

On $\text{CH} + 2^{\aleph_1} \rightarrow (\alpha)_2^2$ for $\alpha < \omega_2$

by

Saharon Shelah¹⁾

1. Introduction

We prove the consistency of

$$\text{CH} + 2^{\aleph_1} \text{ is arbitrarily large} + 2^{\aleph_1} \not\rightarrow (\omega_1 \times \omega)_2^2$$

(Theorem 1). In fact, we can get $2^{\aleph_1} \not\rightarrow [\omega_1 \times \omega]_{\aleph_0}^2$, see 1A. In addition to this theorem, we give generalizations to other cardinals (Theorems 2 and 3). The $\omega_1 \times \omega$ is best possible as CH implies

$$\omega_3 \rightarrow (\omega \times n)_2^2.$$

We were motivated by the question of Baumgartner [B1] on whether CH implies $\omega_3 \rightarrow (\alpha)_2^2$ for $\alpha < \omega_2$ (if $2^{\aleph_1} = \aleph_2$, it follows from the Erdos–Rado theorem). He proved the consistency of positive answer with $\text{CH} + 2^{\aleph_1} > \aleph_3$, and proved in ZFC a related polarized partition relation (from CH)

$$\binom{\aleph_3}{\aleph_2} \rightarrow \binom{\aleph_1}{\aleph_0}^{1,1}.$$

Note: The main proof here is that of Theorem 1. In that proof, in the way things are set up, the main point is proving the \aleph_2 -c.c. The main idea in the proof is using \mathbb{R} (defined in the proof). It turns out that we can use as elements of \mathcal{P} (see the proof) just pairs (a, b) . Not much would be changed if we used $\langle (a_n, \alpha_n) : n < \omega \rangle$, a_n a good approximation of the n^{th} part of the suspected monochromatic set of order type $\omega_1 \times \omega$. In 1A, 2 and 3 we deal with generalizations and in Theorem 4 with complementary positive results.

2. The main result

Theorem 1. Suppose

- (a) CH.
- (b) $\lambda^{\aleph_1} = \lambda$.

Then there is an \aleph_2 -c.c. \aleph_1 -complete forcing notion \mathbb{P} such that

- (i) $|\mathbb{P}| = \lambda$.
- (ii) $\Vdash_{\mathbb{P}} "2^{\aleph_1} = \lambda, \lambda \not\rightarrow (\omega_1 \times \omega)_2^2"$.
- (iii) $\Vdash_{\mathbb{P}} \text{CH}$.
- (iv) Forcing with \mathbb{P} preserves cofinalities and cardinalities.

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Proof. By Erdos and Hajnal [EH] there is an algebra \mathbb{B} with $2^{\aleph_0} = \aleph_1$ ω -place functions, closed under composition (for simplicity only), such that

$$\otimes \quad \text{If } \alpha_n < \lambda \text{ for } n < \omega, \text{ then for some } k \\ \alpha_k \in \text{cl}_{\mathbb{B}}\{\alpha_l : k < l < \omega\}.$$

[\otimes implies that for every large enough k , for every m , $\alpha_k \in \text{cl}_{\mathbb{B}}\{\alpha_l : m < l < \omega\}$.] Let

$$\mathcal{R}_\delta = \{b : b \subseteq \lambda, \text{otp}(b) = \delta, \alpha \in b \Rightarrow b \subseteq \text{cl}_{\mathbb{B}}(b \setminus \alpha)\}.$$

So by \otimes we have

$$\oplus \quad \text{If } \alpha \text{ is a limit ordinal, } b \subseteq \lambda, \text{otp}(b) = \alpha, \\ \text{then for some } \alpha \in b, \quad b \setminus \alpha \in \bigcup_\delta \mathcal{R}_\delta.$$

Let $\mathcal{R}_{<\omega_1} = \bigcup_{\alpha < \omega_1} \mathcal{R}_\alpha$. Let \mathbb{P} be the set of forcing conditions

$$(w, c, \mathcal{P})$$

where w is a countable subset of λ , $c : [w]^2 \rightarrow \{\text{red}, \text{green}\} = \{0, 1\}$ (but we write $c(\alpha, \beta)$ instead of $c(\{\alpha, \beta\})$), and \mathcal{P} is a countable family of pairs (a, b) such that

