

FURTHER CARDINAL ARITHMETIC

BY

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ABSTRACT

We continue the investigations in the author’s book on cardinal arithmetic, assuming some knowledge of it. We deal with the cofinality of $(\mathcal{S}_{\leq \aleph_0}(\kappa), \subseteq)$ for κ real valued measurable (Section 3), densities of box products (Section 5,3), prove the equality $\text{cov}(\lambda, \lambda, \theta^+, 2) = \text{pp}(\lambda)$ in more cases even when $\text{cf}(\lambda) = \aleph_0$ (Section 1), deal with bounds of $\text{pp}(\lambda)$ for λ limit of inaccessible (Section 4) and give proofs to various claims I was sure I had already written but did not find (Section 6).

Annotated Contents

- 1. Equivalence of two covering properties 63
 [We try to characterize when, say, λ has few countable subsets; for a given $\theta \in (\aleph_0, \lambda)$, we try to translate to expressions with pcf’s the cardinal

$$\text{Min} \left\{ |\mathcal{P}| : \mathcal{P} \subseteq \mathcal{S}_{< \mu}(\lambda) \text{ and every } a \in \mathcal{S}_{\leq \theta}(\lambda) \text{ is } \bigcup_{n < \omega} a_n, \text{ such that every } b \in \bigcup_n \mathcal{S}_{\leq \aleph_0}(a_n) \text{ is included in a member of } \mathcal{P} \right\}.$$

This continues and improves [Sh410,§6].

- 2. Equality relevant to weak diamond 70

* Done mainly 1–4/1991.

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[We show that if $\mu > \lambda \geq \kappa$, $\theta = \text{cov}(\mu, \lambda^+, \lambda^+, \kappa)$ and $\text{cov}(\lambda, \kappa, \kappa, 2) \leq \mu$ (or $\leq \theta$), then $\text{cov}(\mu, \lambda^+, \lambda^+, 2) = \text{cov}(\theta, \kappa, \kappa, 2)$. This is used in [Sh-f, Appendix, §1] to clarify the conditions for the holding of versions of the weak diamond.]

3. Cofinality of $\mathcal{S}_{\leq \aleph_0}(\kappa)$ for κ real valued measurable and trees 72

[Dealing with partition theorems on trees, Rubin–Shelah [RuSh117] arrive at the statement: $\lambda > \kappa > \aleph_0$ are regular, $a_\alpha \in \mathcal{S}_{< \kappa}(\mu)$, $\mu < \lambda$; can we find unbounded $W \subseteq \lambda$ such that $|\bigcup_{\alpha \in W} a_\alpha| < \kappa$? Of course, $\bigwedge_{\alpha < \lambda} \text{cov}(\alpha, \kappa, \kappa, 2) < \lambda$ suffice, but is it necessary? By 3.1, yes. Then we answer a problem of Fremlin: e.g. if κ is a real valued measurable cardinal then the cofinality of $(\mathcal{S}_{\leq \aleph_0}(\kappa), \subseteq)$ is κ . Lastly we return to the problem of the existence of trees with many branches (3.3, 3.4).]

4. Bounds for $\text{pp}_{\Gamma(\aleph_1)}$ for limits of inaccessibles 79

[Unfortunately, our results need an assumption: $\text{pcf}(\mathfrak{a})$ does not have an inaccessible accumulation point ($|\mathfrak{a}| < \text{Min } \mathfrak{a}$, $\mathfrak{a} \subseteq \text{Reg}$, of course). Our main conclusion (4.3) is that e.g. if $\langle \lambda_\zeta : \zeta < \omega_4 \rangle$ is the list of the first \aleph_4 inaccessibles then $\text{pp}_{\Gamma(\aleph_1)}\left(\bigcup_{\zeta < \omega_1} \lambda_\zeta\right) < \bigcup_{\zeta < \omega_4} \lambda_\zeta$. This does not follow from the proof of $\text{pp } \aleph_\omega < \aleph_{\omega_4}$ [Sh400, §2], nor do we make our life easier by assuming “ $\bigcup_{\zeta < \omega_1} \lambda_\zeta$ is strong limit”. We indeed in the end quote a variant of [Sh400, §2] (= [Sh410, 3.5]). But the main point now is to arrive at the starting point there: show that for $\delta < \omega_4$, $\text{cf } \delta = \aleph_2$, for some club C of δ , $\text{supp } \text{pcf}_{\aleph_2\text{-complete}}(\{\lambda_\zeta : \zeta \in C\}) \leq \lambda_\delta$. This is provided by 4.2.]

5. Densities of box products* 85

[The behavior of the Tichonov product of topological spaces on densities is quite well understood for ${}^\mu 2$: it is $\text{Min}\{\lambda : 2^\lambda \geq \mu\}$; but less so for the generalization to box products. Let $\mathcal{T}_{\mu, \theta, \kappa}$ be the space with set of points ${}^\mu \theta$, and basis $\{[f] : f \text{ a partial function from } \mu \text{ to } \theta \text{ of cardinality } < \kappa\}$, where $[f] = \{g \in {}^\mu \theta : f \subseteq g\}$. If $\theta \leq \lambda = \lambda^{< \kappa}$, $2^\lambda \geq \mu$ the situation is similar to the Tichonov product. Now the characteristic unclear case is μ strong limit singular of cofinality $< \kappa$, $\theta = 2$, $2^\mu > \mu^+$. We prove that the density is “usually” large (2^μ), i.e. the failure quite limits the cardinal arithmetic involved (we can prove directly consistency results but what we do seems more informative).]

6. Odds and ends 90

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Notation: Let $J_\lambda[\mathfrak{a}]$ be $\{\mathfrak{b} \subseteq \mathfrak{a} : \lambda \notin \text{pcf}(\mathfrak{b})\}$, equivalently $J_{< \lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda[\mathfrak{a}]$. See more in [Sh513], [Sh589].

* There is a paper in preparation on independence results by Gitik and Shelah.

1. Equivalence of Two Covering Properties

1.1 CLAIM: *If $\text{pp } \lambda = \lambda^+$, $\lambda > \text{cf}(\lambda) = \kappa > \aleph_0$ then $\text{cov}(\lambda, \lambda, \kappa^+, 2) = \lambda^+$.*

Proof: Let $\chi = \beth_3(\lambda)^+$; choose $\langle \mathfrak{B}_\zeta : \zeta < \lambda^+ \rangle$ increasing continuous, such that $\mathfrak{B}_\zeta \prec (H(\chi), \in, <^*_\chi)$, $\lambda + 1 \subseteq \mathfrak{B}_\zeta$, $\|\mathfrak{B}_\zeta\| = \lambda$ and $\langle \mathfrak{B}_\xi : \xi \leq \zeta \rangle \in \mathfrak{B}_{\zeta+1}$. Let $\mathfrak{B} =: \bigcup_{\zeta < \lambda^+} \mathfrak{B}_\zeta$ and $\mathcal{P} =: \mathcal{S}_{<\lambda}(\lambda) \cap \mathfrak{B}$. Let $a \in \mathcal{S}_{\leq \kappa}(\lambda)$; it suffices to prove $(\exists A \in \mathcal{P})[a \subseteq A]$. Let f_ξ be the $<^*_\chi$ -first $f \in \prod(\text{Reg} \cap \lambda)$ such that $(\forall g)[g \in \prod(\text{Reg} \cap \lambda) \& g \in \mathfrak{B}_\zeta \Rightarrow g < f \text{ mod } J_\lambda^{bd}]$, such f exists as $\prod(\text{Reg} \cap \lambda)/J_\lambda^{bd}$ is λ^+ -directed.

By [Sh420, 1.5, 1.2] we can find $\langle C_\alpha : \alpha < \lambda^+ \rangle$ such that: C_α is a closed subset of α , $\text{otp } C_\alpha \leq \kappa^+$, $[\beta \in \text{nacc } C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta]$ and $S =: \{\delta < \lambda^+ : \text{cf}(\delta) = \kappa^+ \text{ and } \delta = \sup C_\delta\}$ is stationary.

Without loss of generality $\bar{C} \in \mathfrak{B}_0$.

Now we define for every $\alpha < \lambda^+$ elementary submodels N_α^0, N_α^1 of \mathfrak{B} :

N_α^0 is the Skolem Hull of $\{f_\zeta : \zeta \in C_\alpha\} \cup \{i : i \leq \kappa\}$ and N_α^1 is the Skolem Hull of $a \cup \{f_\zeta : \zeta \in C_\alpha\} \cup \{i : i \leq \kappa\}$, both in $(H(\chi), \in, <^*_\chi)$.

Clearly:

- (a) $N_\alpha^0 \subseteq N_\alpha^1 \subseteq \mathfrak{B}_\alpha \subseteq \mathfrak{B}$ [why? as $f_\zeta \in \mathfrak{B}_{\zeta+1}$ because $\mathfrak{B}_\zeta \in \mathfrak{B}_{\zeta+1}$],
- (b) $\|N_\alpha^\ell\| \leq \kappa + \|C_\alpha\|$,
- (c) $N_\alpha^0 \in \mathfrak{B}_{\alpha+1}$.

[Why? As $\alpha \subseteq \mathfrak{B}_\alpha$ (you can prove it by induction on α) clearly $\alpha \in \mathfrak{B}_{\alpha+1}$, but $\bar{C} \in \mathfrak{B}_0 \subseteq \mathfrak{B}_{\alpha+1}$; hence $C_\alpha \in \mathfrak{B}_{\alpha+1}$, also $\langle \mathfrak{B}_\gamma : \gamma \leq \alpha \rangle \in \mathfrak{B}_{\alpha+1}$ hence $\langle f_\gamma : \gamma \leq \alpha \rangle \in \mathfrak{B}_{\alpha+1}$, hence $\langle f_\gamma : \gamma \in C_\alpha \rangle \in \mathfrak{B}_{\alpha+1}$. Now $N_\alpha^0 \subseteq \mathfrak{B}_\alpha \in \mathfrak{B}_{\alpha+1}$ and the Skolem Hull can be computed in $\mathfrak{B}_{\alpha+1}$.]

- (d) for each α with $\kappa^+ > \text{otp}(C_\alpha)$, for some $\gamma_\alpha < \lambda^+$, letting $\mathfrak{a}_\alpha =: N_\alpha^0 \cap \text{Reg} \cap \lambda \setminus \kappa^{++}$ clearly $\text{Ch}_\alpha \in \prod \mathfrak{a}_\alpha$ where $\text{Ch}_\alpha(\theta) =: \sup(\theta \cap N_\alpha^1)$, and we have: $\text{Ch}_\alpha < f_{\gamma_\alpha} \upharpoonright \mathfrak{a}_\alpha \text{ mod } J_{\mathfrak{a}_\alpha}^{bd}$.

[Why? $\mathfrak{a}_\alpha \in \mathfrak{B}_{\alpha+1}$ as $N_\alpha^0 \in \mathfrak{B}_{\alpha+1}$, and $\prod \mathfrak{a}_\alpha / J_{\mathfrak{a}_\alpha}^{bd}$ is λ^+ -directed (trivially) and has cofinality $\leq \max \text{pcf}_{J_{\mathfrak{a}_\alpha}^{bd}}(\mathfrak{a}_\alpha) \leq \text{pp}(\lambda) = \lambda^+$, so there is $\langle f_\beta^{\mathfrak{a}_\alpha} : \beta < \lambda^+ \rangle$, $<_{J_{\mathfrak{a}_\alpha}^{bd}}$ -increasing cofinal sequence in $\prod \mathfrak{a}_\alpha$, so without loss of generality $\langle f_\beta^{\mathfrak{a}_\alpha} : \beta < \lambda^+ \rangle \in \mathfrak{B}_{\alpha+1}$; also by the ‘‘cofinal’’ above, for some $\beta \in (\alpha, \lambda^+)$, $\text{Ch}_\alpha < f_\beta^{\mathfrak{a}_\alpha} \text{ mod } J_{\mathfrak{a}_\alpha}^{bd}$. We can use the minimal β , now obviously $\beta \in \mathfrak{B}_{\beta+1}$ so $f_\beta^{\mathfrak{a}_\alpha} \in \mathfrak{B}_{\beta+1}$, hence $f_\beta^{\mathfrak{a}_\alpha} < f_{\beta+2} \text{ mod } J_\lambda^{bd}$. Together $\gamma_\alpha =: \beta + 2$ is as required.]

- (d)⁺ for each α with $\text{otp}(C_\alpha) < \kappa^+$ for some $\gamma_\alpha \in (\alpha, \lambda^+)$, for any $\mu \in \text{Reg} \cap N_\alpha^0$, letting $N_\alpha^{0,\mu} =: \text{Ch}_{\mathfrak{B}_\alpha}(N_\alpha^0 \cup \mu)$, $\mathfrak{a}_{\alpha,\mu} = N_\alpha^{0,\mu} \cap \text{Reg} \cap \lambda \setminus \mu^+$ and $\text{Ch}_{\alpha,\mu} \in$

$\Pi a_{\alpha,\mu}$ be

$$\text{Ch}_{\alpha,\mu}(\theta) = \begin{cases} \sup(\theta \cap N_{\alpha}^1) & \text{if } \theta \in N_{\alpha}^1, \\ 0 & \text{otherwise,} \end{cases}$$

we have: $\text{Ch}_{\alpha} < f_{\gamma_{\alpha}} \upharpoonright a_{\alpha,\mu} \text{ mod } J_{a_{\alpha,\mu}}^{bd}$.

[Why? Clearly $\text{Ch}_{\mathfrak{B}_{\alpha}}(N_{\alpha}^0 \cup \mu) \in \mathfrak{B}_{\alpha+1}$, so $a_{\alpha,\mu} \in \mathfrak{B}_{\alpha+1}$, hence there are in $\mathfrak{B}_{\alpha+1}$ elements $\langle b_{\theta}[a_{\alpha,\mu}]: \theta \in \text{pcf}(a_{\alpha,\mu}) \rangle$ and $\langle \langle f_{\alpha}^{a_{\alpha,\mu},\theta}: \alpha < \theta \rangle: \theta \in \text{pcf}(a_{\alpha,\mu}) \rangle$ as in [Sh 371, 2.6, §1]. So for some $\gamma_{\alpha,\mu} \in (\alpha, \lambda^+)$ we have $\text{Ch}_{\alpha} \upharpoonright b_{\lambda^+}[a_{\alpha,\mu}] < f_{\gamma_{\alpha}}$, so it is enough to prove $a_{\alpha,\mu} \setminus b_{\lambda^+}[a_{\alpha,\mu}]$ is bounded below μ but otherwise $\text{pp}(\lambda) = \lambda^+$ will be contradicted. Let $\gamma_{\alpha} = \sup\{\gamma_{\alpha,\mu}: \mu \in N_{\alpha}^0\}$.]

