DENSITIES OF ULTRAPRODUCTS OF BOOLEAN ALGEBRAS[1]

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Abstract. We answer three problems by J. D. Monk on cardinal invariants of Boolean algebras. Two of these are whether taking the algebraic density πA resp. the topological density $\mathrm{d}A$ of a Boolean algebra A commutes with formation of ultraproducts; the third one compares the number of endomorphisms and of ideals of a Boolean algebra.

In set theoretic topology, considerable effort has been put into the study of cardinal invariants of topological spaces, see e.g. [Ju1] and [Ho], [Ju2]. In Monk's book [Mo], similarly a systematic study of cardinal invariants of Boolean algebras is undertaken; in particular, the behaviour of these invariants with respect to algebraic constructions like taking subalgebras, quotients etc. is investigated. One of these is the ultraproduct construction, well known from model theory; cf. [ChK]. Many questions on ultraproducts are highly dependent on set theory; among the more recent results are those in Shelah' s pcf theory dealing with the possible cofinalities cf $(\prod_{\alpha<\kappa} \lambda_\alpha/D)$ where the λ_α are regular cardinals, hence well-ordered in a natural way, and the ultraproduct has the resulting linear order.

Monk's book contains a list of 66 problems, three of which are answered (consistently) in this paper.

Problem 9. Does there exist a system $(A_i)_{i\in I}$ of infinite Boolean algebras and an ultrafilter F on I such that $\mathrm{d}(\prod_{i\in I} A_i/F) < |\prod_{i\in I} \mathrm{d}(A_i)/F|$?

Problem 12. Is it true that always $\pi(\prod_{i\in I} A_i/F) = |\prod_{i\in I} \pi(A_i)/F|$?

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Problem 60. Is there a Boolean algebra A such that $|\mathrm{End}\ A| < |\mathrm{Id}\ A|$?

Here πA and $\mathrm{d}A$ are the "algebraic" and the "topological" density of A, defined by

$$\mathrm{d}A = \min\ \{|Y| : Y \text{ a dense subset of the Stone space of } A\}$$
$$\pi A = \min\ \{|X| : X \text{ a dense subset of } A\}$$

(for more definitions and matters on cardinal functions, see [Mo]). Note that we are dealing only with infinite algebras and that, trivially, $\omega \le \mathrm{d}A \le \pi A$, $\mathrm{d}(\prod_{i\in I} A_i/F) \le |\prod_{i\in I} \mathrm{d}(A_i)/F|$ and $\pi(\prod_{i\in I} A_i/F) \le |\prod_{i\in I} \pi(A_i)/F|$.

In Problem 60, End A is the set of all endomorphisms, Id A the set of all ideals of A.

In section 1, we give a positive answer to Problem 12 under SCH. Here SCH is the Singular Cardinal Hypothesis: if $2^{\mathrm{cf}\,\lambda} < \lambda$ (so λ is singular), then $\lambda^{\mathrm{cf}\,\lambda} = \lambda^+$. However, $\neg$ SCH gives a negative answer to both problems 9 and 12:

Theorem A. *Assume we have cardinals κ, μ, and $(\lambda_\alpha)_{\alpha<\kappa}$ and an ultrafilter D on κ such that: $\kappa < \mu = cf\,\mu$, $\mu^{<\mu} < \lambda_\alpha = cf\,\lambda_\alpha$, and the cofinality of the ultraproduct $\prod_{\alpha<\kappa} \lambda_\alpha/D$ is less than its cardinality. Then there is a forcing notion $\mathbb{R}$ such that*

(a) $\mathbb{R}$ is μ-complete and satisfies the $(\mu^{<\mu})^+$-chain condition; hence forcing with $\mathbb{R}$ preserves all cardinalities and cofinalities outside the interval $[\mu^+, \mu^{<\mu})$

(b) for $K \subseteq \mathbb{R}$ $\mathbb{R}$-generic over V, the following holds in $V[K]$: there are Boolean algebras $(A_\alpha)_{\alpha<\kappa}$ such that $\lambda_\alpha = |A_\alpha| = \pi A_\alpha = dA_\alpha$, but for the ultraproduct $A = \prod_{\alpha<\kappa} A_\alpha/D$,

$$d(A) \le \pi(A) = cf\,(\prod_{\alpha<\kappa} \lambda_\alpha/D) < |\prod_{\alpha<\kappa} \lambda_\alpha/D| = |\prod_{\alpha<\kappa} \pi(A_\alpha)/D| = |\prod_{\alpha<\kappa} d(A_\alpha)/D|.$$

Note that SCH is known to be independent from ZFC, modulo some large cardinal assumption (see [Ma]). And the assumption of Theorem A is a consequence of $\neg$SCH, as follows from pcf theory. A particularly easy case is the classical one for $\neg$SCH: assume λ is strong limit and singular, $\kappa = \mathrm{cf}\,\lambda$ satisfies $2^\kappa < \lambda$, but $\lambda^\kappa > \lambda^+$; let μ be regular such that $\kappa < \mu < \lambda$. Then there are (see [Sh, Ch.II, 1.5]) regular λ_α such that $\lambda = sup_{\alpha<\kappa}\lambda_\alpha$, $\prod_{\alpha<\kappa} \lambda_\alpha/J^{bd}_\kappa$ has true cofinality λ^+ (J^{bd}_κ the ideal of bounded subsets of κ), hence any uniform ultrafilter D on κ gives $\mathrm{cf}\,(\prod_{\alpha<\kappa} \lambda_\alpha/D) = \lambda^+ < |\prod_{\alpha<\kappa} \lambda_\alpha/D|$. More generally if λ violates SCH, i.e. for some κ, we have $2^\kappa < \lambda$ and $\lambda^\kappa > \lambda^+$, let λ' be minimal such that $\lambda'^\kappa = \lambda^\kappa$ (i.e. $\lambda'^\kappa \ge \lambda$); so for every cardinal $\rho < \lambda'$, we have $\rho^\kappa < \lambda'$. Now take $\mu = \kappa^+$ and find, by [Sh, Ch.II, 1.5], an appropriate family $(\lambda'_\alpha)_{\alpha<\kappa}$ with limit λ' and $\mathrm{cf}\,(\prod_{\alpha<\kappa} \lambda'_\alpha/J^{bd}_\kappa) = \lambda'^+$. Moreover we can replace λ'^+ by any regular cardinal in the interval $[\lambda'^+, \lambda'^\kappa]$; similarly for the strong limit case; see [Sh, Ch. VIII, §1].

Theorem 1.1 below and Theorem A show that the answer to Problem 12 is independent from ZFC. However, it has recently been shown in [RoSh 534, 2.6, 2.7] that Problem 9 has a positive answer even in ZFC.

Problem 60 is solved in section 8 by

Theorem B. *Assume μ is a strong limit cardinal satisfying cf $\mu = \omega$ and $2^\mu = \mu^+$. Then there is a Boolean algebra B such that $|B| = |End\ B| = \mu^+$ and $|Id\ B| = 2^{\mu^+}$.*

The organization of sections 2 to 7 is as follows. In section 2, we introduce a first order theory T for Boolean algebras with some extra structure which allows (e.g. in ultraproducts $A = \prod_{\alpha<\kappa} A_\alpha/D$ of models of T) to easily compute πA. In section 3, we construct canonical models $A(p)$ of T from what we call valuation functions p. In sections 4 to 6, we consider the forcing notion $\mathbb{P}$ of valuation functions, determine its completeness and chain conditions, and compute dA and πA for the canonical algebra $A = A(P)$ constructed from a generic valuation function P. In section 7, we prove Theorem A.

