \mu$.
3) $\kappa_{\mu,\theta}(\mathfrak{K})$ [or $\kappa^*_{\mu,\theta}(\mathfrak{K})$] is the set of regular κ such that for some $\leq_{\mathfrak{K}}$-increasing continuous sequence $\langle M_i : i \leq \kappa + 1\rangle$ in K_θ and $b \in M_{\kappa+1}$ for every $i < \kappa$ we have: $\mathrm{tp}(b, M_\kappa, M_{\kappa+1})$ [or $\mathrm{tp}(b, M_{i+1}, M_{\kappa+1})$], μ-divides over M_i, so $\kappa \leq \theta$ (see Definition 4.12 below).

4.10 Remark. 1) A natural question: is there a parallel to forking?
2) Note the difference between $\kappa_\mu(\mathfrak{K})$ and $\kappa^*_\mu(\mathfrak{K})$, e.g., 4.11(2) is not clear for $\kappa_\mu(\mathfrak{K})$. Note that now the "local character" is apparently lost.

4.11 Fact. 1) In Definition 4.9(1) we can equivalently demand: no element realizing $\geq \beth_{(2^\chi)^+}$ of them, where $\chi = \|N\|$.
2) If $\kappa \in \kappa^*_\mu(\mathfrak{K}), \theta = \mathrm{cf}(\theta) \leq \kappa$ then $\theta \in \kappa^*_\mu(\mathfrak{K})$ and similarly of $\kappa^*_{\mu,\theta}(\mathfrak{K})$.
3) $\kappa^*_\mu(\mathfrak{K}) \subseteq \kappa_\mu(\mathfrak{K})$ similarly $\kappa^*_{\mu,\theta}(\mathfrak{K}) \subseteq \kappa_{\mu,\theta}(\mathfrak{K})$.

4.12 Definition. Suppose $M \leq_{\mathfrak{K}} N, p \in \mathscr{S}(N), M \in K_{\leq\mu}, \mu \geq \mathrm{LS}(\mathfrak{K})$.
1) We say p does μ-strongly split over M, if there are $\langle \bar{a}^i : i < \omega\rangle$ such that:

(i) $\bar{a}^i \in {}^{\gamma\geq}\mathfrak{C}$ for $i < \omega$, $\gamma < \mu^+$, $\langle \bar{a}^i : i < \omega\rangle$ is strictly indiscernible over M

(ii) for no b realizing p do we have $\mathrm{tp}(\bar{a}^0{}^\frown\langle b\rangle, M, \mathfrak{C}) = \mathrm{tp}(\bar{a}^1{}^\frown\langle b\rangle, M, \mathfrak{C})$.

2) We say p explicitly μ-strongly splits over M if in addition $\bar{a}^0 \cup \bar{a}^1 \subseteq N$.
3) Omitting μ means any μ (equivalently $\mu = \|N\|$).

4.13 Claim. *1) Strongly splitting implies dividing with models of cardinality $\leq \mu$ if $(*)_\mu$ holds where $(*)_\mu = (*)_{\mu,\aleph_0,\aleph_0}$ and*

$(*)_{\mu,\theta,\sigma}$ *If $\langle \bar{a}^i : i < i^*\rangle$ is a strictly indiscernible sequence, $\bar{a}^i \in {}^\mu\mathfrak{C}, \bar{b} \in {}^{\sigma>}\mathfrak{C}$, then for some $u \subseteq i^*$, $|u| < \theta$ and the isomorphism type of $(\mathfrak{C}, \bar{a}^i{}^\frown\bar{b})$ for all $i \in i^*\backslash u$ is the same.*

4.14 Claim. *1) Let $\mu(*) = \mu + \sigma + \mathrm{LS}(\mathfrak{K})$. Assume $\langle \bar a^i : i < i^* \rangle$ and $\bar b$ form a counterexample to $(*)_{\mu,\theta,\sigma}$ of 4.13 and $\theta \geq \beth_{(2^{\mu(*)})^+}$ then $\mathfrak{K}$ has the $\mu(*)$-order property.*
2) We can also conclude that for every $\chi \geq \mu + \mathrm{LS}(\mathfrak{K})$, for some $M \in K_\chi$ we have $|\mathscr{S}^{\ell g(\bar b)}(M)| > \chi$, note $\ell g(\bar b) < \sigma$.
3) If in (1) we have "$\theta < \beth_{(2^{\mu(*)})^+}$" we can still get that for every $\chi \geq \mu + \sigma + \mathrm{LS}(\mathfrak{K}) + \theta$ for some $M \in K_\chi$, we have $|\mathscr{S}^{\ell g(\bar b)}(M)| \geq \chi^\theta$, moreover $|\mathscr{S}^{\ell g(\bar b)}(M)/E_\mu| \geq \chi^\theta$.
4) In part (1) it suffices to have such an example for every $\theta < \beth_{(2^{\mu(*)})^+}$, of course, for a fixed $\mu(*)$.

Proof. Straight, using 4.15 below.

4.15 Claim. *Assume $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi)$,$\mathrm{LS}(\mathfrak{K}) + \ell g(\bar a_t) \leq \mu$ for $t \in I, \mu \geq |\alpha|$ and $M \leq_{\mathfrak{K}} N, \bar b \in {}^\alpha N$ and*

- $(*)$ *for no $J \subseteq I, |J| < \beth_{(2^\mu)^+}$ do we have for all $t,s \in I \backslash J$, $\mathrm{tp}(\bar a_t \hat{\ } \bar b, \emptyset, N) = \mathrm{tp}(\bar a_s \hat{\ } \bar b, \emptyset, N)$.*

Then

- (A) *we can find Φ' proper for linear orders and a formula φ (not necessarily first order, but $\pm\varphi$ is preserved by $\leq_{\mathfrak{K}}$-embeddings) such that for any linear order I'*
  letting $M' = \mathrm{EM}(I',\Phi')$ having the skeleton $\langle \bar a'_t : t \in I \rangle, \bar a'_t = \bar a^t \hat{\ } \bar b_t, \ell g(\bar a^t) \leq \mu, \ell g(\bar b_t) = \alpha$ and we have:
  $M' \models \varphi[\bar a^t, \bar b_s] \Leftrightarrow t < s$
  (if $\alpha < \omega$, this is half the finitary order property)
- (B) *this implies instability in every $\mu' \geq \mu$ if $\alpha < \omega$*
- (C) *this implies the μ-order property and even the $(\mu,|\alpha|,0)$-order property*
- (D) *if we strengthen the assumption to $\bar b \in {}^\alpha M$ then "$|J| < \mu^+$" and just "$|J| < |\alpha|^+ + \aleph_0$" in $(*)$ suffices*
- (E) *if $\chi \geq \mu$, for some $N' \in K_\chi$, then $|\mathscr{S}^\alpha(N')| > \chi$ moreover $|\mathscr{S}^\alpha(N')/E_\mu| > \chi$.*

Proof. As we can increase I, without loss of generality the linear order I is dense with no first or last element and is $(\beth_{(2^\mu)^+})^+$-strongly saturated, see Definition 4.18 below. So for some p and some interval I_0 of I, the set $Y_0 = \{t \in I_0 :$

$\mathrm{tp}(\bar{a}_t{}^\frown\bar{b},\emptyset,N)=p\}$ is a dense[5] subset of I_0. Also for some $q\in\mathscr{S}^\alpha(M)\backslash\{p\}$, the set $Y_1=\{t\in I:\mathrm{tp}(\bar{a}_t{}^\frown\bar{b},\emptyset,N)=q\}$ has cardinality $\geq\beth_{(2^\mu)^+}$ and let $Y_1'\subseteq Y_1$ have cardinality $\beth_{(2^\mu)^+}$. As we can shrink I_0 without loss of generality I_0 is disjoint from Y_1' and as we can shrink Y_1' without loss of generality $(\forall s\in Y_1')(\forall t\in I_0)(s<^I t)$ or $(\forall s\in Y_1')(\forall t\in I_0)(t<^I s)$.
By the Erdös-Rado theorem, for every $\theta<\beth_{(2^\mu)^+}$ there are $s^\theta_\alpha\in Y_1'$ for $\alpha<\theta$ such that $\langle s^\theta_\alpha:\alpha<\theta\rangle$ is strictly increasing or strictly decreasing; without loss of generality the case does not depend on θ, so as we can invert I without loss of generality it is increasing. Hence (try $(p_1,p_2)=(p,q)$ and $(p_1,p_2)=(q,p)$, one will work)

- $(*)$ we can find $p_1\neq p_2$ such that
- $(**)$ for every $\theta<\beth_{(2^\mu)^+}$ there is an increasing sequence $\langle t^\theta_\alpha:\alpha<\theta+\theta\rangle$ of members of I such that

$$\text{(i)}\qquad \alpha<\theta\Rightarrow\ \mathrm{tp}(\bar{a}_{t^\theta_\alpha}{}^\frown\bar{b},\emptyset,N)=p_0$$

$$\text{(ii)}\qquad \theta\leq\alpha<\theta+\theta\Rightarrow\ \mathrm{tp}(\bar{a}_{t^\theta_\alpha}{}^\frown\bar{b},\emptyset,N)=p_1.$$

[Note that we could have replaced "increasing" by

$$\text{(iii)}\qquad \alpha<\beta<\theta\Rightarrow t^\theta_\alpha<_I t^\theta_\beta<_I t^\theta_{\theta+\alpha}<_I t^\theta_{\theta+\beta}.$$

Why? Let $I_1=\{t\in I:(\forall\alpha<\theta)\,s^\theta_\alpha<t\}$, so every $A\subseteq I_1$ of cardinality $\leq\beth_{(2^\mu)^+}$ has a bound from below in I_1, so for some $q_1\in\mathscr{S}^\alpha(M)$ the set $I_2=\{t\in I_1:\mathrm{tp}(\bar{a}_t{}^\frown\bar{b},\emptyset,N)=q_1\}$ is unbounded from below in I_1. If $q_1\neq q$ then q_1,q can serve as p_1,p_2, so assume $q_1=q$, so p,q_1 can serve as p_1,p_2.]

For every $\theta<\beth_{(2^\mu)^+}$ for every $\alpha<\theta$ we can find an automorphism $f_{\theta,\alpha}$ of $\mathfrak{C}$ such that $f_{\theta,\alpha}(\bar{a}_{t^\theta_\beta})$ is $\bar{a}_{t^\theta_\beta}$ if $\beta<\alpha$ and is $\bar{a}_{t^\theta_{\theta+\beta}}$ if $\beta\in[\alpha,\theta)$, and let $\bar{b}^\theta_\alpha=f^{-1}_{\theta,\alpha}(\bar{b})$. So in $\mathfrak{C}$, for $\theta<\beth_{(2^\mu)^+}$, we have $\langle(\bar{a}_{t^\theta_\alpha},\bar{b}^\theta_\alpha):\alpha<\theta\rangle$ satisfies $\mathrm{tp}(\bar{a}_{t^\theta_\alpha}{}^\frown\bar{b}^\theta_\beta,\emptyset,\mathfrak{C})$ is p_1 iff it is $\neq p_0$ iff $\alpha\geq\beta$.
Now we apply 4.6(5) with h^θ_i listing $\bar{a}^\theta_\alpha{}^\frown\bar{b}^\theta_\alpha$ and letting N^1 be $\mathrm{EM}(I,\Phi)$ (so $\tau(N^1)=\tau(\Phi)$) and we get Ψ as there.

So we have proved clause (A) and clause (B) by 4.8(2), by easy manipulations we get clause (E) and so (C).

We are left with clause (D). Clearly there is $\bar{t}=\langle t_i:i<i^*\rangle$ satisfying $i^*<|\alpha|^++\aleph_0$ such that $\bar{b}=\langle b_\beta:\beta<\alpha\rangle, b_\beta=\tau_\beta(\bar{a}_{t_{i(\beta,0)}},\ldots,\bar{a}_{t_{i(\beta,n(\beta)-1)}})$ where $i(\beta,\ell)<i^*,\tau_\beta$

[5] as $\{\mathrm{tp}(\bar{a},M,\mathfrak{C}):\bar{a}\in{}^\alpha\mathfrak{C}\}$ is $\leq 2^{\|M\|+|\alpha|+\mathrm{LS}(\mathfrak{K})}$ by the amalgamation property and the choice of $\mathfrak{C}$.

a $\tau(\Phi)$-term. So by the version of $(*)$ used in clause (D), necessarily for some $s_1, s_2 \in I \backslash J$ we have:

$$p_1 \neq p_2 \text{ where}$$

$$p_1 = \operatorname{tp}(\bar{a}_{s_1} {}^\frown \bar{b}, \emptyset, N)$$

$$p_2 = \operatorname{tp}(\bar{a}_{s_2} {}^\frown \bar{b}, \emptyset, N)$$

Clearly $s_1 \neq s_2$. By renaming without loss of generality $s_1 <^I s_2$ and initial segments $J_\ell \, (\ell \leq 3)$ of J we have $\emptyset = J_0 \trianglelefteq J_1 \trianglelefteq J_2 \trianglelefteq J_3 = J$ and for every $t \in J, t <^I s_1 \Leftrightarrow t \in J_1$ and $t <^I s_2 \Leftrightarrow t \in J_2$. So for some $0 = i_0 \leq i_1 \leq i_2 \leq i_3 = i^*$ we have $t_i <^I s_1 \Leftrightarrow i < i_1$ and $s_1 <^I t_i <^I s_2 \Leftrightarrow i_1 \leq i < i_2$ and $s_2 <^I t_i \Leftrightarrow i_2 < i < i_3$. As I is $(\beth_{(2^\mu)^+})^+$-strongly saturated we can increase J so adding to J (by the saturation of I) without loss of generality $\beta < \alpha \ \& \ \ell < n(\beta) \Rightarrow i(\beta, \ell) \notin \{t^*_1, t^*_2\}$, and $J_\ell = \{t \in J : t \leq t^*_\ell\}$ for $\ell = 1, 2$ so $t^*_\ell < s_\ell$ for $\ell = 1, 2$. So for every linear order I' we can define a linear order I^* with set of elements

$$J_1 \cup (J \backslash J_2) \cup \{(s,t) : s \in I', t \in J_2 \backslash J_1\}$$

linearly ordered by:

(a) on $J_1 \cup (J \backslash J_2)$ as in J

(b) $t_1 < (s', t') < (s'', t'') < t_2$ <u>if</u> $t_1 \in J_1, t_2 \in J_3 \backslash J_2, s', s'' \in I'$ and $(t', t'' \in J_2 \backslash J_1$ and $(s' <^{I'} s'') \vee (s' = s'' \ \& \ t' <_J t'')$.

In $M = \operatorname{EM}(I^*, \Phi)$ define, for $s \in I'$

$$\bar{c}_s = \bar{a}_{s, t^*_2}$$

$$\bar{b}_s = \langle \tau_\beta(\bar{c}_{s,i(\beta,0)}, \bar{c}_{s,i(\beta,1)}, \ldots, \bar{c}_{s,i(\beta,n(\beta)-1)}) : \beta < \alpha \rangle.$$

Easily

$$s' <^{I'} s'' \Rightarrow \operatorname{tp}(\bar{a}_{s'} {}^\frown \bar{b}_{s''}, \emptyset, M) = p_1$$

$$s'' \leq^{I'} s' \Rightarrow \operatorname{tp}(\bar{a}_{s'} {}^\frown \bar{b}_{s''}, \emptyset, M) = p_2.$$

By easy manipulations we can finish. $\square_{4.15}$

4.16 Claim. *Assume K is categorical in λ and*

(a) $1 \leq \kappa$ *and* $\mathrm{LS}(\mathfrak{K}) < \theta = \mathrm{cf}(\theta) \leq \lambda$ *and* $(\forall \alpha < \theta)(|\alpha|^\kappa < \theta)$

(b) $\bar{a}_i \in {}^\kappa\mathfrak{C}$ *for* $i < \theta$.

Then for some $W \subseteq \theta$ of cardinality θ, the sequence $\langle \bar{a}_i : i \in W \rangle$ is strictly indiscernible.

Proof of 4.16. Let $M' \prec \mathfrak{C}, \|M'\| = \theta$ and $\alpha < \theta \Rightarrow \bar{a}_\alpha \subseteq M'$. There is $M'', M' \prec M'' \prec \mathfrak{C}, \|M''\| = \lambda$. So $M'' \cong \mathrm{EM}(\lambda, \Phi)$ and without loss of generality equality holds. So there is $u \subseteq \lambda, |u| \leq \theta$ such that $M' \subseteq \mathrm{EM}(u, \Phi)$. Hence without loss of generality $M' = \mathrm{EM}(u, \Phi)$. So $\bar{a}_\alpha \in {}^\kappa\, \mathrm{EM}(u_\alpha, \Phi)$ for some $u_\alpha \subseteq u, |u_\alpha| \leq \kappa$.
Without loss of generality: $\mathrm{otp}(u_\alpha) = j^*$, so for $\alpha < \beta$, $\mathrm{OP}_{u_\alpha, u_\beta}$, the order preserving map from u_β onto u_α, induces $f_{\alpha,\beta} : \mathrm{EM}(u_\beta, \Phi) \xrightarrow[\text{onto}]{\text{iso}} EM(u_\alpha, \Phi)$, and without loss of generality $f_{\alpha,\beta}(\bar{a}_\beta) = \bar{a}_\alpha$.

Now as u is well ordered and the assumption (a), (or see below) for some $w \in [\theta]^\theta$ the sequence $\langle u_\alpha : \alpha \in w \rangle$ <u>is indiscernible</u> in the linear order sense (make them a sequence). Now we can create the right Φ.

[Why? Let $u_\alpha = \{\gamma_{\alpha,j} : j < j^*\}$ where $\gamma_{\alpha,j}$ increases with j. For $\alpha < \theta$, let $A_\alpha = \{\gamma_{\beta,j} : \beta < \alpha, j < j^*\} \cup \{ \bigcup\limits_{\beta < \alpha, j} \gamma_{\beta,j} + 1\}$. Let $\gamma^*_{\alpha,j} = \mathrm{Min}\{\gamma \in A_\alpha : \gamma_{\alpha,j} \geq \gamma\}$ and for each $\alpha \in S^*_0 = \{\delta < \theta : \mathrm{cf}(\delta) > \kappa\}$ let $h(\delta) = \mathrm{Min}\{\beta < \delta : \gamma^*_{\delta,j} \in A_\beta\}$ (note that $\langle A_\beta : \beta \leq \delta \rangle$ is increasing continuous, $\mathrm{cf}(\delta) > \kappa \geq |j^*|$ and $\gamma^*_{\delta,j} \in A_\delta$ by the definition of the A_β's).

By Fodor's lemma for some stationary $S_1 \subseteq S_0, h \restriction S_1$ is constantly β^*. As $(\forall \alpha < \theta)(|\alpha|^\kappa < \theta = \mathrm{cf}(\theta))$ for some $S_2 \subseteq S_1$ for each $j < j^*$ and for all $\delta \in S_2$, the truth value of "$\gamma_{\delta,j} \in A_\delta$" (e.g. $\gamma_{\delta,j} = \gamma^*_{\delta,j}$) is the same and $\langle \gamma^*_{\delta,j} : \delta \in S_2 \rangle$ is constant. Now $\langle u_\delta : \delta \in S_2 \rangle$ is as required.] $\square_{4.16}$

That is, see more [Sh 620, §7].

4.17 Observation. If $\theta = \mathrm{cf}(\theta)$ and $(\forall \alpha < \theta)(|\alpha|^\kappa < \theta)$ and $j^* < \kappa$ and $\gamma_{\alpha,j}$ is an ordinal for $\alpha < \theta, j < j^*$ then for some stationary set $S \subseteq \{\alpha < \theta : \mathrm{cf}(\alpha) = \kappa^+\}$ the sequence $\langle \langle \gamma_{\alpha,j} : j < j^* \rangle : \alpha \in S \rangle$ is indiscernible.

4.18 Definition. A model M is λ-strongly saturated <u>if</u>:

- (a) M is λ-saturated
- (b) M is strongly λ-homogeneous which means: if f is a partial elementary mapping from M to M, $|\mathrm{Dom}(f)| < \lambda$ then $(\exists g \in \mathrm{AUT}(M))(f \subseteq g)$.

4.19 Remark. 1) If $\mu = \mu^{<\lambda}, I$ a linear order of cardinality $\leq \mu$, then there is a λ-strongly saturated dense linear order $J, I \subseteq J$.
2) We can even get a uniform bound on $|J|$ (which only depends on μ).

§5 Rank and Superstability

5.1 Definition. For $M \in K_\mu, p \in \mathscr{S}^m(M)$ (and $\mu \geq \text{LS}(\mathfrak{K})$, of course) we define $R(p)$, an ordinal or ∞ as follows: $R(p) \geq \alpha$ iff for every $\beta < \alpha$ there are $M^+, M \leq_{\mathfrak{K}} M^+ \in K_\mu, p \subseteq p^+ \in \mathscr{S}^m(M^+), R(p^+) \geq \beta$ & $[p^+$ μ-strongly splits over $M]$. In case of doubt we write R_μ. This is well defined and has the obvious properties:

- (a) monotonicity, i.e., $p_1 = p_2 \restriction M_1 \Rightarrow R(p_1) \geq R(p_2)$
- (b) if $M \in K_\mu, p \in \mathscr{S}^m(M)$ and $\text{Rk}(p) \geq \alpha$ <u>then</u> for some N, q satisfying $M \leq_{\mathfrak{K}} N \in K_\mu$ and $q \in \mathscr{S}^m(N)$ we have: $q \restriction M = p$ and $\text{Rk}(q) = \alpha$
- (c) automorphisms of $\mathfrak{C}$ preserve everything
- (d) the set of values is $[0, \alpha)$ or $[0, \alpha) \cup \{\infty\}$ for some $\alpha < (2^\mu)^+$, etc.

5.2 Definition. We say $\mathfrak{K}$ is $(\mu, 1)$-superstable <u>if</u>:
$M \in K_\mu$ & $p \in \mathscr{S}(M) \Rightarrow R(p) < \infty$ $\Big(\text{equivalently } < (2^\mu)^+\Big)$.

5.3 Claim. *If $(*)_\mu$ from 4.13 above fails, <u>then</u> $(\mu, 1)$-superstability fails.*

Proof. Straight.

5.4 Claim. *If $\mathfrak{K}$ is not $(\mu, 1)$-superstable, <u>then</u> there are a sequence $\langle M_i : i \leq \omega + 1\rangle$ which is $<_{\mathfrak{K}}$-increasing continuous in K_μ and $m < \omega$ and $\bar{a} \in {}^m(M_{\omega+1})$ such that $(\forall i < \omega)\big[\frac{\bar{a}}{M_{i+1}}$ does μ-strongly split over $M_i\big]$.
Also the inverse holds.*

Proof. As usual.

5.5 Claim. *1) If $\mathfrak{K}$ is not $(\mu, 1)$-superstable <u>then</u> K is unstable in every χ such that $\chi^{\aleph_0} > \chi + \mu + 2^{\aleph_0}$.
2) If $\kappa \in \kappa^*_\mu(\mathfrak{K})$ and $\chi^\kappa > \chi \geq \text{LS}(\mathfrak{K})$, <u>then</u> $\mathfrak{K}$ is not χ-stable.*

Remark. We intend to deal with the following elsewhere; we need stable amalgamation

(A) if $\kappa \in \kappa_\mu(\mathfrak{K})$ and $\chi^\kappa > \chi = \chi^\kappa \geq \mathrm{LS}(\mathfrak{K})$ or just there is a tree with χ nodes and $> \chi$ κ-branches and $\chi \geq \mathrm{LS}(\mathfrak{K})$, then $\mathfrak{K}$ is not χ stable even modulo E_μ

(B) if $\kappa \in \kappa_\mu(\mathfrak{K})$, $\mathrm{cf}(\chi) = \kappa$ and $\bigwedge_{\lambda<\chi} \lambda^\mu \leq \chi$,

then $\mathfrak{K}$ is not χ-stable.

5.6 Remark. 1) Clearly 5.5(1) this implies $I(\mathrm{LS}(\mathfrak{K})^{+(\omega(\alpha_0+\alpha)+n)}, K) \geq |\alpha|$ when $\mu = \aleph_{\alpha_0}$. We conjecture that [GrSh 238] can be generalized to the context of (1) with cardinals which exists by ZFC.
2) Note that for complete first order stable theory $T, \mathfrak{K} =$ MOD(T) so $\leq_{\mathfrak{K}} = \prec \restriction$ MOD(T), for κ regular we have $(*)_1^\kappa \Leftrightarrow (*)_2^\kappa$ where

$(*)_1^\kappa$ for any increasing chain $\langle M_i : i < \kappa\rangle$ of λ-saturated models of length κ, the union $\bigcup_{i<\kappa} M_i$ is λ-saturated,

$(*)_2^\kappa$ $\kappa \in \kappa_r(\mathfrak{K})$.

From this point of view, first order theory T is a degenerated case: $\kappa_r(T)$ is an initial segment so naturally we write the first regular not in it. This is a point where [Sh 300] opens our eyes.
3) In fact in 5.5 not only do we get $\|M\| = \chi, |\mathscr{S}(M)| > \chi$ but also $|\mathscr{S}(M)/E_\mu| > \chi$.
4) Let me try to explain the proof of 5.5, of course, being influenced by the first order case. If the class is superstable, one of the consequences of not having the appropriate order property is that (see 4.15) for a strictly indiscernible sequence $\langle \bar a_t : t \in I\rangle$ over A each $\bar a_t$ of length at most μ and $\bar b$, singleton for simplicity, for all except few of the $t, \bar a_t \hat{\ } \bar b$ realizes the same type (= convergence, existence of average). Of course, we can get better theorems generalizing the ones for first order theories: we can use $\kappa \notin \kappa_\mu(\mathfrak{C})$ and/or demand that after adding to $A, \bar c$ and few of the $\bar a_t$'s the rest is strictly indiscernible over the new A, but this is not used in 5.5. Now if $\mathfrak{C}$ is $(\mu,1)$-superstable the number of exceptions is finite, however, the inverse is not true: for some non $(\mu,1)$-superstable class $\mathfrak{C}$ still the number of exceptions in such situations is finite. In the proof of 5.5(1) this property is used as a dividing line.

Proof of 5.5. 1)

Case I There are $M, N, p, \langle \bar a_i : i < i^*\rangle$ as in 4.13$(*)_\mu$ and $\bar c$, (in fact $\ell g(\bar c) = 1$) such that $\bar c$ realizes $h_i(p)$ for infinitely many i's and fails to realize $h_i(p)$ for infinitely many i's.

