On closed P-sets with *ccc* in the space ω^*

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Abstract. It is proved that – consistently – there can be no *ccc* closed P-sets in the remainder space ω^*.

In this paper we show how to construct a model of set theory in which there are no P-sets satisfying *ccc* (countable antichain condition) in the ultrafilter space $\omega^* = \beta\diamond\omega|\setminus\omega$. The problem of the existence of such sets (which are generalizations of P-points) was known since some time and occurred explicitly in œvM-R—. In the proof we follow the construction from œS— of a model in which there are no P-points. A particular case of P-sets, which are supports of approximative measures has been settled in œM—, where the author shows that there can be no such measures on $P(\omega)/fin$. (Under CH, e.g. the Gleason space $\mathbf{G}(2^\omega)$ of the Cantor space is a ccc P-set in ω^* which carries no approximative measure).

Sec.1. Closed P-sets in the space ω^* can be identified with P-filters F on ω. Thus, the dual ideal $I = \{\omega \setminus A : \ A \in F\}$ has the property:

(1.1) If $A_n \in I$, for $n \in \omega$, then there is an $A \in I$ such that $A_n \subseteq_* A$, for each $n \in \omega$.

Further, the countable chain condition imposed upon F implies that I is fat in the following sense (see œF-Z—):

(1.2) if $A_n \in I$, for $n \in \omega$ and $\lim_n \min A_n = \infty$, then there is an infinite $Z \subseteq \omega$ such that $\bigcup_{n\in Z} A_n \in I$.

Indeed, let $e_n = A_n \setminus A$, where $A \in I$ is as in (1.1). Since $\min A_n$ are arbitrarily large, we can find an infinite set $Y \subseteq \omega$ such that the family $\{e_n : n \in Y\}$ is disjoint. If $\{Y_\alpha : \ \alpha < c\}$ is an almost disjoint family of subsets of Y, then the unions

$$S_\alpha = \bigcup\{e_n : \ n \in Y_\alpha\}, \quad \alpha < c$$

are almost disjoint and hence the closures S_α^* in the space ω^* are disjoint. By *ccc* we have

$$S_\alpha^* \cap \bigcap\{B^* : \ B \in F\} = \emptyset,$$

for some α and consequently $S_\alpha \in I$. It follows that th e union

$$\bigcup_{n\in Y_\alpha} A_n = \bigcup_{n\in Y_\alpha} (A_n \cap A) \cup \bigcup_{n\in Y_\alpha} (A_n \setminus A)$$

is in I as a subset of $S_\alpha \cup A$.

Let us fix a given ccc P-filter F and its dual I. We shall define a forcing $\mathbf{P} = \mathbf{P}(F)$.

A partial ordering $(T, \leq_T)$, where $T \subseteq \omega$, will be called a tree, if for each $i \in T$ the set of predecessors $\{j \in T :\ j \leq_T i\}$ is linearly ordered and

$$i \leq_T j \text{ implies } i \leq j, \text{ for all } i, j \in T.$$

We define a partial ordering for trees

$$T \leq_t S \text{ iff } (S, \leq_S) \text{ is a subordering of } (T, \leq_T)$$
$$\text{and each branch of } T \text{ contains cofinally a (unique) branch of } S.$$

There is a tree T_0 such that $T_0 \in I$ and T_0 is order isomorphic to the full binary tree of height ω.

Deleting the numbers $\leq n$ from T_0 we obtain a subtree denoted by $T_0^{(n)}$ (we have $T_0^{(n)} \leq_t T_0^{(m)}$, for $n \leq m$). Let $\mathcal{T}$ consist of all the trees $T \in I$ such that

$$T \leq_t T_0^{(n)}, \text{ for some } n \in \omega.$$

Note that each tree $T \in \mathcal{T}$ has finitely many roots.

Definition. *Elements of the forcing* $\mathbf{P}$ *are of the form* $p =< T_p, f_p >$, *where* $T_p \in \mathcal{T}$ *and* $f_p :\ T_p \longrightarrow \{0, 1\}$. *The ordering of* $\mathbf{P}$ *is defined thus*

$$p \leq q \text{ iff } T_p \leq_t T_q \text{ and } f_p \supseteq f_q.$$

Let $\{b_\alpha :\ \alpha < c\}$ be a fixed enumeration of all the branches of T_0 in V. For a generic $G \subseteq \mathbf{P}$ let $T_G = \bigcup_{p \in G} T_p$ and $f_G = \bigcup_{p \in G} f_p$.