For definitions and results on set theory, see [Je]; for Boolean algebras, [Ko].

1. Problem 12 under SCH

We give here a positive answer to Monk's problem 12 under SCH. Given an ultraproduct $A = \prod_{i\in\kappa} A_i/D$ of infinite Boolean algebras, we let $\lambda_i = \pi A_i$, so $\omega \leq \lambda_i$. For simplicity of notation, we will denote, in this section, by $\prod_{i\in\kappa} \lambda_i/D$ both the ultraproduct of the λ_i and its cardinality.

Note first that the answer is easy if $\lambda_i \leq 2^\kappa$ for D-almost all $i \in \kappa$ (i.e. if $\{i \in \kappa : \lambda_i \leq 2^\kappa\}$ is in D) and D is regular. For in this case, each A_i has an infinite set of pairwise disjoint elements, so A has cellularity at least 2^κ and, on the other hand, $\prod_{i\in\kappa} \lambda_i/D \leq 2^\kappa$, hence $2^\kappa \leq \mathrm{c}A \leq \pi A \leq \prod_{i\in\kappa} \lambda_i/D \leq 2^\kappa$. Thus Theorem 1.1 covers the interesting case: $2^\kappa < \lambda_i$ for D-almost all i.

1.1 Theorem. *(SCH) Assume $2^\kappa < \lambda_i = \pi A_i$ for all $i \in \kappa$ and D is an ultrafilter on κ; let $A = \prod_{i\in\kappa} A_i/D$. Then $\pi A = \prod_{i\in\kappa} \lambda_i/D$.*

Proof. We know that $\pi A \leq \prod_{i\in\kappa} \lambda_i/D$. Let

$$\lambda = D - \lim\ (\lambda_i : i \in \kappa),$$

i.e. λ is the least cardinal ρ such that $\lambda_i \leq \rho$ holds for all D-almost all i. Without loss of generality, $\lambda_i \leq \lambda$ holds for all $i \in \kappa$.

Claim1. If $\theta < \lambda$, then $\theta^\kappa \leq \lambda$.

To see this, pick i such that $\theta < \lambda_i$. Now if $\theta \leq 2^\kappa$, then $\theta^\kappa = 2^\kappa < \lambda_i \leq \lambda$. Otherwise, $\kappa < 2^\kappa < \theta < \theta^+ \leq \lambda_i$, $(\theta^+)^\kappa = \theta^+$ by SCH, so $\theta^\kappa \leq \theta^+ \leq \lambda_i \leq \lambda$.

Claim 2. $\pi A \geq \lambda$.

Otherwise pick a dense subset Y of A of size ρ, where $\rho < \lambda$, say $Y = \{y_\alpha/D : \alpha < \rho\}$ with $y_\alpha = (y_\alpha(i))_{i\in\kappa}$ in $\prod_{i\in\kappa} A_i$ and $y_\alpha(i) \neq 0$. Since $\rho < \lambda$, we may assume without loss of generality that $\rho < \lambda_i$ for all i. So we can pick, for $i \in \kappa$, $a_i \in A_i \setminus \{0\}$ satisfying $y_\alpha(i) \not\leq a_i$, for all $\alpha < \rho$. The sequence $a = (a_i)_{i\in\kappa}$ is such that $y_\alpha/D \not\leq a/D$ for $\alpha < \rho$, a contradiction.

The theorem now follows immediately from the next three claims.

Claim 3. If $\pi A \geq \lambda^+$, then the assertion of the theorem holds.

For in this case, $\lambda^+ \leq \pi A \leq \prod_{i\in\kappa} \lambda_i/D \leq \lambda^\kappa/D \leq \lambda^\kappa \leq \lambda^+$, where the last inequality follows from SCH and $2^\kappa < \lambda$.

Claim 4. If $\pi A = \lambda$, then every function $f \in \prod_{i\in\kappa} \lambda_i/D$ is bounded below λ, modulo D.

For the proof, work as in Claim 2: fix a dense subset Y of A, $Y = \{y_\alpha/D : \alpha < \lambda\}$, $y_\alpha = (y_\alpha(i))_{i\in\kappa}$, $y_\alpha(i) \neq 0$. Given $f \in \prod_{i\in\kappa} \lambda_i$, we know that $Y_i = \{y_\alpha(i) : \alpha < f(i)\}$ cannot be dense in A_i, since $|Y_i| \leq |f(i)| < \lambda_i = \pi A_i$. So pick $a = (a_i)_{i\in\kappa}$ where $a_i \in A_i \setminus \{0\}$ is such that $y_\alpha(i) \not\leq a_i$, for all $\alpha < f(i)$. Since Y is dense in A, pick $\alpha < \lambda$ such that $y_\alpha/D \leq a/D$. It follows that: $y_\alpha(i) \leq a_i$, for D-almost all i; $\alpha \not< f(i)$ for these i, so $f(i) \leq \alpha$; i.e. $f(i) \leq \alpha$ for D-almost all i. Thus f is bounded by $\alpha < \lambda$.

Claim 5. If $\pi A = \lambda$, then the assertion of the theorem holds.

For Claim 4 says that for every $f \in \prod_{i\in\kappa} \lambda_i$, $f/D = f'/D$ for some $f' : \kappa \to \nu$ and some $\nu < \lambda$. By Claim 1, $\prod_{i\in\kappa} \lambda_i/D \leq \sum_{\nu<\kappa} |\nu|^\kappa \leq \lambda$. It now follows from Claim 2 that $\lambda \leq \pi A \leq \prod_{i\in\kappa} \lambda_i/D \leq \lambda$. $\square$

2. The theory T

We sketch here a first order theory T. Its relevance for solving Problem 12 of [Mo] lies in the fact that the models $\mathfrak{A}$ of T are enlargements $(A, \dots)$ of a Boolean algebra A; the extra structure of $\mathfrak{A}$ allows to easily compute $\pi(A)$ — see 2.1. below. Since every ultraproduct $\mathfrak{U} = (U, \dots)$ of models of T is again a model of T, we can then similarly compute $\pi(U)$.

Let T be the first order theory (in an appropriate language) saying that, for every model $\mathfrak{A} = (A, +, \cdot, -, 0, 1, L, \leq_L, \sim, v, x)$ of T, the following hold true.

(a) $(A, +, \cdot, -, 0, 1)$ is a Boolean algebra.

(b) $L \subseteq A$ is totally ordered by $\leq_L$ and has no greatest element. (We do not require any connection between $\leq_L$ and the Boolean partial order of A, except the one stipulated by (e) below.)

(c) v is a map from A to L; for $l \in L$, $A_l = \{a \in A : v(a) <_L l\}$ is a subalgebra of A. (Hence $(A_l)_{l\in L}$ is an increasing sequence of subalgebras of A whose union is A.)

(d) $\sim$ is an equivalence relation on L and its equivalence classes are convex, with respect to $\leq_L$.