Let I be a $\beth(\chi + \beth_{(2^\mu)^+})^+$-strongly saturated dense linear order (see Definition 4.18) such that even if we omit $\le \beth_{(2^\mu)^+}$ members, it remains so. By the strict indiscernibility we can find $\langle \bar{a}_t : t \in I\rangle, c$ as above.

So there is $u \subseteq I, |u| < \beth_{(2^\mu)^+}$ such that $q =$ tp$(\bar{a}_t \char`^ \bar{c}, \emptyset, \mathfrak{C})$ is the same for all $t \in I \backslash u$; without loss of generality $q =$ tp$(\bar{a}_t \char`^ \bar{c}, \emptyset, \mathfrak{C}) \Leftrightarrow t \in I\backslash u$, so u is infinite. Hence we can find $i_n \in i^* \cap u$ such that $i_n < i_{n+1}$. Let $I' = I\backslash(u\backslash\{i_n : n < \omega\})$, so that I' is still χ^+-strongly saturated. Hence for every $J \subseteq I'$ of order type ω for some $c_J (\in \mathfrak{C})$ we have

$$t \in I'\backslash J \Rightarrow \text{ tp}(\bar{a}_t \char`^ \bar{c}_J, \emptyset, \mathfrak{C}) = q$$

$$t \in J \Rightarrow \text{ tp}(\bar{a}_t \char`^ \bar{c}_J, \emptyset, \mathfrak{C}) \neq q.$$

This clearly suffices.

Case II Not Case I.
As in [Sh 3] (the finitely many finite exceptions do not matter) or see part (2).
2) If $\chi < 2^\kappa$ the conclusion follows from 3.3(2). Possibly decreasing κ (allowable as $\kappa \in \kappa^*_\mu(\mathfrak{K})$ rather than $\kappa \in \kappa_\mu(\mathfrak{K})$ is assumed) we can find a tree $\mathscr{T} \subseteq {}^{\kappa\ge}\chi$, so closed under initial segments such that $|\mathscr{T} \cap {}^{\kappa>}\chi| \le \chi$ but $|\mathscr{T} \cap {}^\kappa\chi| > \chi$. (The assumption "$\chi^\kappa > \chi \ge$ LS$(\mathfrak{K})$" is needed just for this). Let $\langle M_i : i \le \kappa{+}1\rangle, c \in M_{\kappa+1}$ exemplify $\kappa \in \kappa^*_\mu(\mathfrak{K})$ and let $\mathscr{T}' = \mathscr{T} \cup \{\eta\char`^\langle 0\rangle : \eta \in {}^\kappa\text{Ord and } i < \kappa \Rightarrow \eta \restriction i \in \mathscr{T}\}$. Now we can by induction on $i \le \kappa + 1$ choose $\langle h_\eta : \eta \in \mathscr{T}' \cap {}^i\chi\rangle$, such that:

- (a) h_η is a $\le_{\mathfrak{K}}$-embedding from $M_{\ell g(\eta)}$ into $\mathfrak{C}$
- (b) $j < \ell g(\eta) \Rightarrow h_{\eta\restriction j} \subseteq h_\eta$
- (c) if $i = j{+}1, \nu \in \mathscr{T} \cap {}^j\chi$, <u>then</u> $\langle h_\eta(M_i) : \eta \in \text{ Suc}_T(\nu)\rangle$ is strictly indiscernible, and can be extended to a sequence of length $\bar{\kappa}$ such that $\langle h_\eta(p \restriction M_i) : \eta \in \text{Suc}_I(\nu)\rangle$ is contradictory (i.e. as in Definition 4.9(1)).

There is no problem to do this. Let $M \le_{\mathfrak{K}} \mathfrak{C}$ be of cardinality χ and include $\bigcup\{h_\eta(M_i) : i < \kappa \text{ and } \eta \in \mathscr{T} \cap {}^i\chi\}$ hence it includes also $h_\eta(M_\kappa)$ if $\eta \in \mathscr{T} \cap {}^\kappa\chi$ as $M_\kappa = \bigcup\limits_{i<\kappa} M_i$.

For $\eta \in \mathscr{T} \cap {}^\kappa\chi$ let $c_\eta = h_{\eta\char`^<0>}(c)$ and $M_\eta = h_\eta(M_i)$ when $\eta \in \mathscr{T} \cap {}^i\text{Ord}$ and $i \le \kappa + 1$, so by 4.15 clearly (by clause (C))

$(*)$ if $i < \kappa, \eta \in \mathscr{T} \cap {}^i\chi$, and $\eta \triangleleft \eta_1 \in \mathscr{T} \cap {}^\kappa\chi$, <u>then</u>
$\{\rho \in \text{ Suc}_T(\eta) : \text{for some } \rho_1, \rho \triangleleft \rho_1 \in \mathscr{T} \cap {}^\kappa\chi$ and
c_{ρ_1} realizes tp$(c_{\eta_1}, h_{\eta_1 \restriction (i+1)}(m_{i+1}))\}$
has cardinality $< \beth_{(2^{\mu + LS(\mathfrak{K})})^+}$.

Next define an equivalence relation $\mathbf{e}$ on $\mathscr{T} \cap {}^{\kappa}\chi$:

$$\eta_1 \, \mathbf{e} \, \eta_2 \text{ iff } \operatorname{tp}(c_{\eta_1}, M) = \operatorname{tp}(c_{\eta_2}, M).$$

or just

$$\eta_1 \mathbf{e} \eta_2 \text{ iff } (\forall \nu)[\nu \in \mathscr{T} \Rightarrow \operatorname{tp}(c_{\eta_1}, M_\nu) = \operatorname{tp}(c_{\eta_2}, M_\nu)].$$

Now if for some $\eta \in \mathscr{T} \cap {}^{\kappa}\chi, |\eta/\mathbf{e}| > \beth_{(2^{\mu+LS(\mathfrak{K})})^+}$ then for some $\eta^* \in \mathscr{T} \cap {}^{\kappa>}\chi$, we have

$$\{\nu \restriction (\ell g(\eta^* + 1)) : \nu \in \eta/\mathbf{e}\} \text{ has cardinality } > \beth_{(2^{\mu+LS(\mathfrak{K})})^+}$$

which contradicts $(*)$; so if $\chi \geq \beth_{(2^{\mu+LS(\mathfrak{K})})^+}$, we are done.

But if for some $\eta \in \mathscr{T} \cap {}^{\kappa>}\chi$ the set in $(*)$ has cardinality $\geq \kappa$, then we can continue as in case I of the proof of part (1) replacing "infinite" by "of cardinality $\geq \kappa$", so assume this never happens. So above if $|\eta/\mathbf{e}| > 2^\kappa$, we get again a contradiction. So if $|\mathscr{T} \cap {}^{\kappa}\chi| > 2^\kappa$, we conclude $|\mathscr{T} \cap {}^{\kappa}\chi/\mathbf{e}| = |\mathscr{T} \cap {}^{\kappa}\chi|$, so we are done. We are left with the case $\chi < 2^\kappa$, covered in the beginning (note that for $\chi < 2^\kappa$ the interesting notion is splitting). $\square_{5.5}$

5.7 Claim. *If $\lambda > \mu, \mu \geq \operatorname{LS}(\mathfrak{K}, K)$ and $\mathfrak{K}$ is categorical in λ and $\lambda \neq \mu^{+\omega}$, then*
1) K is $(\mu, 1)$-superstable.
2) $\kappa^*_\mu(\mathfrak{K})$ is empty.

Proof. 1) Assume the conclusion fails. If $\lambda > \mu^{+\omega}$, we can use 5.5 + 1.7 so without loss of generality $\mu < \lambda < \mu^{+\omega}$ hence $\operatorname{cf}(\lambda) > \mu \geq \operatorname{LS}(\mathfrak{K})$.

By clause (b) of 1.7 if $M \in K_\lambda$ then M is $\operatorname{cf}(\lambda)$-saturated. On the other hand from the Definition of $(\mu, 1)$-superstable we shall get below a non-μ^+-saturated model.

Let $\chi = \beth_{(2^\lambda)^+}$. Assume $\mathfrak{K}$ is not $(\mu, 1)$- superstable so we can find in K_μ an increasing continuous sequence $\langle M_i : i \leq \omega + 1 \rangle$ and $c \in M_{\omega+1}$ such that $p_{n+1} = \operatorname{tp}(c, M_{n+1}, M_{\omega+1})$ μ-strongly splits over M_n for $n < \omega$. For each $n < \omega$ let $\langle \bar{a}^n_i : i < \omega \rangle$ be a strictly indiscernible sequence over M_n exemplifying p_{n+1} does μ-strongly split over M_n (see Definition 4.12). So we can define $\bar{a}^n_i \in \mathfrak{C}$ for $i \in [\omega, \chi)$ such that $\langle \bar{a}^n_i : i < \chi \rangle$ is strictly indiscernible over M_n. Let $\mathscr{T}_n = \{\eta \in {}^{2n}\chi : \eta(2m) < \eta(2m+1) \text{ for } m < n\}$. For $n < \omega, i < j < \chi$ let $h^n_{i,j} \in \operatorname{AUT}(\mathfrak{C})$ be such that $h^n_{i,j} \restriction M_n = \operatorname{id}_{M_n}, h^n_{i,j}(\bar{a}^n_0 {}^\frown \bar{a}^n_1) = \bar{a}^n_i {}^\frown \bar{a}^n_j$. Now we choose by induction on $n < \omega, \langle f_\eta : \eta \in \mathscr{T}_n \rangle, \langle g_\eta : \eta \in \mathscr{T}_n \rangle, \langle a^\eta_i : i < \chi, \eta \in \mathscr{T}_n \rangle$ such that:

(a) f_η are restrictions of automorphisms of $\mathfrak{C}$
(b) $\mathrm{Dom}(f_\eta) = M_n$
(c) $g_\eta \in \mathrm{AUT}(\mathfrak{C})$
(d) $\bar a^\eta_i = g_\eta(\bar a^n_i)$ if $\eta \in \mathscr{T}_n$
(e) $f_{<>} = \mathrm{id}_{M_0}$,
(f) $f_\eta \subseteq g_\eta$
(g) if $\eta \in {}^{2n}\chi, m < n$ then $f_{\eta \restriction (2m)} \subseteq f_\eta$
(h) if $\eta \in {}^{2n}\chi$ and $i < j < \chi$ then $f_{\eta\char94 <i,j>} \subseteq (g_\eta \circ h^n_{i,j}) \restriction M_{n+1}$.

There is no problem to carry the induction. Now choose by induction on $n, M^*_n, \eta_n, i_n, j_n$ such that

(α) $i_n < j_n < \chi$ and $\eta_n = \langle i_0, j_0, \dots, i_{n-1}, j_{n-1}\rangle$ so $\eta_n \in \mathscr{T}_n$
(β) $M^*_n \in K_\lambda, M^*_n \le_{\mathfrak{K}} M^*_{n+1}$
(γ) $\mathrm{Rang}(f_{\eta_n}) \subseteq M^*_n$
(δ) $\bar a^{\eta_n}_{i_n}, \bar a^{\eta_n}_{j_n}$ realizes the same type over M^*_n
(ε) $\bar a^{\eta_n}_{i_n}, \bar a^{\eta_n}_{j_n} \subseteq M^*_{n+1}$.

There is no problem to carry the induction (using the theorem on existence of strictly indiscernibles to choose $i_n < j_n$).

So $\bigcup\limits_{n<\omega} f_{\eta_n}$ can be extended to $f \in \mathrm{AUT}(\mathfrak{C})$. Let $c^* = f(c), M^*_\omega = \bigcup\limits_n M^*_{\eta_n}, M^*_{\omega+1} <_{\mathfrak{K}} \mathfrak{C}$ includes $M^*_\omega \cup f(M_{\omega+1})$. Clearly $\mathrm{tp}(c, M^*_{n+1}, M^*_{\omega+1})$ does μ-split over M^*_n hence M^*_ω is not μ^+-saturated (as $\mathrm{cf}(\lambda) > \mu$) (see 5.8 below); contradiction.
2) Similar proof. $\square_{5.7}$

5.8 Claim. *If $\mu \ge \mathrm{LS}(\mathfrak{K}), \langle M_i : i \le \delta\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous, $p \in \mathscr{S}^{\le\mu}(M_\delta), p$ μ-strongly splits over M_i for all i (or just μ-splits over M_i) and $\delta < \mu^+$ <u>then</u> M_δ is not μ^+-saturated.*

Proof. Straight.

5.9 Claim. *Assume there is a Ramsey cardinal $> \mu + \mathrm{LS}(\mathfrak{K})$. If $\mathfrak{K}$ is not $(\mu, 1)$-superstable, <u>then</u> for every $\chi > \mu + \mathrm{LS}(\mathfrak{K})$ there are 2^χ pairwise non-isomorphic models in $\mathfrak{K}_\chi$.*

Proof. By [GrSh 238] for χ regular; together with [Sh:e] for all χ.

5.10 Lemma. *1) If for some* $M, |\mathscr{S}(M)/E_\mu| > \chi \geq \|M\| + \beth_{(2^\mu)^+}$ *and* $\mu \geq \mathrm{LS}(\mathfrak{K})$ *then* $\mathfrak{K}$ *is not* $(\mu, 1)$*-superstable.*
2) If $\chi^\kappa \geq |\mathscr{S}(M)/E_\mu| > \chi^{<\kappa} \geq \chi \geq \|M\| + \beth_{(2^\mu)^+}, \mu \geq \mathrm{LS}(\mathfrak{K}) + \kappa$ *then* $\kappa \in \kappa^*_\mu(\mathfrak{K})$.

Proof. No new point when you remember the definition of E_μ (see 1.8).

§6 Existence of Many Non-Splitting

Below alternatively we can start with 6.8.

6.1 Question. Suppose $\kappa + \mathrm{LS}(\mathfrak{K}) \le \mu < \lambda$ and $\bar N = \langle N_i : i \le \delta\rangle$ is $<^1_{\mu,\kappa}$-increasing continuous (we mean for $i < j, j$ non-limit $N_i <^1_{\mu,\kappa} N_j$), $\delta < \mu^+$ (a limit ordinal) and $p \in \mathscr{S}^m(N_\delta)$. Is there $\alpha < \delta$ such that for every $M \in \mathfrak{K}_{\le\lambda}, N_\delta \le_{\mathfrak{K}} M, p$ has an extension $q \in \mathscr{S}^m(M)$ which does not μ-split over N_α (and so in particular p does not μ-split over N_α)?

6.2 Observation. Let $\mu, \lambda, \delta, \bar N$ and M be as in 6.1.
1) If $p \restriction N_{\alpha+1}$ does not μ-split over N_α, then $p \restriction N_{\alpha+1}$ has at most one extension in $\mathscr{S}(M)$ mod E_μ which does not μ-split over N_α because $N_{\alpha+1} \in K_\mu$ is universal over $N_\alpha, N_{\alpha+1} \le_{\mathfrak{K}} M \in K_{\le\lambda}$. So in 6.1 if p does not μ-split over N_α, then there is at most one q/E_μ for q as there.
2) If the asnwer is yes and $p, q \in \mathscr{S}(N_\delta)$ and $i < \delta \Rightarrow p \restriction N_i = q \restriction N_i$ then $p = q$.
3) If the answer is yes and p does not split over N_α and $N_\delta \le_{\mathfrak{K}} M \in K_\mu$ then

- (i) there is $q \in \mathscr{S}^m(M)$ which does not μ-split over N_α and $q \restriction N_{\alpha+1} = p \restriction N_{\alpha+1}$
- (ii) this q is unique and satisfies $p = q \restriction N_\delta$.

Proof. E.g.,
2) For some $i_1 < \delta, p$ does not μ-split over N_{i_1} and there is $i_2 < \delta$, q does not μ-split over N_{i_2}. By monotonicity of non-μ-splitting, without loss of generality $i_1 = i = i_2$. Let $\bar a$ be a sequence of length μ listing N_δ, and let $f \in \mathrm{Aut}(\mathfrak{C})$ extends id_{N_i} and maps N_δ into N_{i+1} and let $\bar a' = f(\bar a)$.
Now if $c_1, c_2 \in \mathfrak{C}$ realizes p, q respectively then $\mathrm{tp}(\langle c_\ell\rangle \hat{}\, \bar a, N_i, \mathfrak{C}) = \mathrm{tp}(\langle c_\ell\rangle \hat{}\, (\bar a', N_i, \mathfrak{C})$ for $\ell = 1, 2$ as p, q does not μ-split over N_i, $\mathrm{tp}(\langle c_1\rangle \hat{}\, \bar a', N_i, \mathfrak{C}) = \mathrm{tp}(\langle c_2\rangle \hat{}\, \bar a', N_i, \mathfrak{C})$ as $p \restriction N_{i+1} = q \restriction N_{i+1}$. Together $\mathrm{tp}(\langle c_1\rangle \hat{}\, \bar a, N_i, \mathfrak{C}) = \mathrm{tp}(\langle c_2\rangle \hat{}\, \bar a, N_i, \mathfrak{C})$ which means $\mathrm{tp}(c, N_\delta, \mathfrak{C}) = \mathrm{tp}(c_2, N_\delta, \mathfrak{C})$. $\square_{6.2}$

6.3 Lemma. *Suppose K is categorical in $\lambda, \mathrm{cf}(\lambda) > \mu \ge \mathrm{LS}(\mathfrak{K})$. Then the answer to question 6.1 is yes.*

6.4 Remark. We intend later to deal with the case $\lambda > \mu \ge \mathrm{cf}(\lambda) + \mathrm{LS}(\mathfrak{K})$ as in [KlSh 362].

Notation. $I \times \alpha$ is $I + I + \ldots (\alpha$ times) (with the obvious meaning).

Proof. Let Φ be proper for linear order such that $|\tau(\Phi)| \leq \mathrm{LS}(\mathfrak{K})$, $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi) \in K$ (of cardinality $|I| + \tau(\mathfrak{K}))$ where I is a linear order, of course and $I \subseteq J \Rightarrow \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi) \leq_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(J, \Phi)$. Let I^* be a linear order of cardinality μ such that $I^* \times (\alpha+1) \cong I^*$ for $\alpha < \mu^+$ and $I^* \times \omega \cong I^*$ and $I^* \models a < b$ implies that I^* is isomorphic to $I^* \restriction (a, b)$, see [Sh:e, AP,§2]. By 1.7 we know that $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \lambda, \Phi)$ is μ^+-saturated.

First assume only $N_i <^0_{\mu,\kappa} N_{i+1}$ for $i < \delta$; (or just N_{i+1} is universal over N_i). Now we choose by induction on i a triple (α_i, N'_i, h_i) for $i \leq \delta$

(a) α_i is an ordinal $< \mu^+$, increasing continuous with i
(b) $N'_i \in K_\mu$ is $\leq_{\mathfrak{K}}$-increasing continuous with i
(c) h_i is an isomorphism from N_i onto N'_i, increasing continuous with i such that
(d) $N'_0 \leq_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \alpha_i, \Phi)$
(e) $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \alpha_i, \Phi) \leq_{\mathfrak{K}} N'_i \leq \mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \alpha_{i+1}, \Phi)$
(f) if i is a limit ordinal then $N'_i = \mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \alpha_i, \Phi)$.

For $i = 0$, as $\mathrm{EM}(I^* \times \lambda, \Phi)$ is μ^+-saturated there is a $\mathfrak{K}$-embedding h_0 of N_0 into $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \lambda, \Phi)$. As $\mathrm{Rang}(h)$ has cardinality μ, there is $u_0 \subseteq \lambda$ of cardinality μ such that $\mathrm{Rang}(h'_0) \subseteq \mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times u_0, \Phi)$. So $\alpha_0 =: \mathrm{otp}(u_0)$ is an ordinal $\in [\mu_i, \mu_{i+1})$ hence $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \mu_0, \Phi) \cong \mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \alpha_i, \Phi)$ so without loss of generality $u_0 = \alpha_i$.

For i limit take union. The case $i = j+1$ is similar to $i = 0$ using amalgamation.

As we have used only N_{i+1} universal over N_i by replacing $\langle N_i : i \leq \delta \rangle$ by a longer sequence and renaming without loss of generality $N'_i = \mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \alpha_i, \Phi)$.

Alternatively,

$\boxtimes_1$ if $\alpha < \mu^+$ then for some $\beta \in (\alpha, \mu^+)$ the model $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \beta, \Phi)$ is $\leq_{\mathfrak{K}}$-universal over $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \beta, \Phi)$.

[Why? We know that there is $N \in K_M$ universal over $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \alpha, \Phi)$. As $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \lambda, \Phi)$ is μ^+-saturated there is a $\leq_{\mathfrak{K}}$-embedding g of N over $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \alpha, \Phi)$ into $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \lambda, \Phi)$. As $|\mathrm{Rang}(g)| \leq \mu$ there is a set $u \subseteq \lambda$ of cardinality μ which includes α and $\mathrm{Rang}(N) \subseteq \mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times u, \Phi)$.

So $\mathrm{otp}(u, <)$ is an ordinal of cardinality μ call it β and let $h : u \to \beta$ be an isomorphism, so $h \restriction \alpha = \mathrm{id}_\alpha$ and let $\hat{h}$ be the isomorphism from $\mathrm{EM}(I^* \times u, \Phi)$ onto $\mathrm{EM}(I^* \times \beta, \Phi)$ which h induces. Clearly it is the identity on $\mathrm{EM}(I \times \alpha, \Phi)$. Now $\beta, \hat{h} \circ g$ are as required.

$\boxtimes_2$ if $\alpha < \mu^+$ and $\kappa \le \mu$ then for some $\beta \in (\alpha, \mu^+)$, $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \alpha, \Phi) <^1_{\mu,\kappa}$ $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \beta, \Phi)$ is ??
[Why? Iterate $\boxtimes_1$]

$\boxtimes_3$ there are $\langle \alpha_i : i \le \delta \rangle$, an increasing continuous sequence of ordinals $< \mu^+$ and $\langle h_i : i \le \delta \rangle$ such that h_{1+i} is an isomorphism from N_i onto $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \alpha_i, \Phi), h_i$ increases with i.
[Why? By $\boxtimes_2$ and the uniqueness for $<^1_{\mu,\kappa}$.]

Now h_δ is defined $h_\delta : N_\delta \overset{\text{onto}}{\to} \mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \alpha_\delta, \Phi)$, so as $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \lambda, \Phi)$ is μ^+-saturated, $h_\delta(p)$ is realized in $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \alpha_\delta, \Phi)$ say by $\bar a$, so let $\bar a = \bar\sigma(x_{(t_1,\gamma_1)}, \dots, x_{(t_n,\gamma_n)})$ where $\bar\sigma$ is a sequence of terms in $\tau(\Phi)$ and (t_ℓ, γ_ℓ) is increasing with ℓ (in $I^* \times \lambda$). Let $\beta < \delta$ be such that:

$$\{\gamma_1, \dots, \gamma_n\} \cap \alpha_\delta \subseteq \alpha_\beta.$$

Let

$$\gamma'_\ell = \begin{cases} \gamma_\ell & \underline{\text{if}} \quad \gamma_\ell < \alpha_\delta \\ \lambda + \gamma_\ell & \underline{\text{if}} \quad \gamma_\ell \ge \alpha_\delta \end{cases}$$

Then in the model $N = \mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \lambda + \lambda, \Phi)$, we shall show that the finite sequence $\bar a' = \bar\sigma\big(x_{(t_1,\gamma'_1)}, \dots, x_{(t_n\gamma'_n)}\big)$ realizes a type as required over $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \lambda, \Phi)$. Why? Let $M_\gamma = \mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times \alpha_\gamma, \Phi)$ for $\gamma < \delta$. Assume toward contradiction that

$(*)$ $\mathrm{tp}(\bar a', M, N)$ does μ-split over $M_{\beta+1}$.

Let $\bar{\mathbf{c}}, \bar{\mathbf{b}} \in {}^\mu M$ realize the same type over $M_{\beta+1}$ but witness splitting.

We can find $w \subseteq \lambda, |w| \le \mu$ such that $\bar{\mathbf{c}}, \bar{\mathbf{b}} \subseteq \mathrm{EM}(I^* \times w, \Phi)$. Choose γ such that

$$\sup(w) < \gamma < \lambda.$$

Let $M^- = \mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times (\alpha_\delta \cup w \cup [\gamma, \lambda)), \Phi) <_{\mathfrak{K}} M$.
Let $N^- = \mathrm{EM}_{\tau(\mathfrak{K})}(I^* \times (\alpha_\delta \cup w \cup [\gamma, \lambda) \cup [\lambda, \lambda + \lambda)), \Phi) <_{\mathfrak{K}} N$.
So still $\bar{\mathbf{c}}, \bar{\mathbf{b}}$ witness that $\mathrm{tp}(\bar a', M^-, N^-)$ does μ-split over $M_{\beta+1}$.