- (i) a, b are subsets of w
- (ii) $b \in \mathcal{R}_{<\omega_1}$ and a is a finite union of members of $\mathcal{R}_{<\omega_1}$
- (iii) $\sup(a) < \min(b)$
- (iv) If $\sup(a) \leq \gamma < \min(b)$, $\gamma \in w$, then $c(\gamma, \cdot)$ divides a or b into two infinite sets.

We use the notation

$$p = (w^p, c^p, \mathcal{P}^p)$$

for $p \in \mathbb{P}$. The ordering of the conditions is defined as follows:

$$p \leq q \iff w^p \subseteq w^q \ \& \ c^p \subseteq c^q \ \& \ \mathcal{P}^p \subseteq \mathcal{P}^q.$$

Let

$$\mathcal{C} = \bigcup \{c^p : p \in \mathbb{G}_{\mathbb{P}}\}.$$

Fact A. \mathbb{P} is \aleph_2 -complete.

Proof. Trivial—take the union. \square

Fact B. For $\gamma < \lambda$, $\{q \in \mathbb{P} : \gamma \in w^q\}$ is open dense.

Proof. Let $p \in \mathbb{P}$. If $\gamma \in w^p$, we are done. Otherwise we define q as follows: $w^q = w^p \cup \{\gamma\}$, $\mathcal{P}^q = \mathcal{P}^p$, $c^q \upharpoonright w^p = c^p$ and $c^q(\gamma, \cdot)$ is defined so that if $(a, b) \in \mathcal{P}^q$, then $c^q(\gamma, \cdot)$ divides a and b into two infinite sets. \square

Fact C. $\Vdash_{\mathbb{P}} "2^{\aleph_1} \geq \lambda \text{ and } \mathcal{C} : [\lambda]^2 \rightarrow \{\text{red, green}\}"$

Proof. The second phrase follows from Fact B. For the first phase, define $\rho_\alpha \in {}^{\omega_1}2$, for $\alpha < \lambda$, by: $\rho_\alpha(i) = \mathcal{C}(0, \alpha + i)$. Easily

$$\Vdash_{\mathbb{P}} "\rho_\alpha \in {}^{\omega_1}2 \text{ and for } \alpha < \beta < \lambda, \rho_\alpha \neq \rho_\beta"; \text{ so } \Vdash_{\mathbb{P}} "2^{\aleph_1} \geq \lambda".$$

□

Fact D. \mathbb{P} satisfies the \aleph_2 -c.c.

Proof. Suppose $p_i \in \mathbb{P}$ for $i < \aleph_2$. For each i choose a countable family \mathcal{A}^i of subsets of w^{p_i} such that $\mathcal{A}^i \subseteq \mathcal{R}_{<\omega_1}$ and $(a, b) \in \mathcal{P}^{p_i}$ implies $b \in \mathcal{A}^i$ and a is a finite union of members of \mathcal{A}^i . For each $\gamma \in c \in \mathcal{A}^i$ choose a function $F_{\gamma, c}^i$ s.t. $F_{\gamma, c}^i(c \setminus (\gamma + 1)) = \gamma$. Let v_i be the closure of w_i (in the order topology).

We may assume that $\langle v_i : i < \omega_2 \rangle$ is a Δ -system (we have CH) and that $\text{otp}(v_i)$ is the same for all $i < \omega_2$. W.l.o.g. for $i < j$ the unique order-preserving function $h_{i, j}$ from v_i onto v_j maps p_i onto p_j , $w^{p_i} \cap w^{p_j} = w^{p_0} \cap w^{p_1}$ onto itself, and

$$F_{\gamma, c}^i = F_{h_{i, j}(\gamma), h_{i, j}c}^j$$

for $\gamma \in c \in \mathcal{A}^i$ (remember: \mathbb{B} has $2^{\aleph_0} = \aleph_1$ functions only). Hence

$$\otimes_1 \quad h_{i, j} \text{ is the identity on } v_i \cap v_j \text{ for } i < j.$$