(e) $E^* =: \{\delta < \lambda^+: \alpha < \delta \ \& \ |C_{\alpha}| \leq \kappa \Rightarrow \gamma_{\alpha} < \delta \text{ and } \delta > \lambda\}$ is a club of λ .

Now as S is stationary, there is $\delta(*) \in S \cap E^*$. Remember $\text{otp } C_{\delta(*)} = \kappa^+$.

Let $C_{\delta(*)} = \{\alpha_{\delta(*),\zeta}: \zeta < \kappa^+\}$ (in increasing order).

Let (for any $\zeta < \kappa^+$) M_{ζ}^0 be the Skolem Hull of $\{f_{\alpha_{\delta(*),\xi}}: \xi < \zeta\} \cup \{i: i \leq \kappa\}$, and let M_{ζ}^1 be the Skolem Hull of $a \cup \{f_{\alpha_{\delta(*),\xi}}: \xi < \zeta\} \cup \{i: i \leq \kappa\}$. Note: for $\zeta < \kappa^+$ non-limit $\{f_{\alpha_{\delta(*),\xi}}: \xi < \zeta\} = \{f_{\xi}: \xi \in C_{\alpha_{\delta(*),\zeta}}\}$. Clearly $\langle M_{\zeta}^0: \zeta < \kappa^+ \rangle, \langle M_{\zeta}^1: \zeta < \kappa^+ \rangle$ are increasing continuous sequences of countable elementary submodels of \mathfrak{B} and $M_{\zeta}^0 \subseteq M_{\zeta}^1$ and for $\zeta < \kappa^+$ a successor ordinal, $N_{\alpha_{\delta(*),\zeta}}^{\ell} = M_{\zeta}^{\ell}$.

Now for each successor ζ , for some $\epsilon(\zeta) \in (\zeta, \omega_1)$ we have $\gamma_{\alpha_{\delta(*),\zeta}} < \alpha_{\delta(*),\epsilon(\zeta)}$ (by the choice of $\delta(*)$) hence $f_{\gamma_{\alpha_{\delta(*),\zeta}}} < f_{\alpha_{\delta(*),\epsilon(\zeta)}} \text{ mod } J_{\lambda}^{bd}$ hence $\text{Ch}_{\alpha_{\delta(*),\zeta}} < f_{\alpha_{\delta(*),\epsilon(\zeta)}} \text{ mod } J_{\lambda}^{bd}$.

Let $E =: \{\delta < \omega_1: \text{for every successor } \zeta < \delta, \epsilon(\zeta) < \delta\}$, clearly E is a club of κ^+ . Let $\lambda = \sum_{i < \kappa} \lambda_i$, $\lambda_i < \lambda$ singular increasing continuous with i , wlog $\{\lambda_i: i < \kappa\} \subseteq \text{Ch}_{\mathfrak{B}}(\{i: i \leq \kappa\} \cup \{\lambda\})$. So for some $\mu_{\zeta,i} < \lambda$, we have:

$$(*) \quad \begin{aligned} i < \kappa, \quad \zeta = \xi + 1 < \kappa^+ \ \& \ \theta \in \text{Reg} \cap \lambda \setminus \mu_{\zeta,i} \ \& \ \theta \in N_{\alpha_{\delta(*),\zeta}}^{0,\lambda_i} \cap N_{\alpha_{\delta(*),\zeta}}^1 \\ \Rightarrow \sup \left(N_{\alpha_{\delta(*),\zeta}}^1 \cap \theta \right) &< f_{\alpha_{\delta(*),\epsilon(\zeta)}}(\theta) \in \theta \cap N_{\alpha_{\delta(*),\zeta+1}}^{0,\lambda_i}. \end{aligned}$$

So for some limit $i(\zeta) < \kappa^+$ we have $\lambda_{i(\zeta)} = \sup\{\mu_{\zeta,j}: j < i(\zeta)\}$. Now as cf $\lambda \leq \kappa^+$ for some $i(*) < \lambda$

$$W =: \{\zeta < \kappa^+: \zeta \text{ successor ordinal and } i(\zeta) = i(*)\}$$

is unbounded in κ^+ . So

$$\otimes \quad \begin{aligned} \text{if } \xi < \kappa^+, \ \xi \in E, \ \xi = \sup(\xi \cap W) \text{ and } \theta \in M_{\xi}^1 \text{ Reg} \cap \lambda \cap M_{\xi}^{0,\lambda_{i(*)}} \setminus \lambda_{i(*)} \\ \text{then } M_{\xi}^{0,\lambda_i} \cap \theta \text{ is an unbound subset of } M_{\xi}^1 \cap \theta. \end{aligned}$$

Hence by [Sh400] 5.1A(1), remembering $M_{\zeta+1}^0 = N_{\alpha_{\delta(*)}, \zeta+1}^0$, we have: $M_\xi^1 \subseteq$ Skolem Hull $\left[\bigcup_{\zeta < \xi} N_{\zeta+1}^0 \cup \lambda_{i(*)} \right] \subseteq$ Skolem Hull $\left(N_{\alpha_{\delta(*)}, \xi+1}^0 \cup \lambda_{i(*)} \right)$ whenever $\xi \in E$ is an accumulation point of W . But $a \subseteq M_\xi^1$ and the right side belongs to \mathfrak{B} (as we can take the Skolem Hull in $\mathfrak{B}_{\delta(*)}$). So we have finished. $\blacksquare_{1.1}$

Remark: Alternatively note: $\text{cov}(\lambda, \lambda, \kappa, 2) \leq \text{cov}(\theta, \lambda, \sigma, 2)$ when $\sigma = \text{cf}(\lambda) < \kappa < \lambda, \sigma \Rightarrow \aleph_0, \theta = \text{pp}_{\Gamma(\kappa, \sigma)}(\lambda)$; remember $\text{cf}(\lambda) < \kappa < \lambda \ \& \ \text{pp}(\lambda) < \lambda^{+\kappa^+} \Rightarrow \text{pp}_{<\lambda}(\lambda) = \text{pp}(\lambda)$.

1.2 CLAIM: For $\lambda > \mu = \text{cf}(\mu) > \theta > \aleph_0$, we have $\lambda(0) \leq \lambda(1) \leq \lambda(2) = \lambda(3)$ and if $\text{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu$ they are all equal, where:

$\lambda(0) =:$ is the minimal κ such that: if $\mathfrak{a} \subseteq \text{Reg} \cap \lambda^+ \setminus \mu, |\mathfrak{a}| \leq \theta$ then we can find $\langle \mathfrak{a}_\ell: \ell < \omega \rangle$ such that $\mathfrak{a} = \bigcup_{\ell < \omega} \mathfrak{a}_\ell$ and $(\forall \mathfrak{b}) [\mathfrak{b} \in \mathcal{S}_{\leq \aleph_0}(\mathfrak{a}_n) \Rightarrow \max \text{pcf}(\mathfrak{b}) \leq \kappa]$.

$\lambda(1) =:$ Min $\{ |\mathcal{P}|: \mathcal{P} \subseteq \mathcal{S}_{<\mu}(\lambda), \text{ and for every } A \subseteq \lambda, |A| \leq \theta \text{ there are } A_n \subseteq A (n < \omega), A = \bigcup_{n < \omega} A_n, A_n \subseteq A_{n+1} \text{ such that: for } n < \omega, \text{ every } a \in \mathcal{S}_{\leq \aleph_0}(A_n) \text{ is a subset of some member of } \mathcal{P} \}$.

$\lambda(2)$ is defined similarly to $\lambda(1)$ as:

Min $\left\{ |\mathcal{P}|: \mathcal{P} \subseteq \mathcal{S}_{<\mu}(\lambda) \text{ and for every } A \in \mathcal{S}_{\leq \theta}(\lambda) \text{ for some } A_n \subseteq A (n < \omega) \right.$
 $A = \bigcup_{n < \omega} A_n \text{ and for each } n < \omega \text{ for some } \mathcal{P}_n \subseteq \mathcal{P}, |\mathcal{P}_n| < \mu,$
 $\sup_{B \in \mathcal{P}_n} |B| < \mu \text{ and every } a \in \mathcal{S}_{\leq \aleph_0}(A_n) \text{ is a subset of some}$
 $\left. \text{member of } \mathcal{P}_n \right\}$.

$\lambda(3)$ is the minimal κ such that: if $\mathfrak{a} \subseteq \text{Reg} \cap \lambda^+ \setminus \mu, |\mathfrak{a}| \leq \theta$, then we can find $\langle \mathfrak{a}_\ell: \ell < \omega \rangle, \mathfrak{a}_\ell \subseteq \mathfrak{a}_{\ell+1} \subseteq \mathfrak{a} = \bigcup_{\ell < \omega} \mathfrak{a}_\ell$ such that: there is $\{ \mathfrak{b}_{\ell, i}: i < i_\ell < \mu \}, \mathfrak{b}_{\ell, i} \subseteq \mathfrak{a}_\ell$ such that $\max \text{pcf} \mathfrak{b}_{\ell, i} \leq \kappa$ and $(\forall \mathfrak{c}) [\mathfrak{c} \subseteq \mathfrak{a}_\ell \ \& \ |\mathfrak{c}| \leq \aleph_0 \Rightarrow \bigvee_i \mathfrak{c} \subseteq \mathfrak{b}_{\ell, i}]$; equivalently: $\mathcal{S}_{\leq \aleph_0}(\mathfrak{a}_n)$ is included in the ideal generated by $\{ \mathfrak{b}_\sigma[\mathfrak{a}_n]: \sigma \in \mathfrak{d} \}$ for some $\mathfrak{d} \subseteq \kappa^+ \cap \text{pcf} \mathfrak{a}_n$ of cardinality $< \mu$.

1.2A Remark: (1) We can get similar results with more parameters: replacing \aleph_0 and/or \aleph_1 by higher cardinals.

(2) Of course, by assumptions as in [Sh410, §6] (e.g. $|\text{pcf } \mathfrak{a}| \leq |\mathfrak{a}|$) we get $\lambda(0) = \lambda(3)$. This (i.e. Claim 1.2) will be continued in [Sh513].

Proof:

$\lambda(1) \leq \lambda(2)$: Trivial.

$\lambda(2) \leq \lambda(3)$: Let $\chi = \beth_3(\lambda(3))^+$ and for $\zeta \leq \mu^+$ we choose $\mathfrak{B}_\zeta \prec (H(\chi), \in, <_\chi)$, $\{\lambda, \mu, \theta, \lambda(2), \lambda(3)\} \in \mathfrak{B}_\zeta$, $\|\mathfrak{B}_\zeta\| = \lambda(3)$ and $\lambda(3) \subseteq \mathfrak{B}_\zeta$, \mathfrak{B}_ζ ($\zeta \leq \mu^+$) increasing continuous and $\langle \mathfrak{B}_\xi: \xi \leq \zeta \rangle \in \mathfrak{B}_{\zeta+1}$ and let $\mathfrak{B} = \mathfrak{B}_{\mu^+}$. Lastly let $\mathcal{P} = \mathfrak{B} \cap \mathcal{S}_{<\mu}(\lambda)$. Clearly

(*)₀ a function $\mathfrak{a} \mapsto \langle \mathfrak{b}_\sigma[\mathfrak{a}]: \sigma \in \text{pcf } \mathfrak{a} \rangle$ as in [Sh371, 2.6] is definable in $(H(\chi), \in, <^*_\chi)$ hence \mathfrak{B} is closed under it.

It suffices to show that \mathcal{P} satisfies the requirements in the definition of $\lambda(2)$.

Let $A \subseteq \lambda$, $|A| \leq \theta$. We choose by induction on $n < \omega$, N_n^a , (for $\ell < \omega$) and N_n^b , f_n such that:

- (a) N_n^a, N_n^b are elementary submodels of $(H(\chi), \in, <^*_\chi)$ of cardinality θ ,
- (b) $f_n \in \prod \mathfrak{a}_n$ where $\mathfrak{a}_n =: N_n^a \cap \text{Reg} \cap \lambda^+ \setminus \mu$, and $f_n(\sigma) > \sup(N_n^b \cap \sigma)$ (for any $\sigma \in \mathfrak{a}_n$),
- (c) $\theta + 1 \subseteq N_n^a \subseteq N_n^b \subseteq \mathfrak{B}$,
- (d) N_n^b is the Skolem Hull of $\bigcup \{\text{Rang } f_\ell: \ell < n\} \cup A \cup (\theta + 1)$,
- (e) N_n^a is the Skolem Hull of $\theta + 1$ in $(H(\chi), \in, <^*_\chi)$,
- (f) N_{n+1}^a is the Skolem Hull of $N_n^a \cup \text{Rang } f_n$,
- (g) there are $\mathcal{P}_{n,\ell} \subseteq \mathcal{S}_{<\mu}(\lambda + 1)$ and $A_{n,\ell} \subseteq N_n^a$ (for $\ell < \omega$) such that:
 - (α) $|\mathcal{P}_{n,\ell}| < \mu$ and $\mu_{n,\ell} =: \sup_{B \in \mathcal{P}_{n,\ell}} |B| < \mu$ and $\mathcal{P}_{n,\ell} \subseteq \mathcal{P}_{n,\ell+1}$,
 - (β) $N_n^a = \bigcup_\ell A_{n,\ell}$, $\mathcal{P}_n = \bigcup_{\ell < \omega} \mathcal{P}_{n,\ell} \subseteq \mathfrak{B}$ and $A_{n,\ell} \subseteq A_{n,\ell+1}$,
 - (γ) for every countable $a \subseteq \lambda \cap A_{n,\ell}$ there is $b \in \mathcal{P}_{n,\ell}$ satisfying $a \subseteq b$,
 - (δ) $\mathcal{P}_{n,\ell} = \mathcal{S}_{\leq \mu_{n,\ell}}(\lambda + 1) \cap (\text{Skolem Hull of } A_{n,\ell} \cup \mathcal{P}_{n,\ell} \cup (\theta + 1))$.