There is an automorphism f of the linear order $I^* \times (\alpha_\delta \cup w \cup [\gamma, \lambda)) \cup [\lambda, \lambda + \lambda))$ such that

$$f \restriction (I^* \times \alpha_{\beta+1}) = \text{ the identity}$$

$$f \restriction (I^* \times [\gamma + 1, \lambda + \lambda)) = \text{ the identity}$$

$$\mathrm{Rang}\big(f \restriction (I^* \times w)\big) \subseteq I^* \times [\alpha_{\beta+1}, \alpha_{\beta+2}).$$

Now f induces an automorphism of N^- naturally called $\hat{f}$.
So

$$\hat{f} \restriction M_\beta = \text{ identity}$$

$$\hat{f}(\bar{a}') = \bar{a}'$$

$$\hat{f}(M^-) = M^-$$

As $\hat{f}$ is an automorphism, $\hat{f}(\bar{\mathbf{c}}), \hat{f}(\bar{\mathbf{b}})$ witness that $\mathrm{tp}(\hat{f}(\bar{a}'), \hat{f}(M^-), \hat{f}(N^-))$ does μ-split over $\hat{f}(M_{\alpha_{\beta+1}})$; i.e. $\mathrm{tp}(\bar{a}', M^-, N^-)$ does μ-split over $M_{\alpha_{\beta+1}}$. So $\mathrm{tp}(\bar{a}', M_{\alpha_{\beta+2}}, N)$ does μ-split over $M_{\alpha_{\beta+1}}$.
Now choose $\alpha_\gamma < \mu^+$ for $\gamma \in (\delta, \mu^+]$, increasing continuous by

$$\alpha_{\delta+i} = \alpha_\delta + i$$

$$M_\gamma = \text{ EM}_{\tau(\mathfrak{K})}(I^* \times \alpha_\gamma, \Phi).$$

So $\langle M_\gamma : \gamma \leq \mu \rangle$ is increasing continuous. So for $\gamma_1 \in [\beta, \mu^+)$ there is $f \in \text{ AUT}(I^* \times (\lambda + \lambda))$ such that

$$f \restriction I^* \times \alpha_\beta = \text{ identity}$$

$$f \text{ takes } I^* \times [\alpha_\beta, \alpha_{\beta+1}) \text{ onto } I^* \times [\alpha_\beta, \alpha_{\gamma_1+1})$$

$$f \text{ takes } I^* \times [\alpha_{\beta+1}, \alpha_{\beta+2}) \text{ onto } I^* \times \{\alpha_{\gamma_1+1}\}$$

$$f \text{ takes } I^* \times [\alpha_{\beta+2}, \alpha_{\gamma_1+2}) \text{ onto } I^* \times \{\alpha_{\gamma_1+2}\}$$

$$f \restriction I^* \times [\alpha_{\gamma_1+2}, \lambda + \lambda) = \text{ identity.}$$

As before this shows (using obvious monotonicity of μ-splitting)

$$\text{tp}(\bar{a}', M_{\gamma_1+2}N)\ \mu\text{-splits over } M_{\gamma_1+1}.$$

So $\{\gamma < \mu : \text{tp}(\bar{a}', M_{\gamma+1}, N)$ does μ-split over $M_\gamma\}$ has order type μ, so without loss of generality is μ. By 3.3(2) we get a contradiction. $\square_{6.3}$

6.5 Theorem. *Suppose K categorical in λ and the model in K_λ is μ^+-saturated (e.g.* $\text{cf}(\lambda) > \mu$*) and* $\text{LS}(\mathfrak{K}) < \mu < \lambda$.
1) $M <^1_{\mu,\kappa} N \Rightarrow N$ is saturated *if* $\text{LS}(\mathfrak{K}) < \mu$.
2) If $\kappa_1, \kappa_2 \le \mu$ are regular cardinals (or just limit ordinals) and for $\ell = 1,2$ we have $M_\ell <^1_{\mu,\kappa_\ell} N_\ell$, *then* *$N_1 \cong N_2$.*
3) There is $M \in K_\mu$ which is saturated.
4) If κ_1, κ_2 are as above $M \le^1_{\mu,\kappa_\ell} N_\ell \Rightarrow N_1 \cong_M N_2$ (in fact this holds for $\mu =$ $\text{LS}(\mathfrak{K})$*, too).*

6.6 Remark. 1) The model we get by (2) we call **the saturated model** of $\mathfrak{K}$ in μ.
2) Formally - we do not use 6.3.
3) If $M <^1_{\mu,\kappa} N$, we call N **brimmed over** M.

Proof. 1) By the uniqueness proofs 2.2 as $M <^1_{\mu,\kappa} N$ there is an $<_{\mathfrak{K}}$-increasing continuous sequence $\langle M_i : i \le \kappa\rangle$ satisfying $M_i <^1_{\mu,\kappa} M_{i+1}, M_0 = M, M_\kappa = N$ and as in the proof of 6.3 without loss of generality $M_i = \text{EM}_{\tau(\mathfrak{K})}(\alpha_i, \Phi)$ where $\alpha_i < \mu^+$.

To prove $N = N_\kappa$ is μ-saturated suppose $p \in \mathscr{S}^1(M^*), M^* \le_{\mathfrak{K}} N, \|M^*\| < \mu$; as we can extend M^* (as long as its power is $< \mu$ and it is $<_{\mathfrak{K}} N$), without loss of generality $M^* = \text{EM}_{\tau(\mathfrak{K})}(J, \Phi), J \subseteq \alpha_\kappa, |J| < \mu$.

So for some γ we have $[\gamma, \gamma+\omega) \cap J = \emptyset$ and $\gamma + \omega \le \alpha_\kappa$. We can replace $[\gamma, \gamma+\omega)$ by a copy of λ; this will make the model μ-saturated. That is we can find a linear order I^* such that $(\alpha_\delta, <) \subseteq I^*$ and $t \in I^* \backslash \alpha_\delta \Rightarrow I^* \models$ "$\gamma < t < (\gamma+1)$" and $\lambda = \text{otp}(I^+ \restriction \{t : t \in I^* \backslash \alpha_\delta\})$ so $\text{EM}_{\tau(\mathfrak{K})}(I^*, \Phi)$ is a $\le_{\mathfrak{K}}$-extension of $N = N_\kappa$ and belongs to K_λ hence is μ^+-saturated [alternatively, use $I^* \times$ ordinal as in a proof of 6.3].
But easily this introduces no new types realized over M^*. So p is realized.
2) In detail assume $M <_{\mu,\kappa_\ell} N_\ell$ for $\ell = 1,2$. So we can find $\langle M_{\ell,i} : i \le \kappa\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous such that $M \le_{\mathfrak{K}} M_{\ell,0}, M_{\ell,i} <^1_{\mu,\aleph_0} M_{\ell,i+1}$ and $M_{\ell,\kappa} = N_\ell$. So let $\alpha_{\ell,i} < \mu$ be increasing continuous for $i \le \kappa_\ell$, divisible by μ and an isomorphism h_ℓ from N_ℓ onto $\text{EM}_{\tau(\mathfrak{K})}(\alpha_{\ell,\kappa_\ell}, \Phi)$ such that $h_1 \restriction M = h_2 \restriction M, h_\ell(N_{\ell,i+1}) = \text{EM}_{\tau(\mathfrak{K})}(\alpha_{\ell,1+i}, \Phi)$. Let $\alpha_{1,\kappa} = \cup\{I_j : j < \kappa_2\}, I_j$ increasing continuous with

$j, |I_{j+1} \backslash I_j| = \mu$. Easily $\langle \mathrm{EM}(I_j, \Phi) : j \le \kappa \rangle$ exemplify that $h_1(N_1)$ is (μ, κ_2)-brimmed over $h(M)$. So "$M <^1_{\mu,\kappa_1} N \Rightarrow M <^1_{\mu,\kappa_1} N$ and so by 2.2(5) we are done.
3) Follows from the proof of part 1) and 2.2(1) + 1.7.
4) Similarly. $\square_{6.5}$

Remark. In part (1) we have used just $\mathrm{cf}(\lambda) > \mu >$ LS($\mathfrak{K}$).

6.7 Claim. *1) Assume K categorical in λ, $\mathrm{cf}(\lambda) > \mu > \mathrm{LS}(\mathfrak{K})$. If $N_i \in K_\mu$ is saturated, increasing with i for $i < \delta$ and $\delta < \mu^+$ then $N = \bigcup_{i<\delta} N_i \in K_\mu$ is saturated.*
2) [K categorical in λ, $\mathrm{cf}(\lambda) > \mu > \mathrm{LS}(\mathfrak{K})$]. Possibly changing Φ (actually as in Definition 8.4(2). If I is a linear order of cardinality μ then $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ is μ-saturated. Moreover, if $I \subset J$ are linear order of cardinality μ then $\mathrm{EM}_{\tau(\mathfrak{K})}(J, \Phi)$ is a universal $\le_{\mathfrak{K}}$-extension of $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ and even brimmed over it.

Proof. 1) We prove this by induction on δ, so by the induction hypothesis without loss of generality $\langle N_i : i < \delta \rangle$ is not just $\le_{\mathfrak{K}}$-increasing and contradicts the conclusion but also is increasing continuous and each N_i saturated. Without loss of generality $\delta = \mathrm{cf}(\delta)$. If $\mathrm{cf}(\delta) = \mu$ the conclusion clearly holds so assume $\mathrm{cf}(\delta) < \mu$. Let $M \le_{\mathfrak{K}} N, \|M\| < \mu$ and $p \in \mathscr{S}(M)$ be omitted in N and let $\theta = \delta + \|M\| + \mathrm{LS}(\mathfrak{K}) < \mu$, and let $p \le q \in \mathscr{S}(N)$. Now we can choose by induction on $i \le \delta, M_i \le_{\mathfrak{K}} N_i$ and $M^+_i \le_{\mathfrak{K}} N$ such that $M_i \in K_\theta, M^+_i \in K_\theta, M_i$ is $\le_{\mathfrak{K}}$-increasing continuous and $M \cap N_i \subseteq M_i, j < i \Rightarrow M^+_j \cap N_i \subseteq M_{i+1}$ and $M_i <^1_{\theta,\omega} M_{i+1}$ and if q does θ-split over M_i then $q \restriction M^+_i$ does θ-split over M_i.

So by 6.3, 6.5 we know that M_δ is saturated, and for some $i(*) < \delta$ we have: $q \restriction M_\delta$ does not θ-split over $M_{i(*)}$. But $M^+_{i(*)} \subseteq N = \bigcup_{i<\delta} N_i, M^+_{i(*)} \cap N_j \subseteq M_{j+1}$ so $M^+_{i(*)} \subseteq M_\delta$. So necessarily $q \in \mathscr{S}(N)$ satisfies $i(*) \le i < \delta$ implies that $q \restriction N_i$ does not θ-split over $M_{i(*)}$.

Now we choose by induction on $\alpha < \theta^+, M_{i(*),\alpha}, b_\alpha, f_\alpha$ such that:
$M_{i(*),\alpha} \in K_\theta, M_{i(*)} \le_{\mathfrak{K}} M_{i(*),\alpha} \le_{\mathfrak{K}} N_{i(*)}, M_{i(*),\alpha}$ is $\le_{\mathfrak{K}}$-increasing continuous in $\alpha, b_\alpha \in N_{i(*)}$ realizes $q \restriction M_{i(*),\alpha}, f_\alpha$ is a function with domain M_δ and range $\subseteq N_{i(*)}$ such that the sequences $\bar c = \langle c : c \in M_\delta \rangle$ and $\bar c^\alpha =: \langle f_\alpha(c) : c \in M_\delta \rangle$ realize the same type over $M_{i(*),\alpha}$ and $\{b_\alpha\} \cup \mathrm{Rang}(f_\alpha) \subseteq M_{i(*),\alpha+1}$. As $N_{i(*)} \in K_\mu$ is saturated and $\mathrm{LS}(\mathfrak{K}) \le \theta < \mu$ we can carry the construction; if some b_α realizes $q \restriction M_\delta$ we are done (as $p = q \restriction M, M \le_{\mathfrak{K}} M_\delta$ and $b_\alpha \in N$ realizes p). Let $d \in \mathfrak{C}$ realize q so

$(*)_1$ $\alpha < \beta < \theta^+ \Rightarrow \bar c^\beta \char`^ \langle b_\alpha \rangle$ does not realize $\mathrm{tp}(\bar c \char`^ \langle d \rangle, M_{i(*)}, \mathfrak{C})$.
[Why? As $\bar c \char`^ \langle b_\alpha \rangle$ does not realize $\mathrm{tp}(\bar c \char`^ \langle d \rangle, M_{i(*)}, \mathfrak{C})$ because d realizes $p = q \restriction \bar c$ whereas b_α does not realize $p = q \restriction \bar c$.]

On the other hand as q does not θ-split over $M_{i(*)}$ we have $\mathrm{tp}(\bar c \char`^ \langle d \rangle, M_{i(*)}, \mathfrak{C}) = \mathrm{tp}(\bar c^\alpha \char`^ \langle d \rangle, M_{i(*)}, \mathfrak{C})$ so by the choice of b_β:

$(*)_2$ if $\alpha < \beta < \theta^+$ then $\bar c^\alpha \char`^ \langle b_\beta \rangle$ realizes $\mathrm{tp}(\bar c \char`^ \langle d \rangle, M_{i(*)}, \mathfrak{C})$.

We are almost done by 4.15.
[Why only almost? We would like to use the "θ-order property fail", now if we could define $\langle \bar c^\beta \char`^ \langle b_\beta \rangle :$ for $\beta < (2^\theta)^+ \rangle$ fine, but we have only $\alpha < \theta^+$, this is too short.] Now we will refine the construction to make $\langle \bar c^\beta \char`^ \langle b_\beta \rangle : \beta < \theta^+ \rangle$ strictly indiscernible which will be enough. As $N_{i(*)}$ is saturated without loss of generality $N_{i(*)} = \mathrm{EM}_{\tau(\mathfrak{K})}(\mu, \Phi)$ and $M_{i(*)} = \mathrm{EM}_{\tau(\mathfrak{K})}(\theta, \Phi)$ (using 6.8 below). As before for some $\gamma < \theta^+$ there are sequences $\bar c', \bar b'$ in $\mathrm{EM}_{\tau(\mathfrak{K})}(\mu + \gamma, \Phi)$ realizing $\mathrm{tp}(\bar c, N_{i(*)}, \mathfrak{C}), q \restriction N_{i(*)}$ respectively, here we use $\mathrm{cf}(\lambda) > \mu$ rather than just $\mathrm{cf}(\lambda) \ge \mu$. For each $\beta < \theta^+$ there is a canonical isomorphism g_β from $\mathrm{EM}_{\tau(\Phi)}(\beta \cup [\mu, \mu + \gamma), \Phi)$ onto $\mathrm{EM}_{\tau(\Phi)}(\beta + \gamma, \Phi)$. So without loss of generality $M_{i(*),\alpha} = \mathrm{EM}_{\tau(\mathfrak{K})}(\theta + \gamma_\alpha, \Phi), \bar c^\alpha = g_{\theta + \gamma_\alpha}(\bar c'), b_\alpha = g_{\theta+\gamma_\alpha}(b')$. So $(*)_1 + (*)_2$ gives the $(\theta, 1, \theta)$-order property contradicting categoricity by 4.8(1) as $\theta^+ \le \mu < \lambda$.
2) By (1) and 6.8 below. $\square_{6.7}$

We really proved, in 6.5 (from λ categoricity):

6.8 Subfact. Assume K is categorical in λ.
1) If $I \subseteq J$ are linear order, of power $<$ cf(λ);

$(*)$ $t \in J \backslash I \Rightarrow \left(\exists^{\aleph_0} s \in J\right)[s \sim_I t]$ where $s \sim_I t$ means "s, t realize the same Dedekind cut",

<u>then</u> every type over $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ is realized in $\mathrm{EM}_{\tau(\mathfrak{K})}(J, \Phi)$.
2) If $I \subseteq J$ are linear orders of cardinality $<$ cf$(\lambda), \kappa \le \mu, |J \backslash I| = |J|$ and $(*)$ above <u>then</u> $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi) <^1_{\mu,\kappa} \mathrm{EM}_{\tau(\mathfrak{K})}(J, \Phi)$ where.
3) We can find Φ' (in fact $\Phi \le^\otimes \Phi' \in \Upsilon^{\mathrm{or}}_{\mathrm{LS}(\mathfrak{K})}$ in the notation of 8.3(3)) such that

(a) Φ' is as in 0.6
(b) if $I \subset J$ and $\mathrm{LS}(\mathfrak{K}) \le |I| \le |J| \le \lambda$ then $\mathrm{EM}_{\tau(\mathfrak{K})}(J, \Phi')$ is a brimmed over $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi')$.

Proof. 1) Why? Use the proof of 6.5(1).
Replace the cut of t in I by λ: we get cf(λ)-saturated model.
2) As in the proof of 6.5(2).
3) For a linear order I we can define $I_* = \{\eta : \eta$ is a finite sequence, and for $i < \ell g(\eta)$ we have i even $\Rightarrow \eta(i) \in I$ and i odd $\Rightarrow \eta(i) \in \{-1,1\}\}$.
Ordered by $\eta <_{I_*} \nu$ iff

$(\exists \text{ even } i)(\eta \restriction i = \nu \restriction i \wedge \eta(i) <_I \nu(i))$ or
$(\exists \text{ odd } i)(\eta \restriction i = \nu \restriction i \wedge \eta(i) = -1 \wedge \nu(i) = 1)$ or
$\ell g(\eta)$ odd $\wedge \ell g(\eta) < \ell g(\nu) \wedge \nu(\ell g(\eta)) = 1$ or
$\ell g(\eta)$ even $\wedge \ell g(\eta) < \ell g(\nu)$.

We can choose $\Phi', |\tau(\Phi')| =$ LS($\mathfrak{K}$) such that for every linear oder I, $\text{EM}(I,\Phi') = \text{EM}(I_*,\Phi)$. Now if $I \subset J, |\text{LS}(\mathfrak{K}) \leq |J|$ then $|J_*| = J, I_* \subset J_*$, moreover for every $t \in J_* \backslash I_*$ the set $\{s \in J_* \backslash I^* : s$ realizes the same cut of I_* as $t\}$ has cardinality $|J|$; moreover we can find $|J|$ pairwise disjoint intervals of J_*, disjoint to I_* so list them as $\langle (a_\alpha, b_\alpha) : \alpha < |J| \times |J| \rangle$ for $i < |J| \times |J|$. Let $I_0 = I, I_{1+i} = J_* \backslash \cup \{|a_\alpha, b_\alpha| : i \leq \alpha < |J| \times |J|\}$ for $i < |J| \times |J|$. So $\langle I_\alpha : \alpha \leq |J| \times |J| \rangle$ is an increasing continuous sequence of suborder of J_*, with $I_0 = I_*, I_{|J| \times |J|} = J_*$. Let $M_\alpha = \text{EM}(I_\alpha,\Phi)$. So $\langle M_\alpha : \alpha \leq |J| \times |J| \rangle$ is increasing continuous, $M_0 = \text{EM}(I,\Phi'), M_{|J|\times|J|} = \text{EM}(J,\Phi'), \langle M'_\alpha = M_\alpha \restriction \tau(\mathfrak{K}) : \alpha \leq |J| \times |J| \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous. By the previous parts (and the choice of the I_α's), every $p \in \mathscr{S}(M'_\alpha)$ is realized in $M'_{\alpha+1}$, hence $M'_{|J|\times|J|} = \text{EM}_{\tau(\mathfrak{K})}(J,\Phi')$ is brimmed over $M'_0 = \text{EM}_{\tau(\mathfrak{K})}(I,\Phi)$ as required. $\square_{6.8}$

§7 More on Splitting

7.1 Hypothesis. As before + conclusions of §6 for $\mu \in [\mathrm{LS}(\mathfrak{K}), \mathrm{cf}(\lambda))$.
So

$(*)(a)$ $\mathfrak{K}$ has a saturated model in μ.

(b) union of increasing chain of saturated models in K_μ of length $\leq \mu$ is saturated.

(c) if $\langle M_i : i \leq \delta \rangle$ increasing continuous in K_μ, each M_{i+1} saturated over M_i (the previous one), $p \in \mathscr{S}(M_\delta)$ then for some $i < \delta$, p does not μ-split over M_i.

7.2 Conclusion. If $p \in \mathscr{S}^m(M)$ and $M \in K_\mu$ is saturated, then for some $M^- <^1_{\mu,\omega} M, M^- \in K_\mu$ is saturated and p does not μ-split over M^-.

Proof. We can find $\langle M_n : n \leq \omega \rangle$ in K_μ, each M_n saturated $M_n \leq^1_{\mu,\omega} M_{n+1}$ and $M_\omega = \bigcup\limits_{n<\omega} M_n$ so as M_ω is saturated, without loss of generality $M_\omega = M$. Now using $(*)(c)$ of 7.1 some M_n is O.K. as M^-. $\square_{7.2}$

7.3 Fact. If $M_0 \leq^1_{\mu,\omega} M_2 \leq^1_{\mu,\omega} M_3, p \in \mathscr{S}^m(M_3)$ and p does not μ-split over M_0, then $R(p) = R(p \restriction M_2)$, see Definition 5.1.

Proof. We can find (by uniqueness) $M_1 \in K_\mu$ such that $M_0 \leq^1_{\mu,\omega} M_1 \leq^1_{\mu,\omega} M_2$ and we can find $M_4 \in K_\mu$ such that $M_3 \leq^1_{\mu,\omega} M_4$.
We can find an isomorphism h_1 from M_3 onto M_2 over M_1 (by the uniqueness properties $<^1_{\mu,\omega}$). By uniqueness 2.2(1) there is an automorphism h of M_4 extending h_1. Also by uniqueness there is $q \in \mathscr{S}(M_4)$ which does not μ-split over M_0 and extend $p \restriction M_1$ (e.g., there is an isomorphism g from M_3 onto M_4 over M_2 and let $q = g(p)$). As $p, q \restriction M_3$ do not μ-split over M_0 and have the same restriction to M_1 and $M_0 \leq^1_{\mu,\omega} M_1$ clearly $p = q \restriction M_3$. Consider q and $h(q)$ both from $\mathscr{S}(M_4)$, both do not μ-split over M_0 and have the same restriction to M_1 and $\mathrm{id}_{M_1} \subseteq h$; as $M_0 <^1_{\mu,\omega} M_1$ it follows that $q = h(q)$.
So $R(p \restriction M_2) = R(q \restriction M_2) = R(h(q \restriction M_3)) = R(q \restriction M_3) = R(p)$ as required. $\square_{7.3}$

7.4 Claim. *[Here?] [K categorical in λ, $\mathrm{cf}(\lambda) > \mu > \mathrm{LS}(\mathfrak{K})$].*

Suppose $m < \omega, M \in K_\mu$ is saturated, $p \in \mathscr{S}^m(M), M \le_{\mathfrak{K}} N \in K_\mu, p \le q \in \mathscr{S}^m(N), N$ brimmed over M, q not a stationarization of p (i.e. for no $M^- <^\circ_{\mu,\omega} M$, q does not μ-split over M^-). Then q does μ-divide over M.

Proof. By 7.5 below and 6.3 (just p does not μ-split over some N_m where $\langle N_\alpha : \alpha \le \omega\rangle$ witness $N_0 <^1_{\mu,\omega} M$).

7.5 Claim. *[$\mathfrak{K}$ categorical in λ and $\mathrm{cf}(\lambda) > \mu > \mathrm{LS}(\mathfrak{K})$]*
Assume $M_0 <^1_{\mu,\omega} M_1 <^1_{\mu,\omega} M_2$ all saturated. If $q \in \mathscr{S}(M_2)$ does not μ-split over M_1 and $q \restriction M_1$ does not μ-split over M_0, then q does not μ-split over M_0.

Proof. Let $M_3 \in K_\mu$ be such that $M_2 <^1_{\mu,\omega} M_3$ and $c \in M_3$ realizes q. Choose a linear order I^* of cardinality μ such that $I^* \times (\mu + \omega^*) \cong I^* \cong I^* \times \mu$, remember that on the product we do not use lexicographic order. I^* has no first nor last element (see [Sh 220, AP]).

Let $I_0 = I^* \times \mu, I_1 = I_0 + I^* \times \mathbb{Z}, I_2 = I_1 + I^* \times \mathbb{Z}, I_3 = I_2 + I^* \times \mu$. Clearly without loss of generality $M_\ell = \mathrm{EM}_{\tau(\mathfrak{K})}(\Phi, I_\ell)$, let $c = \tau(\bar a_{t_0}, \dots, \bar a_{t_k})$ so $t_0, \dots, t_k \in I_3$; let $I_{1,n} = I_0 + I^* \times \{m : \mathbb{Z} \models m < n\}$ and $I_{2,n} = I_1 + I^* \times \{m : \mathbb{Z} \models m < n\}$ and $I_{0,\alpha} = I^* \times \alpha$. So we can find a (negative) integer $n(*)$ small enough and $m(*) \in \mathbb{Z}$ large enough such that $\{t_0, \dots, t_n\} \cap I_{2,n(*)+1} \subseteq I_{1,m(*)-1}$. Let $M_{1,n} = \mathrm{EM}_{\tau(\mathfrak{K})}(I_{1,n}, \Phi)$ and $M_{2,n} = \mathrm{EM}_{\tau(\mathfrak{K})}(I_{2,n}, \Phi)$. Clearly $M_0 <^1_{\mu,\omega} M_{1,n} <^1_{\mu,\omega} M_1 <^1_{\mu,\omega} M_{2,n} <^1_{\mu,\omega} M_2$. Clearly (use automorphism of I_3)

$(*)_0$ $q \restriction M_{2,n}$ does not μ-split over $M_{1,m}$ if $\mathbb{Z} \models n < n(*), m(*) \le m \in \mathbb{Z}$.

As q does not μ-split over M_1 and $M_{2,n+1}$ is brimmed over $M_{2,n}$ for $n \in \mathbb{Z}$, etc., by 7.3 with $q, M_1, M_{2,n}, M_2, q$ here standing for M_0, M_2, M_3, p there we get

$(*)_1$ $R(q) = R(q \restriction M_{2,n})$ if $n \in \mathbb{Z}$.

Similarly

$(*)_2$ $R(q \restriction M_1) = R(q \restriction M_{1,m})$ if $m \in \mathbb{Z}$.

By $(*)_0$ and 7.3 we have

$(*)_3$ $R(q \restriction M_{2,n(*)}) = R(q \restriction M_{1,m(*)})$.

Similarly we can find a successor ordinal $\alpha(*) < \mu$ and $k(*) \in \mathbb{Z}$ such that

$$\{t_0, \ldots, t_k\} \cap I_{1,k(*)+1} \subseteq I_{0,\alpha(*)-1}$$

and then prove

$(*)_4$ $R(q \restriction M_0) = R(q \restriction M_{0,\alpha})$ if $\alpha(*) \leq \alpha < \mu$

$(*)_5$ $R(q \restriction M_{1,\ell(*)}) = R(q \restriction M_{0,\alpha})$ if $\alpha(*) \leq \alpha < \mu$.

Together $R(q) = R(q \restriction M_0)$, hence q does not μ-split over M_0 as required. $\square_{7.4}$

PART II

§8 Existence of nice Φ

We build EM models, where "equality of types over A in the sense of the existence of automorphisms over A" behaves nicely.

8.1 Context.

(a) $\mathfrak{K}$ is an abstract elementary class with models of cardinality $\geq \beth_{(2^{\mathrm{LS}(\mathfrak{K})})^+}$.