(e) x is a map from L into A (we write x_i for $x(i)$) such that $i < l$ implies $x_i \not\leq x_l$. Moreover for $l \in L$, the set $\{x_i : i \sim l\}$ is dense for A_l in the sense that for every $a \in A_l \setminus \{0\}$ there is some $i \sim l$ satisfying $0 < x_i \leq a$. (Hence $\{x_i : i \in L\}$ is a dense subset of A.)

2.1 Remark. *Let $\mathfrak{A} = (A, \dots)$ be a model of T, ρ the cofinality of the linear order $(L, \leq_L)$ and assume that all equivalence classes in L have cardinality at most ρ. Then $\pi(A) = \rho$.*

Proof. To see that $\pi(A) \leq \rho$, fix a cofinal subset M of L of size ρ. The set

$$\{x_i : i \sim m, \text{ for some } m \in M\}$$

has size ρ and is dense in A, by (e). Assume for contradiction that A has a dense subset X of size less than ρ. Without loss of generality, $X \subseteq \{x_i : i \in L\}$; pick

$l \in L$ such that $x_i \in X$ implies $i < l$. X being dense in A, there is $x_i \in X$ such that $0 < x_i \leq x_l$. So $i < l$ which is impossible by (e). $\square$

In Sections 3 and 4, we will construct "standard" models $\mathfrak{A} = (A, \dots)$ of T which will roughly look like this, for some regular cardinal λ: $|A| = \lambda$, so without loss of generality, $\lambda \subseteq A$; we let $L = \lambda$ and $\leq_L$ its natural well-ordering. A will be generated by a sequence $(x_i)_{i \in \lambda}$; we then let A_l be the subalgebra of A generated by $\{x_i : i < l\}$ and define $v(a)$ to be the least i such that $a \in A_{i+1}$. Finally we will have an infinite cardinal $\mu < \lambda$ and define $i \sim l$ iff $i \leq l < i + \mu$ and $l \leq i < l + \mu$ (ordinal addition); thus the equivalence classes will have size μ. Satisfaction of condition (e) above will be guaranteed by a careful choice of the generators x_i — see Proposition 5.1. In particular, πA will be $\lambda = |A|$.

3. Valuation functions

We construct Boolean algebras $A(p)$ from certain functions p, the so-called valuation functions. Later the Boolean algebras $A(P)$, where P will be a generic valuation function, provide the counterexample for Problems 9 and 12 in [Mo] looked for.

We denote the three-element set consisting of the symbols $\geq, \perp, u =$ "undefined" by 3. For any set w with some linear order on it (later w will be a subset of some cardinal λ, hence well-ordered), recall that $[w]^2 = \{(i,j) : i < j \text{ in } w\}$.

Given a Boolean algebra A and a family $(x_i)_{i \in w}$ indexed by w in $A \setminus \{0\}$, we can assign to $(x_i)_{i \in w}$ the function $p : [w]^2 \to 3$ defined by

$$p(i,j) = \geq \text{ if } x_i \geq x_j$$

$$p(i,j) = \perp \text{ if } x_i \perp x_j, \text{ i.e. } x_i \cdot x_j = 0$$

$$p(i,j) = u \text{ otherwise .}$$

Clearly p has then the following properties:

(1) if $p(i,j) = \geq$ and $p(j,k) = \geq$ then $p(i,k) = \geq$ (where $i < j < k$)

(2) if $i < j < k$ and $\{p(i,j), p(i,k)\} = \{\perp, \geq\}$, then $p(j,k) = \perp$; similarly if $i < j < k$ and $p(i,j) = \perp$, $p(j,k) = \geq$, then $p(i,k) = \perp$.

Let us call a function p satisfying (1) and (2) above a *valuation function* and w its domain.

Conversely, given a valuation function $p : [w]^2 \to 3$, we construct a Boolean algebra $A = A(p)$ from p as follows. Let Fr w be the free Boolean algebra on the set $\{u_i : i \in w\}$ of independent generators and let $N(p)$ be the ideal in Fr w generated by the elementary products $u_j \cdot u_i$ where $p(i,j) = \perp$ resp. $u_j \cdot -u_i$ where $p(i,j) = \geq$. Let then $A(p)$ (or A, for short) be the quotient algebra Fr $w/N(p)$ and let c : Fr $w \to A(p)$ be the canonical homomorphism. Setting $x_i = c(u_i)$, for $i \in w$, we find that the x_i generate A. By the very choice of the ideal $N(p)$, $p(i,j) = \geq$ implies that $x_i \geq x_j$ and $p(i,j) = \perp$ implies that $x_i \perp x_j$. To see that no other relations than those imposed by p hold for the x_i, note the following general principle on construction of Boolean algebras via generators with prescribed relations.

3.1 Remark. *Let E be a set of finite partial functions from w to $\{0,1\}$ and let, for $e \in E$, q_e be the elementary product $\prod_{e(i)=1} u_i \cdot \prod_{e(i)=0} -u_i$ in Fr w. Assume N is the ideal of Fr w generated by the q_e, $e \in E$. Then for any function $g : w \to \{0,1\}$, there is an ultrafilter of Fr w/N including $\{x_i : g(i) = 1\} \cup \{-x_i : g(i) = 0\}$ (i.e. $\{x_i : g(i) = 1\} \cup \{-x_i : g(i) = 0\}$ has the finite intersection property) iff no $e \in E$ is extended by g.*

This gives the following properties of the x_i in $A = A(p)$, where p is a valuation function.

3.2 Remark. *x_i is not in the ideal generated by $\{x_j : j > i\}$. In particular, $x_i \neq 0$, the x_i are pairwise distinct, and $i < j$ implies that $x_i \not\leq x_j$.*

To see this, consider the function $g : w \to \{0,1\}$ such that $g(k) = 1$ iff $k = i$ or ($k < i$ and $p(k,i) = \geq$). By Remark 3.1, let s be the ultrafilter of A induced by g. Thus $x_i \in s$ but, for $j > i$, $x_j \notin s$, which shows the claim.

3.3 Remark. *x_i is not in the subalgebra of A generated by $\{x_j : j < i\}$.*

For consider the functions g and h from w to $\{0,1\}$ where g is defined as in the proof of 3.2, $h(k) = g(k)$ for $k \neq i$, but $h(i) = 0$. Let s and t be the corresponding ultrafilters of A, ϕ and ψ the homomorphisms from A to the two-element algebra corresponding to s and t. Now ϕ and ψ coincide on x_j for all $j < i$, but not on x_i.

4. The partial order of valuation functions

For the next sections, fix infinite cardinals λ and μ such that $\mu^{<\mu} = \mu$, $\mu < \lambda$, and λ is regular. We shall choose λ and μ somewhat more carefully in Section 7. Let $\mathbb{P}(\lambda,\mu)$ (or $\mathbb{P}$, for short) be the notion of forcing

$$\mathbb{P} = \{p : p \text{ is a valuation function and dom } p \subseteq \lambda \text{ has size less than } \mu\}$$

ordered by reverse inclusion.