As in previous proofs, if we succeed to carry out the definition, then $\bigcup_n (N_n^a \cap \lambda) = \bigcup_n N_n^b \cap \lambda$, but the former is $\bigcup_{n,\ell} A_{n,\ell} \cap \lambda$, hence $A \subseteq \bigcup_n \bigcup_\ell A_{n,\ell}$, by (g)(α), (β) the $\mathcal{P}'_{n,\ell} = \{a \cap \lambda: a \in \mathcal{P}_{n,\ell}\}$ are of the right form and so by (g)(γ) we finish.

Note that without loss of generality: if $a \in \mathcal{P}_{n,\ell}$ then $a \cap \text{Reg} \cap (\lambda + 1) \setminus \mu \in \mathcal{P}_{n,\ell}$.

For $n = 0$ we can define $N_0^a, N_0^b, A_{n,\ell}$ trivially. Suppose $N_m^a, N_m^b, A_{m,\ell}, \mathcal{P}_{m,\ell}$ are defined for $m \leq n, \ell < \omega$ and f_m ($m < n$) are defined. Now \mathfrak{a}_n is well defined and $\subseteq \text{Reg} \cap \lambda^+ \setminus \mu \subseteq \mathfrak{B}$ and $|\mathfrak{a}_n| \leq \theta$. So $\mathfrak{a}_n = \bigcup_{\ell} \mathfrak{a}_{n,\ell}$ and $\mathfrak{a}_{n,\ell} \subseteq \mathfrak{a}_{n,\ell+1}$ where $\mathfrak{a}_{n,\ell} =: \mathfrak{a}_n \cap A_{n,\ell}$ and, of course, $\mathfrak{a}_{n,\ell} \subseteq \text{Reg} \cap \lambda^+ \setminus \mu$ has cardinality $\leq \theta$. Note that $\mathfrak{a}_{n,\ell}$ is not necessarily in \mathfrak{B} but

(*)₁ every countable subset of $\mathfrak{a}_{n,\ell}$ is included in some subset of \mathfrak{B} which belongs to $\mathcal{P}_{n,\ell}$ and is $\subseteq \text{Reg} \cap \lambda^+ \setminus \mu$.

By the definition of $\lambda(3)$ (see “equivalently” there), for each n, ℓ we can find an increase sequence $\langle \mathfrak{a}_{n,\ell,k} : k < \omega \rangle$ of subsets of $\mathfrak{a}_{n,\ell}$ with union $\mathfrak{a}_{n,\ell}$ and $\mathfrak{d}_{n,\ell,k} \subseteq [\mu, \lambda(3)] \cap \text{pcf}(\mathfrak{a}_{n,\ell,k}), |\mathfrak{d}_{n,\ell,k}| < \mu$ such that:

(*)₂ if $\mathfrak{b} \subseteq \mathfrak{a}_{n,\ell,k}$ is countable then \mathfrak{b} is included in a finite union of some members of $\{\mathfrak{b}_\sigma[\mathfrak{a}_{n,\ell,k}] : \sigma \in \mathfrak{d}_{n,\ell,k}\}$ (hence $\max \text{pcf}(\mathfrak{b}) \leq \lambda(3)$).

By the properties of pcf :

(*)₃ for each $\ell, k < \omega$ and $\mathfrak{c} \subseteq \text{Reg} \cap \lambda^+ \setminus \mu$ such that $\mathfrak{c} \in \mathcal{P}_{n,\ell}$ we can find $\mathfrak{e} = \mathfrak{e}_c^{\ell,k} \subseteq \lambda(3)^+ \cap \text{pcf} \mathfrak{c}, |\mathfrak{e}| \leq |\mathfrak{d}_{n,\ell,k}| < \mu$ such that for every $\sigma \in \mathfrak{d}_{n,\ell,k}$ we have: $\mathfrak{c} \cap \mathfrak{b}_\sigma[\mathfrak{a}_{n,\ell,k}]$ is included in a finite union of members of $\{\mathfrak{b}_\tau[\mathfrak{c}] : \tau \in \mathfrak{e}_c\}$.

By [Sh371, 1.4] we can find $f_n \in \prod_{\sigma \in \mathfrak{a}_n} \sigma$ such that:

(*)₄ (α) $\sup(N_n^b \cap \sigma) < f_n(\sigma)$;
 (β) if $\mathfrak{c} \in \mathcal{P}_{n,\ell}, \ell, k < \omega, \mathfrak{c} \subseteq \text{Reg} \cap \lambda^+ \setminus \mu$ and $\sigma \in \mathfrak{e}_c^{\ell,k} \subseteq \text{pcf}(\mathfrak{c}) \cap [\mu, \lambda(3)]$ (where $\mathfrak{e}_c^{\ell,k}$ is from (*₃)) then for some $m < \omega, \sigma_p \in \sigma^+ \cap \text{pcf}(\mathfrak{c})$ and $\alpha_p < \sigma_p$, (for $p \leq m$) the function $f_n \upharpoonright (\mathfrak{b}_\sigma[\mathfrak{c}])$ is included in $\text{Max}_{p \leq m} f_{\alpha_p}^{\mathfrak{c}, \sigma_p} \upharpoonright \mathfrak{b}_{\sigma_p}[\mathfrak{c}]$ (the Max taken pointwise).

Note

(*)₅ if $\mathfrak{b} \subseteq \mathfrak{a}_{n,\ell,k}$ is countable (where $\ell, k < \omega$) then there is $\mathfrak{c} \in \mathcal{P}_{n,\ell}, |\mathfrak{c}| < \mu, \mathfrak{c} \subseteq \text{Reg} \cap \lambda^+ \setminus \mu$ such that $\mathfrak{b} \subseteq \mathfrak{c}$.

By (*₄):

(*)₆ if $\ell, k < \omega, \mathfrak{c} \in \mathcal{P}_{n,\ell}, \mathfrak{c} \subseteq \text{Reg} \cap \lambda^+ \setminus \mu$, and $\sigma \in \mathfrak{d}_{n,\ell,k} \cap \lambda(3)^+ \cap \text{pcf} \mathfrak{c} \setminus \mu$ then $f_n \upharpoonright \mathfrak{b}_\sigma[\mathfrak{c}] \in \mathfrak{B}$.

You can check that (by (*₂) – (*₆)):

(*)₇ if $\mathfrak{b} \subseteq \mathfrak{a}_{n,\ell,k}$ is countable then there is $f_b^{n,\ell,k} \in \mathfrak{B}, |\text{Dom } f_b^{n,\ell,k}| < \mu$ such that $f_n \upharpoonright \mathfrak{b} \subseteq f_b^{n,\ell,k}$.

Let $\tau_i (i < \omega)$ list the Skolem function of $(H(\chi), \in, <_\chi^*)$. Let

$$A_{n+1,\ell} = \bigcup \{ \text{Rang}(\tau_i \upharpoonright (A_{n,j} \cup \text{Rang } f_n \upharpoonright \mathfrak{a}_{n,j,k})) : i < \ell, j < \ell, k < \ell \},$$

$$\mathcal{P}'_{n+1,\ell} = \bigcup_{m \leq \ell} \mathcal{P}_{n,m} \cup \{ f_n \upharpoonright \mathfrak{a}' : \mathfrak{a}' \in \bigcup_{m \leq \ell} \mathcal{P}_{n,m} \text{ and } f_n \upharpoonright \mathfrak{a}' \in \mathfrak{B} \},$$

and $\mathcal{P}_{n+1,\ell} = \mathcal{S}_{<\mu}(\lambda + 1) \cap (\text{Skolem Hull of } A_{n+1,\ell} \cup \mathcal{P}'_{n+1,\ell} \cup (\theta + 1))$.

So $f_n, \mathcal{P}_{n+1,\ell}$ are as required.

Thus we have carried the induction.

$\lambda(3) \leq \lambda(2)$: Let \mathcal{P} exemplify the definition of $\lambda(2)$. Let $\mathfrak{a} \subseteq \text{Reg} \cap \lambda^+ \setminus \mu$, $|\mathfrak{a}| \leq \theta(<\mu)$. Let $J = J_{\leq \lambda(2)}[\mathfrak{a}]$, and let

$J_1 = \{\mathfrak{b}: \mathfrak{b} \subseteq \mathfrak{a} \text{ and there is } \langle \mathfrak{b}_i: i < i^* \rangle, \text{ satisfying: } \mathfrak{b}_i \subseteq \mathfrak{b}, i^* < \mu, \max \text{pcf } \mathfrak{b}_i \leq \lambda(2) \text{ and any countable subset of } \mathfrak{b} \text{ is in the ideal which } \{\mathfrak{b}_i: i < i^*\} \text{ generates}\}$.

Clearly J_1 is an ideal of subsets of \mathfrak{a} extending J . Let

$$J_2 = \left\{ \mathfrak{b}: \text{for some } \mathfrak{b}_n \in J_1 \text{ (for } n < \omega), \mathfrak{b} \subseteq \bigcup_n \mathfrak{b}_n \right\}.$$

Clearly J_2 is an \aleph_1 -complete ideal extending J_1 (and J). If $\mathfrak{a} \in J_2$ we have that \mathfrak{a} satisfies the requirement thus we have finished so we can assume $\mathfrak{a} \notin J_2$. As we can force by Levy $(\lambda(2)^+, 2^{\lambda(2)})$ (alternatively, replacing \mathfrak{a} by [Sh355, §1]) without loss of generality $\lambda(2)^+ = \max \text{pcf } \mathfrak{a}$ and so $\text{tcf}(\prod \mathfrak{a}/J_2) = \text{tcf}(\prod \mathfrak{a}/J) = \lambda(2)^+$. Let $\bar{f} = \langle f_\alpha: \alpha < \lambda(2)^+ \rangle$ be $<_J$ -increasing, $f_\alpha \in \prod \mathfrak{a}$, cofinal in $\prod \mathfrak{a}/J$. Let $\mathfrak{B} \prec (H(\chi), \in, <_\chi^*)$ be of cardinality $\lambda(2)$, $\lambda(2) + 1 \subseteq \mathfrak{B}$, $\mathfrak{a} \in \mathfrak{B}$, $\bar{f} \in \mathfrak{B}$ and $\mathcal{P} \in \mathfrak{B}$. Let $\mathcal{P}' =: \mathfrak{B} \cap \mathcal{S}_{<\mu}(\lambda)$.

For $B \in \mathcal{P}'$ (so $|B| < \mu$) let $g_B \in \prod \mathfrak{a}$ be $g_B(\sigma) =: \sup(\sigma \cap B)$, so for some $\alpha_B < \lambda$, $g_B <_J f_{\alpha_B}$. Let $\alpha(*) = \sup\{\alpha_B: B \in \mathcal{P}'\}$, clearly $\alpha(*) < \lambda(2)^+$. So $\bigwedge_{B \in \mathcal{P}'} g_B <_J f_{\alpha(*)}$. Note: $\mathcal{P} \subseteq \mathcal{P}'$ (as $\mathcal{P} \in \mathfrak{B}$, $|\mathcal{P}| \leq \lambda(2)$, $\lambda(2) + 1 \subseteq \mathfrak{B}$) and for each $B \in \mathcal{P}$, $\mathfrak{c}_B =: \{\sigma \in \mathfrak{a}: g_B(\sigma) \geq f_{\alpha(*)}(\sigma)\}$ is in J and $J \subseteq J_1 \subseteq J_2$. Apply the choice of \mathcal{P} (i.e. it exemplifies $\lambda(2)$) to $A =: \text{Rang } f_{\alpha(*)}$, get $\langle A_n, \mathcal{P}_n: n < \omega \rangle$ as there. Let $\mathfrak{a}_n =: \{\sigma \in \mathfrak{a}: f_{\alpha(*)}(\sigma) \in A_n\}$, so $\mathfrak{a} = \bigcup_n \mathfrak{a}_n$, hence for some m , $\mathfrak{a}_m \notin J_2$ (as $\mathfrak{a} \notin J_2$, J_2 is \aleph_1 -complete) hence $\mathfrak{a}_m \notin J_1$. As $\mathfrak{a} \in \mathfrak{B}$, $\mathcal{P} \in \mathfrak{B}$ clearly $\mathcal{P}_m \subseteq \mathfrak{B}$. So $\{\mathfrak{c}_B: B \in \mathcal{P}_m\}$ is a family of $< \mu$ subsets of \mathfrak{a} , each in J and every countable $\mathfrak{b} \subseteq \mathfrak{a}_m$ is included in at least one of them (as for some $B \in \mathcal{P}_m$, $\text{Rang}(f_{\alpha(*)} \upharpoonright \mathfrak{b}) \subseteq B$, hence $\mathfrak{b} \subseteq \mathfrak{c}_B$). Easy contradiction.

$\lambda(3) \leq \lambda(0)$ IF $\text{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu$: Let $\mathfrak{a} \subseteq \text{Reg} \cap \lambda^+ \setminus \mu$, $|\mathfrak{a}| \leq \kappa$, let $\langle \mathfrak{a}_\ell: \ell < \omega \rangle$ be as guaranteed by the definition of $\lambda(0)$, let $\mathcal{P}_\ell \subseteq \mathcal{S}_{<\aleph_1}(\mathfrak{a}_\ell)$ exemplify $\text{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu$, for each $\mathfrak{b} \in \mathcal{P}_\ell$ we can find a finite $\mathfrak{e}_\mathfrak{b} \subseteq (\text{pcf } \mathfrak{a}_\ell) \cap \lambda^+ \setminus \mu$ such that $\mathfrak{b} \subseteq \bigcup \{\mathfrak{b}_\sigma[\mathfrak{a}_\ell]: \sigma \in \mathfrak{e}_\mathfrak{b}\}$ and $\{\mathfrak{b}_{\ell,i}: i < i^*\}$ enumerates $\{\mathfrak{e}_\mathfrak{b}: \mathfrak{b} \in \mathcal{P}_\ell\}$.