8.2 Remark. Mostly it suffices to assume $((\alpha), (\beta)$ for 8.6, 8.7 omitting the second clause in 8.6(b), 8.7(3); $(\delta) - (\zeta)$ for $\leq^{\oplus}, \leq^{\otimes}$

(a)$'$ (α) $\mathfrak{K}$ is a class of $\tau(K)$-models, which is $\mathrm{PC}_{2^\kappa,\kappa}$, and

(β) we interpret $\mathrm{LS}(\mathfrak{K})$ as κ such that

(γ) $\mathfrak{K}$ has a model of cardinality $\geq \beth_{(2^{\mathrm{LS}(\mathfrak{K})})^+}$

(δ) $\leq_{\mathfrak{K}}$ is a $\mathrm{PC}_{2^\kappa,\kappa}$ partial order $\leq_{\mathfrak{K}}$ on $\mathfrak{K}$

(ε) $\leq_{\mathfrak{K}}$ is closed under increasing continuous chains (in 8.5(3) hence $\Upsilon^{\mathrm{or}}_\kappa \neq \emptyset$ (see below) for $\kappa \geq \mathrm{LS}(\mathfrak{K})$ is not empty)

(ζ) preserve indiscernible isomorphism.

8.3 Definition. 1) Let $\kappa \geq \mathrm{LS}(\mathfrak{K})$, now $\Upsilon^{\mathrm{or}}_\kappa = \Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$ is the family of Φ proper for linear orders (see [Sh:c, Ch.VII]) such that:

(a) $|\tau(\Phi)| \leq \kappa$ and $\tau_{\mathfrak{K}} \subseteq \tau(\Phi)$

(b) $\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi) = \mathrm{EM}(I,\Phi) \restriction \tau(K) \in K$

(c) $I \subseteq J \Rightarrow \mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi) \leq_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(J,\Phi)$

(d) Φ is as in 6.8(3), (needed only in §9).

2) $\Upsilon^{\mathrm{or}} = \Upsilon^{\mathrm{or}}_{[\mathfrak{K}]}$ is $\Upsilon^{\mathrm{or}}_{\mathrm{LS}(\mathfrak{K})}$.

8.4 Definition. We define partial orders $\leq^{\oplus}_\kappa$ and $\leq^{\otimes}_\kappa$ on $\Upsilon^{\rm or}_\kappa$ (for $\kappa \geq \mathrm{LS}(\mathfrak{K})$):
1) $\Psi_1 \leq^{\oplus}_\kappa \Psi_2$ if $\tau(\Psi_1) \subseteq \tau(\Psi_2)$ and $\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Psi_1) \leq_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(I,\Psi_2)$ and $\mathrm{EM}(I,\Psi_1) = \mathrm{EM}_{\tau(\Psi_1)}(I,\Psi_1) \subseteq \mathrm{EM}_{\tau(\Psi_1)}(I,\Psi_2)$ for any linear order I.
Again for $\kappa = \mathrm{LS}(\mathfrak{K})$ we may drop the κ.
2) For $\Phi_1,\Phi_2 \in \Upsilon^{\rm or}_\kappa$, we say Φ_2 is an inessential extension of Φ_1 and write $\Phi_1 \leq^{\rm ie}_\kappa \Phi_2$ if $\Phi_1 \leq^{\oplus}_\kappa \Phi_2$ and for every linear order I, we have

$$\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi_1) = \mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi_2).$$

(note: there may be more functions in $\tau(\Phi_2)$!)

3) Let $\Upsilon^{\rm lin}_\kappa$ be the class of Ψ proper for linear order and producing linear orders such that:

(a) $\tau(\Psi)$ has cardinality $\leq \kappa$,
(b) $\mathrm{EM}(I,\Psi)$ is a linear order which is an extension of I: in fact $[t \in I \Rightarrow x_t = t]$.

4) $\Phi_1 \leq^{\otimes}_\kappa \Phi_2$ iff there is Ψ such that

(a) $\Psi \in \Upsilon^{\rm lin}_\kappa$
(b) $\Phi_\ell \in \Upsilon^{\rm or}_\kappa$ for $\ell = 1,2$
(c) $\Phi'_2 \leq^{\rm ie}_\kappa \Phi_2$ where $\Phi'_2 = \Psi \circ \Phi_1$, i.e.

$$\mathrm{EM}(I,\Phi'_2) = \mathrm{EM}(\mathrm{EM}(I,\Psi),\Phi_1).$$

(So we allow further expansion by functions definable from earlier ones (composition or even definition by cases), as long as the number is $\leq \kappa$).

8.5 Claim. *1) $(\Upsilon^{\rm or}_\kappa, \leq^{\otimes}_\kappa)$ and $(\Upsilon^{\rm or}_\kappa, \leq^{\oplus})$ are partial orders (and $\leq^{\otimes}_\kappa \subseteq \leq^{\oplus}_\kappa$).*
2) If $\langle \Phi_i : i < \delta\rangle$ is a $\leq^{\otimes}_\kappa$-increasing sequence, $\delta < \kappa^+$, then it has a $<^{\otimes}_\kappa$-l.u.b. Φ; $\mathrm{EM}(I,\Phi) = \bigcup\limits_{i<\delta} \mathrm{EM}(I,\Phi_i)$.
3) Similarly for $<^{\oplus}_\kappa$.

Proof. Easy.

8.6 Lemma. *[The a.e.c. omitting type theorem]*
If $N \le_{\mathfrak{K}} M, \|M\| \ge \beth_{(2^\chi)^+}, \chi \ge \|N\| + \mathrm{LS}(\mathfrak{K})$, <u>then</u> there is $\Phi \in \Upsilon^{\mathrm{or}}_\chi$ so Φ is proper for linear order such that:

- (a) $\mathrm{EM}_{\tau(\mathfrak{K})}(\emptyset, \Phi) = N$
- (b) $N \le_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$, *and recall* $I \subseteq J \Rightarrow \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi) \le_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(J, \Phi)$
- (c) $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ *omits every type* $p \in \mathscr{S}(N)$ *which* M *omits, moreover if* I *is finite <u>then</u>* $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ *can be* $\le_{\mathfrak{K}}$*-embedded into* M
- (d) $|\tau_\Phi| \le \|N\| + \mathrm{LS}(\mathfrak{K})$ *and* Φ *non-trivial, hence,* $|\mathrm{EM}(I, \Phi)| = |I| + |\tau_\Phi|$ *for every linear order* I.

Proof. This is a particular case of 8.7 below when $N_1 = N_0$; which is proved in details (or see straight by [Sh 88, 1.7] or deduce by 4.6). $\square_{8.6}$

8.7 Lemma. *Assume*

- (a) $\mathrm{LS}(\mathfrak{K}) \le \chi \le \lambda$
- (b) $N_0 \le_{\mathfrak{K}} N_1 \le_{\mathfrak{K}} M$
- (c) $\|N_0\| \le \chi, \|N_1\| = \lambda$ *and* $\|M\| \ge \beth_{(2^\chi)^+}(\lambda)$
- (d) $\Gamma_0 = \{p^0_i : i < i^*_0\} \subseteq \mathscr{S}(N_0)$ *each* p^0_i *omitted by* M
- (e) $\Gamma_1 = \{p^1_i : i < i^*_1 \le \chi\} \subseteq \mathscr{S}(N_1)$ *such that for no* $i < i^*_1$ *and* $c \in M$ *does* c *realize* p^1_i/E_χ *[see Definition 1.8; where* c *realizes* p^1_i/E_χ *means that* c *realizes every restriction* $p^1_i \restriction M, M \le_{\mathfrak{K}} N_1, M \in \mathfrak{K}_{\le \chi}$*].*

<u>Then</u> we can find $\langle N'_\alpha : \alpha \le \omega \rangle, \Phi$ *and* $\langle q^1_i : i < i^*_1 \rangle$ *such that*

- (α) Φ *proper for linear orders*
- (β) $N'_\alpha \in \mathfrak{K}_{\le \chi}$ *is* $\le_{\mathfrak{K}}$*- increasing continuous (for* $\alpha \le \omega$*)*
- (γ) $N'_0 = N_0$ *and* $N'_\alpha \le_{\mathfrak{K}} N_1$
- (δ) $q^1_i \in \mathscr{S}(N'_\omega)$ *and* $q^1_i = p^1_i \restriction N'_\omega$
- (ϵ) $\mathrm{EM}_{\tau(\mathfrak{K})}(\emptyset, \Phi)$ *is* N'_0
- (ζ) *for linear order* $I \subseteq J$ *we have* $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi) \le_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(J, \Phi)$
- (η) *for each*[6] *finite linear order* I*, there is a* $\le_{\mathfrak{K}}$*-embedding* h_I *of* $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ *into* M *which extends* $\mathrm{id}_{N'_{|J|}}$

[6]the price for this nice formulation is that it may fail to satisfy $\mathrm{EM}(I_1, \Phi) \cap \mathrm{EM}(I_2, \Phi) = \mathrm{EM}(I_1 \cap I_2, \Phi)$ for $I_1 \cup I_2 \subseteq I_1$, i.e., for some n-place function $f(x_{t_1}, \ldots, x_{t_n})$ may be even constant.

(θ) *[main clause] for any linear order I, $\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi)$ omits every p^0_i for $i < i^*_0$ and omits every q^1_i in a strong sense: for every $a \in \mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi)$ and finite $J \subseteq I$ such that $a \in \mathrm{EM}_{\tau(\mathfrak{K})}(J,\Phi)$ we have $q^1_i \restriction N'_{|J|} \neq \mathrm{tp}(a, N'_{|J|}, h_J(\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi)))$.*

8.8 Remark. 1) So we really can replace q^1_i by $\langle q^1_i \restriction N'_n : n < \omega\rangle$, but for ω-chains by chasing arrows such limit (q^1_i) exists, see 1.12.
2) If $\bar a$ is a sequence in a model $M, c\ell(\bar a, M)$ is the closure of $\mathrm{Rang}(\bar a)$ under the functions of M.

Proof. By [Sh 88, 1.7] (and see 0.5) we can find $\tau_1, \tau(\mathfrak{K}) \subseteq \tau_1, |\tau_1| \le \chi$ (here we can have $|\tau_1| \le \mathrm{LS}(\mathfrak{K}) \le \chi$) and an expansion M^+ of M to a τ_1-model and a set Γ of quantifier free types (so $|\Gamma| \le 2^{\aleph_0 + |\mathrm{LS}(\mathfrak{K})|}$) such that:

(A) (α) M^+ omits every $p \in \Gamma$

(β) if M^* is a τ_1-model omitting every $p \in \Gamma$ <u>then</u> $M^* \restriction \tau(\mathfrak{K}) \in K$ and $N^* \subseteq M^* \Rightarrow N^* \restriction \tau(\mathfrak{K}) \le_{\mathfrak{K}} M^* \restriction \tau(\mathfrak{K})$.

So

(B) for $\bar a \in {}^{\omega>}M$ we let $M^+_{\bar a} = M^+ \restriction c\ell(\bar a, M^+)$ then $M^+_{\bar a} \restriction \tau(\mathfrak{K}) \le_{\mathfrak{K}} M^+ \restriction \tau(\mathfrak{K})$, $\mathrm{Rang}(\bar a) \subseteq \mathrm{Rang}(\bar b) \Rightarrow M^+_{\bar a} \restriction \tau(\mathfrak{K}) \le_{\mathfrak{K}} M^+_{\bar b} \restriction \tau(\mathfrak{K})$ where $\bar a \in {}^{\omega>}(N_\ell) \Rightarrow |M^+_{\bar a}| \subseteq N_\ell$.

Note that $M^+_{\bar a}$ has always cardinality $\le \chi$. Note that further expansion of M^+ to M^*, as long as $|\tau(M^*)| \le \chi$ preserves $(A) + (B)$; so we can add (for clause (E) we use the assumption (e), i.e., M omits p^1_i/E_χ, not just p^1_i)

(C) $N_0, M^+_{\langle\rangle}$ have the same universe and let $M^+_{\bar a,1} = M^+_{\bar a} \restriction (|N_1| \cap |M^+_{\bar a}|)$

(D) $N_0 \le_{\mathfrak{K}} M^+_{\bar a,1} \restriction \tau(\mathfrak{K}) \le_{\mathfrak{K}} M^+_{\bar a} \restriction \tau(\mathfrak{K})$

(E) for $i < i^*_1$, the type $p^1_i \restriction (M^+_{\bar a,1} \restriction \tau(\mathfrak{K}))$ is not realized in $M^+_{\bar a} \restriction \tau(\mathfrak{K})$;

(F) N_0, N_1 are closed under the functions of M^+, so $N^+_0 = M^+ \restriction |N_0|, N^+_1 = M^+ \restriction (N_1)$ are well defined τ_1-models omitting every $p \in \Gamma$.

Now we choose by induction on n, sequence $\langle f^n_\alpha : \alpha < (2^\chi)^+\rangle$ and N'_n such that:

(i) f^n_α is a one-to-one function from $\beth_\alpha(\lambda)$ into M

(ii) $\langle f^n_\alpha(\zeta) : \zeta < \beth_\alpha(\lambda)\rangle$ is n-indiscernible in M^+

(*iii*) moreover, if $\alpha, \beta < (2^\chi)^+$, and $m \le n$ and $\zeta_1 < \ldots < \zeta_m < \beth_\alpha(\lambda)$ and $\xi_1 < \ldots < \xi_m < \beth_\beta(\lambda)$ <u>then</u>: the sequences $\bar a = \langle f^n_\alpha(\zeta_1), \ldots, f^n_\alpha(\zeta_m)\rangle$, $\bar b = \langle f^m_\beta(\xi_1), \ldots, f^m_\beta(\xi_m)\rangle$ realize the same quantifier free type in M^+ over N^+_1, so there is a natural isomorphism $g_{\bar b, \bar a}$ from $M^+_{\bar a}$ onto $M^+_{\bar b}$ (mapping $f_\alpha(\zeta_\ell)$ to $f_\beta(\xi_\ell)$), moreover

(*iv*) $i < i^*_1 \Rightarrow g_{\bar b,\bar a}(p^1_i \restriction (M^+_{\bar a,1} \restriction \tau(\mathfrak{K}))) = p^1_i \restriction (M^+_{\bar b,1} \restriction \tau(\mathfrak{K}))$

(*v*) if $m < n$ then for every $\alpha < (2^\chi)^+$ there are β^* satisfying $\alpha < \beta^* < (2^\chi)^+$ and h^* an increasing function from $\beth_\alpha(\chi)$ to $\beth_{\beta^*}(\chi)$ such that $\zeta < \beth_\alpha(\chi) \Rightarrow f^n_\alpha(\zeta) = f^m_{\beta^*}(h^*(\zeta))$

(*vi*) $N'_n \le_{\mathfrak{K}} M_1$ and for every $m \le n < \omega, \alpha < (2^\kappa)^+$ and $\zeta_0 < \ldots < \zeta_{m_1} < \zeta_\alpha(\lambda), N'_n$ is $(M^+_1 \cap M^+_{\langle f^n_\alpha(\zeta_\ell):\ell<m\rangle}) \restriction \tau(\mathfrak{K}) = M_{\langle f^n_\alpha(\zeta_n):\ell<m\rangle,1}$.

As the indiscernibles in clause (iii) are over N^+_1 we can define, for $n \ge 1, N'_{\ell g(\bar a)} = (N^+_1 \cap M^+_{\bar a,1}) \restriction \tau(\mathfrak{K})$ for any $\bar a$ as in (iii), i.e., this restriction does not depend on n.

For $n = 0$ this is trivial. The induction step $n+1$ first for each $\alpha < (2^\chi)^+$ we apply Erdös Rado theorem for $f^n_{\alpha+\omega}$ getting $Y^n_\alpha \subseteq \beth_{\alpha+\omega}(\lambda)$ of cardinality $\beth_\alpha(\lambda)$ as in (iii) (also for (iv)) for $\alpha = \beta$. Then by the pigeon hole principle for some $X_n \subseteq (2^\lambda)^+$ of cardinality $(2^\chi)^+$ we have: if $\alpha_1, \alpha_2 \in X_n, \zeta^\ell_1 < \zeta^\ell_n$ belongs to Y_{α_ℓ} for $\ell = 1, 2$ then $\langle f^n_{\alpha_1+\omega}(\zeta^1_1), \ldots, f^n_{\alpha_1+\omega}(\zeta^1_n)\rangle, \langle f^n_{\alpha_2+\omega}(\zeta^2_1), \ldots, f^n_{\alpha_2+\omega}(\zeta^2_n)\rangle$ realize the same quantifier free type in M^+ over N^+_1.

Now we choose $f^{n+1}_\alpha(\zeta)$ as $f^n_{\alpha'+\omega}(\zeta')$ where $\alpha' \in X$, otp$(\alpha' \cap X) = \alpha, \zeta' \in Y_\alpha$, otp$(\zeta' \cap Y_\alpha) = \zeta$. Having carried the induction we choose Φ such that for every $n < \omega$,EM(n, Φ) is isomorphic to $M^+_{\bar a}$ whenever $\bar a = \langle f^n_\alpha(\zeta_\ell) : \ell < n\}$, by an isomorphic mapping a_ℓ to $f^n_\alpha(\zeta_\ell)$ for $\ell < n$ wherever $\alpha < (2^\kappa)^+, \zeta_0 < \ldots < \zeta_{n-1} < \beth_\alpha(\lambda)$. $\square_{8.7}$

8.9 Definition. 1) Let $K^{\mathrm{or}(+)}$ be the class of I a linear order expanded by the unary relations P^I_1, P^I_2 such that P^I_1 is an initial segment of I and $P^I_2 = I \backslash P^I_1$. Let $\tau(*)$ be the vocabulary $\{<, P_1, P_2\}$.
2) For $\kappa \ge$ LS$(\mathfrak{K})$ let $\Upsilon^{\mathrm{or}(+)}_\kappa = \Upsilon^{\mathrm{or}(+)}_\kappa[\mathfrak{K}]$ be the family of Φ proper for $K^{\mathrm{or}(+)}$ (see [Sh:c, Ch.VII]), such that

(*a*) $\tau(\Phi)$ extends $\tau_{\mathfrak{K}}$ and has cardinality $\le \kappa$

(*b*) for every $I \in K^{\mathrm{or}(+)}$, EM(I, Φ) is a $\tau(\Phi)$-model which is the closure (by the functions $F^M, F \in \tau(\Phi)$ a function symbol) of the skeleton $\langle \bar a_t : t \in I\rangle$; for simplicity $\bar a_t = \langle a_t\rangle$ and, of course, $s \ne t \Rightarrow a_s \ne a_t$; and let EM$_\tau(I, \Phi)$ be the τ-reduct of EM(I, Φ) for $\tau \subseteq \tau(\Phi)$

(*c*) $\langle a_t : t \in I\rangle$ is qf-indiscernible in EM(I, Φ) which means that: if $t_0 <_I \ldots <_I t_{n+1}, s_0 <_I \ldots <_I s_{n-1}$ and $s_\ell \in P^I_1 \Leftrightarrow t_\ell \in P^I_1$ for $\ell < n$ then $\langle a_{t_\ell} : \ell < n\rangle, \langle a_{s_\ell} : \ell < n\rangle$ realizes the same quantifier free type in EM(I, Φ)

(d) if $I \subseteq J \in K^{\text{or}(+)}$ then $\text{EM}(I,\Phi) \subseteq \text{EM}(J,\Phi)$ and $\text{EM}_{\tau(\mathfrak{K})}(I,\Phi) \leq_{\mathfrak{K}} \text{EM}_{\tau(\mathfrak{K})}(J,\Phi)$, so both are in $\mathfrak{K}$.

3) If $\kappa = \text{LS}(\mathfrak{K})$ we may omit it.
4) If $I \in K^{\text{or}(+)}, \Phi \in \Upsilon_\kappa^{\text{or}(+)}$ and h is a partial automorphism of I, then $\hat{h}$ is the following function; letting $I_0 = \text{Dom}(h)$ we have: if $n < \omega, t_0 <_I \ldots <_I t_{n-1}$ are from I_0 and $\sigma(x_0,\ldots,x_{n-1})$ is a $\tau(\Phi)$-term then $\hat{h}(\sigma(a_{t_0},\ldots,a_{t_{n-1}}) = \sigma(a_{h(t_0)},\ldots,a_{h(t_{n-1})})$.

8.10 Definition. 1) Let $\Upsilon_\kappa^{\text{lin}(+)}$ be the class of Φ proper for $K^{\text{or}(+)}$ such that τ_Φ has cardinality $\leq \kappa$, the two-place relation $<$ and unary predicates P_1, P_2 belong to $\tau(\Phi)$ and $\text{EM}_{\tau(*)}(I,\Phi) \in K^{\text{or}(+)}$ for $I \in K^{\text{or}(+)}$ and $t \mapsto a_t$ embeds I into $\text{EM}_{\tau(*)}(I,\Phi)$ (so $t \in P_\ell^I \Leftrightarrow a_t \in P_\ell^{\text{EM}(I,\Phi)}$ hence we may identify $t \in I$ with $a_t \in \text{EM}(I,\Phi)$).
2) $I \in K^{\text{or}(+)}$ is strongly $\aleph_0$-homogeneous when: if $n < \omega, s_0 <_I \ldots <_I s_{n-1}, t_0 <_I \ldots <_I t_{n-1}$ and $s_\ell \in P_1^I \Leftrightarrow t_\ell \in P_1^I$ for $\ell < n$ <u>then</u> there is an automorphism h of I (so mapping P_1^I onto P_1^I hence it maps P_2^I onto P_2^I) satisfying $h(s_\ell) = t_\ell$ for $\ell < n$.
3) If $\Phi \in \Upsilon_\kappa^{\text{or}(+)}[\mathfrak{K}]$ and $I_0, I_1 \subseteq I \in K^{\text{or}(+)}$ and h is an isomorphism from I_0 onto I_1 <u>then</u> $\hat{h}$ is an isomorphism from $\text{EM}(I_0,\Phi)$ onto $\text{EM}(I_1,\Phi)$.

8.11 Observation. 1) $I \in K^{\text{or}(+)}$ is strongly $\aleph_0$-homogeneous iff the linear orders $(P_1^I, <^I)$ and $(P_2^I, <^I)$ are strongly $\aleph_0$-homogeneous.
2) There is $\Psi \in \Upsilon_{\aleph_0}^{\text{lin}(+)}$ such that $\text{EM}_{\tau(*)}(I,\Psi)$ is strongly $\aleph_0$-homogeneous for every $I \in K^{\text{or}(+)}$ and $\text{EM}_{\tau(*)}(I,\Phi) = \sum\limits_{\ell=1}^{2} \text{EM}_{\tau(*)}(I \restriction P_\ell^I, \Psi)$.

Proof. Easy, e.g.
2) Let $\tau(\Psi) \backslash \tau(*) = \{F_n : n < \omega\}, F_n$ a $(2n+1)$-place function and we demand that in any $M = \text{EM}(I,\Psi)$ we have:

(a) $P_1(x_\ell) \wedge P_2(x_m) \to F_n(x_0,\ldots,x_{2n}) = x_0$,

(b) $\bigwedge\limits_{i=0}^{2n} P_\ell(x_i) \to P_\ell(F_n(x_0,\ldots,x_{2n}))$

(c) if $a_1 <^M \ldots <^M a_n$ and $b_1 <^M \ldots <^M b_n$ and $\{a_1,\ldots,a_n,b_1,\ldots,b_n\} \subseteq P_\ell^M$ then $x \mapsto F_n(x,a_j,\ldots,a_n,b_1,\ldots,b_n)$ is an automorphism of $(P_\ell^M, <^M)$ mapping a_m to b_m for $m = 1,\ldots,n$. $\square_{8.11}$

8.12 Claim. *M is saturated if: $\mathfrak{K}$ is categorical in λ and*

⊛ (a) $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$

(b) $\Phi \in \Upsilon^{\mathrm{or}(+)}_\kappa[\mathfrak{K}]$

(c) $I \in K^{\mathrm{or}(+)}$ *satisfies* $\mathrm{LS}(\mathfrak{K}) < |I| \le \mathrm{cf}(\lambda)$

(d) *for any $\theta < |I|$ in I there is a monotonic sequence of length θ^+.*

Proof. As in [Sh 394, §6] but we prove in details. To prove that M is saturated, let $N \le_{\mathfrak{K}} M$ be of cardinality $< |I|$ and $p \in \mathscr{S}_{\mathfrak{K}}(N)$; then there is $J \subseteq I$ of cardinality $< |I|$ such that $N \subseteq \mathrm{EM}_{\tau(\mathfrak{K})}(J, \Phi)$, so without loss of generality equality holds. Let the cardinality of J be θ so $\theta < |I|$ hence there is a monotonic sequence $\langle t_i : i < \theta^+\rangle$; without loss of generality it is increasing. So there is an ordinal $i(*) < \theta^+$ such that the interval $[t_{i(*)}, t_j)_I$ is disjoint to J whenever $i(*) < j < \theta^*$ and clearly for some j this interval is infinite.

Let I^* be like I when we add a copy of λ just above $t_{i(*)}$. Let $M^* = \mathrm{EM}_{\tau(\mathfrak{K})}(I^*, \Phi)$ so $M \le_{\mathfrak{s}} M^*$ and the latter is $|I|$-saturated (as $\mathrm{LS}(\mathfrak{K}) < |I| \le \mathrm{cf}(\lambda)$ and we know then $M \in K_\lambda \Rightarrow M$ is $\mathrm{cf}(\lambda)$-saturated by [Sh 394, 6.7=6.4tex]) hence p is realized by some member of M^*. By a claim from [Sh 394], every type over N realized in M^* is already realized in M so we are done. $\square_{8.12}$

Below we can manage using only $<^{\oplus,1}_\kappa, \le^{\oplus,2}_\kappa$, see remarks.

8.13 Remark. Clause (d) in 8.12 is not really necessary but not harmful here. Why not necessary? E.g., let $I' = I \times \mathbb{Q}$ ordered lexicographically. Now

(∗) if $J \subseteq I'$ has cardinality $< |I|$ then for some $t^* \in I$ we have $\{t^*\} \times \mathbb{Q}$ is disjoint to J, hence we can proceed as above.

Now given Φ we can find Φ' such that $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi')$ is isomorphic to $\mathrm{EM}_{\tau(\mathfrak{K})}(I \times \mathbb{Q}, \Phi)$.