Clearly by the definition of $\mathcal{R}_{<\omega_1}$ and the condition on $F_{\gamma, c}^i$:

$$\otimes_2 \quad \text{If } a \in \mathcal{A}^i, i \neq j \text{ and } a \not\subseteq w^{p_i} \cap w^{p_j}, \\ \text{then } a \setminus (w^{p_i} \cap w^{p_j}) \text{ is infinite.}$$

We define q as follows.

$$w^q = w^{p_0} \cup w^{p_1}.$$

$$\mathcal{P}^q = \mathcal{P}^{p_0} \cup \mathcal{P}^{p_1}.$$

c^p extends c^{p_0} and c^{p_1} in such a way that, for $e \in \{0, 1\}$,

(*) for every $\gamma \in w^{p_e} \setminus w^{p_{1-e}}$ and every $a \in \mathcal{A}^{1-e}$, $w^p(\gamma, \cdot)$ divides a into two infinite parts, provided that

(**) $a \setminus w^{p_e}$ is infinite.

This is easily done and $p_0 \leq q, p_1 \leq q$, provided that $q \in \mathbb{P}$. For this the problematic part is c^q and, in particular, part (iv) of the definition of \mathbb{P} . So suppose $(a, b) \in \mathcal{P}^q$, e.g., $(a, b) \in \mathcal{P}^{p_0}$. Suppose also $\gamma^* \in w^q$ so that $\text{sup}(a) \leq \gamma^* < \text{sup}(b)$. If $\gamma^* \in w^{p_0}$, there is no problem, as $p_0 \in \mathbb{P}$. So let us assume $\gamma^* \in w^q \setminus w^{p_0} = w^{p_1} \setminus w^{p_0}$. If $a \setminus w^{p_1}$ or $b \setminus w^{p_1}$ is infinite, we are through in view of condition (*) in the definition of c^q . Let us finally assume $a \setminus w^{p_1}$ is finite. But $a \subseteq w^{p_0}$. Hence $a \setminus (w^{p_0} \cap w^{p_1})$ is finite and \otimes_2 implies it is empty, i.e. $a \subseteq w^{p_0} \cap w^{p_1}$. Similarly, $b \subseteq w^{p_0} \cap w^{p_1}$. So $h_{0,1} \upharpoonright (a \cup b)$ is the identity. But $(a, b) \in \mathcal{P}^{p_0}$. But $h_{i,j}$ maps p_i onto p_j . Hence $(a, b) \in \mathcal{P}^{p_1}$. As $p_1 \in \mathbb{P}$, we get the desired conclusion. □

Fact E. $\Vdash_{\mathbb{P}}$ “There is no c -monochromatic subset of λ of order-type $\omega_1 \times \omega$.”

Proof. Let p force the existence of a counterexample. Let G be \mathbb{P} -generic over V with $p \in G$. In $V[G]$ we can find $A \subseteq \lambda$ of order-type $\omega_1 \times \omega$ such that $c \upharpoonright [A]^2$ is constant. Let $A = \bigcup_{n < \omega} A_n$ where $\text{otp}(A_n) = \omega_1$ and $\text{sup}(A_n) \leq \min(A_{n+1})$. We can replace A_n by any $A'_n \subseteq A_n$ of the same cardinality. Hence we may assume w.l.o.g.

$$(*)_1 \quad A_n \in \mathcal{R}_{\omega_1} \quad \text{for } n < \omega.$$

Let $\delta_n = \text{sup}(A_n)$ and

$$\beta_n = \min\{ \beta : \delta \leq \beta < \lambda, d(\beta, \cdot) \text{ does not} \\ \text{divide } \bigcup_{l \leq n} A_l \text{ into two infinite sets} \},$$

where $d = \underline{c}^G$. Clearly $\beta_n \leq \min(A_{n+1})$. Hence $\beta_n < \beta_{n+1}$. Let $d_n \in \{0, 1\}$ be such that $d(\beta_n, \gamma) = d_n$ for all but finitely many $\gamma \in \bigcup_{l \leq n} A_l$. Let u be an infinite subset of ω such that $\{ \beta_n : n \in u \} \in \mathcal{R}_\omega$. Let $A_l = \{ \alpha_i^l : i < \omega_1 \}$ in increasing order. So p forces all this on suitable names

$$\langle \beta_n : n < \omega \rangle, \langle \alpha_i^l : i < \omega_1 \rangle, \langle \delta_n : n < \omega \rangle.$$