$\lambda(0) \leq \lambda(1)$: Similar to the proof of $\lambda(3) \leq \lambda(2)$. ■_{1.2}

1.3 CLAIM: Assume $\aleph_0 < \text{cf } \lambda \leq \theta < \lambda < \lambda^*$, $\text{pp}(\lambda) \leq \lambda^*$ and

$$\text{cov}(\lambda^*, \lambda^+, \theta^+, 2) < \lambda^*.$$

Then $\text{cov}(\lambda, \lambda, \theta^+, 2) < \lambda^*$.

Proof: Easy.

1.3A Definition: Assume $\lambda \geq \theta = \text{cf } \theta > \kappa = \text{cf } \kappa > \aleph_0$.

(1) $(\bar{C}, \bar{\mathcal{P}}) \in T^\oplus[\theta, \kappa]$ if $(\bar{C}, \bar{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$ (see [Sh420, Def 2.1(1)]), and $\delta \in S(\bar{C}) \Rightarrow \delta = \sup(\text{acc } C_\delta)$ (note: $\text{acc } C_\delta \subseteq C_\delta$), and we do not allow (viii)⁻ (in [Sh420, Definition 2.1(1)]), or replace it by:

(viii)* for some list $\langle a_i : i < \theta \rangle$ of $\bigcup_{\alpha \in S(\bar{C})} \mathcal{P}_\alpha$, we have: $\delta \in S(\bar{C})$, $\alpha \in \text{acc } C_\delta$ implies $\{a \cap \beta : a \in \mathcal{P}_\delta, \beta \in a \cap \alpha\} \subseteq \{a_i : i < \alpha\}$.

(2) For $(\bar{C}, \bar{\mathcal{P}}) \in T^\oplus[\theta, \kappa]$ we define a filter $\mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}^\oplus(\lambda)$ on $[\mathcal{S}_{<\kappa}(\lambda)]^{<\kappa}$ (rather than on $\mathcal{S}_{<\kappa}(\lambda)$ as in [Sh420, 2.4]) (let $\chi = \beth_{\omega+1}(\lambda)$):

$Y \in \mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}^\oplus(\lambda)$ iff $Y \subseteq (\mathcal{S}_{<\kappa}(\lambda))^{<\kappa}$ and for some $x \in H(\chi)$ for every $\langle N_\alpha, N_\alpha^* : \alpha < \theta, a \in \bigcup_{\delta \in S} \mathcal{P}_\delta \rangle$ satisfying condition \otimes from [Sh420, 2.4], and also $[a \in \mathcal{P}_\delta \ \& \ \delta \in S \ \& \ \alpha < \theta \Rightarrow x \in N_\alpha^* \ \& \ x \in N_\alpha]$ there is $A \in \text{id}^a(\bar{C})$ such that $\delta \in S(\bar{C}) \setminus A \Rightarrow \langle \bigcup_{\alpha \in \mathcal{P}_\delta} N_\alpha^* \cap \lambda \cap N_\alpha : \alpha \in \text{acc } C_\delta \rangle \in Y$.

Remark: For 1.3B below, see Definition of $\mathcal{T}^\ell(\theta, \kappa)$ and compare with [Sh420, Definition 2.1(2), (3)].

1.3B CLAIM:

(1) If $(\bar{C}, \bar{\mathcal{P}}) \in T^\oplus[\theta, \kappa]$ (so $\lambda > \kappa$ are regular uncountable) then $\mathcal{D}_{(\bar{C}, \bar{\mathcal{P}})}^\oplus(\lambda)$ is a non-trivial ideal on $[\mathcal{S}_{<\kappa}(\lambda)]^{<\kappa}$.

(2) If $\bar{C} \in \mathcal{T}^0[\theta, \kappa]$, $[\delta \in S(\bar{C}) \Rightarrow \delta = \sup \text{acc } C_\delta]$, $\mathcal{P}_\delta = \{C_\delta \cap \alpha : \alpha \in C_\delta\}$ then $(\bar{C}, \bar{\mathcal{P}}) \in T^\oplus[\theta, \kappa]$. If $\bar{C} \in \mathcal{T}^1[\theta, \kappa]$, $[\delta \in S(\bar{C}) \Rightarrow \delta = \sup \text{acc } C_\delta]$ and $\mathcal{P}_\delta = \mathcal{S}_{<\aleph_0}(C_\delta)$ then $(\bar{C}, \bar{\mathcal{P}}) \in T^\oplus[\theta, \kappa]$.

(3) If θ is successor of regular, $\sigma = \text{cf } \sigma < \kappa$, there is $\bar{C} \in \mathcal{T}^0[\theta, \kappa] \cap \mathcal{T}^1[\theta, \kappa]$ with: for $\delta \in S(\bar{C})$, C_δ is closed, $\text{cf } \delta = \sigma$ and $\text{otp } C_\delta$ divisible by ω^2 (hence $\delta = \sup \text{acc } C_\delta$).

(4) Instead of “ θ successor of regular”, it suffices to demand

$$(*) \quad \theta > \kappa \text{ regular uncountable, and } \bigwedge_{\alpha < \theta} \bigvee_{\kappa_1 \in [\kappa, \theta]} \text{cov}(\alpha, \kappa_1, \kappa, 2) < \theta.$$

Replacing 2 by σ , “ C_δ closed” is weakened to “ $\{\text{otp}(\alpha \cap C_\delta) : \alpha \in C_\delta\}$ is stationary”.

Proof: Check.

1.3C CLAIM: Let $\lambda > \kappa = \text{cf } \kappa > \aleph_0$, $\theta = \kappa^+$, $(\bar{C}, \bar{P}) \in \mathcal{T}^\oplus[\theta, \kappa]$ then the following cardinals are equal:

$$\begin{aligned} \mu(0) &= \text{cf}(\mathcal{S}_{<\kappa}(\lambda), \subseteq), \\ \mu(4) &= \text{Min} \left\{ |Y| : Y \in \mathcal{D}_{(\bar{C}, \bar{P})}^\oplus(\lambda) \right\}. \end{aligned}$$

Proof: Included in the proof of [Sh420, 2.6].

1.3D CLAIM: Let $\lambda_1 \geq \lambda_0 > \kappa = \text{cf } \kappa > \aleph_0$, $\theta = \kappa^+$ and $(\bar{C}, \bar{P}) \in \mathcal{T}^\oplus[\theta, \kappa]$. Let \mathfrak{B}_{λ_1} be a rich enough model with universe λ_1 and countable vocabulary which is rich enough (e.g. all functions (from λ_1 to λ_1) definable in $(H(\beth_\omega(\lambda_1)^+), \in, <^*)$ with any finite number of places). Then the following cardinals are equal:

$$\begin{aligned} \mu^*(0) &= \text{cov}(\lambda_1, \lambda_0^+, \kappa, 2), \\ \mu^+(4) &= \text{Min} \left\{ |Y| \approx_{\mathfrak{B}_{\lambda_1}}^{\lambda_0} : Y \in \mathcal{D}_{(\bar{C}, \bar{P})}^\oplus(\lambda_1) \right\} \text{ where } \langle a'_i : i \in \text{acc } C_\delta \rangle \approx_{\mathfrak{B}}^{\lambda_0} \\ &\quad \langle a''_i : i \in \text{acc } C_\delta \rangle \text{ iff } \bigwedge_{i \in \text{acc } C_\delta} \text{Skolem Hull}_{\mathfrak{B}_{\lambda_1}}(a'_i \cup \lambda_0) = \\ &\quad \text{Skolem Hull}_{\mathfrak{B}_{\lambda_1}}(a''_i \cup \lambda_0). \end{aligned}$$

Proof: Like the proof of [Sh420], 2.6, but using [Sh400, 3.3A].

2. Equality Relevant to Weak Diamond

It is well known that:

$$\kappa = \text{cf } \kappa \ \& \ \theta > 2^{<\kappa} \Rightarrow \text{cov}(\theta, \kappa, \kappa, 2) = \theta^{<\kappa} = \text{cov}(\theta, \kappa, \kappa, 2)^{<\kappa}.$$

Now we have

2.1 CLAIM:

(1) If $\mu > \lambda \geq \kappa$, $\theta = \text{cov}(\mu, \lambda^+, \lambda^+, \kappa)$, $\text{cov}(\lambda, \kappa, \kappa, 2) \leq \mu$ (or $\leq \theta$) then

$$\text{cov}(\mu, \lambda^+, \lambda^+, 2) = \text{cov}(\theta, \kappa, \kappa, 2).$$

(2) If in addition $\lambda \geq 2^{<\kappa}$ (or just $\theta \geq 2^{<\kappa}$) then

$$\text{cov}(\mu, \lambda^+, \lambda^+, 2)^{<\kappa} = \text{cov}(\mu, \lambda^+, \lambda^+, 2).$$

2.1A Remark:

- (1) A most interesting case is $\kappa = \aleph_1$.
- (2) This clarifies things in [Sh-f,AP1.17].

Proof: (1) Note that $\theta \geq \mu$ (because $\mu > \lambda \geq \kappa$). First we prove “ \leq ”. Let \mathcal{P}_0 be a family of θ subsets of μ each of cardinality $\leq \lambda$, such that every subset

of μ of cardinality $\leq \lambda$ is included in the union of $< \kappa$ of them (exists by the definition of $\theta = \text{cov}(\mu, \lambda^+, \lambda^+, \kappa)$). Let $\mathcal{P}_0 = \{A_i: i < \theta\}$. Let \mathcal{P}_1 be a family of $\text{cov}(\theta, \kappa, \kappa, 2)$ subsets of θ , each of cardinality $< \kappa$ such that any subset of θ of cardinality $< \kappa$ is included in one of them.

Let $\mathcal{P} =: \{\bigcup_{i \in a} A_i: a \in \mathcal{P}_1\}$; clearly \mathcal{P} is a family of subsets of μ each of cardinality $\leq \lambda$, $|\mathcal{P}| \leq |\mathcal{P}_1| = \text{cov}(\theta, \kappa, \kappa, 2)$, and every $A \subseteq \mu$, $|A| \leq \lambda$ is included in some union of $< \kappa$ members of \mathcal{P}_0 (by the choice of \mathcal{P}_0), say $\bigcup_{i \in b} A_i$, $b \subseteq \theta$, $|b| < \kappa$; by the choice of \mathcal{P}_1 , for some $a \in \mathcal{P}_1$ we have $b \subseteq a$, hence $A \subseteq \bigcup_{i \in b} A_i \subseteq \bigcup_{i \in a} A_i \in \mathcal{P}$. So \mathcal{P} exemplify $\text{cov}(\mu, \lambda^+, \lambda^+, 2) \leq \text{cov}(\theta, \kappa, \kappa, 2)$.

Second we prove the inequality \geq . If $\kappa \leq \aleph_0$ then $\text{cov}(\mu, \lambda^+, \lambda^+, 2) = \theta$ and $\text{cov}(\theta, \kappa, \kappa, 2) = \theta$ so \geq trivially holds; so assume $\kappa > \aleph_0$. Obviously $\text{cov}(\mu, \lambda^+, \lambda^+, 2) \geq \theta$. Note, if κ is singular then, as cf $\lambda^+ > \lambda \geq \kappa$ for some $\kappa_1 < \kappa$, we have $\theta = \text{cov}(\mu, \lambda^+, \lambda^+, \kappa) = \text{cov}(\mu, \lambda^+, \lambda^+, \kappa')$ whenever $\kappa' \in [\kappa_1, \kappa]$ is a successor (by [Sh355, 5.2(8)]); also $\text{cov}(\theta, \kappa, \kappa, 2) \leq \sup\{\text{cov}(\theta, \kappa, \kappa', 2): \kappa' \in [\kappa_1, \kappa] \text{ is a successor cardinal}\}$ and $\text{cov}(\theta, \kappa, \kappa', 2) \leq \text{cov}(\theta, \kappa', \kappa', 2)$ when $\kappa' < \kappa$, so without loss of generality κ is regular uncountable. Hence for any $\theta_1 < \theta$ we have

(*) $_{\theta_1}$ we can find a family $\mathcal{P} = \{A_i: i < \theta_1\}$, $A_i \subseteq \mu$, $|A_i| \leq \lambda$, such that any subfamily of cardinality $\leq \lambda^+$ has a transversal. [Why? By [Sh355, 5.4], (=+) and [Sh355,1.5A] even for $\leq \mu$.]

Hence if $\theta_1 \leq \theta$, cf $\theta_1 < \lambda^+$ (or even cf $\theta_1 \leq \mu$) then (*) $_{\theta_1}$. Now we shall prove below

$$(\otimes_1) \quad (*)_{\theta_1} \Rightarrow \text{cov}(\theta_1, \kappa, \kappa, 2) \leq \text{cov}(\mu, \lambda^+, \lambda^+, 2)$$

and obviously

$$(\otimes_2) \quad \text{if cf } \theta \geq \kappa \text{ then } \text{cov}(\theta, \kappa, \kappa, 2) = \sum_{\alpha < \theta} \text{cov}(\alpha, \kappa, \kappa, 2)$$

together; (as $\theta \leq \text{cov}(\theta, \lambda^+, \lambda^+, 2)$ which holds as $\lambda < \mu \leq \theta$) we are done.