8.14 Definition. We define partial orders $\le^{\oplus,1}_\kappa, \le^{\oplus,2}_\kappa$ and $\le^{\oplus,3}_\kappa$ on $\Upsilon^{\mathrm{or}(+)}_\kappa$ (for $\kappa \ge \mathrm{LS}(\mathfrak{K})$) as follows:

1) $\Psi_1 \le^{\oplus,\ell}_\kappa \Psi_2$ if:

(a) $\tau(\Psi_1) \subseteq \tau(\Psi_2)$ and

(b) $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Psi_1) \le_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Psi_2)$ and

(c) $\mathrm{EM}(I, \Psi_1) = \mathrm{EM}_{\tau(\Psi_1)}(I, \Psi_1) \subseteq \mathrm{EM}_{\tau(\Psi_1)}(I, \Psi_2)$

(d) if $\ell = 2, 3$ then $\mathrm{EM}(I \restriction P^I_1, \Psi_1) = \mathrm{EM}_{\tau(\Psi_1)}(I \restriction P^I_1, \Psi_2)$ for any $I \in K^{\mathrm{or}(+)}$

(e) if $\ell = 3$ then any Ψ_1-automorphism scheme $\mathfrak{t}_1$ there is a Ψ_2-automorphism scheme $\mathfrak{t}_2$ which extends it, see definition below.

2) We say that $\mathfrak{t}$ is a Φ-automorphism scheme <u>when</u> it is a Φ-automorphism over $P_1, (n, n_1)$-scheme for some $n_1 \le n < \omega$ which means that

- (a) $\mathfrak{t}$ is a set of tuples of the form $\langle m, m_1, u, \sigma_1(x_0, \ldots, x_{m-1}), \sigma_2(x_0, \ldots, x_{m-1})\rangle$ such that $m_1 \le m < \omega, u \subseteq m, |u| = n, |u \cap m_1| = n_1$ and σ_1, σ_2 are $\tau(\Phi)$-terms
- (b) for every m, m_1, u and $\sigma_1(x_0, \ldots, x_{m-1})$ as above for some $\sigma_2(x_0, \ldots, x_{m-1})$ as above the tuple $\langle m, m_1, u, \sigma_1(x_0, \ldots, x_{m-1}), \sigma_2(x_0, \ldots, x_{m-1})\rangle$ belongs to $\mathfrak{t}$
- (c) if $\sigma(x_0, \ldots, x_{m_1-1})$ is a $\tau(\Phi)$-term, $m_1 \le m$ so $x_{m_1}, \ldots, x_{m-1}$ are dummy variables when we use below $\sigma(x_0, \ldots, x_{m-1})$, and $u \subseteq m, |u| = n, |u \cap m_1| = n_1$ then $\langle m, m_1, u, \sigma, \sigma\rangle = \langle m, m_1, u, \sigma(x_0, \ldots, x_{m-1}), \sigma(x_0, \ldots, x_{m-1})\rangle$ belongs to $\mathfrak{t}$
- (d) for every $I \in K^{\text{or}(+)}_\kappa$ and $t_0 <_I \ldots <_I t_{n-1}$ satisfying $t_\ell \in P^I_1 \Leftrightarrow \ell < n_1$ the set of pairs $f^{\mathfrak{t}}_{\Phi,I}[t_0, \ldots, t_{n-1}]$ defined below is an automorphism of $\text{EM}_{\tau(\Phi)}(I, \Phi)$.

2A) If we omit "over P_1" we omit clause (c).
3) For $\Phi, I, \mathfrak{t}, n_1$ and $t_0 <_I \cdots <_I t_{n-1}$ as above $f^{\mathfrak{t}}_{\Phi,I}[t_0, \ldots, t_{n-1}]$ is (where $M = \text{EM}(I, \Phi)$) the set of pairs

$$\begin{aligned}\big\{(\sigma^M_1(a_{s_0}, \ldots, a_{s_{m-1}}), \sigma^M_2(a_{s_0}, \ldots, a_{s_{m-1}})) : \text{ there is } &\langle m, m_1, u, \sigma_1(x_0, \ldots, x_{m-1}),\\ &\sigma_2(x_0, \ldots, x_{m-1})\rangle \in \mathfrak{t}, \text{ and}\\ &s_0 <_I \cdots <_I s_{m-1} \text{ such that } s_\ell \in P^I_1 \Leftrightarrow \ell < m_1\\ &\text{and } \ell \in u \rightarrow s_\ell = t_{|u \cap \ell|}\big\}.\end{aligned}$$

4) Assume that for $m = 1, 2, \Phi_m$ satisfies clauses (a),(b),(d) of part (1) and $\mathfrak{t}_m$ is an Φ_m-automorphism scheme for $m = 1, 2$. We say that $\mathfrak{t}_2$ extend $\mathfrak{t}_1$ when $\mathfrak{t}_1 \subseteq \mathfrak{t}_2$. Again for $\kappa = \text{LS}(\mathfrak{K})$ we may drop the κ.

8.15 Claim. *1) $(\Upsilon^{\text{or}(+)}_\kappa, \le^{\oplus,\ell}_\kappa)$ are partial orders for $\ell = 1, 2, 3$.*
2) If $\langle \Phi_i : i < \delta\rangle$ is a $\le^{\oplus,\ell}_\kappa$-increasing sequence, $\delta < \kappa^+$, <u>then</u> it has a $<^{\oplus,\ell}_\kappa$-l.u.b. Φ; $\text{EM}(I, \Phi) = \bigcup\limits_{i<\delta} \text{EM}(I, \Phi_i)$.
3) Assume that

- (a) *f is an automorphism of $EM_{\tau(\mathfrak{K})}(I, \Phi)$*
- (b) *P^I_1, P^I_2 are dense (in particular with neither first nor last element)*

(c) *f commutes with every partial automorphism $\hat{h}$ where h is a (finite) partial automorphism of I extending the identity on $\{t_0, \dots, t_{n-1}\}$ where $t_\ell \in P_1^I \Leftrightarrow \ell < n_1$*

(d) *if $J \subseteq K^{\text{or}(+)}$ is finite and include $\{t_0, \dots, t_{n-1}\}$ then f maps $EM(J, \Phi)$ onto itself.*

Then for some Φ-automorphism (n, n_1)-scheme $\mathfrak{t}$ we have $f = f^{\mathfrak{t}}_{\Phi,I}[t_0, \dots, t_{n-1}]$.
4) If in (3) we can add "some Φ-automorphism over P_1" when we add the assumption

(e) *f is the identity on $EM(I \cap P_1^I, \Phi)$.*

5) In clause (d) of Definition 8.14(2) it is enough that for every $i < \omega$ there is such I with every $E^I_{\{t_0,\dots,t_{n-1}\}}$-equivalence class having $\geq k$ members where $s^1 E^I_{\{t_0,\dots,t_{n-1}\}} s^2$ iff $s^1, s^2 \in I \backslash \{t_0, \dots, t_{n-1}\}$ and $s_1 \in P_1^I \Leftrightarrow s_2 \in P_1^I, \ell < n \Rightarrow s_1 <_I t_\ell \Leftrightarrow s_2 <_I^{t_2}$.

Proof. Easy. E.g., 3),4) Let $\mathfrak{t} = \{\mathfrak{x} : \mathfrak{x}$ has the form $(n, n_1, u, \sigma_2(x_0, \dots, x_{m-1}), \sigma_2(x_0, \dots, x_{m-1}), n_1 \leq n, u \subseteq n, \sigma_1, \sigma_2$ are $\tau(\Phi)$ terms and there are $s^i_0 <_I \dots <_I s^i_{m-1}$ for $i = 1, 2$ such that $\ell \in u \wedge i \in \{1, 2\} \Rightarrow s^i_\ell = t_{(u \cap \ell)}$ and $f(a_1) = a_2$ when we let $a_i = \sigma_i(a_{s^i_0}, \dots, a_{s^i_{n-1}})$. It is enough to check the clauses (a)-(d) of Definition 8.14(2) and the equality in the end of the conclusion of part (3).

Clause (a): By inspection of $\mathfrak{t}$ is a set of tuples of the right form.

Clause (b): By clause (d) of the assumption.

Clause (c): (For part (2)) by assumtion (e).

Clause (d): By part (5) of the claim it suffices to prove this for our present I. But then this is the equality we have promised and proved below.

The equalities: $f = f^{\mathfrak{t}}_{\Phi,I}[t_0, \dots, t_{n-1}]$.

The inclusion $\subseteq$: Assume $f(a_1) = a_2$ so for some finite $J \subseteq I$ we have $a, b \in$ EM(J, Φ) and without loss of generality $t_0, \dots, t_{n-1} \in J$. Let $s_0 <_I \dots <_I t_{m-1}$ list J hence tere are $\tau(\Phi)$-terms $\sigma_i(x_0, \dots, x_{m-1})$ such that $a_i = \sigma_i(a_{s_{m-1}})$ for $i = 1, 2$. Let m_1 be such that $\ell < m_1 \equiv s_\ell \in P_1^I$, let $u = \{\ell < m : s_\ell \in \{t_0, \dots, t_{n-1}\}$, so clearly $(m, m_1, u, \sigma_1, \sigma_2) \in \mathfrak{t}$ hence $(a_1, a_i) = (\sigma_1(a_{s_0}, \dots, a_{s_{m-1}}), \sigma_2(a_{s_0}, \dots, a_{s_{m-1}})) \in f^{\mathfrak{t}}_{\Phi,I}[t_0, \dots, t_{n-1}]$, so we have proved the inclusion $\subseteq$.

The inclusion $\supseteq$: If $(a_1, a_2) \in f^{\mathfrak{t}}_{\Phi,I}[t_0, \dots, t_{n-1}]$ then there are $\mathfrak{x} = (m, m_1, u, \sigma_1, \sigma_2) \in \mathfrak{t}$ and $s^1_0 <_I \dots <_I s^1_{m-1}$ which witnesses it, so $\ell < m \Rightarrow \ell < m_1 \Leftrightarrow s^1_\ell \in P_1^I$ and

$a_i = \sigma_i(a_{s^1_0}, \ldots, a_{s^1_{m-1}})$ for $i = 1, 2$ and $\ell \in u \Rightarrow s^1_\ell = t_{(\ell \cap u)}$. But why $\mathfrak{x} \in \mathfrak{t}$? There should be witnesses $s^2_0 <_I \ldots <_I s^2_{m-1}$ for this hence $\ell < m \Rightarrow [\ell < m_1 \Leftrightarrow s^2_\ell \in P^I_1]$ and $\ell \in u \Rightarrow s^2_\ell = t_{(\ell \cap u)}$. Let $h = \{(s^1_\ell, s^2_\ell) : \ell < m\}$ so h is a partial automorphism of I which is the identity on $\{t_0, \ldots, t_{n-1}\}$. By clause (c) of the assumption we are done.

8.16 Claim. *1) For $\kappa \geq \mathrm{LS}(\mathfrak{K})$ we have $\Upsilon^{\mathrm{or}(+)}_\kappa[\mathfrak{K}] \neq \emptyset$.*
2) If N is a model, $b^\ell_n \in N$ for $\ell = 1, 2$ and $n < \omega$ such that $\langle b^\ell_n : n < \omega\rangle$ is an indiscernible sequence over $\{b^{3-\ell}_n : n < \omega\}$ then we can find Φ proper for $K^{\mathrm{or}(+)}$ such that $\tau(\Phi) = \tau_N$ and

- ⊛ *if $N' = \mathrm{EM}(I, \Phi), s_0 <_I \ldots <_I s_{m-1}, t_0 <_I \ldots <_I t_{n-1}$ and $s_\ell \in P^I_1, t_\ell \in P^I_2$ then the quantifier-type which $\langle a_{s_0}, \ldots, a_{s_{m-1}}, a_{t_0}, \ldots, a_{t_{n-1}}\rangle$ realizes in N' is equal to the quantifier free type which $\langle b^1_0, \ldots, b^1_{m-1}, b^2_0, \ldots, b^2_{n-1}\rangle$ realizes in N.*

3) If in addition $\tau(\mathfrak{K}) \subseteq \tau_N$ and for $n_1 \leq n_2 < \omega, m_1 \leq m_2 < \omega$ we have $(N \restriction \tau(\mathfrak{K})) \restriction c\ell_N\{b^1_0, \ldots, b^1_{m_1-1}, b^2_0, \ldots, b^2_{n_1-1}\}$ is a $\leq_{\mathfrak{K}}$-submodel of $N \restriction \tau(\mathfrak{K})$ (or just of $(N \restriction \tau(\mathfrak{K})) \restriction c\ell_N(\{b^1_0, \ldots, b^1_{m_2-1}, b^2_0, \ldots, b^2_{n_2-1}\}))$ then $\Phi \in \Upsilon^{\mathrm{or}(+)}_\kappa[\mathfrak{K}]$.
4) Assume $\Phi_1 \leq^{\oplus,1}_\kappa \Phi'_2$.
Then we can find $\Phi_2 \in \Upsilon^{\mathrm{or}(+)}_\kappa[\mathfrak{K}]$ such that

- (α) $\Phi_1 \leq^{\oplus,2}_\kappa \Phi_2$
- (β) *if $I \in K^{\mathrm{or}(+)}$ satisfies $P^I_2 \neq \emptyset$ then $EM_{\tau(\Phi_1)}(I, \Phi_2) = EM_{\tau(\Phi_1)}(I, \Phi'_2)$*
- (γ) *if $I \in K^{\mathrm{or}(+)}$, then $EM_{\tau(\Phi_1)}(I \restriction P^I_1, \Phi_2) = EM(I \restriction P^I_1, \Phi_1)$ actually this follows from clause (α)*
- (δ) *if $\mathfrak{t}_1$ is a Φ_1-automorphism over $P_1, (n, n_1)$-scheme and $\mathfrak{t}'_2$ is a Φ'_2-automorphism (n, n_1)-scheme extending $\mathfrak{t}_1$ then there is $\mathfrak{t}_2$ a Φ_2-automorphism over $P_1, (n, n_1)$-scheme extending $\mathfrak{t}_1$.*

Proof. 1) Because $\Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}] \neq \emptyset$ by [Sh 394].
2) Think.
3) The main possibility implies the "or just of ..." by $\mathfrak{K}$ being an a.e.c. The statement itself is easy to check (as we can use just finite $I \subseteq J$ and then by the axioms of a.e.c. the case of n_1, m_2, is enough).
4) Let the vocabulary τ_2 (intended to be $\tau(\Phi_2)$) have the same predicates and function symbols as $\tau(\Phi'_2)$ except that for any function symbol $F \in \tau(\Phi'_2) \backslash \tau(\Phi_1)$ we change its arity: $\mathrm{arity}_{\tau_2}(F) = \mathrm{arity}_{\tau(\Phi'_2)}(F) + 1$. For $I \in K^{\mathrm{or}(+)}$ we define $M_2 = \mathrm{EM}(I, \Phi_2)$ as follows: let $M'_2 = \mathrm{EM}(I, \Phi'_2)$, now M_2, a τ_2-model is the same

as M_2' except that if $F \in \tau(\Phi_2)\backslash\tau(\Phi_1), n =$ arity$_{\tau(\Phi_2')}(F)$ and $a_0, \dots, a_n \in M_2'$ we define $F^{M_2}(a_0, \dots, a_n)$ as follows:

if $\{a_0, \dots, a_n\} \subseteq$ EM$(I \restriction P_1^I, \Phi_1)$ then $F^{M_2}(a_0, \dots, a_n) = a_0$

and if otherwise then $F^{M_2}(a_0, \dots, a_n) = F^{M_2'}(a_0, \dots, a_{n-1})$.

Now it is easy to check. $\square_{8.16}$

For the version (using $<_\kappa^{\oplus,\ell}, \ell = 1, 2$ only) we need

8.17 Definition. 1) We say that $\mathfrak{t}$ is a weak Φ-automorphism (n, n_1)-scheme over P_1 when:

$(a), (b), (c)$ as in Definition 8.14(2) above

(d) for every $I \in K^{\text{or}(+)}$ and $t_0 <_I \dots <_I t_{n-1}$ satisfying $t_\ell \in P_1^I \Leftrightarrow \ell < n_1$ the set $f = f_{\Phi,I}^{\mathfrak{t}}[t_0, \dots, t_{n-1}]$ satisfies

- (α) f is a one to one function
- (β) $M_f^1 := (\text{EM}_{\tau(\mathfrak{K})}(I, \Phi)) \restriction$ Dom(f) is a $\leq_{\mathfrak{K}}$-submodel of EM$_{\tau(\mathfrak{K})}(I, \Phi)$
- (γ) $M_f^2 := (\text{EM}_{\tau(\mathfrak{K})}(I, \Phi)) \restriction$ Range(f) is a $\leq_{\mathfrak{K}}$-submodel of EM$_{\tau(\mathfrak{K})}(I, \Phi)$
- (δ) f is an isomorphism from M_f^1 onto M_f^2.

2) We say that $\mathfrak{x}$ is a Φ-task if $\mathfrak{x}$ has the form $(n, n_1, \sigma_1(x_0, \dots, x_{n-1}), \sigma_2(x_0, \dots, x_{n-1})$ where σ_1, σ_2 are $\tau(\Phi)$-terms.
3) We say that the [weak] Φ-automorphism over $P_1, (n, n_1)$-scheme $\mathfrak{t}$ solves the Φ-task $\mathfrak{x} = (n^{\mathfrak{x}}, n_1^{\mathfrak{x}}, \sigma_1^{\mathfrak{x}}(x_0, \dots, x_{n-1}), \sigma_2(x_0, \dots, x_{n-1}))$ <u>when</u> $(n, n_1) = (n^{\mathfrak{x}}, n_1^{\mathfrak{x}})$ and the tuple $(n^{\mathfrak{x}}, n_1^{\mathfrak{x}}, n^{\mathfrak{x}}, \sigma_1^{\mathfrak{x}}(x_0, \dots, x_{n-1}), \sigma_2^{\mathfrak{x}}(x_0, \dots, x_{n-1}))$ belongs to $\mathfrak{t}$.
4) We say that the Φ-task $\mathfrak{x}$ is [weakly] solvable or Φ-solvable if some [weak] Φ-automorphism (n, n_1)-scheme solves it.

8.18 Observation. 0) A Φ-automorphism over $P_1, (n, n_1)$-scheme is a weak Φ-automorphism over $P_1, (n, n_1)$-scheme.
1) If $\Phi_1 \leq_\kappa^{\oplus,1} \Phi_2$ and $\mathfrak{x}$ is a Φ_1-task then $\mathfrak{x}$ is a Φ_2-task.
2) If $\Phi_1 \leq_\kappa^{\oplus,2} \Phi_2$ and $\mathfrak{x}$ is a weakly Φ_1-solvable Φ_1-task <u>then</u> $\mathfrak{x}$ is a weakly Φ_2-solvable Φ_2-task.
3) If a Φ-task is solvable then it is weakly solvable.

8.19 Remark. For the weak version, in 8.20 it suffices if we weaken the conclusion

$\otimes$ (α) $\Phi_1 \leq_\kappa^{\oplus,2} \Phi_2$

(β) the Φ_1-task $\mathfrak{x}$ is Φ_2-solvable.

[This simplifies the proof.]

8.20 Main Claim. *Assume*

(a) $\Phi_1 \in \Upsilon^{\rm or(+)}_\kappa$

(b) $n_1 \le n < \omega$ *and* $\sigma_1(\dots, x_\ell, \dots)_{\ell<n}, \sigma_2(\dots, x_\ell, \dots)_{\ell<n}$ *are* $\tau(\Phi_1)$*-terms; for convenience assume* $n_1 < n$. *Let* $\mathfrak{x} = (n, n_1, \sigma_2(\dots, x_\ell, \dots)_{\ell<n}, \sigma_2(x_0, \dots, x_{n-1})_{\ell<n})$

(c) *for every* $\alpha < (2^\kappa)^+$ *there are* I *and* $t_0, \dots, t_{n-1}$ *such that:*

$\boxdot_\alpha$ (α) $I \in K^{\rm or(+)}$ *is strongly* $\aleph_0$*-homogeneous*

(β) $|P^I_1| \ge \beth_\alpha$

(γ) $|P^I_2| \ge \beth_\alpha(|P^I_1|)$

(δ) $t_0 <_I \dots <_I t_{n-1}$ *so are from* I

(ε) $t_\ell \in P^I_1 \Leftrightarrow \ell < n_1$

(ζ) *there is an automorphism of* $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi_1)$ *which is the identity on* $\mathrm{EM}_{\tau(\mathfrak{K})}(I \restriction P^I_1, \Phi_1)$ *and maps* $\sigma_1(\dots, a_{t_\ell}, \dots)_{\ell<n}$ *to* $\sigma_2(\dots, a_{t_\ell}, \dots)_{\ell<n}$.

<u>Then</u> *there are is* Φ_2 *such that*

⊗ (α) $\Phi_1 \le^{\oplus,3} \Phi_2$

(β) *there is* $\mathfrak{t}$ *such that:*
if $I \in K^{\rm or(+)}, M = EM_{\tau(\mathfrak{K})}(I, \Phi_1)$
$t_0 <_I \dots <_I t_{n-1}$ *and* $t_\ell \in P^I_1 \Leftrightarrow \ell < n_1$
<u>then</u> *there is* f *such that*

(i) f *is an automorphism of* M *which is the identity on* $\mathrm{EM}_{\tau(\mathfrak{K})}(I \restriction P^I_1, \Phi_1)$

(ii) $f(\sigma_1(\dots, a_{t_\ell}, \dots)) = \sigma_2(\dots, a_{t_\ell}, \dots)$

(iii) f *is* $f^{\mathfrak{t}}_{\Phi_2, I}[t_0, \dots, t_{n-1}]$ *and* $\mathfrak{t}$ *is a* Φ_2*-automorphism over* $P_1, (n, n_1)$*-scheme* $\mathfrak{t}$.

Remark. 1) Note that $\mathrm{EM}_{\tau(\mathfrak{K})}(I \restriction P^I_1, \Phi_2)$ is equal to $\mathrm{EM}_{\tau(\mathfrak{K})}(I \restriction P^I_1, \Phi_1)$.
2) In the clause (c) of the assumption, "there are $t_0 <_I$ such that" is equivalent to "for every $t_0 <_I \dots$ such that".

Proof. For each $\alpha < (2^\kappa)^+$ we choose $I_\alpha, \langle t^\alpha_\ell : \ell < n\rangle$ and f_α exemplifying clause (c) of the assumption. We expand $M_\alpha = \mathrm{EM}(I_\alpha, \Phi_1)$ to a model M^+_α as follows:

$\circledast_1$ (*a*) $Q^{M^+_\alpha}_\ell = \{a_t : t \in P^{I_\alpha}_\ell\}$ for $\ell = 1, 2$ and $Q^{M^+_\alpha} = Q^{M^+_\alpha}_1 \cup Q^{M^+_\alpha}_2$ and $Q^{M^+_\alpha}_3 = \mathrm{EM}(I_\alpha \restriction P^{I_\alpha}_1, \Phi_0)$

(*b*) the relation $<^{M^+_\alpha}_*$ chosen as $\{(a_s, a_t) : s <_{I_\alpha} t\}$

(*c*) $H^{M^+_\alpha}_m$, $(2m+1)$-place functions (for $m < \omega$) are chosen in $\boxtimes_0$ below together witnessing I_α is strongly $\aleph_0$-homogeneous respecting inverses, that is $H^\alpha_m(H^\alpha_m(x, \bar s, \bar t), \bar t, \bar s) = x$

(*d*) the function $F^{M^+_\alpha}$ chosen in $\boxtimes_1$ below

(*e*) individual constants $c^{M^+_\alpha}_\ell = a_{t^\alpha_\ell}$ for $\ell < n$

(*f*) the predicates $\langle P^{M^+}_\sigma : \sigma$ a $\tau(\Phi_1)$-term$\rangle$ and functions $G^{M^+}_\ell (\ell < \omega)$ as in $\boxtimes_2$ below

(*g*) Skolem functions (see $\boxtimes_3$ below).

Now

$\boxtimes_0$ for $m < \omega$ let

(*i*) h^α_m be a $(\geq m+1)$-place function from I_α to I_α such that: if $s_0 <_I \ldots <_{I_\alpha} s_{m-1}$ and $t_0 <_{I_\alpha} \ldots <_{I_\alpha} t_{m-1}$ and $s_\ell \in P^{I_\alpha}_1 \equiv t_\ell \in P^{I_\alpha}_1$ then the function $x \mapsto h^\alpha_m(x, s_0, \ldots, s_{m-1}, t_0, \ldots, t_{m-1})$ is an automorphism of I_α mapping t_ℓ to s_ℓ for $\ell < n$; we can add: if $\{t_0, \ldots, t_{m-1}\} \subseteq P^{I_\alpha}_\ell$ then $x \notin P^I_\ell \Rightarrow x = h^\alpha_m(x, s_0, \ldots, s_{m-1}, t_0, \ldots, t_{m-1})$

(*ii*) $H^{M^+_\alpha}_m$ is the $(2m+1)$-place function from M_α to M_α defined by: if $s_0 <_{I_\alpha} \ldots <_{I_\alpha} s_{m-1}, t_0 <_{I_\alpha} \ldots <_{I_\alpha} t_{m-1}, s_\ell \in P^{I_\alpha}_1 \equiv t_\ell \in P^{I_\alpha}_1$ and $M_\alpha \models$ "$a = \sigma(a_{r_0}, \ldots, a_{r_{k-1}})$" and $r'_\ell = h^\alpha_m(r_\ell, s_0, \ldots, s_{m-1}, t_0, \ldots, t_{m-1})$, <u>then</u> $H^{M^+_\alpha}_m(a, a_{s_0}, \ldots, a_{s_{m-1}}, a_{t_0}, \ldots, a_{t_{m-1}}) = \sigma(a_{r'_0}, \ldots, a_{r'_{k-1}})$, in other cases $H^{M^+_\alpha}_m(a, b_0, \ldots) = a$

$\boxtimes_1$ recall that $f_\alpha \in \mathrm{Aut}(\mathrm{EM}_{\tau(\mathfrak{K})}(I_\alpha, \Phi_1))$ is as in (ζ) of clause (c) of the assumption of the claim. Let $F^{M^+_\alpha}$ be a unary function, $F^{M^+_\alpha}(b) = f_\alpha(b)$ for every $b \in \mathrm{EM}_{\tau(\mathfrak{K})}(I_\alpha, \Phi_1)$.