As \mathbb{P} is \aleph_1 -complete, we can find $p_0 \in \mathbb{P}$ with $p \leq p_0$ so that p_0 forces $\beta_l = \beta_l$ and $\delta_n = \delta_n$ for some β_l and δ_n . We can choose inductively conditions $p_k \in \mathbb{P}$ such that $p_k \leq p_{k+1}$ and there are $i_k < j_k$ and α_i^l (for $i < j_k$) with

$$p_{k+1} \Vdash \text{“} \alpha_{i_k}^l > \text{sup}\{ i : \alpha_i^l \in w^{p_k} \}, \\ \alpha_i^l \in w^{p_{k+1}} \text{ for } i < j_k, \\ \{ \alpha_i^l : i < i_k \} \subseteq \text{cl}_{\mathbb{B}}\{ \alpha_i^l : i_k < i < j_k \}, \\ \alpha_i^l = \alpha_i^l \text{ for } i < j_k \text{ and} \\ \gamma \in [\delta_m, \beta_m) \cap w^{p_k} \text{ implies } \underline{c}(\gamma, \cdot) \\ \text{divides } \{ \alpha_i^l : i < j_k, l \leq m \} \text{ into} \\ \text{two infinite sets.} \text{”}$$

(remember our choice of β_m). Let

$$l(*) = \min(u) \\ a = \{ \alpha_i^l : l \leq l(*), i < \bigcup_k j_k \} \\ b = \{ \beta_l : l \in u \} \\ q = \left(\bigcup_k w^{p_k}, \bigcup_k c^{p_k}, \bigcup_k \mathcal{P}^{p_k} \cup \{(a, b)\} \right).$$

Now $q \in \mathbb{P}$. To see that q satisfies condition (iv) of the definition of \mathbb{P} , let $\sup(a) \leq \gamma < \min(b)$. Then $\sup\{\alpha_{i_k}^{l(*)} : k < \omega\} \leq \gamma < \beta_{l(*)}$. But $\gamma \in w^p = \bigcup_k w^{p_k}$, so for some k , $\gamma \in w^{p_k}$. This implies

$$\gamma \notin \left(\alpha_{i_{k+1}}^{l(*)}, \delta_{l(*)} \right),$$

whence $\gamma \geq \delta_{l(*)}$ and

$$\{\alpha_i^l : l \leq l(*), i < j_k\} \subseteq a,$$

which implies the needed conclusion.

Also $q \geq p_k \geq p$. But now, if $r \geq q$ forces a value to $\alpha_{\bigcup_k j_k}^{l(*)}$; we get a contradiction. \square

Remark 1A. Note that the proof of Theorem 1 also gives the consistency of $\lambda \not\vdash [\omega_1 \times \omega]_{\aleph_0}^2$: replace “ $c(\gamma, \cdot)$ divides a set x into two infinite parts” by “ $c(\gamma, \cdot)$ gets all values on a set x .”

3. Generalizations to other cardinals

How much does the proof of Theorem 1 depend on \aleph_1 ? Suppose we replace \aleph_0 by μ .

Theorem 2. Assume $2^\mu = \mu^+ < \lambda = \lambda^\mu$ and $2 \leq \kappa \leq \mu$. Then for some μ^+ -complete μ^{++} -c.c. forcing notion \mathbb{P} of cardinality 2^μ :

$$\Vdash_{\mathbb{P}} 2^\mu = \lambda, \quad \lambda \not\vdash [\mu^+ \times \mu]_\kappa^2.$$

Proof. Let \mathbb{B} and \mathcal{R}_δ be defined as above (for $\delta \leq \mu^+$). Clearly

\oplus If $a \subseteq \lambda$ has no last element, then for some $\alpha \in a$, $a \setminus \alpha \in \bigcup_\delta \mathcal{R}_\delta$.