4.1 Remark. *$\mathbb{P}$ is μ-closed.*

We now build up some machinery for constructing elements of $\mathbb{P}$ with prescribed properties. Given a set r of relations of the form $x_i \geq x_j$, $x_i \perp x_j$ (where $i, j \in \lambda$; the relations may be thought of as being formulas in some formal language in the variables x_i, $i \in \lambda$), we define when a relation ρ can be derived from r and we write $r \vdash \rho$: if ρ has the form $x_k \geq x_l$, $r \vdash \rho$ iff there are $i_1, \ldots, i_m \in \lambda$ such that the relations $x_k \geq x_{i_1}$, $x_{i_1} \geq x_{i_2}$, ..., $x_{i_m} \geq x_l$ are all in r (in particular, $r \vdash x_i \geq x_i$); if ρ has the form $x_k \perp x_l$, $r \vdash \rho$ iff there are $\alpha, \beta \in \lambda$ such that $x_\alpha \perp x_\beta$ is in r and $r \vdash x_\alpha \geq x_k$, $r \vdash x_\beta \geq x_l$.

Call r *consistent* if no relation of the form $x_j \geq x_i$ where $i < j$ and no relation of the form $x_k \perp x_k$ is derivable from r. Given $p \in \mathbb{P}$, define rel p, the relevant part of p, by

$$\text{rel } p = \{x_i \geq x_j : p(i,j) = \geq\} \cup \{x_i \perp x_j : p(i,j) = \perp\}.$$

4.2 Proposition. *If $|r| < \mu$, then r is consistent iff $r \subseteq$ rel p for some $p \in \mathbb{P}$.*

Proof. Assume first that $p \in \mathbb{P}$ and $r \subseteq \text{rel}\, p$ where $\text{dom}\, p = w \subseteq \lambda$. Then in the Boolean algebra $A(p)$ constructed in Section 3, all relations in r and hence all relations derivable from r are satisfied by the canonical generators $\{x_i : i \in w\}$; moreover, these generators are non-zero. Hence no relation $x_k \perp x_k$ and no relation of the form $x_j \geq x_i$, $i < j$, can be derived from r.

Conversely, if r is consistent, let w be any subset of λ such that $|w| < \mu$ and $\{i \in \lambda : x_i \text{ occurs in } r\} \subseteq w$. Define $p : [w]^2 \to 3$ by

$p(i,j) = \geq$ iff $r \vdash x_i \geq x_j$
$p(i,j) = \perp$ iff $r \vdash x_i \perp x_j$
$p(i,j) = u$ otherwise.

p is a well-defined function (i.e. r does not derive both $x_i \geq x_j$ and $x_i \perp x_j$, for $i < j \in w$) since otherwise, $r \vdash x_j \perp x_j$, contradicting the consistency of r. By the above definition of derivability from r, p is a valuation function. □

For further reference, call $p \in \mathbb{P}$ defined from a consistent set r and $w \subseteq \lambda$ as in the proof above the *canonical extension* of r over w.

We give one trivial and one not-so-trivial application of this machinery. If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over our universe V of set theory, then clearly $P_G = \bigcup G$ is a valuation function with $\text{dom}\, P_G = \bigcup_{p \in G} \text{dom}\, p$.

4.3 Remark. *If G is generic, then dom $P_G = \lambda$.*

To see this, we have to make sure that, for $i \in \lambda$, the set $D_i = \{p \in \mathbb{P} : i \in \text{dom}\, p\}$ is dense in $\mathbb{P}$. But given $q \in \mathbb{P}$, let $w \subseteq \lambda$ be such that $|w| < \mu$ and $\text{dom}\, q \cup \{i\} \subseteq w$. Now by 4.2, rel q is consistent; let p be the canonical extension of rel q over w. Then $p \in D_i$ and $q \subseteq p$.

4.4 Proposition. *If $p, q \in \mathbb{P}$ coincide on dom $p \cap$ dom q, then they are compatible in $\mathbb{P}$.*

Proof. This follows from a number of claims. We write $p \vdash \dots$ instead of rel $p \vdash \dots$ and we say that a relation, e.g. $x_i \geq x_j$, is in p if $p(i,j) = \geq$ etc.

Claim 1. If $p \vdash x_i \geq x_j$ where $i < j$, then $i, j \in \text{dom}\, p$ and the relation $x_i \geq x_j$ is in p. Similarly for q and for relations of the form $x_i \perp x_j$. — The claim holds because rel p, for $p \in \mathbb{P}$, is closed under derivations.

By 4.2 we are left with showing that the set

$$r = \text{rel}\, p \cup \text{rel}\, q$$

is consistent.

Claim 2. If $r \vdash x_i \geq x_j$, then $p \vdash x_i \geq x_j$ or $q \vdash x_i \geq x_j$ or, for some α, ($p \vdash x_i \geq x_\alpha$ and $q \vdash x_\alpha \geq x_j$) or, for some α, ($q \vdash x_i \geq x_\alpha$ and $p \vdash x_\alpha \geq x_j$).

Claim 3. If $r \vdash x_i \perp x_j$, then $p \vdash x_i \perp x_j$ or $q \vdash x_i \perp x_j$ or, for some α, ($p \vdash x_i \perp x_\alpha$ and $q \vdash x_\alpha \geq x_j$) or, for some α, ($q \vdash x_i \perp x_\alpha$ and $p \vdash x_\alpha \geq x_j$) (or similarly with i interchanged with j).

Claim 4. If $r \vdash x_i \geq x_j$ and $i, j \in \text{dom}\, p$, then $p \vdash x_i \geq x_j$. Similarly for q and for relations of the form $x_i \perp x_j$.

The proofs are easy but require consideration of a number of cases. We give two typical examples. In Claim 3, assume e.g. that $p \vdash x_\gamma \perp x_\delta$, $q \vdash x_\gamma \geq x_i$ and

$q \vdash x_\delta \geq x_j$. Then γ and δ are in dom $p \cap$ dom q, $x_\gamma \perp x_\delta$ is (by Claim 1) in p, hence in q, because p and q coincide on dom $p \cap$ dom q, and $q \vdash x_i \perp x_j$.

Similarly in Claim 4, assume e.g. that $p \vdash x_i \geq x_\alpha$ and $q \vdash x_\alpha \geq x_j$ where $i, j \in$ dom p. Since α is in dom $p \cap$ dom q, it follows that $x_\alpha \geq x_j$ is in p, hence $p \vdash x_i \geq x_j$.

Claim 5. r is consistent. — For otherwise by Claim 3, we may assume that, e.g., for some α, $p \vdash x_k \perp x_\alpha$ and $q \vdash x_\alpha \geq x_k$. Then k and α are in dom $p \cap$ dom q, $x_\alpha \geq x_k$ is in q and $x_k \perp x_k$ is in p, a contradiction. □

4.5 Proposition. *$\mathbb{P}$ satisfies the μ^+-chain condition.*

Proof. If X is a subset of $\mathbb{P}$ of size μ^+, then by $\mu^{<\mu} = \mu$ and the Δ-lemma there are p and q in X coinciding on dom $p \cap$ dom q. So we are finished by Proposition 4.4. □

5. Computing $\pi(A(P))$

In this and the following section, let G be a $\mathbb{P}$-generic filter over V and P the resulting generic valuation function (see 4.3). Write A for $A(P)$. We prove condition (e) of section 2 for A, thus being able to compute $\pi(A)$ in $V[G]$.

5.1 Proposition. *The following holds in $V[G]$. Let $\alpha < \lambda$ be an ordinal, $a \subseteq \alpha$ finite, $e : a \to \{0, 1\}$ and*

$$y = \prod_{e(i)=1} x_i \cdot \prod_{e(i)=0} -x_i > 0 \qquad (\textit{in } A).$$

Then there is $i^* \in [\alpha, \alpha+\mu)$ (ordinal addition) such that $x_{i^*} \leq y$. - In particular, the set $\{x_{i^*} : i^* \in [\alpha, \alpha+\mu)\}$ is dense for the subalgebra of A generated by $\{x_i : i < \alpha\}$.