$\boxtimes_2$ For every $a \in M_\alpha$ there are $n, t_0 <_{I_\alpha} \ldots <_{I_\alpha} t_{n-1}$ and $\tau(\Phi_1)$-term $\sigma(x_0, \ldots, x_{n-1})$ such that $M_\alpha \models$ "$a = \sigma(a_{t_0}, \ldots, a_{t_{n-1}})$". Let $\mathbf{n} : M_\alpha \to \omega$ be such that $\mathbf{n}(a)$ is the minimal n for which there are $t_0 <_{I_\alpha} \ldots <_{I_\alpha} t_{n-1}$ and σ as above. Let

$<^*_{\alpha,n}$ be a well ordering of the set of $\tau(\Phi_1)$-terms of the form $\sigma(x_0,\dots,x_{n-1})$. For $a \in M_\alpha$ let σ_a be the $<^*_{\alpha,\mathbf{n}(a)}$-minimal $\tau(\Phi_1)$-term $\sigma(x_0,\dots,x_{n-1})$ such that for some $t_0 <_{I_\alpha} \dots <_{I_\alpha} t_{n-1}$ we have $M_\alpha \models$ "$a = \sigma(a_{t_0},\dots,a_{t_{n-1}})$". Define $G^{M^+_\alpha}_\ell : M_\alpha \to Q^{M_\alpha}_3$ for $\ell < \omega$ such that if $a \in M_\alpha, n = \mathbf{n}(a), \sigma = \sigma_a$ then $G^{M^+_\alpha}_0(a) <^{M^+_\alpha}_* \dots <^{M^+_\alpha}_* G^{M^+_\alpha}_{n-1}(a)$ so they all belong to $\{a_t : t \in I_\alpha\}$ and $M_\alpha \models a = \sigma_a(G^{M^+_\alpha}_0(a),\dots,G^{M^+_\alpha}_{\mathbf{n}(a)-1}(a))$. Let $G^{M^+_\alpha}_\ell(a) = a$ when $\ell \in [\mathbf{n}(a),\omega)$. Now lastly, let $P^{M^+_\alpha}_\sigma = \{a \in M : \sigma_a = \sigma\}$.

$\boxtimes_3$ we further add Skolem functions in particular we have $A \subseteq M_\alpha \Rightarrow (M_\alpha \restriction \tau(\mathfrak{K})) \restriction c\ell_{M^+_\alpha}(A) \le_{\mathfrak{K}} M_\alpha \restriction \tau(\mathfrak{K})$.

The model we get we call M^+_α and without loss of generality $\tau(M^+_\alpha)$ does not depend on α. Now use a variant of the a.e.c. omitting types theorem, 8.6. So there are $\alpha(*) < (2^\kappa)^+$ and a model N^+ and $\langle b^\ell_m : m < \omega\rangle$ in it for $\ell = 1,2$ such that

(a) $b^\ell_m \in Q^{N^+}_\ell, \langle b^2_m : m < \omega\rangle$ is indiscernible over $Q^{N^+}_3$ which include $\{b^1_m : m < \omega\}$ and

(b) $\langle b^1_m : m < \omega\rangle$ is indiscernible over $\{b^2_m : m < \omega\}$ and

(c) $\mathrm{Th}_{\mathbb{L}}(N^+) = \mathrm{Th}_{\mathbb{L}}(M^+_{\alpha(*)})$ (recalling that $M^+_{\alpha(*)}$ has Skolem functions),

(d) N^+ omits all quantifier free types which $M^+_{\alpha(*)}$ omits and

(e) for every $m < \omega$ for arbitrarily large $\alpha < (2^\kappa)^+$,
  - (α) $\mathrm{Th}_{\mathbb{L}}(M^+_\alpha) = \mathrm{Th}_{\mathbb{L}}(N^+)$ and
  - (β) N^+ omits all the quantifier free types which M^+_α omits

(f) for some $s^\ell_0 <_{I_\alpha} \dots <_{I_\alpha} s^\ell_{m-1}$ from $P^{I_\alpha}_\ell$ for $\ell = 1,2$ the quantifier free type of $\langle b^1_0,\dots,b^1_{m-1},b^2_0,\dots,b^2_{m-1}\rangle$ in N^+ is equal to the quantifier free type of $\langle a_{s^1_0},\dots,a_{s^1_{m-1}},a_{s^2_0},\dots,a_{s^2_{m-1}}\rangle$ in M^+_α.

Now there is $\Phi' \in \Upsilon^{\mathrm{or}(+)}_\kappa$ such that

$\circledast_2$ for any $I^* \in K^{\mathrm{or}(+)}$, $\mathrm{EM}(I^*,\Phi')$ is a $\tau(N^+)$-model generated by $\{a_s : s \in I^*\}$ satisfying $s_0 < \dots < s_{m-1} \in P^{I^*}_1, t_0 < \dots < t_{k-1} \in P^{I^*}_2 \Rightarrow$ the quantifier free type of $\langle a_{s_0},\dots,a_{s_{m-1}},a_{t_0},\dots,a_{t_{n-1}}\rangle$ in $\mathrm{EM}(I^*,\Phi')$ is equal to the quantifier free type of $\langle b^1_i : i < m\rangle\hat{\ }\langle b^2_i : i < k\rangle$ in N^+.

[Why? There is Φ' proper for $K^{\mathrm{or}(+)}_\kappa$ by 8.16(2). By the choice of $N^+, \langle b^\ell_m : m < \omega, \ell = 1,2\rangle$ and 8.16(3) we know that $\Phi' \in \Upsilon^{\mathrm{or}(+)}_\kappa[\mathfrak{K}]$.]

$(*)_0$ $\mathrm{EM}(I,\Phi') \equiv N^+$ for $I \in K^{\mathrm{or}(+)}$.

[Why? As $M^+_{\alpha(*)}$ has Skolem functions.]
Now

$\circledast_3$ $\Phi_1 <^{\oplus,1} \Phi'$.
Why? (We prove more). Let $I^* \in K^{\text{or}(+)}$.

Assume $N^* = \text{EM}(I^*, \Phi')$ and let $J[N^*] \in K^{\text{or}(+)}$ be defined as follows: it is the set $\{a : N^* \models Q(a)\}$ linearly ordered by $<^{N^*}_*$ and let $P^{J[N^*]}_\ell = Q^{N^+}_\ell$ for $\ell = 1, 2$. So identifying $t \in I^*$ with a_t we have $I^* \subseteq J[N^*]$.
As $N^* \equiv N^+$ clearly

$(*)_1$ if $k \le m$ and $N^* \models$ "$a_0 <_* a_1 <_* \ldots <_* a_{m-1}$ and $Q_1(a_\ell)$ for $\ell < k, Q_2(a_\ell)$ for $\ell \in [k, m)$" then the $\mathbb{L}(\tau_{\Phi_1})$-quantifier free type which $\langle a_0, \ldots, a_{m-1}\rangle$ realizes in N^* is equal to the $\mathbb{L}(\tau_{\Phi_1})$-type which $\langle b^1_0, \ldots, b^1_{k-1}, b^2_k, \ldots, b^2_{m-1}\rangle$ realizes in N^+; this type is determined by Φ_α.

We define an embedding $\mathbf{j}$ of $\text{EM}(J[N^*], \Phi_1)$ into $N^* \restriction \tau(\Phi_1)$ as follows for $a_0 <_{J[N^*]} \ldots <_{J[N^*]} a_{m-1}$ and $\tau(\Phi_1)$-term $\sigma(x_0, \ldots, x_{m-1})$, we decide: $\sigma(a_0, \ldots, a_{m-1})$ as interpreted in $\text{EM}(J[N^*], \Phi_1)$ is mapped to $\sigma(a_0, \ldots, a_{m-1})$ as interpreted in N^* by $(*)_1$ this is an embedding (for τ_{Φ_1}). This embedding is "onto" as

$(*)_2$ every $c \in N^*$ is in the closure of $\{a_t : t \in J[N^*]\}$ under the $\tau(\Phi_1)$-functions (as interpreted in N^*)

which holds as

$(*)_3$ $M^+_{\alpha(*)}$ so N^+ hence N^* omit the type

$$p(x) = \big\{\neg(\exists y_0, \ldots, y_{k-1})(\bigwedge_{\ell<n} Q(y_\ell) \ \ \& \ \ x = \sigma(y_0, \ldots, y_{k-1})) : \sigma(x_0, \ldots, x_{k-1}) \in \tau_{\Phi_1})\big\}.$$

Also

$(*)_4$ $\mathbf{j}$ is an isomorphism from $\text{EM}(J[N^*], \Phi_1)$ onto $\text{EM}_{\tau(\Phi_1)}(I^*, \Phi')$ mapping a_t to a_t for $t \in I^*$.

Now by the choice of the M^+_α's it follows that

$(*)_5$ $\text{EM}(I^*, \Phi_1) \subseteq \text{EM}_{\tau(\Phi_1)}(I^*, \Phi')$ and $\text{EM}_{\tau(\mathfrak{K})}(I^*, \Phi_1) \le_{\mathfrak{K}} \text{EM}_{\tau(\mathfrak{K})}(I^*, \Phi')$ for every $I \in K^{\text{or}(+)}$.

Now $\circledast_3$ follows.

Note that for some Ψ proper for $K^{\text{or}(+)}, |\tau_\Psi| \le \kappa$ we have $J[N^*] = \text{EM}_{\tau(*)}(I^*, \Psi) \in K^{\text{or}(+)}$ recalling $\tau(*) = \{<, P_1, P_2\}$ and $|\tau(\Psi)| \le \kappa$.
Next there is Φ_2 such that

$\circledast_4$ $\Phi_1 \le^{\oplus,2}_\kappa \Phi_2$ and for every $I \in K^{\rm or(+)}$ if $P^I_2 \ne \emptyset$ then $\mathrm{EM}_{\tau(\Phi_1)}(I,\Phi_2) = \mathrm{EM}_{\tau(\Phi_1)}(I,\Phi')$.

[Why? By 8.16(4).]

Lastly, we have to prove $\otimes$, that is the conclusion of 8.20.

CLAUSE (α) of the conclusion of the claim:

WHY? By $\circledast_4$ we have just to lift a Φ_1-automorphism over $P_1,(n,n_1)$-scheme $\mathfrak{t}_1$ to a Φ_2-automorphism over $P_1,(n,n_1)$-scheme.
So let $I^* \in K^{\rm or(+)}$ be dense and $t_0 <_{I^*} \ldots <_{I^*} t_{n-1}, t_\ell \in P^{I^*}_1 \Leftrightarrow \ell < n_1$. So $f_1 = f^{\mathfrak{t}_1}_{\Phi_1,I^*}[t_0,\ldots,t_{n-1}]$ is a well defined an automorphism of $\mathrm{EM}_{\tau(\mathfrak{K})}(I^*,\Phi_1)$ over $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \restriction P^I_1,\Phi_1)$. As in the proof of $\circledast_3$ let $N^* = \mathrm{EM}(I^*,\Phi')$ and let $J = J[N^*]$ be as there. So $I^* \subseteq J$ hence

$(*)$ $\mathrm{EM}(I^*,\Phi_1) \subseteq \mathrm{EM}(J,\Phi_1) = \mathrm{EM}_{\tau(\Phi_1)}(I^*,\Phi') = \mathrm{EM}_{\tau(\Phi_1)}(I^*,\Phi_2)$.

Let f_2 be $f^{\mathfrak{t}_1}_{\Phi_1,J}[t_0,\ldots,t_{n-1}]$, it is an automorphism of $\mathrm{EM}_{\tau(\mathfrak{K})}(J,\Phi_1)$ extending f_1 hence it is an automorphism of $\mathrm{EM}_{\tau(\mathfrak{K})}(I^*,\Phi_2)$ extending f_1 and it is the identity of $\mathrm{EM}_{\tau(\mathfrak{K})}(J \restriction P^J_1,\Phi_1)$ which is equal to $\mathrm{EM}_{\tau(\mathfrak{K})}(J^* \restriction p^{I^*},\Phi_2)$.

Now assume $s_0 <_{I^*} \ldots <_{I^*} s_{m-1}, s_{m-1} \notin p^{I^*}_1$ and $\sigma = \sigma(x_0,\ldots,x_{m-1})$ is a $\tau(\Phi')$-term and $\{t_0,\ldots,t_{n-1}\} \subseteq \{s_0,\ldots,s_{m-1}\}$. So we can find $k < \omega$ and $r_0 <_J \ldots <_J r_{k-1}$ and $\tau(\Phi_2)$-terms $\tau_\ell(x_0,\ldots,x_{m-1})$ such that $r_\ell = \tau_\ell(a_{s_0},\ldots,a_{s_{m-1}})$ for $\ell < k$ and $\tau(\Phi_1)$-term $\tau^*(x_0,\ldots,x_{m-1})$ such that $\tau(a_{s_0},\ldots,a_{s_{m-1}}) = \tau^*(a_{s_0},\ldots,a_{s_{k-1}})$.

Without loss of generality $\{t_0,\ldots,t_{n-1}\} \subseteq \{r_0,\ldots,r_{k-1}\}$ and let $u = \{\ell < k : r_\ell \in \{t_0,\ldots,t_{n-1}\}\}$, so for each $\ell < k$ for some $\tau^{**}(x_0,\ldots,x_{k-1})$ we have $(n,n_1,u,\tau(x_0,\ldots,x_{k-1}),\tau^{**}(x_0,\ldots,x_{k-1})$. So let $f_2(\tau^*(a_{s_0},\ldots,a_{s_{m-1}})) = f_2(\tau^{**}(a_{r_0},\ldots,a_{r_{k-1}})) = f_2(\tau^{**}(\tau_0(a_{s_0},\ldots,a_{s_{m-1}}),\ldots,\tau_{k-1}(a_{s_0},\ldots,a_{s_{m-1}})))$.

So by 8.15(3), clearly there is $\mathfrak{t}_2$ as required except that we should replace Phi' by Φ_i but as $a_{s_{m-1}} \notin \mathrm{EM}(I^* \restriction P^{I^*}_1,\Phi_2)$, there is no problem to correct this.

CLAUSE (β) of the conclusion:

So assume that $I^* \in K^{\rm or(+)}$ and $I^* \models t_0 < \ldots < t_{n-1}$ and $t_\ell \in P^{I^*}_1 \Leftrightarrow \ell < n_1$ and let $N^* = \mathrm{EM}(I^*,\Phi'), N^2 = \mathrm{EM}(I,\Phi_2)$.

As $I^* \subseteq J[N^*]$, we have $J[N^*] \models$ "$t_0 < \ldots < t_{n-1}$" and $t_\ell \in P^{J[N^*]}_1 \Leftrightarrow \ell < n_1$. If $n_1 = n$ the identity can serve as the automorphism, so without loss of generality $n_1 < n$ hence $P^{I^*}_2 \ne \emptyset$. Let t^*_ℓ be $c^{N^*}_\ell$ for $\ell = 0,\ldots,n-1$, recalling clause (e) of $\circledast_1$. By the choice of the functions $H^{M^+}_n$ (see clause (c) of $\circledast_1$ above) there is an automorphism h of the linear order $J[N^*]$ such that

$\circledast_5$ $h^{-1}(t_\ell) = t^*_\ell$

Clearly

$\circledast_6$ h induces an automorphism $\hat{h}$ of the model $\mathrm{EM}_{\tau(\Phi_1)}(I^*, \Phi')$ which is equal to $\mathrm{EM}(J[N^*], \Phi_1)$ by $\hat{h}(\sigma(a_{s_0}, \ldots, a_{s_{k-1}}) = \sigma(a_{h(s_0)}, \ldots, a_{h(s_{k-1})})$ when $s_0, \ldots, s_{k-1} \in J[N^*]$ and σ is a term in $\tau(\Phi_1)$.

So clearly

$\circledast_7$ $h^{-1}(t_0), \ldots, h^{-1}(t_{n-1})$ are as in clause (c) of the assumption of 8.20 for the linear order $J[N^*] \in K^{\mathrm{or}(+)}$.

Now the property $\boxdot_\alpha(\zeta)$ which is inherited by N^* as $N^+ \equiv N^* \equiv M^+_{\alpha(*)}$ (and the choice of $F^{M^+_{\alpha(*)}}$ in $\boxtimes_1$ above and the choices of t^*_ℓ (and of the $c^{M^+_\alpha}_\ell$ above) gives

$\oplus'$ for some automorphism f of $\mathrm{EM}_{\tau(\mathfrak{K})}(J[N^*], \Phi_1)$ we have

- (α) $f \restriction \mathrm{EM}_{\tau(\mathfrak{K})}(J[N^*] \restriction Q^{N^*}_1, \Phi_1)$ is the identity
- (β) $f(\sigma_1(\ldots, a_{h^{-1}(t_\ell)}, \ldots)_{\ell<n}) = \sigma_2(\ldots, a_{h^{-1}(t_\ell)}, \ldots)_{\ell<n}$ because $f(\sigma_1(\ldots, a_{t^*_\ell}, \ldots)_{\ell<n}) = \sigma_2(\ldots, a_{t^*_\ell}, \ldots)_{\ell<n}$
- (γ) $f(b) = F(b)$ that is $f(b) = F^{\mathrm{EM}(I^*, \Phi')}(b)$ for every $b \in \mathrm{EM}(I^*, \Phi')$; see the choice of $F^{M^+_{\alpha(*)}}$ above.

Hence also $f' = \hat{h} \circ f \circ \hat{h}^{-1}$ is an automorphism of $N^* \restriction \tau(\mathfrak{K}) = \mathrm{EM}_{\tau(\mathfrak{K})}(J[N^*], \Phi_1) = \mathrm{EM}_{\tau(\mathfrak{K})}(I^*, \Phi')$.

Let us check the demands listed in $\otimes(\beta)$ of 8.20. First half of Subclause $(\beta)(i)$ holds by $\oplus'$, and Subclause $(\beta)(ii)$ there holds by clause (β) of $\oplus'$ above. Next Subclause $(\beta)(iii)$ there holds for Φ' by our choice of f and $F^{M^+_{\alpha(*)}}$, i.e., clause (γ) of $\oplus'$ and by h being definable by the H_n's and M^+_α having Skolem functions.

More fully, let $\bar{t} = \langle t_\ell : \ell < n \rangle, \bar{t}^* = \langle t^*_\ell : \ell < n \rangle, a^*_1 = \sigma_1(a_{t_0}, \ldots, a_{t_{n-1}})$. For every $b \in N^*$ clearly for some $\sigma = \sigma(x_0, \ldots, x_{k-1})$, a $\tau(\Phi_1)$-term and $s_\ell <_I \ldots <_J s_{k-1}$ we have $b = \sigma(a_{s_0}, \ldots, a_{s_{k-1}})$ where $b \in P^{N^*}_\sigma$ and $s_\ell = G^{N^*}_\ell(b)$, so[7] by $\circledast_6$

$$\begin{aligned} b_1 =: \hat{h}^{-1}(b) &= H^{N^*}_n(b, a_{t_0}, \ldots, a_{t^*_0}, \ldots) \\ &= H^{N^*}_n(b, G^{N^*}_0(a^*_1), G^{N^*}_1(a^*_1), \ldots, G^{N^*}_{n-1}(a^*_1), c^*_0, \ldots, c^*_{n-1}) \end{aligned}$$

$$b_2 =: f(b_1) = F^{N^*}(b_1)$$

[7]we ignore the case that in $\sigma_1(x_1, \ldots, x_{n-1})$ some of the variables are dummy variables just use a_{t_ℓ} instead $G^{N^*}_1(a^*_1)$.

and recalling the end of clause $\circledast_6$ we have

$$\begin{aligned} b_3 =: \hat{h}(b_2) &= H_n^{N^*}(b_2, a_{t_0^*}, \ldots, a_{t_{n-1}^*}, a_{t_0}, \ldots, a_{t_{n-1}}) \\ &= H_n^{N^*}(b_2, c_0^*, \ldots, c_{n-1}^*, G_0^{N^*}(a_1^*), \ldots, G_{n-1}^{N^*}(a_1^*)). \end{aligned}$$

Composing clearly $f'(b) = (\hat{h} \circ f \circ \hat{h}^{-1})(b) = \hat{h}(f(\hat{h}^{-1}(b)) = \hat{h}(f(b_1)) = \hat{h}(b_2) = b_3$ is equal to $\sigma^*(b)$ for the suitable term $\sigma^*(x) = \sigma(x, G_0(a_1^*), \ldots, G_{n-1}(a_1^*), c_0^*, \ldots, c_{n-1}^*)$ of $\tau(\Phi')$ which does not depend on b. As $M^*_{\alpha(*)}$ has Skolem function we can replace $\sigma^*(x)$ by $F'(x, \bar{y})$ for some function symbol $F' \in \tau(\Phi')$. But we need "f' is the identity on $\mathrm{EM}(P_1^{I^*}, \Phi_1)$" which is for the second half of Subclause $(\beta)(i)$; let $b \in \mathrm{EM}(I^* \restriction P_1^{I^*}, \Phi_1)$ or just $b \in \mathrm{EM}(P_1^{J[N^*]}, \Phi_1)$; so for some term σ of $\tau(\Phi_1)$ and $s_0, \ldots, s_{m-1} \in P_1^{J[N^*]} = Q_1^{N^*}$ we have $N^* \models$ "$b = \sigma(a_{s_0}, \ldots, a_{s_{m-1}})$", hence (as $N^* \equiv N^+$ and the choice of F^{M^+})

$\circledast_8$ $f'(b) = (\hat{h} f \hat{h}^{-1})(\sigma(a_{s_0}, \ldots, a_{s_{m-1}})) =$
$\hat{h}(f(\hat{h}^{-1}(\sigma(a_{s_0}, \ldots, a_{s_{m-1}})))) =$
$(\hat{h}(f(\sigma(a_{h^{-1}(s_0)}, \ldots, a_{h^{-1}(s_{m-1})})))).$
[The last equality by the definition of $\hat{h}$ from h, see $\circledast_6$.]

But we are assuming that $s_0, \ldots, s_{m-1} \in Q_1^{[N^*]}$ and by the choice of h in $\circledast_5$ we conclude that $h^{-1}(s_0), \ldots, h^{-1}(s_{m-1}) \in Q_1^{N^*}$ hence by clause (α) of $\oplus'$ above we have

$\circledast_9$ $f(\sigma(a_{h^{-1}(s_0)}, \ldots, a_{h^{-1}(s_{m-1})})) = (\sigma(a_{h^{-1}(s_0)}, \ldots, a_{h^{-1}(s_{m-1})})).$

By $\circledast_8 + \circledast_9$

$$\begin{aligned} f'(b) = \hat{h}(\sigma(a_{h^{-1}(s_0)}, \ldots, \bar{a}_{h^{-1}(s_{m-1})}) &= \sigma(\bar{a}_{hh^{-1}(s_0)}, \ldots, \bar{a}_{hh^{-1}(s_{m-1})}) \\ &= \sigma(\bar{a}_{s_0}, \ldots, \bar{a}_{s_{m-1}}) = b. \end{aligned}$$

As b was any member of $\mathrm{EM}_{\tau(\mathfrak{K})}(I^* \restriction P_1^{I^*}, \Phi_1)$ we are done proving the second half of subclause $(\beta)(i)$ of $\otimes$.

We have shown above strongly version of definability for Φ', so by the way Φ_2 was constructed from Φ' it follows that also subclause $(\beta)(iii)$ of $\otimes$ holds that is 8.16(4)(δ). $\square_{8.20}$

8.21 Claim. *Assume*

(a) $\mathfrak{K}$ *is an a.e.c. with amalgamation, categorical in* λ
(b) *the* $M \in K_\lambda$ *is* χ^+*-saturated (holds if* $\mathrm{cf}(\lambda) > \chi$*)*
(c) $\chi \geq \mathrm{LS}(\mathfrak{K})$.

Then every $M \in K$ *of cardinality* $\geq \beth_{(2^\chi)^+}$ *(or just* $\geq \beth_{\mu(\chi)}$ *if* $\chi \geq 2^{\mathrm{LS}(\mathfrak{K})}$*) is* χ^+*-saturated.*

Proof. If M is a counterexample, let $N \leq_{\mathfrak{K}} M, \|N\| \leq \chi$ and $p \in \mathscr{S}(N)$ be omitted by N. By the omitting type theorem for abstract elementary classes (see 8.6, i.e. [Sh 88]), we get $M' \in K_\lambda, N \leq_{\mathfrak{K}} M', M'$ omitting p a contradiction. $\square_{1.9}$

8.22 Claim. *Assume*

(a) $\mathrm{LS}(\mathfrak{K}) \leq \chi$
(b) *for every* $\alpha < (2^\chi)^+$ *there are* $M_\alpha <_{\mathfrak{K}} N_\alpha$ *(so* $M_\alpha \neq N_\alpha$*),* $\|M_\alpha\| \geq \beth_\alpha$ *and* $p \in \mathscr{S}(M_\alpha)$ *such that* $c \in N_\alpha \Rightarrow \neg p E_\chi \mathrm{tp}(c, M_\alpha, \mathfrak{C})$.

1) For every $\theta > \chi$ *there are* $M <_{\mathfrak{K}} N$ *in* K_θ *and* $p \in \mathscr{S}(M_\alpha)$ *as in clause (b).*
2) Moreover, if Φ *is proper for orders as usual,* $|\tau(\Phi)| \leq \chi, \beth_{(2^\chi)^+} \leq \lambda, K$ *categorical in* λ *and* I *a linear order of cardinality* θ, *then we can demand* $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$.

Proof. Straight.

8.23 Conclusion. 1) For $\kappa \geq$ LS($\mathfrak{K}$) and $\Phi \in \Upsilon_\kappa^{\rm or(+)}$ there are $\alpha(*) < (2^\kappa)^+$ and Φ^* such that

(a) $\Phi \leq_\kappa^{\oplus,3} \Phi^*$ so $\Phi^* \in \Upsilon_\kappa^{\rm or(+)}$

(b) if $\mathfrak{x} = \langle n, n_1, \sigma_2(\dots, x_m, \dots)_{m<n}, \sigma_2(x, \dots, x_m, \dots)_{m<n}\rangle$ and Φ_2 and $\langle t_\ell : \ell < n\rangle$ satisfies $\boxdot_{\alpha(*)}$ of clause (c) of 8.20 holds with $(\mathfrak{x}, \Phi^*)$ here standing for $(\mathfrak{x}, \Phi_1)$ there (so $\mathfrak{x}$ is a Φ^*-task) <u>then</u> $\otimes(\beta)$ from the conclusion of 8.20 holds.

2) We can replace $\alpha(*) < (2^\kappa)^+$ by $\alpha(*) < \delta_\kappa$; on δ_κ see, e.g., [Sh:c, VII,§5].