Hence, if $\delta = \text{otp}(a)$ is additively indecomposable, then $a \setminus \alpha \in \mathcal{R}_\delta$ for some $\alpha \in a$.

Let \mathbb{P}_μ be the set of forcing conditions

$$(w, c, \mathcal{P})$$

where $w \subseteq \lambda$, $|w| \leq \mu$, $c : [w]^2 \rightarrow \{\text{red}, \text{green}\}$, and \mathcal{P} is a set of $\leq \mu$ pairs (a, b) such that

- (i) a, b are subsets of w .
- (ii) $b \in \mathcal{R}_\mu$, and a is a finite union of members of $\bigcup_{\mu \leq \delta < \mu^+} \mathcal{R}_\delta$.
- (iii) $\sup(a) < \min(b)$.
- (iv) If $\sup(a) \leq \gamma < \min(b)$, $\gamma \in w$, then the function $c(\gamma, \cdot)$ gets all values ($< \kappa$) on a or on b .

With the same proof as above we get

\mathbb{P}_μ satisfies the μ^{++} -c.c.,

\mathbb{P}_μ is μ^+ -complete,

(so cardinal arithmetic is clear) and

$$\Vdash_{\mathbb{P}_\mu} \lambda \not\vdash [\mu \times \mu]_\kappa^2.$$

\square

What about replacing μ^+ by an inaccessible θ ? We can manage by demanding

$$\{ a \cap (\alpha, \beta) : (a, b) \in \mathcal{P}, \bigcup_n \text{otp}(a \cap (\alpha, \beta)) \times n = \text{otp}(a) \\ (\alpha, \beta) \text{ maximal under these conditions} \}$$

is free (meaning there are pairwise disjoint end segments) and by taking care in defining the order. Hence the completeness drops to θ -strategical completeness. This is carried out in Theorem 3 below.

Theorem 3. Assume $\theta = \theta^{<\theta} > \aleph_0$ and $\lambda = \lambda^{<\theta}$. Then for some θ^+ -c.c. θ -strategically complete forcing \mathbb{P} , $|\mathbb{P}| = \lambda$ and

$$\Vdash_{\mathbb{P}} 2^\theta = \lambda, \lambda \not\rightarrow (\theta \times \theta)_2^2.$$

Proof. For W a family of subsets of λ , each with no last element, let

$$\text{Fr}(W) = \{ f : f \text{ is a choice function on } W \text{ s.t.} \\ \{ a \setminus f(a) : a \in W \} \text{ are pairwise disjoint} \}.$$

If $\text{Fr}(W) \neq \emptyset$, W is called *free*.

Let $\mathbb{P}_{<\theta}$ be the set of forcing conditions

$$(w, c, \mathcal{P}, W)$$

where $w \subseteq \lambda$, $|w| < \theta$, $c : [w]^2 \rightarrow \{\text{red}, \text{green}\}$, W is a free family of $< \theta$ subsets of w , each of which is in $\bigcup_{\delta < \theta} \mathcal{R}_\delta$, and \mathcal{P} is a set of $< \theta$ pairs (a, b) such that

- (i) a, b are subsets of w .
- (ii) $b \in \mathcal{R}_w$.
- (iii) $\sup(a) < \min(b)$ and for some $\delta_0 < \delta_1 < \dots < \delta_n$, $\delta_0 < \min(a)$, $\sup(a) \leq \delta_n$, $a \cap [\delta_l, \delta_{l+1}) \in W$.
- (iv) If $\sup(a) \leq \gamma < \min(b)$, $\gamma \in w$, then $c(\gamma, \cdot)$ divides a or b into two infinite sets.