Proof. We do not distinguish notationally between elements of $V[G]$ and their $\mathbb{P}$-names; in particular since a and e, being finite, are in the ground model. Pick $p \in G$ such that

$$p \Vdash y = \prod_{e(i)=1} x_i \cdot \prod_{e(i)=0} -x_i > 0;$$

it suffices to prove that

$$D = \{t \in \mathbb{P} : t \leq p, \text{ and } t \Vdash x_{i^*} \leq y \text{ for some } i^* \in [\alpha, \alpha + \mu)\}$$

is dense below p. To this end, let $q \leq p$ be arbitrary. By 4.3, we can fix $r \leq q$ such that $a \subseteq$ dom r. Then fix $i^* \in [\alpha, \alpha + \mu) \setminus$ dom r; this is possible by $|\text{dom } r| < \mu$. We define a function s with domain $a \cup \{i^*\}$ by putting

$$\begin{aligned} s \restriction [a]^2 &= r \restriction [a]^2 \\ s(i, i^*) &= \geq \text{ if } i \in a \text{ and } e(i) = 1 \\ s(i, i^*) &= \perp \text{ if } i \in a \text{ and } e(i) = 0. \end{aligned}$$

Claim $s \in \mathbb{P}$, i.e. s is a valuation function.

Let us check just one case. Note that, for $u \in \mathbb{P}$, $u(i,j) = \geq$ implies that $u \Vdash x_i \geq x_j$ and similarly for $\perp$ instead of $\geq$ since for any generic $H \subseteq \mathbb{P}$ containing u, $u \subseteq P_H$ and thus $x_i \geq x_j$ will hold in $A(P_H)$. Assume e.g. $i < j$ in a, $s(i,j) = \geq$ and $s(j,i^*) = \geq$; we have to show that $s(i,i^*) = \geq$. The assumptions say that $r(i,j) = \geq$ (since $i, j \in a$) and $e(j) = 1$; we have to show that $e(i) = 1$. But if $e(i) = 0$, then: $p \Vdash 0 \neq -x_i \cdot x_j$ (because $p \Vdash 0 < y \leq -x_i \cdot x_j$), $r \Vdash 0 \neq -x_i \cdot x_j$ (since $r \leq p$), $r \Vdash x_i \geq x_j$ (by the above assumption), $r \Vdash -x_i \cdot x_j = 0$, a contradiction. Now r and s coincide on $a = \text{dom } r \cap \text{dom } s$, so by 4.4, pick $t \in \mathbb{P}$ extending both r and s. Then $t \leq q$ and $s \Vdash x_{i^*} \leq y$, by the very definition of s above, so $t \in D$. $\square$

5.2 Corollary. *$\pi(A) = \lambda$ (in $V[G]$).*

Proof. This follows from Remark 2.1 and the sketch of the model $\mathfrak{A} = (A, \dots) \models T$ following it, plus 5.1. Let us remark that 6.1 gives another proof, since $\text{d}A = \lambda$, $\text{d}A \leq \pi A$ holds for all Boolean algebras and $\pi A \leq |A| = \lambda$. $\square$

5.3 Example. Our construction of $A = A(P)$ and 5.1 above give a counterexample to the assertion in 4.1 of [Mo], in $V[G]$. For this, let A_α be the subalgebra of A generated by $\{x_i : i < \alpha\}$; so if $\alpha \in I = \{\alpha < \lambda : \text{cf } \alpha = \mu\}$, then by Remark 2.1 and 5.1 above, we have $\pi A_\alpha = \mu$. Moreover $A = \bigcup_{\alpha \in I} A_\alpha$ and $\pi A = \lambda$ where λ can be larger than μ^+. — In fact, the argument given in [Mo, 4.1] depends on the assumption that the chain $(A_\alpha)_{\alpha \in I}$ is continuous which is not the case here.

6. Computing $\mathbf{d}(A(P))$

Our single theorem here is the following.

6.1 Theorem. *In $V[G]$, $A = A(P)$ satisfies $d(A) = \lambda$.*

Proof. Otherwise, the cardinal $\sigma = \text{d}(A)^{V[G]}$ is less than λ. There are a $\mathbb{P}$-name u and a condition $p \in \mathbb{P}$ (in fact, $p \in G$) such that

$$p \Vdash u \text{ is a sequence } (u_\nu)_{\nu<\sigma}, \text{ each } u_\nu \text{ is an ultrafilter of } A, \text{ and } A \setminus \{0\} = \bigcup_{\nu<\sigma} u_\nu.$$

For $\alpha < \lambda$, fix $p_\alpha \in \mathbb{P}$ and $\nu_\alpha < \sigma$ such that $p_\alpha \leq p$ and

$$p_\alpha \Vdash x_\alpha \in u_{\nu_\alpha}$$

(x_α the (name of the) α'th generator of A). In the next steps, we construct stationary subsets $S_1 \supseteq S_2 \supseteq S_3 \supseteq S_4$ of λ.

Step 1. $S_1 = \{\alpha \in \lambda : \text{cf } \alpha = \mu\}$ is stationary in λ because $\mu < \lambda$ and λ is regular.

Step 2. Since $\sigma < \lambda = \text{cf } \lambda$, there are $\nu^* < \sigma$ and a stationary $S_2 \subseteq S_1$ such that $\nu_\alpha = \nu^*$, for all $\alpha \in S_2$.

Step 3. Write $w_\alpha = \text{dom } p_\alpha$, for $\alpha \in \lambda$. We find $\alpha^* \in \lambda$ and a stationary $S_3 \subseteq S_2$ such that for all $\alpha \in S_3$, $\alpha^* < \alpha$ and $w_\alpha \cap \alpha \subseteq \alpha^*$ hold. To this end, note that cf $\alpha = \mu$ for $\alpha \in S_2$ and $|w_\alpha \cap \alpha| < \mu$; so pick $j_\alpha < \alpha$ satisfying $w_\alpha \cap \alpha \subseteq j_\alpha$. Apply Fodor's theorem to obtain S_3.

Step 4. We find a stationary set $S_4 \subseteq S_3$ such that $\alpha < \beta$ in S_4 implies $w_\alpha \subseteq \beta$. To do this, define by induction $f : \lambda \to \lambda$ strictly increasing and continuous such that, for all α, $\bigcup_{\nu<\alpha} w_\nu \subseteq f(\alpha)$ and let $S_4 = S_3 \cap C$ where $C = \{\alpha : f(\alpha) = \alpha\}$ is closed unbounded. Then S_4 is stationary and, for $\alpha < \beta$ in S_4, we have $w_\alpha \subseteq f(\beta) = \beta$.

Now $\mu^+ \leq \lambda$ and $\mathbb{P}$ satisfies the μ^+-chain condition. So we can find $\alpha < \beta$ in S_4 such that p_α and p_β are compatible in $\mathbb{P}$. Let r be the following set of relations:

$$r = \text{rel }(p_\alpha) \cup \text{rel }(p_\beta) \cup \{x_\beta \perp x_\alpha\}$$

(see the machinery in section 4).

Claim. r is consistent.