Proof of \otimes_1 : Let $\{A_i: i < \theta_1\}$ exemplify (*) $_{\theta_1}$ and \mathcal{P}_2 exemplify the value of $\text{cov}(\mu, \lambda^+, \lambda^+, 2)$. Now for every $a \subseteq \theta_1$, $|a| < \kappa$, let $B_a =: \bigcup_{i \in a} A_i$; so $B_a \subseteq \mu$, $|B_a| \leq \lambda$ hence there is $A_a \in \mathcal{P}_2$ such that: $B_a \subseteq A_a$. Now for $A \in \mathcal{P}_2$ define $b[A] =: \{i < \theta_1: A_i \subseteq A\}$; it has cardinality $\leq \lambda$ (as any subfamily of $\{A_i: A_i \subseteq A\}$ of cardinality $\leq \lambda^+$ has a transversal). Note $a \subseteq b[A_a]$ (just read

the definitions of $b[A]$ and A_a ; note $a \in \mathcal{S}_{<\kappa}(\theta_1)$). For $A \in \mathcal{P}_2$ let \mathcal{P}_A be a family of $\leq \text{cov}(\lambda, \kappa, \kappa, 2)$ subsets of $b[A]$ each of cardinality $< \kappa$ such that any such set is included in one of them (exists as $|b[A]| \leq \lambda$ by the definition of $\text{cov}(\lambda, \kappa, \kappa, 2)$). So for any $a \in \mathcal{S}_{<\kappa}(\theta_1)$ for some $c \in \mathcal{P}_{A_a}$, $a \subseteq c$. We can conclude that $\bigcup\{\mathcal{P}_A: A \in \mathcal{P}_2\}$ is a family exemplifying $\text{cov}(\theta_1, \kappa, \kappa, 2) \leq \text{cov}(\mu, \lambda^+, \lambda^+, 2) + \text{cov}(\lambda, \kappa, \kappa, 2)$ but the last term is $\leq \mu$ (by an assumption) whereas the first is $\geq \mu$ (as $\mu > \lambda$) hence the second term is redundant.

(2) By the first part it is enough to prove $\text{cov}(\theta, \kappa, \kappa, 2)^{<\kappa} = \text{cov}(\theta, \kappa, \kappa, 2)$, which is easy and well known (as $\theta \geq \mu > \lambda \geq 2^{<\kappa}$). ■_{2.1}

2.1B Remark: So actually if $\mu > \lambda \geq \kappa$, $\theta = \text{cov}(\mu, \lambda^+, \lambda^+, \kappa)$ then ($\theta \geq \mu > \lambda \geq \kappa$ and)

$$\begin{aligned} \text{cov}(\mu, \lambda^+, \lambda^+, 2) &\leq \text{cov}(\mu, \lambda^+, \lambda^+, \kappa) + \text{cov}(\theta, \kappa, \kappa, 2) \\ &= \theta + \text{cov}(\theta, \kappa, \kappa, 2) = \text{cov}(\theta, \kappa, \kappa, 2) \end{aligned}$$

and

$$\text{cov}(\theta, \kappa, \kappa, 2) \leq \text{cov}(\mu, \lambda^+, \lambda^+, 2) + \text{cov}(\lambda, \kappa, \kappa, 2),$$

hence, $\text{cov}(\theta, \kappa, \kappa, 2) = \text{cov}(\mu, \lambda^+, \lambda^+, 2) + \text{cov}(\lambda, \kappa, \kappa, 2)$.

3. Cofinality of $S_{\leq \aleph_0}(\kappa)$ for κ Real Valued Measurable and Trees

In Rubin–Shelah [RuSh117] two covering properties were discussed concerning partition theorems on trees, the stronger one was sufficient, the weaker one necessary so it was asked whether they are equivalent. [Sh371, 6.1, 6.2] gave a partial positive answer (for λ successor of regular, but then it gives a stronger theorem); here we prove the equivalence.

In Gitik–Shelah [GiSh412] cardinal arithmetic, e.g. near a real valued measurable cardinal κ , was investigated, e.g. $\{2^\sigma: \sigma < \kappa\}$ is finite (and more); this section continues it. In particular we answer a problem of Fremlin: for κ real valued measurable, do we have $\text{cf}(\mathcal{S}_{<\aleph_1}(\kappa), \subseteq) = \kappa$? Then we deal with trees with many branches; on earlier theorems see [Sh355, §0], and later [Sh410, 4.3].

3.1 THEOREM: *Assume λ, θ, κ are regular cardinals and $\lambda > \theta = \kappa > \aleph_0$. Then the following conditions are equivalent:*

- (A) *for every $\mu < \lambda$ we have $\text{cov}(\mu, \theta, \kappa, 2) < \lambda$,*
- (B) *if $\mu < \lambda$ and $a_\alpha \in \mathcal{S}_{<\kappa}(\mu)$ for $\alpha < \lambda$ then for some $W \subseteq \lambda$ of cardinality λ we have $|\bigcup_{\alpha \in W} a_\alpha| < \theta$.*

3.1A Remark: (1) Note that (B) is equivalent to: if $a_\alpha \in \mathcal{S}_{<\kappa}(\lambda)$ for $\alpha < \lambda$, then for some unbounded $S \subseteq \{\delta < \lambda: \text{cf}(\delta) \geq \kappa\}$ and $b \in \mathcal{S}_{<\theta}(\lambda)$, for $\alpha \neq \beta$ in S , $a_\alpha \cap a_\beta \subseteq b$ (we can start with any stationary $S_0 \subseteq \{\delta < \lambda: \text{cf} \delta \geq \kappa\}$, and use Fodour Lemma).

(2) We can replace everywhere θ by κ , but want to prepare for a possible generalization. By the proof we can strengthen “ $W \subseteq \lambda$ of cardinality λ ” to “ $W \subseteq \lambda$ is stationary” (for $\neg(A) \rightarrow \neg(B)$ this is trivial, for $(A) \rightarrow (B)$ real), so these two versions of (B) are equivalent.

Proof:

(A) \Rightarrow (B):

Trivial [for $\mu < \lambda$ let $\mathcal{P}_\mu \subseteq \mathcal{S}_{<\theta}(\mu)$ exemplify $\text{cov}(\mu, \theta, \kappa, 2) < \lambda$; suppose $\mu < \lambda$ and $a_\alpha \in \mathcal{S}_{<\kappa}(\mu)$ for $\alpha < \lambda$ are given, for each α for some $A_\alpha \in \mathcal{P}_\mu$ we have $a_\alpha \subseteq A_\alpha$; as $|\mathcal{P}_\mu| < \lambda = \text{cf} \lambda$ for some A^* we have $W =: \{\alpha < \lambda: A_\alpha = A^*\}$ has cardinality λ , so S is as required in (B)].

$\neg(A) \Rightarrow \neg(B)$:

FIRST CASE: For some $\mu \in [\theta, \lambda)$, $\text{cf} \mu < \kappa < \mu$ and $pp_{<\kappa}^+(\mu) > \lambda$. Then we can find $\mathfrak{a} \subseteq \text{Reg} \cap \mu \setminus \theta$, $|\mathfrak{a}| < \kappa$, $\sup \mathfrak{a} = \mu$ and $\max \text{pcf}_{J_{\mathfrak{a}}^{bd}} \mathfrak{a} \geq \lambda$. So by [Sh355, 2.3] without loss of generality $\lambda = \max \text{pcf} \mathfrak{a}$; let $\langle f_\alpha: \alpha < \lambda \rangle$ be $<_{J_{<\lambda}[\mathfrak{a}]}$ -increasing cofinal in $\prod \mathfrak{a}$.

Let $a_\alpha = \text{Rang}(f_\alpha)$, so for $\alpha < \lambda$, a_α is a subset of $\mu < \lambda$ of cardinality $< \kappa$. Suppose $W \subseteq \lambda$ has cardinality λ , hence is unbounded, and we shall show that $\mu = |\bigcup_{\alpha \in W} a_\alpha|$; as $\mu \geq \theta$ this is enough. Clearly $a_\alpha = \text{Rang} f_\alpha \subseteq \sup \mathfrak{a} = \mu$, hence $\bigcup_{\alpha \in W} a_\alpha \subseteq \mu$. If $|\bigcup_{\alpha \in W} a_\alpha| < \mu$ define $g \in \prod \mathfrak{a}$ by: $g(\sigma)$ is $\sup(\sigma \cap \bigcup_{\alpha \in W} a_\alpha)$ if $\sigma > |\bigcup_{\alpha \in W} a_\alpha|$ and 0 otherwise. So $g \in \prod \mathfrak{a}$ hence for some $\beta < \lambda$ $g < f_\beta \text{ mod } J_{<\lambda}[\mathfrak{a}]$. As the f_β 's are $<_{J_{<\lambda}[\mathfrak{a}]}$ -increasing and $W \subseteq \lambda$ unbounded, without loss of generality $\beta \in W$, hence by g 's choice $[\sigma \in \mathfrak{a} \setminus |\bigcup_{\alpha \in W} a_\alpha|^+ \Rightarrow f_\beta(\sigma) \leq g(\sigma)]$ but $\{\sigma: \sigma \in \mathfrak{a}, \sigma > |\bigcup_{\alpha \in W} a_\alpha|^+\} \notin J_{<\lambda}[\mathfrak{a}]$ (as μ is a limit cardinal and $\max \text{pcf}_{J_{\mathfrak{a}}^{bd}}(\mathfrak{a}) \geq \lambda$), contradiction.

The main case is:

SECOND CASE: For no $\mu \in [\theta, \lambda)$ is $\text{cf} \mu < \kappa < \mu$, $pp_{<\kappa}^+(\mu) > \lambda$. Let $\chi =: \beth_2(\lambda)^+$, \mathfrak{B} be the model with universe λ and the relations and functions definable in $(H(\chi), \in, <_\chi^*)$ possibly with the parameters κ, θ, λ . We know that $\lambda > \theta^+$ (otherwise $\lambda = \theta^+$ and (A) holds). Let $S \subseteq \{\delta < \lambda: \text{cf} \delta = \theta\}$ be stationary and

in $I[\lambda]$ (see [Sh420, 1.5]) and let $S \subseteq S^+$, $\bar{C} = \langle C_\alpha: \alpha \in S^+ \rangle$ be such that: C_α closed, $\text{otp } C_\alpha \leq \theta$, $[\beta \in \text{nacc } C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta]$, $[\text{otp } C_\alpha = \kappa \Leftrightarrow \alpha \in S]$ and for $\alpha \in S^+$ limit, C_α is unbounded in α (see [Sh420, 1.2]).

Without loss of generality \bar{C} is definable in $(\mathfrak{B}, \kappa, \theta, \lambda)$. Let $\mu_0 \in [\theta, \lambda)$ be minimal such that $\text{cov}(\mu_0, \theta, \kappa, 2) \geq \lambda$, so $\mu_0 > \theta$, $\kappa > \text{cf } \mu_0$. We choose by induction on $\alpha < \lambda$, $\mathfrak{A}_\alpha, a_\alpha$ such that:

- (α) $\mathfrak{A}_\alpha \prec (H(\chi), \in, <_\chi^*)$, $\|\mathfrak{A}_\alpha\| < \lambda$ and $\mathfrak{A}_\alpha \cap \lambda$ is an ordinal and $\{\lambda, \mu_0, \theta, \kappa, \mathfrak{B}, \bar{C}\} \in \mathfrak{A}_\alpha$.
- (β) $\mathfrak{A}_\alpha (\alpha < \lambda)$ is increasing continuous and $\langle \mathfrak{A}_\beta: \beta \leq \alpha \rangle \in \mathfrak{A}_{\alpha+1}$.
- (γ) $a_\alpha \in \mathcal{S}_{<\kappa}(\mu_0)$ is such that for no $A \in \mathcal{S}_{<\theta}(\mu_0) \cap \mathfrak{A}_\alpha$ is $a_\alpha \subseteq A$.
- (δ) $\langle a_\beta: \beta \leq \alpha \rangle \in \mathfrak{A}_{\alpha+1}$.

There is no problem to carry the definition and let $\mathfrak{A} = \bigcup_{\alpha < \lambda} \mathfrak{A}_\alpha$. Clearly it is enough to show that $\bar{a} = \langle a_\alpha: \alpha < \lambda \rangle$ contradict (B). Clearly $\mu_0 \in (\theta, \lambda)$ and $a_\alpha \in \mathcal{S}_{<\kappa}(\mu_0)$. So let $W \subseteq \lambda$, $|W| = \lambda$ and we shall prove that $|\bigcup_{\alpha \in W} a_\alpha| \geq \theta$. Note:

- (*) if $\mathfrak{a} \subseteq [\theta, \lambda)$, $|\mathfrak{a}| < \kappa$, $\mathfrak{a} \in \mathfrak{A}_\alpha$ (and $\mathfrak{a} \subseteq \text{Reg}$, of course) then $(\prod \mathfrak{a}) \cap \mathfrak{A}_\alpha$ is cofinal in $\prod \mathfrak{a}$ (as $\max \text{pcf } \mathfrak{a} < \lambda$).

Let $R = \{(\alpha, \beta): \beta \in a_\alpha, \alpha < \lambda\}$ and

$$E =: \{\delta < \lambda: (\mathfrak{A}_\delta, R \upharpoonright \delta, W \cap \delta, \mu_0) \prec (\mathfrak{A}, R, W, \mu_0) \text{ and } \mathfrak{A}_\delta \cap \lambda = \delta\}.$$

Clearly E is a club of λ , hence we can find $\delta(*) \in S \cap \text{acc}(E)$. Let $C_{\delta(*)} = \{\gamma_i: i < \theta\}$ (in increasing order). We now define by induction on $n < \omega$, $M_n, \langle N_\zeta^n: \zeta < \theta \rangle, f_n$ such that:

- (a) M_n is an elementary submodel of (\mathfrak{A}, R, W) , $\|M_n\| = \theta$,
- (b) $\langle N_\zeta^n: \zeta < \theta \rangle$ is an increasing continuous sequence of elementary submodels of \mathfrak{B} ,
- (c) $\|N_\zeta^n\| < \theta$,
- (d) $N_\zeta^n \in \mathfrak{A}_{\delta(*)}$,
- (e) $\bigcup_{\zeta < \kappa} |N_\zeta^n| \subseteq |M_n|$,
- (f) $f_n \in \prod (\text{Reg} \cap M_n)$,
- (g) $f_n(\sigma) > \sup(M_n \cap \sigma)$ for $\sigma \in \text{Dom}(f_n) \setminus \theta^+$,
- (h) for every $\zeta < \theta$, $f_n \upharpoonright (\text{Reg} \cap N_\zeta^n \setminus \theta^+) \in \mathfrak{A}_{\delta(*)}$,
- (i) N_ζ^0 is the Skolem Hull in \mathfrak{B} of $\{\gamma_i, i: i < \zeta\}$,
- (j) N_ζ^{n+1} is the Skolem Hull in \mathfrak{B} of $N_\zeta^n \cup \{f_n(\sigma): \sigma \in \text{Reg} \cap N_\zeta^n \setminus \theta^+\}$,
- (k) M_n is the Skolem Hull in (\mathfrak{A}, R, W) of $\bigcup_{\ell < n} M_\ell \cup \bigcup_{\zeta < \theta} N_\zeta^n$.