We order $\mathbb{P}_{<\theta}$ as follows:

$$p \leq q \text{ iff } w^p \subseteq w^q, c^p \subseteq c^q, \mathcal{P}^p \subseteq \mathcal{P}^q, W^p \subseteq W^q \\ \text{and every } f \in \text{Fr}(W^p) \text{ can be extended} \\ \text{to a member of } \text{Fr}(W^q).$$

□

4. A provable partition relation

Claim 4. Suppose $\theta > \aleph_0$, $n, r < \omega$ and $\lambda = \lambda^{<\theta}$. Then

$$(\lambda^+)^r \times n \rightarrow (\theta \times n, \theta \times r)_2^2.$$

Proof. We prove this by induction on r . Clearly the claim holds for $r = 0, 1$. So w.l.o.g. we assume $r \geq 2$. Let c be a 2-place function from $(\lambda^+)^r \times n$ to $\{\text{red}, \text{green}\}$. Let $\chi = \text{beth}_2(\lambda)^+$. Choose by induction on l a model N_l such that

$$N_l \prec (H(\chi), \in, <^*),$$

$|N_l| = \lambda$, $\lambda + 1 \subseteq N_l$, $N_l^{<\theta} \subseteq N_l$, $c \in N_l$ and $N_l \in N_{l+1}$. Here $<^*$ is a well-ordering of $H(\chi)$. Let

$$A_l = [(\lambda^+)^r \times l, (\lambda^+)^r \times (l + 1)],$$

and let $\delta_l \in A_l \setminus N_l$ be such that $\delta_l \notin x$ whenever $x \in N_l$ is a subset of A_l and $\text{otp}(x) < (\lambda^+)^{r-1}$. W.l.o.g. we have $\delta_l \in N_{l+1}$. Now we shall show

- (*) If $Y \in N_0$, $Y \subseteq A_m$, $|Y| = \lambda^+$ and $\delta_m \in Y$,
then we can find $\beta \in Y$ such that $c(\beta, \delta_m) = \text{red}$.

Why () suffices?* Assume (*) holds. We can construct by induction on $i < \theta$ and for each i by induction on $l < n$ an ordinal $\alpha_{i,l}$ s.t.

- (a) $\alpha_{i,l} \in A_l$ and $j < i \Rightarrow \alpha_{j,l} < \alpha_{i,l}$.
- (b) $\alpha_{i,l} \in N_0$.
- (c) $c(\alpha_{i,l}, \delta_m) = \text{red}$ for $m < n$.
- (d) $c(\alpha_{i,l}, \alpha_{i_1,l_1}) = \text{red}$ when $i_1 < i$ or $i_1 = i \ \& \ l_1 < l$.

Accomplishing this suffices as $\alpha_{i,l} \in A_l$ and

$$l < m \Rightarrow \sup A_l \leq \min A_m.$$

Arriving in the inductive process at (i, l) , let

$$Y = \{\beta \in A_l : c(\beta, \alpha_{j,m}) = \text{red} \text{ if } j < i, m < n, \text{ or } j = i, m < l\}.$$

Now clearly $Y \subseteq A_\lambda$. Also $Y \in N_0$ as all parameters are from N_0 , their number is $< \theta$ and $N_0^{<\theta} \subseteq N_0$. Also $\delta_l \in Y$ by the induction hypothesis (and $\delta_l \in A_l$). So by (*) we can find $\alpha_{i,l}$ as required.

Proof of (*): $Y \not\subseteq N_0$, because $\delta_m \in Y$ and $Y \in N_0$. As $|Y| = \lambda^+$, we have $\text{otp}(Y) \geq \lambda^+$. But $\lambda^+ \rightarrow (\lambda^+, \theta)^2$, so there is $B \subseteq A_m$ s.t. $|B| = \lambda^+$ and $c \upharpoonright B \times B$ is constantly red or there is $B \subseteq A_m$ s.t. $|B| = \theta$ and $c \upharpoonright B \times B$ is constantly green. In the former case we get the conclusion of the claim. In the latter case we may assume $B \in N_0$, hence $B \subseteq N_0$, and let $k \leq n$ be maximal s.t.