By the claim and 4.2, pick then $q \in \mathbb{P}$ such that $r \subseteq \text{rel}(q)$. This q will force the following statements:

$x_\beta \perp x_\alpha$

$x_\alpha \in u_{\nu_\alpha} = u_{\nu^*}$ and $x_\beta \in u_{\nu_\beta} = u_{\nu^*}$

u_{ν^*} has the finite intersection property (being an ultrafilter),

and this contradiction finishes the proof.

Proof of the Claim. Clearly no relation $x_i \geq x_j$ where $j < i$ can have a derivation from r, since such a derivation would not use the relation $x_\beta \perp x_\alpha$; hence $x_i \geq x_j$ would be derivable from rel $(p_\alpha) \cup$ rel (p_β), contradicting the compatibility of p_α and p_β.

Now assume $r \vdash x_k \perp x_k$, for some $k \in \lambda$. A derivation witnessing this starts, without loss of generality, with the relation $x_\beta \perp x_\alpha$. So in $p_\alpha \cup p_\beta$ there are relations

$$x_{i_0} \geq x_{i_1}, \ldots, x_{i_{r-1}} \geq x_{i_r} \text{ where } i_0 = \alpha, i_r = k$$

$$x_{j_0} \geq x_{j_1}, \ldots, x_{j_{s-1}} \geq x_{j_s} \text{ where } j_0 = \beta, j_s = k.$$

Note that $\alpha = i_0 < i_1 < \cdots < i_r = k$ (since if $x_j \geq x_i$ is in $p_\alpha \cup p_\beta$, then $j < i$); similarly, $\beta = j_0 < j_1 < \cdots < j_s = k$.

We prove by induction on $t \in \{0, \ldots, r\}$ that $i_t \notin w_\beta = \text{dom } p_\beta$; for $t = r$ this gives a contradiction because then $k = i_r \notin w_\beta$, so $k \in w_\alpha$ and $k \geq \beta$, but $w_\alpha \subseteq \beta$. First, $i_0 \notin w_\beta$: otherwise, by Step 3, $i_0 = \alpha \in w_\beta \cap \beta \subseteq \alpha^*$, contradicting $\alpha^* < \alpha$ for $\alpha \in S_3$. If $i_t \notin w_\beta$ but $i_{t+1} \in w_\beta$, then the relation $x_{i_t} \geq x_{i_{t+1}}$ must be in p_α. But then $i_{t+1} \in w_\alpha \subseteq \beta$ and again $i_{t+1} \in w_\beta \cap \beta \subseteq \alpha^* < \alpha$, a contradiction. □

7. Proof of Theorem A

7.1 Proof of Theorem A. Fix κ, μ, λ_α and D as given in the theorem; $\mathbb{R}$ will be the iteration of two forcing notions. In the first step, collapse $\mu^{<\mu}$ to μ with $\mathbb{Q} = \mathrm{Fn}\,(\mu, \mu^{<\mu}, <\mu)$ in Kunen's notation ([Ku]). This forcing is μ-closed and satisfies the $(\mu^{<\mu})^+$-chain condition; in the resulting generic model $V[H]$, $\mu^{<\mu} = \mu$ holds. The notions of ultrafilters on κ, the cartesian product $\prod_{\alpha<\kappa} \lambda_\alpha$ etc. are absolute for this forcing by μ-closedness of $\mathbb{Q}$ and $\kappa < \mu$; thus all assumptions of the theorem continue to hold in $V[H]$.

Working now in $V[H]$, let, for $\alpha \in \kappa$, $\mathbb{P}_\alpha$ be the forcing notion $\mathbb{P}(\lambda_\alpha, \mu)$ defined in section 4; let $\mathbb{P}$ be the full cartesian product $\mathbb{P} = \prod_{\alpha<\kappa} \mathbb{P}_\alpha$ with the coordinate-wise partial order. For $G \subseteq \mathbb{P}$ generic over V, $G_\alpha = {\mathrm{pr}_\alpha}^{-1}[G]$ is $\mathbb{P}_\alpha$- generic over $V[H]$ (pr_α the α'th projection). $\mathbb{P}$ is clearly μ- closed, moreover, as in the proof of 4.5, the Δ-lemma implies that $\mathbb{P}$ satisfies the μ^+-chain condition since $\mu^{<\mu} = \mu$. Thus the assumptions of the theorem, as well as $\mu^{<\mu} = \mu$, continue to hold in $V[H][G]$.

In $V[H][G]$, $P_\alpha = \bigcup G_\alpha : [\lambda_\alpha]^2 \to 3$ is a generic valuation function. Let $A_\alpha = A(P_\alpha)$ be its associated Boolean algebra; by sections 5 and 6, $\pi(A_\alpha) = \mathrm{d}(A_\alpha) = \lambda_\alpha$. In the standard model $\mathfrak{A}_\alpha = (A_\alpha, \dots)$ of T (see section 2), the predicate L is interpreted by λ_α and the equivalence classes of $\sim_L$ have size μ. So in the ultraproduct $\mathfrak{A} = \prod_{\alpha<\kappa} \mathfrak{A}_\alpha/D$, L is interpreted by $\prod_{\alpha<\kappa} \lambda_\alpha/D$ and the equivalence classes of $\sim_L$ have size $\le |\mu^\kappa/D| = \mu$ (by $\kappa < \mu$ and $\mu^{<\mu} = \mu$). Now Remark 2.1 says that $\pi(A) = \mathrm{cf}\,\prod_{\alpha<\kappa} \lambda_\alpha/D$ and hence $\mathrm{d}(A) \le \pi(A) = \mathrm{cf}\,(\prod_{\alpha<\kappa} \lambda_\alpha/D) < |\prod_{\alpha<\kappa} \lambda_\alpha/D| = |\prod_{\alpha<\kappa} \pi(A_\alpha)/D| = |\prod_{\alpha<\kappa} \mathrm{d}(A_\alpha)/D|$. $\square$

We can prove a little more:

7.2 Remark. *In $V[H][G]$, let $A = \prod_{\alpha<\kappa} A_\alpha/D$ be the algebra constructed in 7.1 and let $\lambda = cf\ \prod_{\alpha<\kappa} \lambda_\alpha/D$. Then $d(A) = \lambda$.*

Proof. Our proof will closely follow that of 6.1.

Fix a sequence $(f_\gamma)_{\gamma\in\lambda}$ in $\prod_{\alpha<\kappa} \lambda_\alpha$ such that $(f_\gamma/D)_{\gamma\in\lambda}$ is strictly increasing and cofinal in the ultraproduct $\prod_{\alpha<\kappa} \lambda_\alpha/D$. By [Sh, Ch.II], the set

$$\begin{aligned} S = \{\gamma \in \lambda : \mathrm{cf}\ \gamma = \mu^+ &\text{ , and there is } g \in \prod_{\alpha<\kappa} \lambda_\alpha \\ &\text{such that } g/D \text{ is the least upper bound of } \{f_\delta/D : \delta < \gamma\} \\ &\text{and cf } g(\alpha) = \mu^+ \text{ for all } \alpha \in \kappa\} \end{aligned}$$

is stationary; so we may assume that, for $\gamma \in S$, f_γ satisfies the requirements for g above.