Proof. 1) We iterate 8.20 recalling that $\Phi_1 \leq_\kappa^{\oplus,3} \Phi_2 \in \Upsilon_\kappa^{\rm or(+)}[\mathfrak{K}]$ implies that $I \in K^{\rm or(+)}, I = I \restriction P_1^I \Rightarrow {\rm EM}_{\tau(\Phi_1)}(I \restriction P_1^I, \Phi_2) = {\rm EM}(I \restriction P_1^I, \Phi_1)$. We choose Φ_α by induction on $\alpha \leq \lambda$ such that $\langle \Phi_\alpha : \alpha \leq \lambda\rangle$ is $\leq_\kappa^{\oplus,3}$-increasing continuous, in $\Upsilon_\kappa^{\rm or(+)}$. Let $\Phi_0 = \Phi$ and in limit stages we take unions. In the induction step, $\alpha = \beta + 1$ is by 8.20, with $(\Phi_\beta, \Phi_\alpha)$ here standing for (Φ_1, Φ_2) there and $\mathfrak{x} = \mathfrak{x}_\beta$ as in clause (b) of the assumption of 8.20 is chosen such that for every $\alpha < \lambda$ and any Φ_α-tasks $\mathfrak{x}$ (i.e., as in clause (b) for Φ_α) for some $\beta < \lambda, \mathfrak{x}_B = \mathfrak{x}$, this is done by bookkeeping.
2) Reflect. $\square_{8.23}$

8.24 Definition. 1) For $\Phi^* \in \Upsilon_\kappa^{\rm or(+)}[\mathfrak{K}]$ let $\alpha(\Phi^*)$ be the minimal ordinal $\alpha(*)$ such that $\beth_{\alpha(*)} >$ LS($\mathfrak{K}$) and if $\mathfrak{x}$ is a Φ^*-task (see Definition 8.17(2)) and $\boxdot_\alpha$ of clause (c) of the assumption of 8.20 fails for some α then it fails for some $\alpha \leq \alpha(*)$ (hence for $\alpha = \alpha(*)$).
2) For $\ell = 0, 1$ let $\chi_\ell(\Phi^*) = \chi_\ell(*) = \beth_{\alpha(\Phi^*)+\alpha(\Phi^*)\times\ell}$ and $\chi(*) = \chi_1(\Phi^*)$.

8.25 Observation. 1) If Φ_1 is from $\Upsilon_\kappa^{\rm or(+)}[\mathfrak{K}]$ <u>then</u> $\alpha(\Phi_1) < \delta_\kappa < (2^\kappa)^+$.

8.26 Remark. 1) Actually because of "$\mathfrak{K}$ has amalgamation", it is easier to prove the following weaker variant of 8.23 replacing "there is an automorphism f of ${\rm EM}_{\tau(\mathfrak{K})}(I, \Phi^*)$ over ${\rm EM}_{\tau(\mathfrak{K})}(I \restriction P_1^I, \Phi^*)$ which maps b^1 to b^2" by "there are models $N_\ell, \{b_1^1, b^2\} \subseteq N_\ell$ and ${\rm EM}_{\tau(\mathfrak{K})}(I \restriction P_1^I, \Phi^*) \leq_{\mathfrak{K}} N_\ell \leq_{\mathfrak{K}} {\rm EM}_{\tau(\mathfrak{K})}(I, \Phi^*)$" and isomorphism f from N_1 onto N_2 over ${\rm EM}_{\tau(\mathfrak{K})}(I \restriction P_1^I, \Phi^*)$ mapping b^1 to b^2. This is actually enough.
2) We can get also somewhat stronger results, see [Sh:F657].

§9 Small Pieces are Enough and Categoricity

Using the context below is justified by the previous sections.

9.1 Context. a) $\mathfrak{K}$ is categorical in λ.
b) Φ^* as in 8.23.

9.2 Main Lemma=Local Lemma. *If $M^* \in K$ is a saturated model of cardinality $\chi, \chi_1(*) \le \chi < \mathrm{cf}(\lambda) \le \lambda$, see 8.24(2) then $\mathscr{S}(M^*)$ has character or locality $\le \chi_1(*)$, (see Definition 1.8, i.e., if $p_1 \ne p_2$ are in $\mathscr{S}(M^*)$ then for some $N \le_{\mathfrak{K}} M, N \in K_{\chi_1(*)}$ we have $p_1 \restriction N \ne p_2 \restriction N$).*

Proof. So Φ^* as in 8.23. Choose I such that (easily exists by 8.11(2))

$\circledast_1$ (a) $I \in K^{\mathrm{or}(+)}$ is strongly $\aleph_0$-homogeneous
(b) P^I_1 has cardinality χ
(c) I has cardinality λ
(d) in P_ℓ there is a monotonic sequence of length θ^+ whenever $\theta < |P^I_\ell|$ for $\ell = 1, 2$.

Let $N^* = \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi^*) \in K_\lambda$ and $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I \restriction P^I_1, \Phi^*)$ so $M \in K_\chi$ is saturated (see 8.12 above) hence without loss of generality $M^* = M$. Similarly in N^* every $p \in \mathscr{S}(M^*)$ is realized. Assume toward contradiction that $p_1 \ne p_2$ are from $\mathscr{S}(M^*)$ but $p_1/E_{\chi_1(*)} = p_2/E_{\chi_1(*)}$, i.e., $M' \le_{\mathfrak{K}} M, M' \in K_{\chi_1(*)} \Rightarrow p_1 \restriction M' = p_2 \restriction M'$. We can find $b_1, b_2 \in N^*$ which realize p_1, p_2 respectively. As we can add dummy variables we can find n and $t_0 <_I < \ldots <_I t_{n-1}$ and terms σ_1, σ_2 of $\tau(\Phi^*)$ such that $b_\ell = \sigma_\ell(a_{t_0}, \ldots, a_{t_{n-1}})$. Let $n_1 \le n$ be such that $t_\ell \in P^I_1 \Leftrightarrow \ell < n_1$; again without loss of generality, $n_1 < n$.

Recall (8.11(1)) that the linear order $(P^I_1, \le_I)$ is strongly $\aleph_0$-homogeneous. We can find a strongly $\aleph_0$-homogeneous $J_1 \subseteq P^I_1$ of cardinality $\chi_1(*)$ which includes $\{t_\ell : \ell < n\} \cap P^I_1$. Let $J_2 \subseteq P^J_2$ be of cardinality χ such that it contains a monotonic sequence of length θ^+ for every $\theta < \chi$ and is strongly $\aleph_0$-homogeneous and $\{t_\ell : \ell < n\} \cap P^I_2 \subseteq J_2$.

Let $J = I \restriction (J_1 \cup P^I_2)$ and let $M' = \mathrm{EM}_{\tau(\mathfrak{K})}(J \restriction P^J_1, \Phi^*) = \mathrm{EM}(J_1, \Phi^*)$ and $N^1 = \mathrm{EM}_{\tau(\mathfrak{K})}(J, \Phi^*)$. Easily $M' \le_{\mathfrak{K}} N' \in \mathfrak{K}_\lambda, b_1, b_2 \in N', M' \in \mathfrak{K}_{\chi(*)}, J$ is strongly $\aleph_0$-homogeneous (see 8.10(1)) and by the choice of p_1, p_2, b_1, b_2 we have $\mathrm{tp}_{\mathfrak{K}}(b_1, M', N') = \mathrm{tp}_{\mathfrak{K}}(b_2, M', N')$. But N' is saturated by 8.12 hence there is an automorphism of N' over M' mapping b_1 to b_2. As $\lambda \ge \chi_1(*)$ by the choice of Φ^*, i.e., by 8.23 we can conclude that there is an automorphism of N^* over M

mapping b_1 to b_2, contradiction to $p_1 \neq p_2$. We have to check $\boxdot_\alpha$; so note that $|P^J_1| = |J_1| = \chi_0(\Phi^*), P^J_2 = |J_2| = \chi \geq \chi_1(\Phi^*)$, etc. $\square_{9.2}$

9.3 Claim. *If T is categorical in λ, $\mathrm{LS}(\mathfrak{K}) \leq \chi(*) \leq \mu < \lambda$ and $\langle M_i : i < \delta\rangle$ an $<_{\mathfrak{K}}$-increasing sequence of μ^+-saturated models <u>then</u> $\bigcup\limits_{i<\delta} M_i$ is μ^+-saturated.*

Remark. 1) Hence this holds for limit cardinals $>$ $\mathrm{LS}(\mathfrak{K})$.
2) The addition compared to 6.7 are the cases $\mathrm{cf}(\lambda) = \mu^+, \mu^{++}$, e.g. $\lambda = \mu^+$. The only case needed is $\lambda = \mu^{++}$ (used in $(*)_8$ of the proof of 9.5).
[Saharon: check again! as $\mu^+ = \lambda$ is trivial.]

Proof. Let $M_\delta = \bigcup\limits_{i<\delta} M_i$ and assume M_δ is not μ^+-saturated. So there are $N \leq_{\mathfrak{K}} M_\delta$ of cardinality $\leq \mu$ and $p \in \mathscr{S}(N)$ which is not realized in M_δ. Choose $p_1 \in \mathscr{S}(M_\delta)$ such that $p_1 \restriction N = p$.
Without loss of generality N is saturated (by 6.7, or think).

Let $\chi = \chi(*)$, without loss of generality $\delta =$ $\mathrm{cf}(\delta) \leq \mu$ and each M_i has cardinality μ^+ hence $i < \delta \Rightarrow M_i$ is saturated.
We claim

$\bigotimes$ there are $i(*) < \delta$ and $N^* \leq_{\mathfrak{K}} M_{i(*)}$ of cardinality χ such that p_1 does not χ-split over N^*.

Why? Assume toward contradiction that this fails. The proof of $\bigotimes$ splits to two cases.

<u>Case I</u>: $\mathrm{cf}(\delta) \leq \chi$.

We can choose by induction on $i < \delta =$ $\mathrm{cf}(\delta)$ models N^0_i, N^1_i such that

- (a) $N^0_i \leq_{\mathfrak{K}} M_i$ has cardinality χ
- (b) N^0_i is increasing continuous
- (c) $N^0_i <^1_{\chi,\omega} N^0_{i+1}$
- (d) $N^0_i \leq_{\mathfrak{K}} N^1_i \leq_{\mathfrak{K}} M_\delta$
- (e) N^1_i has cardinality $\leq \chi$
- (f) $p_1 \restriction N^1_i$ does χ-split over N^0_i
- (g) for $\varepsilon, \zeta < i, N^1_\varepsilon \cap M_\zeta \subseteq N^0_i$.

There is no problem to carry the definition and then $N^1_i \subseteq \bigcup\limits_{j<\delta} N^0_j$ and $\langle N^0_i : i < \delta\rangle, p_1 \restriction \bigcup\limits_{i<\delta} N^0_i$ contradicts 6.3.

Case II: $\mathrm{cf}(\delta) > \chi$.

Now by 3.3

(∗) for some $N^* \leq_{\mathfrak{K}} M_\delta$ of cardinality χ we have p_1 does not χ-split over N^*.

As $\delta = \mathrm{cf}(\delta) \geq \mu > \chi$, for some $i(*) < \delta$ we have $N^* \leq_{\mathfrak{K}} M_{i(*)}$. This ends the proof of $\bigotimes$.

So $i(*), N^*$ are well defined. Without loss of generality N^* is saturated. Let $c \in \mathfrak{C}$ realize p_1. We can find a $\leq_{\mathfrak{K}}$-embedding h of $\mathrm{EM}_{\tau(\mathfrak{K})}(\mu^+ + \mu^+, \Phi^*)$ into $\mathfrak{C}$ such that

(a) N^* is the h-image of $\mathrm{EM}_{\tau(\mathfrak{K})}(\chi, \Phi^*)$

(b) h maps $EM_{\tau(\mathfrak{K})}(\mu^+, \Phi^*)$ onto some $M' \leq_{\mathfrak{K}} M_{i(*)}$

(c) every $d \in N$ and c belong to the range of h.

By renaming, h is the identity, clearly for some $\alpha < \mu^+$ we have $N \cup \{c\} \subseteq EM_{\tau(\mathfrak{K})}(\alpha \cup [\mu^+, \mu^+ + \alpha), \Phi^*)$, so some list $\bar b$ of the members of N is $\bar\sigma(\dots, \bar a_i, \dots, a_{\mu^+ + j}, \dots)_{i<\alpha, j<\alpha}$ (assume $\alpha > \mu$ for simplicity) and $c = \sigma^*(\dots, \bar a_i, \dots, a_{\mu^+ + j}, \dots)_{i \in u, j \in w}$ ($u, w \subseteq \mu^+$ finite as, of course, only finitely many $\bar a_i$'s are needed for the term σ^*) and without loss of generality $u \cup w \subseteq \alpha$.

For each $\gamma < \mu^+$ we define $\bar b^\gamma = \bar\sigma(\dots, \bar a_i, \dots, a_{(1+\gamma)\alpha + j}, \dots)_{i<\alpha, j<\alpha}$ and $c^\gamma = \sigma^*(\dots, \bar a_i, \dots, a_{(1+\gamma)\alpha+j}, \dots)_{i,j}$ and stipulate $\bar b^{\mu^+} = \bar b, c^{\mu^+} = c$ and let $q = \mathrm{tp}(\bar b \hat{\ } c, N^*, \mathfrak{C})$. Clearly $\langle \bar b^\gamma \hat{\ } c^\gamma : \gamma < \mu^+\rangle \hat{\ } \langle \bar b \hat{\ } c\rangle$ is a strictly indiscernible sequence over N^* and $\subseteq M_\delta \cup \{c\}$, so also $\{\bar b^\gamma : \gamma \leq \mu^+\} \subseteq M_\delta$ is strictly indiscernible over N.

[Why? Use $I \supseteq \mu^+ + \mu^+$ which is a strongly μ^{++} saturated dense linear order and use automorphisms of $\mathrm{EM}(I, \Phi^*)$ induced by an automorphism of I.]

As c realizes p_1 clearly $\mathrm{tp}(c, M_\delta)$ does not χ-split over N^* hence by 9.2 recalling that $M_{i(*)}$ is a saturated model of cardinality μ^+ necessarily $\mathrm{tp}(\bar b^\gamma \hat{\ } c, N^*, \mathfrak{C})$ is the same for all $\gamma \leq \mu^+$, hence $\gamma < \mu^+ \Rightarrow \mathrm{tp}(\bar b^\gamma \hat{\ } c^{\mu^+}, N^*, \mathfrak{C}) = q$, so by the indiscernibility $\beta < \gamma \leq \mu^+ \Rightarrow \mathrm{tp}(\bar b^\beta \hat{\ } c^\gamma, N^*, \mathfrak{C}) = q$.

Similarly for some q_1,

$$\beta < \gamma \leq \mu^+ \Rightarrow \mathrm{tp}(\bar b^\gamma \hat{\ } c^\beta, N^*, \mathfrak{C}) = q_1.$$

If $q \neq q_1$, we get the $(\chi, 1, 0)$-order property (see Definition 4.3) contradiction to (or 4.8).

Hence necessarily $\beta \le \mu^+$ & $\gamma \le \mu^+$ & $\beta \ne \gamma \Rightarrow$ $\text{tp}(\bar b^\beta \hat{} c^\gamma, N^*, \mathfrak{C}) = q$. For $\beta = \mu^+, \gamma = 0$ we get that $c^\gamma \in M_{i(*)} \le_{\mathfrak{K}} M_\delta$ realizes $\text{tp}(c^\gamma, N, \mathfrak{C}) = p_1 \restriction N$ as desired. $\square_{9.3}$

We could have mentioned earlier parts (1) + (2) of the following observation.

9.4 Observation. 1) If M is θ-saturated, $\theta >$ LS($\mathfrak{K}$) and $\theta < \lambda$ and $N \le_{\mathfrak{K}} M, N \in K_{\le\theta}$ <u>then</u> there is $N', N' \le_{\mathfrak{K}} M, N' \in K_\theta$ and every $p \in \mathscr{S}(N)$ realized in M is realized in N'.
2) Assume $\langle N_i : i \le \delta\rangle$ is $\le_{\mathfrak{K}}$-increasingly continuous, $\delta < \theta^+$ is divisible by $\theta, N_i \in K_{\le\theta}, N_i \le_{\mathfrak{K}} M, M$ is θ-saturated, and every $p \in \mathscr{S}(N_i)$ realized in M is realized in N_{i+1} <u>then</u>

- (a) if $\delta = \theta \times \sigma$, LS($\mathfrak{K}$) $< \sigma =$ cf(σ) $\le \theta$, then N_δ is σ-saturated
- (b) if $\delta = \theta \times \theta, \theta >$ LS($\mathfrak{K}$), <u>then</u> N_δ is saturated.

3) In part (1) we can add: N' is saturated and even saturated over N (here we use 9.3).

9.5 Theorem. *(The Downward Łos theorem for λ successors).*
If λ is successor $\beth_{(2^{\chi(*)})^+} \le \chi < \lambda$, <u>then</u> K is categorical in χ.

9.6 Remark. 0) In fact, we can replace $\beth_{(2^{\chi(*)})^+}$ by $\mu(\chi(*))$.
1) We intend to try to prove in future work that also in $K_{>\lambda}$ we have categoricity and deal with λ not successor. This calls for using $\mathscr{P}^-(n)$-diagrams as in [Sh 87a], [Sh 87b], etc.
2) We need [and can have one but not here] some theory of orthogonality and regular types parallel to [Sh:a, Ch.V] = [Sh:c, Ch.V], as done in [Sh:h] and then [MaSh 285] (which appeared) and then (without the upward categoricity) [KlSh 362], [Sh 472]. Then the categoricity can be proved as in those papers.

Proof. Let $K' = \{M \in K : M$ is $\chi(*)^+$-saturated hence of cardinality $> \chi(*)\}$. So

- $(*)_0$ there is $M \in K_\lambda$ which is λ-saturated
  [why? by 2.6, 1.7, as λ is regular]
- $(*)_1$ K' is closed under $\le_{\mathfrak{K}}$-increasing unions
  [Why? By 9.3 (or 6.7)]
- $(*)_2$ if $\chi \ge \beth_{(2^{\chi(*)})^+}$ and $M \in K_\chi$ <u>then</u> $M \in K'_\chi$, moreover M is $\beth_{(2^{\chi(*)})^+}$-saturated

[Why? Otherwise by 8.6 there is a non $(\chi(\Phi^*))^+$-saturated $M \in K_\lambda$ contradicting $(*)_0$, or use 8.21. For the "Moreover" use 8.7 instead of 8.6. Assume $M \in K_\chi, \chi \geq \beth_{(2^{\chi(*)})^+}$ and M is not $\beth_{(2^{\chi(*)})^+}$-saturated. Let $N \leq_{\mathfrak{K}} M$ be of minimal cardinality such that some $p \in \mathscr{S}(N)$ is omitting so $\beth_{(2^{\chi(*)})^+} > \|N\| > \chi(*)$ hence $\beth_{(2^{\chi(*)})^+}(\|N\|) = \beth_{(2^{\chi(*)})^+} \leq \|M\|$. By 9.4, without loss of generality N is saturated, hence by 9.2 for every $c \in M$, for some $N_c \leq_{\mathfrak{K}} M$ of cardinality $\chi(*)$ we have: $\mathrm{tp}_{\mathfrak{K}}(c, N_c, M) \neq p \restriction N_c$. Now 8.7 applies and gives contradiction.]

$(*)_3$ if $M \in K$ and $p \in \mathscr{S}(M)$ then for some $M_0 \leq_{\mathfrak{K}} M, M_0 \in K'_{\chi(\Phi^*)}$ and p does not $\chi(\Phi^*)$-split over M_0
[why? by 3.3, 1.7]

$(*)_4$ Definition: for $\chi \in [\chi(*), \lambda)$ and $M \in K'_\chi$ and $p \in \mathscr{S}(M)$ we say p is minimal if:

- (a) p is not algebraic which means p is not realized by any $c \in M$
- (b) if $M \leq_{\mathfrak{K}} M' \in K'_\chi$ and $p_1, p_2 \in \mathscr{S}(M')$ are non-algebraic extending p, then $p_1 = p_2$

$(*)_5$ Fact: if $M \in K'_\chi$ is saturated, $\chi \in [\chi(*), \lambda)$, then some $p \in \mathscr{S}(M)$ is minimal
[Why? If not, we choose by induction on $\alpha \leq \chi$ for every $\eta \in {}^\alpha 2$ a triple (M_η, N_η, a_η) and $h_{\eta, \eta \restriction \beta}$ for $\beta \leq \alpha$ such that:

- (a) $M_\eta <_{\mathfrak{K}} N_\eta$ and $a_\eta \in N_\eta \backslash M_\eta$
- (b) $\langle M_{\eta \restriction \beta} : \beta \leq \alpha \rangle$ is $\leq_{\mathfrak{K}}$-increasingly continuous
- (c) $M_{\eta \restriction \beta} <^1_{\chi, \omega} M_{\eta \restriction (\beta+1)}$
- (d) $h_{\eta, \eta \restriction \beta}$ is a $\leq_{\mathfrak{K}}$- embedding of $N_{\eta \restriction \beta}$ into N_η which is the identity on $M_{\eta \restriction \beta}$ and maps $a_{\eta \restriction \beta}$ to a_η
- (e) if $\gamma \leq \beta \leq \alpha, \eta \in {}^\alpha 2$, then $h_{\eta, \eta \restriction \gamma} = h_{\eta, \eta \restriction \beta} \circ h_{\eta \restriction \beta, \eta \restriction \gamma}$
- (f) $M_{\eta\hat{}\langle 0 \rangle} = M_{\eta\hat{}\langle 1 \rangle}$ but $\mathrm{tp}(a_{\eta\hat{}\langle 0 \rangle}, M_{\eta\hat{}\langle 0 \rangle}, N_{\eta\hat{}\langle 0 \rangle}) \neq \mathrm{tp}(a_{\eta\hat{}\langle 1 \rangle}, M_{\eta\hat{}\langle 1 \rangle}, N_{\eta\hat{}\langle 1 \rangle})$
- (g) $M_\eta <_{\mathfrak{K}} \mathfrak{C}$.

No problem to carry the definition and let $\kappa = \mathrm{Min}\{\kappa : 2^\kappa > \chi\}$ and choose $M <_{\mathfrak{K}} \mathfrak{C}$ such that $\|M\| \leq \chi$ and $\eta \in {}^{\kappa >}2 \Rightarrow M_\eta \subseteq M$ hence $\eta \in {}^\kappa 2 \Rightarrow M_\eta \subseteq M$ so $\{\mathrm{tp}(a_\eta, M, \mathfrak{C}) : \eta \in {}^\kappa 2\}$ is a subset of $\mathscr{S}(M)$ of cardinality $2^\kappa > \chi$. So we can get a contradiction to stability in χ, i.e., to 1.7].

$(*)_6$ Fix a saturated $M^* \in K_{\chi(\Phi^*)}$ and minimal $p^* \in \mathscr{S}(M^*)$

$(*)_7$ if $M^* \leq_{\mathfrak{K}} M \in K'_{<\lambda}$, <u>then</u> p^* has a non-algebraic extension $p \in \mathscr{S}(M)$, moreover; if M is saturated, it is unique and also p is minimal
[Why? Let $M' \in K_{<\lambda}$ be saturated such that $M \leq_{\mathfrak{K}} M', \|M'\| = \|M\|$; and if $\|M\| = \chi(*), M'$ is brimmed over M^* (see Definition 2.7). Existence of non-algebraic $p' \in \mathscr{S}(M')$ holds by as (M^*, M) is isomorphic to $(\mathrm{EM}_\tau(\chi(\Phi^*), \Phi^*), \mathrm{EM}_\tau(\|M'\|, \Phi^*)$.
Now for (p', M') uniqueness modulo $E_{\chi(*)}$ follows from the definition of "p^* is minimal" hence uniqueness for (M', p') holds by the locality lemma 9.2. So we have proved the "moreover". When M is not saturated applying what we have proved to a saturated extension M' of M of cardinality $\|M\|$ we get a non-algebraic $p' \in \mathscr{S}(M')$ extending p, now $p' \restriction M$ is as required].

Let λ_1 be the predecessor of λ.

$(*)_8$ there are no M_1, M_2 such that:

(a) $M^* \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2$

(b) M_1, M_2 are saturated of cardinality λ_1

(c) $M_1 \neq M_2$

(d) no $c \in M_2 \backslash M_1$ realizes p^*

[Why? If there is such a pair (M_1, M_2), we choose by induction on $\zeta < \lambda, N_\zeta \in K_{\lambda_1}$ which is $\leq_{\mathfrak{K}}$-increasing continuous, each N_ζ is saturated, $N_0 = M_1, N_\zeta \neq N_{\zeta+1}$ and no $c \in N_{\zeta+1} \backslash N_\zeta$ realizes p^*. If we succeed, then $N = \bigcup_{\zeta<\lambda} N_\zeta$ is in K_λ (as $N_\zeta \neq N_{\zeta+1}$!) but no $c \in N \backslash N_0$ realizes p^* (why? as $\{\zeta : c \notin N_\zeta\}$ is an initial segment of λ, non-empty as 0 is in so it has a last element ζ, so $c \in N_{\zeta+1} \backslash N_\zeta$ so realizes p^*, contradiction); hence N is not saturated, contradiction to categoricity in λ by $(*)_0$. For $\zeta = 0, N_0 = M_1$ is okay by clause (b). If ζ is limit $< \lambda$, let $N_\zeta = \bigcup_{\varepsilon<\zeta} N_\varepsilon$, clearly $N_\zeta \in K_{\lambda_1}$ and it is saturated by 9.3. If $\zeta = \varepsilon+1$, note that as N_ε, M_1 are saturated and in K_{λ_1} and $\leq_{\mathfrak{K}}$-extends M^* which has smaller cardinality, there is an isomorphism f_ζ from M_1 onto N_ε which is the identity on M^*. We define N_ζ such that there is an isomorphism f^+_ζ from M_2 onto N_ζ extending f_ζ. By assumption (b), $N_\zeta \in K_{\lambda_1}$ is saturated and by assumption (c), $N_\zeta \neq N_{\zeta+1}$, and by assumption (d), no $c \in N_{\zeta+1} \backslash N_\zeta$ realizes p^* (as $f_\zeta \restriction M^* =$ the identity). So as said above, we have derived the desired contradiction].