Each branch B of T_G contains cofinally a unique b_α. Let us write $B = B_\alpha$ and define

$$X_\alpha = \{i \in \omega :\ i \in B_\alpha \text{ and } f_G(i) = 1\}$$

Since $T_p \in I$, for any $p \in \mathbf{P}$, hence $\omega \setminus T_p \cap A$ is infinite, for each $A \in F$. It follows that the sets

$$D_{n\varepsilon}^{A\alpha} = \{p \in \mathbf{P} :\ \exists i > n \diamond i \in b_\alpha^p \text{ and } f_p(i) = \varepsilon|\}$$

are dense, for each $A \in F$, $n \in \omega$, $\alpha < c$ and $\varepsilon = 0, 1$ (here b_α^p denotes the branch of T_p extending b_α).

Thus, $\mathbf{P}$ adds uncountably many almost disjoint Gregorieff-like sets.

Sec.2. Let $\mathbf{Q} = \mathbf{Q}(F)$ be a countable product of $\mathbf{P} = \mathbf{P}(F)$. Thus the elements $q \in \mathbf{Q}$ can be written in the form

$$q =< f_0^q, f_1^q, \ldots >, \text{ where } < \mathrm{dm}(f_i^q), f_i^q >\in \mathbf{P}, \text{ for each } i < \omega.$$

By $q^{(n)}$ we denote the condition $< g_i :\ i < \omega >$ where

$$g_i = \begin{cases} f_i^q \mid \mathrm{dm}(f_i^q)^{(n)}, & \text{for } i < n \\ f_i^q & \text{for } i \geq n \end{cases}$$

Here $T^{(n)}$ is a tree obtained from T by deleting the numbers $\leq n$.

LEMMA 2.1. *For each decreasing sequence $p_0 \geq p_1 \geq \ldots$ there is a q and an infinite $Z \subseteq \omega$ such that*

$$q \leq p_n^{(n)}, \text{ for each } n \in Z.$$

PROOF. Let $T_{ni} = \mathrm{dm}(f_i^{p_n})$, where $p_n =< f_i^{p_n} : i < \omega >$. Since $\min T_{ni}^{(n)} \geq n$, we may use (1.2) to define by induction a descending sequence $Z_0 \supseteq Z_1 \supseteq \ldots$ of infinite subsets of ω such that

$$\bigcup_{n \in Z_i} T_{ni}^{(n)} \text{ is in } I, \text{ for each } i < \omega.$$

There is an infinite $Z \subseteq \omega$, such that $Z \subseteq_* Z_i$, for each $i < \omega$. Define

$$T_i = T_{ii} \cup \bigcup_{n \in Z} T_{ni}^{(n)}$$

and

$$f_i^q = f_i^{p_i} \cup \bigcup_{n \in Z} f_i^{p_n} \mid T_{ni}^{(n)}.$$

Then, $\mathrm{dm}(f_i^q) = T_i$ and $q =< f_i^q : i < \omega >$ is as required. QED

For $q \in \mathbf{Q}$ and $n \in \omega$ let $S(q,n)$ be the set of all sequences $s =< s_0, \ldots, s_{n-1} >$ satisfying the following properties

1. $s_0, \ldots, s_{n-1}$ are finite zero-one functions.
2. The domains $t_0 = \mathrm{dm}(s_0), \ldots, t_{n-1} = \mathrm{dm}(s_{n-1})$ are finite trees such that

$$t_0 \cap T_0^{(n)} = \ldots = t_{n-1} \cap T_{n-1}^{(n)} = \emptyset,$$

where $T_0 = \mathrm{dm}(f_0^q), \ldots, T_{n-1} = \mathrm{dm}(f_{n-1}^q)$

3. Ordered sums $t_0 \oplus T_0^{(n)}, \ldots, t_{n-1} \oplus T_{n-1}^{(n)}$ are trees belonging to $\mathcal{T}$.

Note that from the definition of $\mathcal{T}$ it follows that $S(q,n)$ is always finite. Let us denote

$$s * q^{(n)} =< s_0 \cup f_0^q, \ldots, s_{n-1} \cup f_{n-1}^q, f_n^q, \ldots >$$

for q, n, s as above. Obviously, we have

(2.2) the set $\{s * q^{(n)} : s \in S(q,n)\}$ is predense below $q^{(n)}$

(i.e. the boolean sum $\sum_{s \in S(q,n)} s * q^{(n)} = q^{(n)}$).

Now, we obtain easily an analogue of VI, 4.5 in œS—.