Now note that, in $V[H][G]$, $\mathrm{d}A \le \pi A = \lambda$ as shown in the proof of 7.1; so assume for contradiction that $\mathrm{d}A < \lambda$. Thus , in $V[H][G]$,there are a $\mathbb{P}$-name u, $\sigma < \lambda$ and $p \in \mathbb{P}$ such that

$$p \Vdash u = (u_\nu)_{\nu<\sigma} \text{ is a sequence of ultrafilters of } A \text{ covering } A \setminus \{0\}.$$

For $\gamma \in S$, fix $p_\gamma \ge p$ and $\nu_\gamma \in \sigma$ such that

$$p_\gamma \Vdash y_\gamma/D \in u_{\nu_\gamma}$$

where y_γ is (a $\mathbb{P}$-name for) $(x_{f_\gamma(\alpha)})_{\alpha<\kappa}/D$ and x_i is (a $\mathbb{P}$-name for) the i'th canonical generator of A_α, for $i < \lambda_\alpha$. There is a stationary subset S_1 of S such that ν_γ is some fixed ν^*, for $\gamma \in S_1$ (because $\nu_\gamma < \sigma < \lambda$ and λ is regular). As in Step 3 in the proof of 6.1, there exists , for $\gamma \in S_1$, some $\beta_\gamma < \gamma$ such that, for D-almost all α,

$$\text{dom } p_\gamma(\alpha) \cap f_\gamma(\alpha) \subseteq f_{\beta_\gamma}(\alpha).$$

Without loss of generality (i.e. by passing to a stationary subset), β_γ is some fixed β^*, for all $\gamma \in S_1$. Now $K_\gamma = \{\alpha \in \kappa : \text{dom } p_\gamma(\alpha) \cap f_\gamma(\alpha) \subseteq f_{\beta^*}(\alpha)\} \in D$, for $\gamma \in S_1$; since $2^\kappa < \lambda$, we may assume without loss of generality that K_γ is some fixed $K^* \in D$, for $\gamma \in S_1$.

As in Step 4 of the proof of 6.1, we may assume that $\gamma < \delta$ in S_1 implies that

$$K_{\gamma\delta} = \{\alpha \in \kappa : \text{dom } p_\gamma(\alpha) \subseteq f_\delta(\alpha)\} \in D$$

because $(f_\delta/D)_{\delta\in\lambda}$ is cofinal in $\prod_{\alpha<\kappa} \lambda_\alpha/D$.

Now $\mathbb{P}$ satisfies the μ^+-chain condition and S_1 has size $\lambda \geq \mu^+$; so fix $\gamma < \delta$ in S_1 such that p_γ and p_δ are compatible in $\mathbb{P} = \prod_{\alpha\in\kappa} \mathbb{P}_\alpha$, i.e. $p_\gamma(\alpha)$ and $p_\delta(\alpha)$ are compatible in $\mathbb{P}_\alpha$, for all $\alpha \in \kappa$.

We conclude as in 6.1: for all $\alpha \in K^* \cap K_{\gamma\delta}$, the set

$$r_\alpha = \text{rel } p_\gamma(\alpha) \cup \text{rel } p_\delta(\alpha) \cup \{x_{f_\delta(\alpha)} \perp x_{f_\gamma(\alpha)}\}$$

is consistent; so pick $q_\alpha \in \mathbb{P}_\alpha$ satisfying $r_\alpha \subseteq \text{rel } q_\alpha$. Choose $q \in \mathbb{P}$ having α'th coordinate q_α, for $\alpha \in K^* \cap K_{\gamma\delta}$; then q forces that: $y_\delta/D \perp y_\gamma/D$, $y_\gamma/D \in u_{\nu_\gamma} = u_{\nu^*}$ and $y_\delta/D \in u_{\nu_\delta} = u_{\nu^*}$, u_{ν^*} is an ultrafilter. This gives a contradiction. $\square$

8. Proof of Theorem B

To abbreviate the main body of the proof, we state in advance two easy lemmas. The proofs are left to the reader.

8.1 Lemma. *Assume $h : C \to D$ is a homomorphism of Boolean algebras, $\{c_n : n \in \omega\}$ is a partition of unity in C, and also $\{h(c_n) : n \in \omega\}$ is a partition of unity in D. Then, if $x_n \in C$ are such that $\sum_{n\in\omega}^{C} x_n \cdot c_n$ exists, we have $h(\sum_{n\in\omega}^{C} x_n \cdot c_n) = \sum_{n\in\omega}^{D} h(x_n \cdot c_n)$.*

Given a subalgebra C of D and $x \in D$, let $I_C(x) = \{c \in C : c \cdot x = 0\}$, an ideal of C. Call $x, y \in D$ equivalent over C (and write $x{\sim_C}y$) if both $I_C(x) = I_C(y)$ and $I_C(-x) = I_C(-y)$ hold, i.e. if x and y realize the same quantifier-free type over C.

8.2 Lemma. *If $x, y \in D$ are equivalent over C, then there is no $c \in C\setminus\{0\}$ disjoint from $x + -y$.*

We break up the proof of Theorem B into eight preparatory steps in which certain objects are constructed or notation is fixed, plus four claims. Let $C \leq D$ denote that C is a subalgebra of D; $\overline{A}$ is the completion of A.

Step 1. Take μ as assumed in the theorem, fix a set U of cardinality μ, and let $A = \mathrm{Fr}\ U$, the free Boolean algebra over U. Since $|\overline{A}| = \mu^\omega \geq \mu^+ = 2^\mu$, we have $|\overline{A}| = \mu^+$. The algebra B promised in the theorem will be a subalgebra of $\overline{A}$, generated by A and pairwise distinct elements b_i of $\overline{A}$, $i < \mu^+$. So $|B| = \mu^+$ and we know in advance that $\mu^+ \leq |\mathrm{End}\ B|$ and $|\mathrm{Id}\ B| \leq 2^{\mu^+}$.

Step 2. Fix an enumeration $\{g_j : j < \mu^+\}$ of all homomorphisms from A into $\overline{A}$. This is possible since $|A| = \mu$ and $|\overline{A}| = \mu^+ = (\mu^+)^\mu$.

Step 3. Fix a sequence $(\mu_n)_{n\in\omega}$ of cardinals such that $\mu = sup_{n\in\omega}\mu_n$ and $2^{\mu_n} < \mu_{n+1}$.

Step 4. For each ordinal $i < \mu^+$, fix subsets S_{in} of i such that $i = \bigcup_{n\in\omega} S_{in}$, $S_{in} \subseteq S_{i,n+1}$ and $|S_{in}| \leq \mu_n$. This is possible since $|i| \leq \mu$.

Step 5. Fix a sequence $(A_n)_{n\in\omega}$ of subalgebras of A such that $A = \bigcup_{n\in\omega} A_n$, $A_n \subseteq A_{n+1}$ and $|A_n| \leq \mu_n$.

Step 6. Define a tree $T = \bigcup_{n\in\omega} T_n$ with n'th level $T_n = \mu_0 \times \cdots \times \mu_{n-1}$ where $t \leq s$ in T means that s extends t; so $|T| = \mu$. The cartesian product $F = \prod_{n\in\omega} \mu_n$ has size $\mu^\omega = \mu^+$; fix a one-one enumeration $\{f_i : i < \mu^+\}$ of F.

Split $U \subseteq A = \mathrm{Fr}\ U$ (cf. Step 1) into two disjoint subsets X and Z such that $|X| = |Z| = \mu$; then split both X and Z into pairwise disjoint subsets X_t, $t \in T$, and Z_t, $t \in T$, such that $|X_t| = \mu$ and $Z_t \neq \emptyset$.