There is no problem to carry the definition: for $n = 0$ define N_ζ^0 by (i) [trivially (b) holds and also (c), as for (d), note that $\bar{C} \in \mathfrak{A}_0 \prec \mathfrak{A}_{\delta(*)}$ and $\{\gamma_i: i < \zeta\} \in \mathfrak{A}_{\delta(*)}$ as \bar{C} is definable in \mathfrak{B} hence $\{\langle \alpha, \gamma, \zeta \rangle: \alpha \in S^+, \zeta < \theta, \text{ and } \gamma \text{ is the } \zeta\text{-th member of } C_\alpha\}$ is a relation of \mathfrak{B} hence each $C_{\gamma_{\zeta+1}} (\zeta < \theta)$ is in $\mathfrak{A}_{\delta(*)}$ hence each $\{\gamma_i: i < \zeta\}$ is and we can compute the Skolem Hull in \mathfrak{A}_{γ_j} for $j < \theta$ large enough].

Next, choose M_n by (k), it satisfies (e) + (a). If $\langle N_\zeta^n: \zeta < \theta \rangle$, M_n are defined, we can find f_n satisfying (f) + (g) + (h) by [Sh371,1.4] (remember (*)). For $n + 1$ define N_ζ^{n+1} by (j) and then M_{n+1} by (k).

Next by [Sh400, 3.3A or 5.1A(1)] we have

$$(*) \quad \bigcup_{n < \omega} M_n \cap \delta(*) = \bigcup_{\substack{n < \omega \\ \zeta < \theta}} N_\zeta^n \cap \delta(*) \quad \text{hence} \quad \bigcup_{\substack{n < \omega \\ \zeta < \theta}} N_\zeta^n \cap W \text{ is unbounded in } \delta(*),$$

hence for some n

$$(*)_n \quad \bigcup_{\zeta < \theta} N_\zeta^n \cap W \text{ is unbounded in } \delta(*) .$$

Remember $N_\zeta^n \in \mathfrak{A}_{\delta(*)} = \bigcup_{\alpha < \delta(*)} \mathfrak{A}_\alpha = \bigcup_{i < \theta} \mathfrak{A}_{\gamma_i}$. So for some club e of θ we have:

$$(\otimes) \quad \text{if } \zeta \in e, \quad \xi < \zeta \text{ then: } N_\xi^n \in \mathfrak{A}_{\gamma_\zeta}, \text{ and } \gamma_\zeta \in E \cap C_{\delta(*)}$$

(remember $\delta(*) \in \text{acc}(E)$).

Hence, for $\zeta \in e$, we have: $\mathfrak{A}_{\gamma_\zeta} \cap \lambda = \gamma_\zeta$, and $W \cap N_\zeta^n \setminus \sup N_\xi^n \neq \emptyset$ for every $\xi < \zeta$. Let $e = \{\zeta(\epsilon): \epsilon < \theta\}$, $\zeta(\epsilon)$ strictly increasing continuous in ϵ . Now for every $\epsilon < \theta$, $N_{\zeta(\epsilon)}^n \in \mathfrak{A}_{\gamma_{\zeta(\epsilon+1)}}$ (and $\langle a_\beta: \beta \leq \sup(\lambda \cap N_{\zeta(\epsilon)}^n) \rangle \in \mathfrak{A}_{\gamma_{\zeta(\epsilon+1)}}$) hence $A_1 =: \bigcup \{a_\beta: \beta \in W \cap N_{\zeta(\epsilon)}^n\} \subseteq A_2 =: \bigcup \{a_\beta: \beta \in N_{\zeta(\epsilon+1)}^n\} \cap \mu_0 \in \mathfrak{A}_{\gamma_{\zeta(\epsilon+1)}}$ and A_2 is a subset of μ_0 of cardinality $< \theta$ hence (by the choice of the a_γ 's above) $a_{\gamma_{\zeta(\epsilon+1)}} \not\subseteq A_2$ hence $a_{\gamma_{\zeta(\epsilon+1)}} \not\subseteq \bigcup \{a_\beta: \beta \in W \cap N_{\zeta(\epsilon)}^n\}$; moreover, similarly $\gamma_{\zeta(\epsilon+1)} \leq \gamma < \lambda \Rightarrow a_\gamma \not\subseteq \bigcup \{a_\beta: \beta \in W \cap N_{\zeta(\epsilon)}^n\}$.

But $W \cap N_{\zeta(\epsilon+2)}^n \setminus \gamma_{\zeta(\epsilon+1)} \neq \emptyset$, hence $\langle \bigcup \{a_\beta: \beta \in W \cap N_{\zeta(\epsilon)}^n\}: \epsilon < \theta \rangle$ is not eventually constant, hence

$$\bigcup \left\{ a_\beta: \beta \in W \cap \bigcup_{\epsilon < \theta} N_{\zeta(\epsilon)}^n \right\} = \bigcup \left\{ a_\beta: \beta \in W \cap \bigcup_{\zeta < \theta} N_\zeta^n \right\}$$

has cardinality θ . Hence $\bigcup_{\beta \in W} a_\beta$ has cardinality $\geq \theta$, as required. ■_{3.1}

3.2 Conclusion: (1) If λ is real valued measurable then $\kappa = \text{cf}[\mathcal{S}_{<\aleph_1}(\lambda), \subseteq]$ (equivalently, $\text{cov}(\lambda, \aleph_1, \aleph_1, 2) = \lambda$).

(2) Suppose λ is regular $> \kappa = \text{cf } \kappa > \aleph_0$, I is a λ -complete ideal on λ extending J_λ^{bd} and is κ -saturated (i.e. we cannot partition λ to κ sets not in I). Then for $\alpha < \lambda$, $\text{cf}(\mathcal{S}_{<\kappa}(\alpha), \subseteq) < \lambda$, equivalently $\text{cov}(\alpha, \kappa, \kappa, 2) < \lambda$.

3.2A Remark: (1) So for regular $\theta \in (\kappa, \lambda)$ (in the above situation) we have $\bigwedge_{\alpha < \lambda} \text{cov}(\alpha, \theta, \theta, 2) < \lambda$; actually $\kappa \leq \text{cf } \theta \leq \theta < \lambda$ suffices by the proof.

Proof: (1) Follows by (2).

(2) The conclusion is (A) of Theorem 3.1, hence it suffices to prove (B). Let $\mu < \lambda$ and $a_\alpha \in \mathcal{S}_{<\kappa}(\mu)$ for $\alpha < \lambda$ be given. As $\kappa < \lambda = \text{cf } \lambda$ without loss of generality for some $\sigma < \kappa$, $\bigwedge_{\alpha < \lambda} |a_\alpha| = \sigma$. Let f_α be a function from σ onto a_α , so $\text{Rang } f_\alpha \subseteq \mu$. Now for each $i < \sigma$, $\langle \{\alpha < \lambda: f_\alpha(i) = \gamma\}: \gamma < \mu \rangle$ is a partition of λ to μ sets; as I is κ -saturated, $b_i =: \{\gamma < \mu: \{\alpha < \lambda: f_\alpha(i) = \gamma\} \notin I\}$ has cardinality $< \kappa$, hence $b =: \bigcup_{i < \sigma} b_i$ has cardinality $< \kappa + \sigma^+ \leq \kappa$ (remember $\sigma < \kappa = \text{cf } \kappa$). For each $i < \sigma$, $\gamma \in \mu \setminus b_i$ the set $\{\alpha < \lambda: f_\alpha(i) = \gamma\}$ is in I ; so as I is λ -complete, $\lambda > \mu$ we have: $\{\alpha < \lambda: f_\alpha(i) \notin b_i\}$ is in I . Now let

$$W =: \{\alpha < \lambda: \text{for some } i < \sigma, f_\alpha(i) \notin b_i\} \subseteq \bigcup_{i < \sigma} \{\alpha < \lambda: f_\alpha(i) \notin b_i\}.$$

This is the union of $\leq \sigma < \lambda$ sets each in I , hence is in I , so $|\lambda \setminus W| = \lambda$, and clearly

$$\bigcup_{\alpha \in \lambda \setminus W} a_\alpha = \{f_\alpha(i): \alpha \in \lambda \setminus W, i < \sigma\} \subseteq \{f_\alpha(i): \alpha < \lambda, \neg f_\alpha(i) \notin b_i, i < \sigma\} \subseteq b,$$

and $|b| < \kappa$ so $\lambda \setminus W$ is as required in (B) of Theorem 3.1. ■_{3.2}

3.3 LEMMA: For every λ there is μ , $\lambda \leq \mu < 2^\lambda$ such that (A) or (B) or (C) below holds (letting $\kappa = \text{Min}\{\theta: 2^\theta = 2^\lambda\}$):

(A) $\mu = \lambda$ and for every regular $\chi \leq 2^\lambda$ there is a tree T of cardinality $\leq \lambda$ with $\geq \chi$ $\text{cf}(\kappa)$ -branches (hence there is a linear order of cardinality $\geq \chi$ and density $\leq \lambda$).

(B) $\mu > \lambda$ is singular, and:

(α) $\text{pp}(\mu) = 2^\lambda$ (even $\lambda = \kappa \Rightarrow \text{pp}^+(\mu) = (2^\lambda)^+$), $\text{cf } \mu \leq \lambda$, $(\forall \theta)[\text{cf } \theta \leq \lambda < \theta < \mu \Rightarrow \text{pp}_\lambda \theta < \mu]$ (and $\mu \leq 2^{<\kappa}$)

hence

- (α)' for every successor* $\chi \leq 2^\lambda$ there is a tree from [Sh355, 3.5]: cf μ levels, every level of cardinality $< \mu$ and χ (cf μ)-branches,
 - (β) for every $\chi \in (\lambda, \mu)$, there is a tree T of cardinality λ with $\geq \chi$ branches of the same height,
 - (γ) cf $\mu \geq$ cf κ and even cf $\kappa > \aleph_0 \Rightarrow \text{pp}_{\Gamma(\text{cf } \mu)}(\mu) =^+ 2^\lambda$.
- (C) Like (B) but we omit (α) and retain (α)'.

Proof:

FIRST CASE: $\kappa = \aleph_0$. Trivially (A) holds.

SECOND CASE: κ is regular uncountable. So $\kappa \leq \lambda$ and $2^\kappa = 2^\lambda$ and [$\theta < \kappa \Rightarrow 2^\theta < 2^\kappa$] hence $2^{<\kappa} < 2^\kappa$ (remember $\text{cf}(2^\kappa) > \kappa$). Try to apply [Sh410, 4.3], its assumptions (i) + (ii) hold (with κ here standing for λ there) and if possibility (A) here fails then the assumption (iii) there holds, too; so there is μ as there; so (α), (γ) of (B) of 3.3 holds** and let us prove (β), so assume $\chi \in (\lambda, \mu)$, without loss of generality, is regular, and we shall prove the statement in (β) of 3.3(B). Without loss of generality χ is regular and $\mu' \in (\lambda, \chi) \& \text{cf } \mu' \leq \lambda \Rightarrow \text{pp}_\lambda(\mu') < \chi$; i.e. χ is $(\lambda, \lambda^+, 2)$ -inaccessible. [Why? If χ is not as required, we shall show how to replace χ by an appropriate regular $\chi' \in [\chi, \mu)$.]

Let $\mu' \in (\lambda, \chi)$ be minimal such that $\text{pp}_\lambda(\mu') \geq \chi$, (so $\text{cf } \mu' \leq \lambda$) now $\text{pp}(\mu') < \mu$ (by the choice of μ) and $\chi' =: \text{pp}(\mu')^+$, by [Sh355, 2.3] is as required].

Let θ be minimal such that $2^\theta \geq \chi$. So trivially $\theta \leq \kappa \leq \lambda < \chi$ and $(2^{<\kappa})^\kappa = 2^\kappa$ hence $\mu \leq 2^{<\kappa}$ hence $\chi < 2^{<\kappa}$; as χ is regular $< 2^{<\kappa}$ but $> \lambda \geq \kappa$, clearly $\theta < \kappa \leq \lambda$; also trivially $2^{<\theta} \leq \chi \leq 2^\theta$ but χ is regular $> \lambda \geq \kappa > \theta$ and [$\sigma < \theta \Rightarrow 2^\sigma < \chi$], so $2^{<\theta} < \chi \leq 2^\theta$. Try to apply [Sh410, 4.3] with θ here standing for λ there; assumptions (i), (ii) there hold, and if assumption (iii) fails we get a tree with $\leq \theta$ nodes and $\geq \chi$ θ -branches as required. So assume (iii) holds and we get there μ' ; if $\mu' \leq \lambda$ we have a tree as required; if

* If $\lambda = \kappa$, just regular, and we can change λ for this.

** Alternatively to quoting [Sh410, 4.3], we can get this directly, if $\text{cov}(2^{<\kappa}, \lambda^+, (\text{cf } \kappa)^+, \text{cf } \kappa) < 2^\lambda$ we can get (A); otherwise by [Sh355, 5.4] for some $\mu_0 \in (\lambda, 2^{<\kappa}]$, $\text{cf}(\mu_0) = \text{cf } \kappa$ and $\text{pp}(\mu_0) = (2^\lambda)$. Let $\mu \in (\lambda, 2^{<\kappa}]$ be minimal such that $\text{cf } \mu \leq \lambda$ & $\text{pp}_\lambda(\mu) > 2^{<\kappa}$. Necessarily ([Sh355, 2.3] and [Sh371, 1.6(2), (3), (5)]) $\text{pp}_\lambda(\mu) = \text{pp } \mu = \text{pp}(\mu_0) = (2^\lambda)$ and (again using [Sh355, 2.3]) we have $(\forall \theta)[\text{cf } \theta \leq \lambda < \theta < \mu \Rightarrow \text{pp}_\lambda(\theta) < \mu]$; together (α) of (B) holds. Also $\mu \leq 2^{<\kappa}$, hence $\text{cf}(\mu) < \kappa \Rightarrow \text{pp } \mu \leq \mu^{<\kappa} \leq 2^{<\kappa}$, contradiction, so (γ) of (B) follows from (α). Note that if we replace λ by κ (changing the conclusion a little; or $\lambda = \kappa$) then by [Sh355, 5.4(2)] if 2^λ is regular the conclusion holds for $\chi = 2^\lambda$ too.