(2.3) For arbitrary $p \in \mathbf{Q}$, $n < \omega$ and $\tau \in V^{(Q)}$ such that $\mathbf{Q} \Vdash$ "τ is an ordinal" there is a $q \leq p$ and ordinals $\{\alpha(s) :\ s \in S(p,n)\}$ so that

$$q^{(n)} \Vdash \text{“} \bigvee_s \tau = \alpha(s) \text{”}$$

Indeed, if $S(q,n) = \{s^0, \dots, s^{m-1}\}$, then we define inductively conditions $p_0, \dots, p_m$ so that $p_0 = p$ and $p_{k+1} \leq s^k * p_k^{(n)}$ is such that

$$p_{k+1} \Vdash \text{“}\tau = \alpha\text{”, for some ordinal } \alpha = \alpha(s^k)$$

Now, $q = s * p_m^{(n)}$, where s is such that $p = s * p^{(k)}$ (we may assume $s \in S(p,n)$), satisfies (2.3).

2.4. Theorem. **Q** *is α-proper, for every $\alpha < \omega_1$, and has the strong PP-property.*

Proof. Let countable $N \prec H(\kappa)$ for sufficiently large κ, be such that $\mathbf{Q} \in N$ and suppose that $p \in \mathbf{Q} \cap N$. To prove that $\mathbf{Q}$ is proper we have to find a $q \leq p$, which is N-generic. Let $\{\tau_n :\ n < \omega\}$ be an enumeration of all the $\mathbf{Q}$-names for ordinals, such that $\tau_n \in N$, for $n < \omega$. Using (2.3) we define inductively a sequence $p_0 = p \geq p_1 \geq \dots$ and ordinals $\alpha(n,s)$ so that

$$p_n^{(n)} \Vdash \text{“} \bigwedge_{i \leq n} \bigvee_s \tau_i = \alpha(n,s)\text{” for each } n < \omega$$

(i.e. in the n-th step we apply (2.3) for all names $\tau_0, \dots, \tau_n$). Note that the $p's$ and $\alpha's$ can be found in N, since $N \prec H(\kappa)$. By Lemma 2.1 there is a q and an infinite $Z \subseteq \omega$ such that

$$q \leq p_n^{(n)}, \text{ for each } n \in Z.$$

Hence also $q^{(m)} \leq p_n^{(n)}$ holds for arbitrarily large n and all $m < \omega$ and thus

$$q^{(m)} \Vdash \text{“}\tau_n \in N\text{”},$$

for all $n, m < \omega$.

By III, 2.6 of œS—, each $q^{(m)}$ is N-generic.

To see that $\mathbf{Q}$ is α-proper let $< N_\xi :\ \xi \leq \alpha >$ be a continuous sequence of elementary countable submodels of $H(\kappa)$ such that $\mathbf{Q} \in N_0$ and

$$< N_\xi :\ \xi \leq \eta > \in N_{\eta+1}, \text{ for each } \eta < \alpha.$$

Assume that $\mathbf{Q}$ is β-proper, for each $\beta < \alpha$ and let $q_0 \in \mathbf{Q} \cap N_0$. If $\alpha = \beta + 1$, we have a $q \leq q_0$ which is N_ξ-generic, for each $\xi \leq \beta$ and we may assume that the $q^{(n)}$ have the same property, for all $n < \omega$. since $N_\alpha \prec H(\kappa)$ and all the parameters are in N_α, such a q can be found in N_α and as above we construct a $q_\alpha \leq q$ which is N_α-generic and so are the $q_\alpha^{(n)}$, for $n < \omega$. Thus, q_α and all the $q_\alpha^{(n)}$ are N_ξ-generic for all $\xi \leq \alpha$. If α is a limit ordinal, we fix an increasing sequence $< \xi_n : n < \omega >$ such that $\alpha = \sup_{n<\omega} \xi_n$ and by the inductive hypothesis there is a sequence $q_0 \geq q_{\xi_0} \geq q_{\xi_1} \geq \ldots$ such that, for each $n < \omega$, q_{ξ_n} is N_ξ-generic, for each $\xi \leq \xi_n$ and $q_{\xi_n} \in N_{\xi_n+1}$ and that $q_{\xi_n}^{(m)}$ have the same property for each $m < \omega$. By Lemma 2.1 there is a $q \in \mathbf{Q}$ such that $q \leq q_{\xi_n}^{(n)}$, for infinitely many $n < \omega$. Thus, $q \leq q_0$ and q is N_ξ-generic for each $\xi < \alpha$ and hence also for each $\xi \leq \alpha$.