Step 7. Here we define, for $i < \mu^+$, the elements b_i of $\overline{A}$ and then let B be the subalgebra of $\overline{A}$ generated by $A \cup \{b_i : i < \mu^+\}$. b_i is constructed out of certain elements x_{in}, y_{in}, z_{in}, $n \in \omega$, of U by putting

$$s_{in} = x_{in} + -y_{in}$$

$$d_{i,n} = s_{i,n} \cdot \prod_{m<n} -s_{im}$$

$$b_i = \sum_{n\in\omega} z_{in} \cdot d_{in}.$$

To choose the x_{in}, y_{in}, z_{in}, fix $i < \mu^+$ and $n \in \omega$; thus $t = f_i \restriction n$ is an element of the tree T. Pick $z_{in} \in Z_t$ (see Step 6) arbitrarily. x_{in} and y_{in} are chosen much more carefully: we want them to be distinct elements of X_t satisfying

(*) for all $j \in S_{in}$, $g_j(x_{in}) \sim_{A_n} g_j(y_{in})$

(cf. Steps 4, 2, 5, and the definition of $\sim_{A_n}$ before 8.2). This is possible since:
$|A_n| \leq \mu_n$
there are at most 2^{μ_n} equivalence classes in $\overline{A}$, with respect to $\sim_{A_n}$, since there are at most 2^{μ_n} ideals in A_n
$|S_{in}| \leq \mu_n$
the set $\{(g_j(x)/\sim_{A_n})_{j\in S_{in}} : x \in X_t\}$ has size at most 2^{μ_n}
$2^{\mu_n} < \mu = |X_t|$.

Step 8. (Remark) For $b \in A$, let us denote by supp b (the support of b) the smallest subset of U generating b. Now for $i < \mu^+$, the supports $\{\text{supp } s_{in} : n \in \omega\}$ are pairwise disjoint and thus $\sum^{\overline{A}} s_{in} = 1$. It follows that the pairwise disjoint set $\{d_{in} : n \in \omega\}$ is a partition of unity in $\overline{A}$ and all d_{in} are non-zero. — Similarly, for any homomorphism $g : A \to \overline{A}$, the sets $\{g(d_{in}) : n \in \omega\}$ and $\{g(s_{in}) : n \in \omega\}$ have the same upper bounds in A resp. $\overline{A}$.

Claim 1. If $j < i < \mu^+$, then $\{g_j(d_{in}) : n \in \omega\}$ is a partition of unity (in $\overline{A}$). — Otherwise, assume $a \in A^+$ and $a \cdot g_j(s_{in}) = 0$ for all n (cf. Step 8). Pick n so large that $a \in A_n$ and $j \in S_{in}$. Then $a \cdot g_j(x_{in} + -y_{in}) = 0$, so $a \cdot (g_j(x_{in}) + -g_j(y_{in})) = 0$, contradicting (*) and Lemma 8.2.

Claim 2. Let g be an endomorphism of B, say $g \restriction A = g_j$ (see Step 2). Then for all $i > j$, $g(b_i) = \sum^{\overline{A}} g_j(z_{in}) \cdot g_j(d_{in})$ holds. Hence g is uniquely determined by its action on $A \cup \{b_i : i \leq j\}$. — This follows from Claim 1 and Lemma 8.1.

Claim 3. $|\text{End } B| \leq \mu^+$. — To completely describe some $g \in \text{End } B$, we have only μ^+ choices for $g \restriction A$ (Step 2) and, for $j < \mu^+$, at most $(\mu^+)^{|j|} \leq 2^\mu = \mu^+$ choices for $(g(b_i))_{i\leq j}$, so we are finished by Claim 2.

Claim 4. The generators $\{b_i : i < \mu^+\}$ are ideal-independent; hence $|\text{Id } B| = 2^{\mu^+}$. — We prove that, for $i \in \mu^+$ and J a finite subset of $\mu^+ \setminus \{i\}$, $b_i \not\leq \sum_{j\in J} b_j$. (It follows that the ideals I_K generated by $\{b_i : i \in K\}$ for $K \subseteq \mu^+$, are all distinct, so B has 2^{μ^+} ideals.) The argument is elementary but a little tedious and we give it in some detail. Assume for contradiction that $b_i \leq \sum_{j\in J} b_j$.

For arbitrary $n \in \omega$, we have the following situation. d_{in} is non-zero and for $j \in J$, $\{d_{jm} : m \in \omega\}$ is a partition of unity; hence there are elements $m(j) \in \omega$, for $j \in J$, such that $p = d_{in} \cdot \prod_{j\in J} d_{jm(j)}$ is non-zero. Now $b_i \cdot d_{in} \leq z_{in}$ and thus $b_i \cdot p \leq z_{in}$; similarly $b_j \cdot p \leq z_{jm(j)}$ holds for $j \in J$. It follows from $b_i \leq \sum_{j\in J} b_j$ that $z_{in} \cdot p \leq b_i \cdot p \leq \sum_{j\in J} z_{jm(j)}$. But supp $p \subseteq X$ and z_{in}, $z_{jm(j)}$ are in Z; hence $z_{in} \leq \sum_{j\in J} z_{jm(j)}$. So $z_{in} = z_{jm(j)}$, for some $j \in J$, since $Z \subseteq U$ is independent. Since z_{in} was chosen in Step 7 from Z_t, where $t = f_i \restriction n$, and $(Z_t)_{t\in T}$ was a disjoint family, it follows that $n = m(j)$ and $f_i \restriction n = f_j \restriction n$.

We have thus shown that for every $n \in \omega$, there is some $j \in J$ satisfying $f_i \restriction n = f_j \restriction n$. But then $f_i \in \{f_j : j \in J\}$ and $i \in J$ (since the enumeration $\{f_i : i < \mu^+\}$ in Step 6 was one-one), a contradiction. $\square$

References

[ChK] C. C. Chang and H. J. Keisler, *Model Theory, 3rd edition*, North Holland, 1990.

[Ho] R. Hodel, *Cardinal functions I*, Handbook of set-theoretic topology (K. Kunen and J. E. Vaughan, eds.), North Holland, Amsterdam, 1984.

[J] T. Jech, *Set Theory*, Academic Press, 1978.

[Ju1] I. Juhasz, *Cardinal functions in topology - ten years later*, Math. Center Tracts 123, 1980.

[Ju2] I. Juhasz, *Cardinal functions II*, Handbook of set-theoretic topology (K. Kunen and J. E. Vaughan, eds.), North Holland, Amsterdam, 1984.

[Ko] S. Koppelberg, *General Theory of Boolean algebras. Handbook of Boolean algebras, Part I*, North Holland, 1989.

[Ku] K. Kunen, *Set Theory*, North Holland, 1980.

[Ma] M. Magidor, *On the singular cardinals problem*, J. Israel Journal of Mathematics **28** (1977), 517–547.

[Mo] J. D. Monk, *Cardinal functions on Boolean algebras*, Birkhäuser, 1990.

[RoSh 534] A. Rosłanowski, S. Shelah, *F-99: Notes on cardinal invariants and ultraproducts of Boolean algebras, preprint.*

[Sh] S. Shelah, *Cardinal Arithmetic*, Oxford University Press (in press).

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