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Postscript to Shelah & Fremlin p90

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1. The statement (‡) of Shelah & Fremlin p90, 2G, can be strengthened, as follows. Write (‡*) for the statement

there is a closed negligible set $Q \subseteq [0,1]$ such that $(\mu_L)_*(Q^{-1}[D]) \ge \mu_L^* D$ for every $D \subseteq [0,1]$.

Proposition If $\mathbb{P}$ is a partially ordered set as in Shelah & Fremlin p90, then $1\!\!1_{\mathbb{P}} \Vdash_{\mathbb{P}} (‡^*)$.

proof (a) Take X and $\mathbb{P}$ and μ and Ψ as in Shelah & Fremlin p90, §1. Then if $D \subseteq X$ and $\epsilon > 0$, there is a closed set $F \subseteq X$, with $\mu F \ge \mu^* D - \epsilon$, such that

$$1\!\!1_{\mathbb{P}} \Vdash_{\mathbb{P}} \Psi \cap \ulcorner F \urcorner \subseteq \ulcorner R \urcorner^{-1}[D],$$

writing $\ulcorner F \urcorner$ for the $\mathbb{P}$-name for a closed subset of X corresponding to F.

P Choose k_0' such that $2^{-k_0'} \le \frac{1}{4}\epsilon$, and a closed set F_0 such that $\mu F_0 = \mu^*(D \cap F_0) \ge \mu^* D - \frac{1}{2}\epsilon$. Set

$$F = \{x : \forall\ k \ge k_0',\ \mu^*(D \cap \{w : w \restriction k = x \restriction k\}) > 2^{-k+1}\mu\{w : w \restriction k = x \restriction k\}\}.$$

Then F is closed, and

$$F_0 \setminus F \subseteq \{x : x \in F_0,\ \exists\ k \ge k_0',\ \mu(F_0 \cap \{w : w \restriction k = x \restriction k\}) \le 2^{-k+1}\mu\{w : w \restriction k = x \restriction k\}\}$$

has measure at most $2^{-k_0'+1} \le \frac{1}{2}\epsilon$, so $\mu F \ge \mu^* D - \epsilon$.

Now suppose that σ is a $\mathbb{P}$-name such that

$$1\!\!1_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma \in \Psi \cap \ulcorner F \urcorner.$$

Set $r = 1$, $L_k = n_k$ for every $k \in \mathbb{N}$ and follow the argument of Lemma 1R of Shelah & Fremlin p90 down to the end of part (c), but insisting at the beginning that $k_0 \ge k_0'$.

Observe that

$$p_3 \Vdash_{\mathbb{P}} \sigma \supseteq s,$$

where $s = \langle H_i(\mathbf{v}_i^*)\rangle_{i<k_1}$, as in part (d) of the proof of Lemma 1R. Also $\#(\tilde{J}_k) \le 2^{-k} n_k$ for all $k \ge k_0$, so that

$$\mu\{x : s \subseteq x,\ x(k) \notin \tilde{J}_k\ \ \forall\ k \ge k_1\} > (1 - 2^{-k_1+1})\mu\{x : s \subseteq x\};$$

but as

$$p_3 \Vdash_{\mathbb{P}} \sigma \in \ulcorner F \urcorner,\ s \subseteq \sigma,$$

we must have

$$\mu^*(D \cap \{x : s \subseteq x\}) > 2^{-k_1+1}\mu\{x : s \subseteq x\},$$

and there is a $\tilde{z} \in D$ such that $s \subseteq \tilde{z}$ and $\tilde{z} \notin \tilde{J}_k$ for every $k \ge k_1$.

Now continue the argument as in (e)-(i) of the proof of Lemma 1R to get $p_5 \le p_3$ such that $p_5 \Vdash_{\mathbb{P}} (\sigma, \tilde{z}) \in \ulcorner R \urcorner$. **Q**

(b) Now we find that

$$1\!\!1_{\mathbb{P}} \Vdash_{\mathbb{P}} \forall D \subseteq \ulcorner X \urcorner,\ \ulcorner \mu \urcorner_*(\ulcorner R \urcorner^{-1}[D]) \ge \ulcorner \mu \urcorner^* D.$$

P Let Δ_0 be a $\mathbb{P}$-name for a subset of X, and ϵ, ϵ' (ground-model) rationals such that

$$1\!\!1_{\mathbb{P}} \Vdash_{\mathbb{P}} \ulcorner \mu \urcorner^* \Delta_0 > \epsilon > \epsilon'.$$

Take $\beta < \kappa$ such that whenever Γ is a $\mathbb{P}_\beta$-name for a closed subset of X and

$$1\!\!1_{\mathbb{P}_\beta} \Vdash_{\mathbb{P}_\beta} \ulcorner \mu \urcorner \Gamma > 1 - \epsilon$$

then there is a $\mathbb{P}_\beta$-name for a member of $\Gamma \cap \Delta_0$. Now taking Δ to be a $\mathbb{P}_\beta$-name for the subset of X consisting of those members of Δ_0 which can be represented by $\mathbb{P}_\beta$-names, we see that

$$1\!\!1_{\mathbb{P}_\beta} \Vdash_{\mathbb{P}_\beta} \ulcorner \mu \urcorner^* \Delta \ge \epsilon.$$

Using part (a) in $V^{\mathbb{P}_\beta}$, we have a $\mathbb{P}_\beta$-name Γ for a closed subset of X such that

$$1\!\!1_{\mathbb{P}_\beta} \Vdash_{\mathbb{P}_\beta} \left(\ulcorner \mu \urcorner \Gamma \ge \epsilon' \ \ \&\ \ 1\!\!1_{\mathbb{P}'} \Vdash_{\mathbb{P}'} \Psi^{(\beta)} \cap \Gamma \subseteq \ulcorner R \urcorner^{-1}[\Delta]\right),$$

expressing $\mathbb{P}$ as an iteration $\mathbb{P}_\beta * \mathbb{P}'$ as in the proof of Theorem 1S of Shelah & Fremlin p90. But this must mean that

$$1\!\!1_{\mathbb{P}} \Vdash_{\mathbb{P}} \ulcorner \mu \urcorner_*(\ulcorner R \urcorner^{-1}[\Delta_0]) \ge \epsilon',$$

and as ϵ' was arbitrary this proves the claim. **Q**

(c) This proves the result for (X, μ) rather than for $([0,1], \mu_L)$. But as remarked in 2G of Shelah & Fremlin p90, we have a continuous inverse-measure-preserving $f : X \to [0,1]$ such that $\mu^* f^{-1}[D] = \mu_L^*[D]$

for every $D \subseteq [0,1]$; so that setting $Q = \{(f(x), f(y)) : (x,y) \in \overline{R}\}$ we obtain the result for Lebesgue measure, as stated.

2. In fact wse can go a little further: in $V^{\mathbb{P}}$, Q has the property that

whenever $D \subseteq [0,1]$ and E is a measurable set such that $\mu_*(E \setminus D) = 0$, then $\mu(E \setminus Q^{-1}[D]) = 0$.

To see this, follow the arguments above, observing that it is enough to consider closed E, and that the set F of part (a) of the proof can be taken to be a subset of E.

Reference

Shelah S. & Fremlin D.H. [p90] 'Pointwise compact and stable sets of measurable functions', to appear in J. Symbolic Logic.