Finally, to prove the PP-property let $h : \omega \longrightarrow \omega$ diverge to infinity and suppose that $p \Vdash$ "$f : \omega \longrightarrow \omega$". Define

$$k_n = \min\{i : h(i) > 2^n \cdot \mathrm{card} S(p,n)\}, \text{ for } n < \omega$$

and, using (2.3), define inductively the sequence $p = p_0 \geq p_1 \geq \ldots$ such that

$$p_n^{(n)} \Vdash \text{“} \bigwedge_{i<k_n} \bigvee_{s \in S(p_i,i)} f(i) = \alpha(s,i)\text{” for each } n < \omega \text{ and some integers } \alpha(s,i) < \omega.$$

Let T be the tree built up of integers

$$\{\alpha(s,i) : i < \omega \text{ and } s \in S(p_i,i)\}$$

.

If $q \leq p_n^{(n)}$, for infinitely many n, then we have $q \Vdash$ "$f \in \mathrm{Lim}\ T$" and $T \cap \omega^{k_n}$ has less elements than $h(k_n)$, for all $n < \omega$, which finishes the proof. QED

The last point to be discussed is how does $\mathbf{Q} = \mathbf{Q}(F)$ act in the course of iteration.

2.5.LEMMA.*If* $\mathbf{R}$ *is* ω^ω*-bounding (i.e. the set of old functions* $: \omega \longrightarrow \omega$ *dominates) and* $\mathbf{Q}(F)$ *is a complete subforcing of* $\mathbf{R}$*, then in* $V^{(\mathbf{R})}$ *the filter* F *cannot be extended to a ccc* P*-filter.*

PROOF. Let X_α^n be the α-th set added by n-th factor of the product $\mathbf{Q} = \mathbf{P}^\omega$. Suppose that for some $r \in \mathbf{R}$ and a *ccc* P-filter $E \in V^{(\mathbf{R})}$ we have

$$r \Vdash \text{“}F \subseteq E\text{”}$$

Note that for each $n < \omega$, the relation $X_\alpha^n \in E$ hold for at most countably many α's, since E is *ccc*. Hence, there is an α such that for all $n < \omega$ we have $\omega \setminus X_\alpha^n \in E$ and, since E is a P-filter, there is an $A \in E$ and a function g, so that $A \subseteq \bigcap_{n<\omega}(\omega \setminus X_\alpha^n) \cup \diamond 0, g(n))$ i.e. for some $r_1 \leq r$ we have

$$r_1 \Vdash \text{“} \bigcap_{n<\omega} (\omega \setminus X_\alpha^n) \cup \diamond 0, g(n)) \in E\text{”} \tag{2.6}$$

Since $\mathbf{R}$ is ω^ω-bounding we may assume that $g \in V$. By the assumption, $\mathbf{Q}$ is a complete subforcing of $\mathbf{R}$ and hence there is a $q \in \mathbf{Q}$ such that r is compatible with each $q' \leq q$.

On the other hand, since $T_n = \mathrm{dm}(f^q_n) \in \mathcal{T}$, there is a set $B \in I$ and an increasing sequence $a_0 < a_1 < \ldots$ such that $T_n \setminus \diamond 0, a_n) \subseteq B$, $g(n) < a_n$ and $\diamond a_n, a_{n+1}) \setminus B \neq \emptyset$, for each $n < \omega$. Define $q' \leq q$ as follows. For a given n extend T_n by adding elements of $\diamond a_n, a_{n+1}) \setminus B$ on the α-th branch b^q_α and put $f^{q'}_n(i) = 1$, for each $i \in \diamond a_n, a_{n+1}) \setminus B$. Obviously, we have

$$q' \Vdash \text{“}(\omega \setminus X^n_\alpha) \cup \diamond 0, g(n)) \cap \diamond a_n, a_{n+1}) \setminus B = \emptyset\text{”}, \text{ for each } n$$

and hence

$$q' \Vdash \text{“} \bigcap_{n<\omega} (\omega \setminus X^n_\alpha) \cup \diamond 0, g(n)) \setminus B \cap \bigcup_{n<\omega} \diamond a_n, a_{n+1}) = \emptyset\text{”}$$

Consequently $q' \Vdash$ “$\bigcap_{n<\omega}(\omega \setminus X^n_\alpha) \cup \diamond 0, g(n)) \subseteq_* B$”, which contradicts (2.6). QED

The rest of the proof is routine. Beginning with a model V of $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$ we iterate with countable supports the forcings $\mathbf{Q}(F)$, for all *ccc* P-filters F booked at each stage $\alpha < \omega_2$ of the iteration. From œS—, V.4 we know that the resulting forcing $\mathbf{R}$ (obtained after ω_2 stages) is proper and ω^ω-bounding. Hence, in $V \diamond G|$ there are no *ccc* P-sets.

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