$\mu' \in (\lambda, 2^{<\theta}] \subseteq (\lambda, \chi)$ we get contradiction to “ χ is $(\lambda, \lambda^+, 2)$ -inaccessible” which, without loss of generality, we have assumed above.

THIRD CASE: κ is singular (hence $2^{<\kappa}$ is singular, $\text{cf}(2^{<\kappa}) = \text{cf } \kappa$). Let $\mu =: 2^{<\kappa}$ and we shall prove (C); easily (B)(γ) holds. Now ${}^\kappa 2$ is a tree with $2^{<\kappa} = \mu$ nodes and $2^\kappa = 2^\lambda$ κ -branches, so $(\alpha)'$ of (C) holds. As for (β) of (B), if κ is strong limit checking the conclusion is immediate, otherwise it follows from 3.4 part (3) below.

Clearly if $\text{cf } \kappa > \aleph_0$, also (B) holds. ■_{3.3}

3.4 CLAIM:

(1) Assume $\theta_{n+1} = \text{Min } \{\theta : 2^\theta > 2^{\theta_n}\}$ for $n < \omega$ and $\sum_{n < \omega} \theta_n < 2^{\theta_0}$ (so θ_{n+1} is regular, $\theta_{n+1} > \theta_n$). Then: for infinitely many $n < \omega$, for some $\mu_n \in [\theta_n, \theta_{n+1})$ (so $2^{\mu_n} = 2^{\theta_n}$) we have:

(*) $_{\mu_n, \theta_n}$ for every regular $\chi \leq 2^{\theta_n}$ there is a tree of cardinality μ_n with $\geq \chi$ θ_n -branches; if $\mu_n > \theta_n$ then $\text{cf}(\mu_n) = \theta_n$, μ_n is $(\theta_n, \theta_n^+, 2)$ -inaccessible.

(2) Moreover

(α) for every $n < \omega$ large enough for some μ_n :

$$\theta_n \leq \mu_n < \sum_{m < \omega} \theta_m \quad \text{and } (*)_{\mu_n, \theta_n} \quad \text{and } \text{cf}(\mu_n) = \theta_n,$$

$$[\mu_n > \theta_n \Rightarrow \mu_n \text{ is } [(\theta_n, \theta_n^+, 2)\text{-inaccessible, pp}(\mu_n) = 2^{\theta_n}].$$

(β) Moreover, for infinitely many m we can demand: for every $n < m$, $\chi = \text{cf } \chi \leq 2^{\theta_n}$ the tree T_χ^n (witnessing $(*)_{\mu_n, \theta_n}$ for χ) has cardinality $< \theta_{m+1}$ (i.e. $\mu_m < \theta_{m+1}$).

(3) If κ is singular, $\kappa < 2^{<\kappa} < 2^\kappa$ then for every regular $\chi \in (\kappa, 2^{<\kappa})$, there is a tree with $< \kappa$ nodes and $\geq \chi$ branches (of same height). Also for some $\theta^* \in (\kappa, \text{pp}^+(\kappa)) \cap \text{Reg}$, for every regular $\chi \leq 2^\kappa$ there is a tree T , $|T| \leq \kappa^{\text{cf } \kappa}$, with $\geq \chi$ θ^* -branches.

Proof: Clearly (2) implies (1) and (3) (for (3) second sentence use ultraproduct). Let $\theta =: \sum_{n < \omega} \theta_n$. Let $S_0 =: \{n < \omega : (*)_{\theta_n, \theta_n} \text{ fails}\}$. Let for $n \in \omega \setminus S_0$, $\mu_n = \theta_n$ and note that (α) of 3.4(2) holds and if S_0 is co-infinite, also (β) of 3.4(2) holds. We can assume that S_0 is infinite (otherwise the conclusion of 3.4(2) holds). By [Sh355, 5.11], fully [Sh410, 4.3] for $n \in S_0$ there is μ_n such that:

(α) $_n$ $\theta_n = \text{cf } \mu_n < \mu_n \leq 2^{<\theta_n}$,

(β) $_n$ $\text{pp}_{\Gamma(\theta_n)}(\mu_n) \geq 2^{\theta_n}$ (hence equality holds and really $\text{pp}_{\Gamma(\theta_n)}^+(\mu_n) = (2^{\theta_n})^+$)
and

$(\gamma)_n \theta_n < \mu' < \mu_n \& \text{cf } \mu' \leq \theta_n \Rightarrow \text{pp}_{\leq \theta_n}(\mu') < \mu_n$ hence $\text{pp}_{\theta_n}^+(\mu_n) = \text{pp}_{\Gamma(\theta_n)}^+(\mu_n) = (2^{\theta_n})$.

Note that $2^{< \theta_n} = 2^{\theta_{n-1}}$ so $\mu_n \leq 2^{\theta_{n-1}}$. By [Sh355, 5.11] for $n \in S_0$, part (α) (of 3.4(2)) holds except possibly $\mu_n < \theta$.

Remember $\text{cf}(\mu_n) = \theta_n$.

Let $n < m$ be in S_0 and $\mu_n > \theta_m$, so $\text{Max}\{\text{cf } \mu_n, \text{cf } \mu_m\} = \text{Max}\{\theta_n, \theta_m\} < \text{Min}\{\mu_n, \mu_m\}$ so by $(\gamma)_n$ (and [Sh355, 2.3(2)]) we have $\mu_n \geq \mu_m$. Note $\text{cf } \mu_n = \theta_n$, $\text{cf } \mu_m = \theta_m$ (which holds by $(\alpha)_n, (\alpha)_m$) hence $\mu_n > \mu_m$. As the class of cardinals is well ordered we get $S_1 =: \{n < \omega: n \in S_0, \mu_n \geq \theta_{n+1}\}$ is co-infinite and $S =: \{n: \mu_n \geq \theta\}$ is finite (so (α) of 3.4(2)(b) holds).

So for some $n(*) < \omega$, $S \subseteq n(*)$ hence for every $n \in [n(*), \omega)$ for some $m \in (n, \omega)$, $\mu_n < \theta_m$. Note: $n \neq m \Rightarrow \mu_n \neq \mu_m$ (as their cofinalities are distinct) and $[n \notin S_0 \Rightarrow \mu_n \notin \{\theta_m: m < \omega\}]$. Assume $n \geq n(*)$, if $\mu_n > \theta_{n+1}$, let $m = m_n = \text{Min}\{m: \mu_{m+1} > \mu_n \text{ and } m \geq n\}$ (it is well defined as $\bigvee_k \mu_n < \theta_k$ and $\theta_k < \mu_k < \theta = \bigcup_{\ell < \omega} \theta_\ell$) and we shall show $\mu_m < \theta_{m+1}$; assume not, hence $m \in S_0$; so $\mu_{m+1} \leq 2^{\theta_m} = \text{pp}_{\Gamma(\theta_m)}(\mu_m) \leq \text{pp}_{\theta_{m+1}}(\mu_m)$ but $\mu_m \leq \mu_n$ (by the choice of m) so as $\text{cf}(\mu_m) = \theta_m \neq \theta_{m+1}$, necessarily $\mu_m > \theta_{m+1}$ and if $m+1 \notin S_0$ trivially and if $m+1 \in S_0$ by one of the demands on μ_{m+1} (in its choice) and [Sh355, 2.3] we have $\mu_{m+1} \leq \mu_m$; but $\mu_m < \mu_n$, so $\mu_{m+1} < \mu_n$ contradicting the choice of m . So by the last sentence, $n \geq n(*) \Rightarrow \mu_{m_n} < \theta_{m_n+1}$. By [Sh355, 5.11] we get the desired conclusion (i.e. also part (β) of 3.4(2)). ■_{3.4}

Remark: It seemed that we cannot get more as we can get an appropriate product of a forcing notion as in Gitik and Shelah [GiSh344].

4. Bounds for $\text{pp}_{\Gamma(\aleph_1)}$ for Limits of Inaccessibles*

4.1 Convention: For any cardinal μ , $\mu > \text{cf } \mu = \aleph_1$ we let \mathcal{Y}_μ, Eq_μ be as in [Sh420, 3.1], $\bar{\mu}$ is a strictly increasing continuous sequence of singular cardinals of cofinality \aleph_0 of length ω_1 , $\mu = \sum_{i < \aleph_1} \mu_i$.

So μ stands here for μ^* in [Sh420, §3, §4, §5]. (Of course, \aleph_1 can be replaced by “regular uncountable”.)

* In previous versions these sections have been in [Sh410], [Sh420] hence we use \mathcal{Y} , etc. (and not the context of [Sh386]); see 4.2B below.

4.2 THEOREM (Hypothesis [Sh420, 6.1C]*):

(1) Assume

- (a) $\mu > \text{cf } \mu = \aleph_1, \mathcal{Y} = \mathcal{Y}_\mu, Eq'_\mu \subseteq Eq_\mu,$
- (b) every $D \in \text{FIL}(\mathcal{Y})$ is nice (see [Sh420, 3.5]), $E = \text{FIL}(\mathcal{Y})$ (or at least there is a nice \mathcal{E} (see [Sh420, 5.2-5], $E = \bigcup \mathcal{E} = \text{Min } \mathcal{E}$, \mathcal{E} is μ -divisible having weak μ -sums, but we concentrate on the first case),
- (c) $\mu < \lambda < \text{pp}_E^+(\mu), \lambda$ inaccessible.

Then there are $e \in Eq_\mu$ and $\langle \lambda_x : x \in \mathcal{Y}/e \rangle$, a sequence of inaccessibles $< \mu$ and a $D \in \text{FIL}(e, \mathcal{Y}) \cap E$ nice to $\mu, D \in \text{FIL}(e, \mathcal{Y}_\mu)$ such that:

- (α) $\prod_{x \in \mathcal{Y}_\mu/e} \lambda_x/D$ has true cofinality $\lambda,$
- (β) $\mu = \text{tlim}_D \langle \lambda_x : x \in \mathcal{Y}_\mu \rangle.$
- (2) We can weaken “(b)” to “ $E \subseteq \text{FIL}(Eq, \mathcal{Y})$ and for $D \in E$, in the game $wG(\mu, D, e, \mathcal{Y})$ the second player wins choosing filters only from E .”
- (3) Moreover, for given $e_0, D_0, \langle \lambda_x^0 : x \in \mathcal{Y}/e_0 \rangle$, if $\prod_{x \in \mathcal{Y}/e_0} \lambda_x^0/D_0^e$ is λ -directed, then without loss of generality $e_0 \leq e, D_0 \leq D$ and $\lambda_x \leq \lambda_{x[e_0]}$.

4.2A Remark: (1) We could have separated the two roles of μ (in the definition of \mathcal{Y} , etc. and in $\lambda \in (\mu, \text{pp}_E^+(\mu))$) but the result is less useful; except for the unique possible cardinal appearing later.

(2) Compare with a conclusion of [Sh386] (see in particular 5.8 there):

THEOREM: Suppose $\lambda > 2^{\aleph_1}, \lambda$ (weakly) inaccessible.

- (1) If $\aleph_1 < \lambda_i = \text{cf } \lambda_i < \lambda$ for $i < \omega_1, D$ is a normal filter on $\omega_1, \prod_{i < \omega_1} \lambda_i/D$ is λ -directed, then for some $\lambda'_i, \aleph_1 < \lambda'_i = \text{cf } \lambda'_i \leq \lambda_i$ and normal filter D' extending $D, \lambda = \text{tcf}(\prod_{i < \omega_1} \lambda'_i/D')$ and $\{i : \lambda_i \text{ inaccessible}\} \in D'.$
- (2) If $\aleph_1 = \text{cf } \mu < \mu < \lambda, \text{pp}_{\Gamma(\aleph_1)}(\mu) \geq \lambda$ then for some $\langle \lambda_i : i < \omega_1 \rangle, \aleph_1 < \lambda_i = \text{cf } \lambda_i < \mu,$ each λ_i inaccessible and $\lambda \in \text{pcf}_{\Gamma(\aleph_1)}\{\lambda_i : i < \omega_1\}.$

Proof of 4.2: (1) By the definition of $\text{pp}_E^+(\mu)$ (and assumption (c), and [Sh355, 2.3 (1) + (3)]) there are $D \in E$ and $f \in {}^{\mathcal{Y}_\mu/e}\mu$ such that:

- (A) $_f$ $\mu > f(x) = \text{cf}[f(x)] > \mu_{\iota(x)},$
- (B) $_{f,D}$ $\lambda = \text{tcf} \left[\prod_{x \in \mathcal{Y}/e} f(x)/D \right].$

Let $K_0 = \{(f, D) : D \in E, f \in {}^{\mathcal{Y}_\mu/e}\mu \text{ and conditions (A)}_f \text{ and (B)}_{f,D} \text{ hold}\},$ so $K_0 \neq \emptyset.$ Now if $(f, D) \in K_0,$ for some γ

- (C) $_{f,D,\gamma}$ in $G^\gamma(D, f, e, \mathcal{Y})$ the second player wins (see [Sh420, 3.4(2)])

* I.e.: if $\mathfrak{a} \subset \text{Reg}, |\mathfrak{a}| < \min(\mathfrak{a}), \lambda$ inaccessible then $\lambda > \sup(\lambda \cap \text{pcf } \mathfrak{a}).$

hence $K_1 \neq \emptyset$ where $K_1 =: \{(f, D, \gamma) \in K_0 \text{ condition (C)}_{f,D,\gamma} \text{ holds}\}$.
 Choose $(f^1, D_1, \gamma_{\langle \rangle}) \in K_1$ with $\gamma_{\langle \rangle}$ minimal. By the definition of the game

(*) for every $A \neq \emptyset \text{ mod } D_1$ we have $(f^1, D_1 + A, \gamma_{\langle \rangle}) \in K_1$.