$$B' = \{\xi \in B : \bigwedge_{l < k} c(\delta_l, \xi) = \text{red}\}$$

has cardinality θ . If $k = n$, any member of B' is as required in (*). So assume $k < n$. Now $B' \in N_k$, since $B \in N_0 \prec N_k$ and $\{\delta_l, A_l\} \in N_k$ and $\delta_l \in N_k$ for $l < k$. Also

$$\{\xi \in B' : c(\delta_k, \xi) = \text{red}\}$$

is a subset of B' of cardinality $< \theta$ by the choice of k . So for some $B'' \in N_0$, $c \upharpoonright \{\delta_k\} \times (B' \setminus B'')$ is constantly green (e.g., as $B' \subseteq N_0$, and $N_0^{<\theta} \subseteq N_0$). Let

$$Z = \{ \delta \in A_k : c \upharpoonright \{\delta\} \times (B' \setminus B'') \text{ is constantly green} \}$$

and

$$Z' = \{ \delta \in Z : (\forall \alpha \in B' \setminus B'')(\delta < \alpha \Leftrightarrow \delta_k < \alpha) \}.$$

So $Z \subseteq A_k$, $Z \in N_k$, $\delta_k \notin N_k$ and therefore $\text{otp}(Z) = \text{otp}(A_k) = (\lambda^+)^r$. Note that $k \neq l \Rightarrow Z' = Z$ and $k = l \Rightarrow Z' = Z \setminus \sup(B' \setminus B'')$, so Z' has the same properties. Now we apply the induction hypothesis: one of the following holds (note that we can interchange the colours): (a) there is $Z'' \subseteq Z'$, $\text{otp}(Z'') = \theta \times n$, $c \upharpoonright Z'' \times Z''$ is constantly red, wlog $Z'' \in N_k$, or (b) there is $Z'' \subseteq Z'$, $\text{otp}(Z'') = \theta \times (r-1)$, $c \upharpoonright Z'' \times Z''$ green and wlog $Z'' \in N_k$. If (a), we are done; if (b), $Z'' \cup (B' \setminus B'')$ is as required. \square

Remark 4A. So $(\lambda^+)^{n+1} \rightarrow (\theta \times n)^2$ for $\lambda = \lambda^{<\theta}$, $\theta = \text{cf}(\theta) > \aleph_0$ (e.g., $\lambda = 2^{<\theta}$).

Remark 4B. Suppose $\lambda = \lambda^{<\theta}$, $\theta > \aleph_0$. If c is a 2-colouring of $(\lambda^{+r})^s \times n$ by k colours and every subset of it of order type $(\lambda^{+(r-1)})^s \times n$ has a monochromatic subset of order type θ for each of the colours, one of the colours being red, then by the last proof we get

- (a) There is a monochromatic subset of order type $\theta \times n$ and of colour red or
- (b) There is a colour d and a set Z of order type $(\lambda^{+r})^s$ and a set B of order type θ s.t. $B < Z$ or $Z < B$ and

$$\{ (\alpha, \beta) : \alpha \in B, \beta \in Z \text{ or } \alpha \neq \beta \in B \}$$

are all coloured with d .

So we can prove that for 2-colourings by k colours c

$$(\lambda^{+r})^s \times n \rightarrow (\theta \times n_1, \dots, \theta \times n_k)^2$$

when r, s, n are sufficiently large (e.g., $n \geq \min\{n_l : l = 1, \dots, k, s \geq \sum_{l=1}^k n_l\}$) by induction on $\sum_{l=1}^k n_l$.

Note that if c is a 2-colouring of λ^{+2k} , then for some $l < k$ and $A \subseteq \lambda^{+2k}$ of order type $\lambda^{+(2l+2)}$ we have

- (*) If $A' \subseteq A$, $\text{otp}(A') = \lambda^{+2l}$, and d is a colour which appears in A , then there is $B \subseteq A'$ of order type θ s.t. B is monochromatic of colour d .

We can conclude $\lambda^{+2k} \rightarrow (\theta \times n)_k^2$.

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Institute of Mathematics
The Hebrew University
Jerusalem
Israel

Department of Mathematics
Rutgers University
New Brunswick, NJ
USA