$(*)_9$ if $M \in K'_{<\lambda}$ and $M^* \leq_{\mathfrak{K}} M <_{\mathfrak{K}} N, M$ has cardinality $\geq \theta^* = \beth_{(2^{\chi(*)})^+}$, <u>then</u> some $c \in N \backslash M$ realizes p^*.
[Why? Assume this fails, hence by $(*)_2$, M, N are θ^*-saturated. So we

can find saturated $M' \leq_{\mathfrak{K}} M, N' \leq_{\mathfrak{K}} N$ of cardinality θ^* such that $M' = N' \cap M, M^* \neq N'$ (why? by observation 9.4(1)-(3)). So still no $c \in N' \backslash M'$ realizes p^*. We would like to transfer (using the appropriate omitting type theorem) this situation from θ^* to λ_1; the least trivial point is preserving the saturation. But this can be expressed as: "is isomorphic to $\mathrm{EM}(I, \Phi^*)$ for some linear order I" for appropriate Φ, and this is easily transferred].

$(*)_{10}$ if $M \in K'_{\leq\lambda}$ has cardinality $\geq \theta^* = \beth_{(2^{\chi(*)})^+}$ then it is θ^*-saturated (so $\in K'_{\leq\lambda}$).
[Why? By $(*)_2$.]

$(*)_{11}$ if $M \in K'_{\leq\lambda}$ has cardinality $\geq \theta^*$, then M is saturated
[why? Assume not; by $(*)_{10}$, M is θ^*-saturated let θ be such that M is θ-saturated but not θ^+-saturated; by $(*)_{10}$, $\theta \geq \theta^*$, without loss of generality $M^* \leq_{\mathfrak{K}} M$. Let $M_0 \leq_{\mathfrak{K}} M$ be such that $M_0 \in K_\theta$ and some $q \in \mathscr{S}(M_0)$ is omitted by M and without loss of generality $M^* \leq_{\mathfrak{K}} M_0$.
Now choose by induction on $i < \theta^+$ a triple (N_i^0, N_i^1, f_i) such that:

- (a) $N_i^0 \leq_{\mathfrak{K}} N_i^1$ belong to K_θ and are saturated
- (b) N_i^0 is $\leq_{\mathfrak{K}}$-increasing continuous
- (c) N_i^1 is $\leq_{\mathfrak{K}}$-increasing continuous
- (d) $N_0^0 = M_0$ and $d \in N_0^1$ realizes q
- (e) f_i is a $\leq_{\mathfrak{K}}$-embedding of N_i^0 into M and $f_0 = \mathrm{id}_{M_0}$
- (f) for each i, for some $c_i \in N_i^1 \backslash N_i^0$ we have $c_i \in N_{i+1}^0$
- (g) f_i is increasing continuous.

If we succeed, let $E = \{\delta < \theta^+ : \delta$ limit and for every $i < \delta$ and $c \in N_i^1$ we have $(\exists j < \theta^+)(c_j = c) \Rightarrow (\exists j < \delta)(c_j = c)\}$. Clearly E is a club of θ^+, and for each $\delta \in E, c_\delta$ belongs to $N_\delta^1 = \bigcup_{i<\delta} N_i^1$ so there is $i < \delta$ such that $c_\delta \in N_i^1$, so for some $j < \delta, c = c_j$ so $c_\delta = c_j \in N_{j+1}^0 \leq_{\mathfrak{K}} N_\delta^0$, contradiction to clause (f).
So we are stuck for some ζ, now $\zeta \neq 0$ trivially. Also ζ not limit by 9.3, so $\zeta = \varepsilon + 1$. Now if $N_\varepsilon^0 = N_\varepsilon^1$, then $f_\varepsilon(d) \in M$ realizes q, where d is from clause (d), a contradiction, so $N_\varepsilon^0 <_{\mathfrak{K}} N_\varepsilon^1$. Also $f_\varepsilon(N_\varepsilon^0) <_{\mathfrak{K}} M$ by cardinality consideration. Now by $(*)_9$ some $c_\varepsilon \in N_\varepsilon^1 \backslash N_\varepsilon^0$ realizes p^*.
We can find $N'_\zeta \leq_{\mathfrak{K}} M$ such that $f_\varepsilon(N_\varepsilon^0) <_{\mathfrak{K}} N'_\zeta \in K_\theta, N'_\zeta$ saturated (why? by 9.4(3)).
Again by $(*)_9$ we can find $c'_\varepsilon \in N'_\zeta \backslash f_\varepsilon(N_\varepsilon^0)$ realizing p^*. By $(*)_{5,7}$, the uniqueness part clearly $\mathrm{tp}(c'_\varepsilon, f_\varepsilon(N_\varepsilon^0), M) = f_\varepsilon(\mathrm{tp}(c_\varepsilon, N_\varepsilon^0, N_\varepsilon^1))$ so we can find $N_\zeta^1 \in K_\theta$ which is a $\leq_{\mathfrak{K}}$-extension of N_ε^1 and a $\leq_{\mathfrak{K}}$-embedding g_ε of N'_ζ into N_ζ^1 which extends f_ε^{-1} and maps c'_ε to c_ε. Without loss of generality

N^1_ζ is saturated. Let $N^0_\zeta = g_\varepsilon(N'_\zeta)$ and N^1_ζ, c_ε were already defined. So we can carry the construction, contradiction, so $(*)_{11}$ holds].

$(*)_{12}$ K_λ is categorical in every $\chi \in [\beth_{(2^{\chi(*)})^+}, \lambda)$
[why? by $(*)_{11}$ every model is saturated and the saturated model is unique].

$\square_{9.5}$

Glossary

§0 Introduction

Definition 0.1: $\mathfrak{K}$ is an a.e.c.

Definition 0.2: 1) K_μ, h a $\leq_{\mathfrak{K}}$-embedding, $\mathfrak{K}$ has amalgamation, λ-amalgamation, JEP and $M <_{\mathfrak{K}} N$

Definition 0.3: $\mathrm{LS}(\mathfrak{K})$, $\mathrm{LS}'(\mathfrak{K})$

Claim 0.4: directed unions

Claim 0.5: Representing an a.e.c. by $\mathrm{EC}(\Gamma, \emptyset)$

Claim 0.6: Existence of EM models see 8.6, CHECK!

§1

Hypothesis 1.1: (a) $\mathfrak{K}$ a.e.c.
(b) $\mathfrak{K}$ has amalgamation and JEP
(c) $K_\lambda \neq \emptyset$ for every λ

Convention 1.2: There is monster $\mathfrak{C}$

Definition 1.3: 1) $\mathfrak{K}$ is categorical in λ
2) $I(\lambda, K)$

Definition 1.4: $\mathrm{tp}(a, M, N)$
2) $\mathrm{tp}(a, M) = \mathrm{tp}(\bar{a}, M, \mathfrak{C}) = \frac{\bar{a}}{M} = \bar{a}/m$; define M is κ-saturated where $\kappa > \mathrm{LS}(\mathfrak{K})$
3),4) $\mathscr{S}^\alpha(M), \mathscr{S}^*(M) = \mathscr{S}^1(M)$
5) $p_0 = p_1 \restriction M$

Definition 1.5: $\mathfrak{K}$ is stable in λ

Convention 1.6: Φ as in 0.6

Claim 1.7: If K is cateogorical in λ then $\mathfrak{K}$ is stable in every μ, $\mathrm{LS}(\mathfrak{K}) \leq \mu < \lambda$ and any $M \in K_\lambda$ is cf$(*)$-saturated [proof] CHECK

Definition 1.8: 1) For $\mu \geq \mathrm{LS}(\mathfrak{K}), \mathbb{E}_\mu = \mathbb{E}^1_\mu[\mathfrak{K}] = \{(p_1, p_2) : p_1, p_2 \in S^\Upsilon(M)$ and $(\forall N \in K_{\leq \mu})(N \leq_{\mathfrak{K}} M \rightarrow p_1 \restriction N = p_2 \restriction N)$
2) $p \in \mathscr{S}^m(M)$ is μ-local if $p/\mathbb{E}_\mu$ is a singleton
3) $\mathfrak{K}$ is μ-local if every $p \in \mathscr{S}^{<\omega}(M)$ is μ-local
4) $\bar{c}$ realizes $p/\mathbb{E}_\mu$ is in M^* if $M \leq_{\mathfrak{K}} M^*, c \in M^*$, $\mathrm{tp}(c, M, M^*) E_\mu p$.

Remark 1.9: 1) $\mathbb{E}$ is an equivalence relation

2) In previous contents $\mathbb{E}_{\mathrm{LS}(\mathfrak{K})}$ is equality; a place to encounter some of the difficulty

3) μ-local $\not\Rightarrow$ μ-compactness

Claim 1.10: 1) N_0 is $\leq_{\mathfrak{K}}$-maximal member

2) $p_s \restriction M_1$ is well defined, unique when $M_1 \leq_{\mathfrak{K}} M_2, p_2 \in \mathscr{S}^\alpha(M_0)$

3) types have extension

4) $(p \restriction M_1) \restriction M_0 = p \restriction M_0$

Claim 1.11: If $M_i(i \leq w)$ is $\leq_{\mathfrak{K}}$-increasing continuous, $p_n \in \mathscr{S}^\alpha(M_n), p_n = p_{n+1} \restriction M_n$ <u>then</u> there s a limit $p_w \in \mathscr{S}(M_w), n < \omega \Rightarrow p_n = p_\omega \restriction M_n$

Remark 1.12:

§2 Variants of saturated

Definition 2.1: Assume $\mathfrak{K}$ is stable in $\mu, \alpha < \mu^+$

1) $M <^0_{\mu,\alpha} N$

2) $M <^1_{\mu,\alpha} N$

3) default for α is...

Lemma 2.2: Basic properties; assume $\mathfrak{K}$ stable in μ and $\alpha < \mu^*$

0) when $\leq^\ell_{\mu,\alpha_2} \subseteq \leq^| ell_{\mu,\alpha}$

1) For $M \ inK_\mu$ there are N_α's such that $N \leq^\ell_{\mu,\alpha} N$

2) Monotonicity

3) Preservation under increasing

4),7) $\leq^0_\mu$ implies universality

5),6) Uniqueness for $\leq^1_\mu$

8) $M <^1_{\mu,\kappa} \Rightarrow N$ is cf(κ)-saturated [proof later]

Discussion 2.3: On [Sh 300], [Sh 87a], [Sh 87b]

Remark 2.4: On $\leq^1_{\mu,\kappa}$

Definition 2.5: Recall κ-saturated

Proof of 2.2(8):

Claim 2.6: If K is categorical in $\lambda, M \in K_\lambda$, cf$(\lambda) > \mu$ then M is (λ, μ^+)-model homogeneous

Definition 2.7: N is (μ, κ)-brimmed

Claim 2.8: 1) Restating properties for brimmness

Discussion:

§3 Splitting

(7 line intro)

Content 3.1: $\mathfrak{C}$

Definition 3.2: $\mathscr{S}(M)$ does μ-split over $N \leq_{\mathfrak{K}} M$

Claim 3.3: From stability to non-splitting [proof]

Conclusion 3.4: When p is the $\mathbb{E}_\mu$-unique extension of $p \restriction N_1$ which does not μ-split over N_0

§4 Indiscernibility and E.M. Models

Notation 4.1: A function $h : Y \to M$ is the same as sequence $\langle h(t) : t \in Y \rangle$

Definition 4.2: 1) Let $h_i : Y \to \mathfrak{C}$ for $i < i^*$
1) $\langle h_i : i < i^* \rangle$ is an indiscernible sequence of character $< \kappa$ (...use automorphisms
2) $\langle h_i : i < i^* \rangle$ is an indiscernible set
3) $\langle h_i : i < i^* \rangle$ is strictly indiscernible sequence if.... (there are Φ...)
4) $\langle h_i : i < i^* \rangle$ has localness θ

Definition 4.3: 1) $\mathfrak{K}$ has the (κ, θ)-order property
2) $\mathfrak{K}$ has the $9\kappa_1, \kappa_2, \theta)$-order property

Claim 4.5: 1) Any strictly indiscernible sequence (over) A is an indiscernible set (over A)
2) We can omit strictly , we can add "of character $< \kappa$"

Claim 4.6: 1) Existence of $\Phi \in \Upsilon^{\mathrm{or}}_{\mathrm{LS}(\mathfrak{K})+|Y|}[\mathfrak{K}]$ immitating $\bar{h} = (\langle h_i : i < i^* \rangle)$ if $i^* < \beth_{1,1}(\mathrm{LS}(\mathfrak{K}) + |Y|)$, (or use $h^\theta = \langle h^\theta_i : i < \theta$ for $\theta < \beth_{1,1}(\mathrm{LS}(\mathfrak{K}) + |Y|)$
2) If $\langle h_i : i < i^* \rangle$ is an indiscernible sequence of character $\aleph_0$, greater similarly
3) Apparent weaking of assumption in 4.3
4) Variants of 4.3(1),4.7(2)
5) Another building of Φ

Lemma 4.7: If there is a strictly indiscernible sequence which is not an indiscernible set of character $\aleph_0$ with sequence of length γ <u>then</u> $\mathfrak{K}$ hsa the (γ)-order property

Claim 4.8: 1) If $\mathfrak{K}$ has the κ-order property then $\dot{I}(\chi, \mathfrak{K}) = \hat{I}$ for every $\dot{I}(\chi > (\kappa + \mathrm{LS}(\mathfrak{K}))^+$ (and more)
2) If $\mathfrak{K}$ has the $(\kappa_1, \kappa_2, \theta)$-order property and $\chi \geq \kappa =: \kappa_1 + \kappa_2 + \theta$ then for some $M \in \mathfrak{K}_\chi$ we have $|\mathscr{S}^{\kappa_2}(M)/\mathbb{E}_\kappa| > \chi$

Definition 4.9: 1) $p \in \mathscr{S}^m(N)$ divides or mu-divides over $M \leq_{\mathfrak{K}} N$ (indiscernible copies of N over M)
2) $\kappa_\mu(\mathfrak{K}), \kappa^*_\mu(\mathfrak{K})$ (long diviidng + types for $\mathfrak{K}_\mu$), sets of cardinals

3) $\kappa_{\mu,\tau}(\mathfrak{K})$ and $\kappa^*_{M,\theta}(\mathfrak{K})$, similarly with models in K_θ and μ-dividing

Remark 4.10:

Fact 4.11: 1) Equivalent variant of 4.9(1)
2) $\kappa \in K^*_\mu(\mathfrak{K}), \theta = \text{cf}(\theta) \le \kappa \Rightarrow \theta \in K^*_\mu(\mathfrak{K})$; similarly for $\kappa^*_{\mu,\theta}(\mathfrak{K})$
3) $\mathfrak{K}^*_\mu(\mathfrak{K}) \subseteq \kappa_\mu(\mathfrak{K})$, similarly for $\kappa_{\mu,\theta}$

Definition 4.12: Assume $M \le_{\mathfrak{K}} N, p \in \mathscr{S}(N), M \in \mathfrak{K}_{\le\mu}, \mu \ge \text{LS}(\mathfrak{K})$
1) p does μ-strongly split over M
2) p explicitly μ-strongly splits over M

Claim 4.13: 1) Strongly splitting implies dividing if $(*)_{\mu,\aleph_0,\aleph_0}$ (see below)

Claim 4.14: 1) If $(*)_{\mu,\theta,\sigma}$ fails then $\mathfrak{K}$ has the $(\mu, \sigma + \text{LS}(\mathfrak{K}))$-order property
2) for $\chi \ge \mu + \text{LS}(\mathfrak{K})$ then for some $M \in K_u, \mathscr{S}(M)$ for some $\alpha < \sigma$
3) weaker version
4) a variant

Claim 4.15: EM model M, if the skeleton converges in any $N, M \le_{\mathfrak{K}} N$ then we get e.g. instability, order property [proof]

Claim 4.16: for $\mathfrak{K}$ categorical, existence of strictly indiscernible sub-sequences [proof]

Observation 4.17: indiscernible in ${}^\sigma\alpha$

Definition 4.18: M is λ-strongly saturaetd (also automorphic)

§4 (Second version)

Definition 4.2: indiscernible, strictly indiscernible (by EM)

Definition 4.3: (κ, θ)-order, $\ell g(a_i) = \theta, |A| = \kappa, (\kappa_1, \kappa_2, \theta)$-order

Observation 4.4: monotonicity

Claim 4.5: triviality

Claim 4.6: existence of indiscernibles toward EM

Claim 4.7: indiscernible sequence not set implies order

Claim 4.8: κ-order $\Rightarrow I(\chi, K) = 2^\chi$

Definition 4.9: p-divide, μ-divide, $\kappa_{\mu,\theta}[K]$

Definition 4.12: μ-split, μ-strongly split

Claim 4.13: implication

Claim 4.14: getting order property, unstability

Claim 4.15: almost $t \in I, \bar{a}_t/\bar{b}$ equal

Claim 4.16: every $\langle \bar{a}_i : i < \theta \rangle$ contains large strictly indiscernible sub-sequence

Definition 4.18: M strongly saturated

§5 Rank and superstability

Definition 5.1: $R(p)$

Definition 5.2: $(\mu, 1)$-superstable

Claim 5.3: failure from 4.13 $\Rightarrow (\mu, 1)$-superstability fail

Claim 5.4: if $\mathfrak{K}$ not $(\mu, 1)$-superstable then $\langle \mu_i : i \le w + 1 \rangle$, etc

Claim 5.5: in 5.4 get unstable in $\chi < \chi^{\aleph_0}$

Remark 5.6: discussion

Claim 5.7: categoricity $\Rightarrow (\mu, 1)$-superstable, $\kappa_\mu(\mathfrak{K}) = \emptyset$

Claim 5.8: long splitting $\Rightarrow$ not saturated

Claim 5.9: from Ramsey cardinal....getting $I(\chi, \mathfrak{K})$ large

Claim 5.10: $|\mathscr{S}(M)/E_M|, \chi \ge \|M\| \ge \beth_{(2^\mu)^+}, \mu \ge$ LS$(\mathfrak{K})$ implies not $(\mu, 1)$-superstable and even $\kappa^*_\mu(\mathfrak{K})$ large.

§6 Existence of many non-splitting

Question 6.1: union of $<^1_{\mu,\kappa}$-increasing chains of every type in $\mathscr{S}(N_\delta)$ does not split over some N_i (and have many extensions)

Observation 6.1: Implications related to 6.1 $\Rightarrow p \ne q \in \mathscr{S}(N_\delta) \Rightarrow (\exists i < \delta)(p \restriction i \ne q \restriction i)$

Lemma 6.3: yes, by categoricity but $\mu < \lambda$

Theorem 6.5: Assume categoricity in λ and the $M \in K_\lambda$ is μ^+-saturated, LS$(\mathfrak{K}) < \mu < \lambda$.
1) $M <^1_{\mu,\kappa} \Rightarrow N$ saturated.
2) There is a saturated $M \in K_\lambda$.
3) $M \le^1_{\mu,\kappa_\ell} N_\ell \Rightarrow N_1 \cong_M N_2$.

Claim 6.7: union of saturated chains is saturated [2003/10/13, changes 6.5-6.7]

Claim 6.8: ($\mathfrak{K}$ cateogorical in λ, cf$(\lambda) > \mu \ge$ LS$(\mathfrak{K})$ or just the $M \in \mathfrak{K}_\lambda$ are μ^+-saturated).

1) If $I \subset J$ every cut if I realized in $J \backslash I$ is realized infinitely after (or slightly closed Φ) then every $p \in \mathscr{S}(\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi)$ is realized in $\mathrm{EM}_{\tau(\mathfrak{K})}(J,\Phi)$, when $I \in K^{\mathrm{lin}}_\mu$.

§7 More on splitting

Hypothesis 7.1: increasing union of μ-saturated is μ-saturated when $\mu \in (\mathrm{LS}(\mathfrak{K}),\lambda)$

Conclusion 7.2: $M \in K_\mu$ saturated, $p \in \mathscr{S}(M)$, then p does not μ-split over some $M^- <_{\mu,\omega} M$

Fact 7.3: M_ℓ is $\le^1_{\mu,\omega}$-increasing, $p \in \mathscr{S}(M_3)$ does not split over some M_0 then $R(p) = R(p \restriction M_2)$

Claim 7.4: [categoricity] q not stationary $\Rightarrow q$ split

Claim 7.5: additivity of non-splitting.

§8 Existence of ncie Φ

Context 8.1: $\mathfrak{K}$ a.e.c. $\forall\lambda(K_\lambda \ne \emptyset)$

Remark 8.2: On variants

Definition 8.3: $\Upsilon^{\mathrm{or}}_\kappa = \Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}], \Upsilon^{\mathrm{or}} = \Upsilon^{\mathrm{or}}_{[\mathfrak{K}]} = \Upsilon_{\mathrm{LS}(\mathfrak{K})}$

Definition 8.4: 1) $\le^{\otimes}_\kappa$, partial order on $\Upsilon^{\mathrm{or}}_\kappa$.
2) Φ_2 is an inessential extension of $\Phi_1, \Phi_1 \le^{\mathrm{ie}}_\kappa \Phi_2$.
3) $\Upsilon^{\mathrm{lin}}_\kappa$.
4) $\le^{\otimes}_\kappa$, a partial order on $\Upsilon^{\mathrm{or}}_\kappa$.

Claim 8.5: basic properties

Lemma 8.6: The a.e.c. omitting type theories

Lemma 8.7: A two cardinal version [proof]

Remark 8.8: 1) Can use $\langle q^1_1 \restriction N'_n : n < \omega\rangle$.
2) $c\ell(\bar a, M)$.

Definition 8.9: 1) $K^{\mathrm{or}(*)}$: order expanded by P^I_ℓ, P^I_1 an initial segment P^I_2 its compliment.
2) $\Upsilon^{\mathrm{or}(*)}_\kappa[\mathfrak{K}]$, class of relevant Φ

Definition 8.10: 1) $\Upsilon^{\mathrm{or}(*)}_\kappa$ producing reasonable extension of $I \in K^{\mathrm{or}(*)}$.
2) $I \in K^{\mathrm{or}(+)}$ is strongly $\aleph_0$-homogenous.
3) Φ for h isomorphic from I_0 to I_1 both $\subseteq I \in K^{\mathrm{or}(*)}, \hat h$ is the induced homomorphism on the models

Observation 8.11: 1) When $I \in K^{\rm or(+)}$ is strongly $\aleph_0$-homogeneous.
2) There is $\Phi \in \Upsilon^{\rm lin(+)}_{\aleph_0}$ guaranteeing this [short proof]

Claim 8.12: M is saturated when $\mathfrak{K}$ is categorical in $\lambda, M \in$ rm EM(I,Φ), etc. [proof]

Remark 8.13: The "I wide not necessary"

Definition 8.14: 1) We define $\le^{\oplus,\ell}_\kappa$ partial orders on $\Upsilon^{\rm or(+)}_\kappa$
2) $\mathfrak{t}$ is a Φ-automorphism scheme over p_1
2A) Without over P_1
3) Define $f^{\mathfrak{t}}_{\Phi,I}[t_0,\dots,t_{n-1}]$ the automorphism
4) We use $\mathfrak{t}_1 \subseteq \mathfrak{t}_2$

Claim 8.15: 1) Basic properties of the objects from 8.14, [proof]

Claim 8.16: Existence of Φ's [proof]

Definition 8.17: 1) $\mathfrak{t}$ is a weak Φ-automorphism
2) $\mathfrak{x}$ is a Φ-task
3) $\mathfrak{t}$ sovles $\mathfrak{x}$
4) $\mathfrak{x}$ is Φ-solvable

Observation 8.18: Basic properties

Remark 8.10: Variant of 8.20, simplifying

Main Claim 8.20: Solving one task, if possible [long proof]

Claim 8.21: Under categoricity in λ, etc., even $M \in K$ of cardinality $\ge \beth_{(2^\chi)^+}$ is χ^+-saturated

Claim 8.22: variants of 8.4

Conclusion 8.23: 1) For $\kappa \ge {\rm LS}(\mathfrak{K})$, there is $\alpha(*) < (2^\kappa)^+$ and $\Phi^* \in \Upsilon^{\rm or(+)}_\kappa$ which "satisfies all the tasks" it can [proof]
2) On bounds on $\alpha(\Phi)$

Definition 8.24: 1) Φ^* as in 8.23
2) $\chi_\ell(\Phi^*)$

Remark 8.26: It was enough to prove less

§9 Small pieces are enough and categoricity

Context 9.1: $\mathfrak{K}$ categorical in λ, Φ^* as in 8.23

Main Lemma = Local Lemma 9.2: For saturated $M^* \in K_\chi, \chi(*) \le \chi <$ rm cf$(\lambda) \le \chi$ <u>then</u> every $p \in \mathscr{S}(M^*)$ is $\chi_1(*)$-local [proof]

Claim 9.3: If T is cateogorical in λ, $\mathrm{LS}(\mathfrak{K}) \le \chi(*) \le \mu < \lambda$ and $\langle M_i : i < \delta \rangle$ is an increasing sequence of μ^+-saturated models <u>then</u> $\cup\{M_i : i < \delta\}$ is μ^+-saturated [CHECK:!! $\chi(*)$?
$\mu < \mathrm{cf}(\lambda)$? quote 9.2? [proof]

Observation 9.4: 1) If M if θ-saturated, $\theta >$ rm $\mathrm{LS}(\mathfrak{K}), \theta < \lambda, N \le_{\mathfrak{K}} N, N \in K_{\le \theta}$ <u>then</u> for some $N' \le_{\mathfrak{K}} M, N' \in K_0$ and every $p \in \mathscr{S}(N)$ realized in M is realized in N'.
2) When increasing union is θ-saturated.

Theorem 9.5: If λ is successor, $\beth_{(2^{\chi(*)})^+} \le \chi < \lambda$ <u>then</u> K is categorical in χ [proof]

Remark 9.6:

<u>Assignments</u>:

1) See 0.5(d),p.6, [Saharon?]
2) See 1.12,p.8, (fill?)
3) Concerning Definition 2.1, $<^d_{\mu,\alpha}$ see more [Sh 600, §4], $<^1_{\mu,\kappa}$ is called there $\mu, \mathrm{cf}(\alpha))$-brimmed over M.
4) This applies to Definition 2.8 as well.
5) $\mathrm{PC}_{\kappa^+,\omega}$, 8.3??
6) 8.10 not repetition of §1 or §2. ??
7) II,§1; redo for 734. !!

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