Let $e_1 = e(D_1)$.

CASE A: $\{x: f^1(x) \text{ inaccessible}\} \neq \emptyset \text{ mod } D_1$. We can get the desired conclusion (by increasing D_1).

CASE B: $\{x: f^1(x) \text{ successor cardinal}\} \neq \emptyset \text{ mod } D_1$. By (*), without loss of generality $f^1(x) = g(x)^+$, $g(x)$ a cardinal (so $\geq \mu_{\iota(x)}$) for every $x \in \mathcal{Y}/e$. By [Sh355, 1.3] for every regular $\kappa \in (\mu, \lambda)$ there is $f_\kappa \in {}^{\mathcal{Y}/e}\text{Ord}$ satisfying:

- (a) $f_\kappa < f^1$, each $f_\kappa(x)$ regular,
- (b) $\text{tlim}_{D_1} f_\kappa = \mu$,
- (c) $\prod_x f_\kappa(x)/D_1$ has true cofinality κ .

By (a) we get

- (d) $f_\kappa \leq g$.

By (b) we get, by the normality of D_1 , that for the D_1 -majority of $x \in \mathcal{Y}/e$, $f_\kappa(x) \geq \mu_{\iota(x)}$; as $f_\kappa(x)$ is regular (by (a)) and $\mu_{\iota(x)}$ singular (see 4.1) we get

- (e) for the D_1 -majority of $x \in \mathcal{Y}/e$, we have $f_\kappa(x) > \mu_{\iota(x)}$.

Let χ be large enough, let N be an elementary submodel of $(H(\chi), \in, <^*_\chi)$, $\lambda \in N$, $D_1 \in N$, $N \cap \lambda$ is the ordinal $\|N\|$ (singular for simplicity) and $\{\mu, \langle f^1, g, f_\kappa: \kappa \in \text{Reg} \cap (\mu, \lambda) \rangle\}$ belongs to N . Choose $\kappa \in \text{Reg} \cap \lambda \setminus (\sup \lambda \cap N)$, now in $\prod_{x \in \mathcal{Y}/e_1} f_\kappa(x)/D_1$, there is a cofinal sequence $\langle f_{\kappa, \zeta}: \zeta < \kappa \rangle$; as $\kappa > \sup(\lambda \cap N)$, so for some $\zeta(*) < \kappa$:

$$\otimes h \in N \cap {}^{\mathcal{Y}/e_1}\text{Ord} \Rightarrow \{x \in \mathcal{Y}/e_1: f_{\kappa, \zeta(*)}(x) \leq h(x) < f_\kappa(x)\} = \emptyset \text{ mod } D_1.$$

[Why? For any such h define $h' \in {}^{\mathcal{Y}/e_1}\text{Ord}$ by: $h'(x)$ is $h(x)$ if $h(x) < f_\kappa(x)$ and zero otherwise, so for some $\zeta_h < \kappa$, $h' < f_{\kappa, \zeta_h} \text{ mod } D_1$. Let $\zeta(*) = \sup \{\zeta_h: h \in N \cap {}^{\mathcal{Y}/e_1} N\}$; it is $< \kappa$ as $\|N\| < \kappa$, and it is as required.]

Let $f_* = f_{\kappa, \zeta(*)}$. The continuation imitates [Sh371, §4], [Sh410, §5].

Let

$$K_2 = \left\{ (D, \bar{B}, \langle j_x: x \in \mathcal{Y}/e_1 \rangle): D_1 \subseteq D \in E, \text{ player II wins } G_E^{\gamma_{\langle \rangle}}(f^1, D), \right. \\
 \left. e_1 = e(D), \bar{B} = \langle \langle B_{x,j}: j < j_x^0 \leq \mu_{\iota(x)} \rangle : x \in \mathcal{Y}/e_1 \rangle \in N, \right. \\
 |B_{x,j_x}| \leq g(x) \text{ and } j_x < j_x^0 \leq \mu_{\iota(x)}, \\
 \left. \{x \in \mathcal{Y}/e_1: f_*(x) \text{ is in } B_{x,j_x}\} \in D \right\}.$$

Clearly $K_2 \neq \emptyset$. For each $(D, \bar{B}, \langle j_x: x \in \mathcal{Y}/e_1 \rangle) \in K_2$:

(*)₁ letting $h \in {}^{\mathcal{Y}/e_1}\text{Ord}$, $h(x) = |B_{x,j_x}|$, for some $\bar{h} = \langle \langle \langle \rangle, f^1 \rangle, \langle \langle 0 \rangle, h \rangle \rangle$, for some $\gamma_{<0>} < \gamma_{<>}$ and D player II wins in $G_E^{\langle \gamma_{<>}, \gamma_{<0>} \rangle}(D, \bar{h}, e_1, \mathcal{Y}_\mu)$.

So choose $(D, \bar{B}, \langle j_x: x \in \mathcal{Y}/e_1 \rangle, \gamma_{\langle 0 \rangle})$ such that:

(*)₂ $(D, \bar{B}, \langle j_x: x \in \mathcal{Y}/e_1 \rangle) \in K_2$, (*)₁ for $\gamma_{\langle 0 \rangle}$ holds and (under those restrictions) $\gamma_{\langle 0 \rangle}$ is minimal.

So (as player I can “move twice”), for every $A \in D^+$, if we replace D by $D + A$, then (*)₂ still holds.

So without loss of generality (for the first and third members use normality):

(*)₃ one of the following sets belongs to D :

$$A_{0,\zeta} = \{x \in \mathcal{Y}/e_1: \text{cf } |B_{x,j_x}| > \mu_{\iota(x)} \text{ and } j_x^0 < \mu_\zeta \}$$

$$\text{(for some } \zeta < \omega_1 \text{ such that } |\mathcal{Y}/e_1| < \mu_\zeta \text{),}$$

$$A_1 = \{x \in \mathcal{Y}/e_1: \text{cf } |B_{x,j_x}| < \mu_{\iota(x)} \leq |B_{x,j_x}|\},$$

$$A_{2,\zeta} = \{x \in \mathcal{Y}/e_1: |B_{x,j_x}| \leq \mu_\zeta \text{ and } j_x < \mu_\zeta \} \quad \text{(for some } \zeta < \omega_1 \text{).}$$

If $A_{2,\zeta} \in D$ then (for $x \in \mathcal{Y}/e_1$)

$$B_x^* =: \bigcup \{B_{x,j}: x \in \mathcal{Y}/e_1, j < j_x^0 \text{ and } |B_{x,j_x}| < \mu_\zeta \text{ and } j < \mu_\zeta \}$$

is a set of $\leq \mu_\zeta$ ordinals and

$$\{x \in \mathcal{Y}/e_1: f_*(x) \in B_x^*\} \in D$$

and $\langle B_x^*: x \in \mathcal{Y}/e_1 \rangle$ belongs to N (as $(D, \bar{B}, \langle j_x: x \in \mathcal{Y}/e_1 \rangle) \in K_2$ and the definition of K_2), contradiction to the choice of f_* (see \otimes , remember $D_1 \subseteq D$ by the definition of K_2).

If $A_1 \in D$, we can find $\bar{B}^1 \in N$, $\bar{B}^1 = \langle \langle B_{x,j}^1: j < j_x^1 \leq \mu_{\iota(x)} \rangle: x \in \mathcal{Y}/e_1 \rangle$, $|B_{x,j}^1| \leq g(x)$ and $\bigwedge_{j < j_x^1} [\text{cf } |B_{x,j}^1| \geq \mu_{\iota(x)} \vee |B_{x,j}^1| = 1]$ and each $B_{x,j}$ satisfying $\text{cf } |B_{x,j}| < \mu_{\iota(x)}$ is a union of $\text{cf } |B_{x,j}|$ sets of the form B_{x,j^1}^1 of smaller cardinality and so for some $j_x^2 < j_x^1$, $f_*(x) \in B_{x,j_x} \Rightarrow f_*(x) \in B_{x,j_x^2} \ \& \ |B_{x,j_x^2}| < |B_{x,j_x}|$. Now playing one move in $G_E^{\langle \gamma_{<>}, \gamma_{<0>} \rangle}(D, \bar{h}, e, \mathcal{Y})$ we get contradiction to choice of $\gamma_{\langle 0 \rangle}$.

We are left with the case $A_{0,\zeta} \in D$, so without loss of generality $\bigwedge_{x,j} \text{cf } |B_{x,j}| > \mu_{\iota(x)}$. Let

$$\mathfrak{a} = \{ \text{cf } |B_{x,j}|: \text{cf } |B_{x,j}| > \mu_{\iota(x)}, x \in \mathcal{Y}/e_1, j < j_x^0, j < \mu_\zeta \text{ and } \iota(x) > \zeta \},$$

so \mathfrak{a} is a set of regular cardinals, and (remember $|\mathcal{Y}/e_1| < \mu_\zeta$) we have $|\mathfrak{a}| < \text{Min } \mathfrak{a}$, so let $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\theta[\mathfrak{a}]: \theta \in \text{pcf } \mathfrak{a} \rangle$ be as in [Sh371, 2.6]. So as (by the Definition of K_2), $\langle \langle B_{x,j}: j < j_x^0 \rangle: x \in \mathcal{Y}/e_1 \rangle \in N$, clearly $\mathfrak{a} \in N$ hence without loss of generality $\bar{\mathfrak{b}} \in N$. Let $\lambda^* = \sup[\lambda \cap \text{pcf } \mathfrak{a}]$, so by Hypothesis [420, 6.1(C)], $\lambda^* < \lambda$, but $\lambda^* \in N$, so $\lambda^* + 1 \subseteq N$.

By the minimality of the rank we have for every $\theta \in \lambda^* \cap \text{pcf } \mathfrak{a}$, $\{x \in \mathcal{Y}/e_1: \text{cf } |B_{x,j_x}| \in \mathfrak{b}_\theta\} = \emptyset \pmod D$ hence $\prod_x \text{cf } |B_{x,j_x}|/D$ is λ -directed, hence we get contradiction to the minimality of the rank of f_1 .

(2), (3) Proof left to the reader. ■_{4.2}

4.2B Remark:

- (1) The proof of 4.3 below shows that in [Sh386] the assumption of the existence of nice filters is very weak, removing it will cost a little for at most one place.
- (2) We could have used the framework of [Sh386] but not for 4.3 (or use forcing).

4.3 CLAIM (Hypothesis 6.1(C) of [Sh420] even in any $K[A]$): Assume $\mu > \text{cf } \mu = \aleph_1$, $\mu > \theta > \aleph_1$, $\text{pp}_{\Gamma(\theta, \aleph_1)}(\mu) \geq \lambda > \mu$, λ inaccessible. Then for some $e \in \text{Eq}_\mu$, $D \in \text{FIL}(e, \mathcal{Y}_\mu)$ and sequence of inaccessibles $\langle \lambda_x: x \in \mathcal{Y}_\mu/e \rangle$, we have $\text{tlim}_D \lambda_x = \mu$ and $\lambda = \text{tcf}(\prod \lambda_x/D)$ except perhaps for a unique λ in V (not depending on μ) and then $\text{pp}_{\Gamma(\theta, \aleph_1)}^+(\mu) \leq \lambda^+$.

Proof: By the Hyp. (see [Sh513, 6.12]) for some $\mathfrak{a} \subseteq \text{Reg} \cap \mu$, $|\mathfrak{a}| < \text{Min}(\mathfrak{a})$, $\lambda = \max \text{pcf}(\mathfrak{a})$, and

$$(\forall \lambda' < \lambda)(\exists \mathfrak{b})[\mathfrak{b} \subseteq \mathfrak{a} \ \& \ |\mathfrak{b}| < \theta \ \& \ \lambda > \sup_{\aleph_1\text{-complete}} \text{pcf } (\mathfrak{b}) > \lambda'],$$

$J = J_{< \lambda}[\mathfrak{a}]$. First assume “in $K[A]$ there is a Ramsey cardinal $> \lambda^\theta$ when $A \subseteq \lambda^\theta$ ”. Choose $A \subseteq \lambda^\theta$ such that ${}^\theta \lambda \subseteq L[A]$ and for every $\alpha < \lambda^\theta$, there is a one to one function f_α from $|\alpha|$ (i.e. $|\alpha|^V$) onto α , $f_\alpha \in L[A]$, so $\text{Card}^{L[A]} \cap (\lambda^\theta + 1) = \text{Card}^V$, and apply 4.2 to the universe $K[A]$ (its assumption holds by [Sh420, 5.6]).

Second assume $(*)_\lambda$ “in $K[A]$ there is a Ramsey cardinal $> \lambda$ when $A \subseteq \lambda^+$ ” and assume our desired conclusion fails. Let $S \subseteq \lambda$ be stationary $[\delta \in S \Rightarrow \text{cf } \delta = \theta^+]$, $\langle a_\alpha: \alpha < \lambda \rangle$, exemplify $S \in I[\lambda]$ (exist by [Sh420, §1]). We can find \mathfrak{a} , J as described above. Let $\langle f_\alpha: \alpha < \lambda \rangle$ exemplify $\lambda = \text{tcf}(\prod \mathfrak{a}/J)$, now by [Sh355, 1.3] without loss of generality $\lambda = \max \text{pcf } \mathfrak{a}$. Let $A_0 \subseteq \lambda$ be such that \mathfrak{a} , $\langle f_\alpha: \alpha < \lambda \rangle$, $\langle \mathfrak{b}_\sigma[\mathfrak{a}]: \sigma \in \text{pcf } \mathfrak{a} \rangle$ are in $L[A_0]$. Hence in $L[A_0]$ for suitable J , $\langle f_\alpha/J: \alpha < \lambda \rangle$ is increasing, and without loss of generality for some $\langle \langle \mathfrak{c}_\alpha^\delta: \alpha \in a_\delta \rangle: \delta \in S \rangle \in L[A_0]$,

we have: for $\delta \in S$, $\text{cf } \delta = |\mathfrak{a}|^+$, a_δ a club of δ and $\langle f_\alpha \upharpoonright (\mathfrak{a} \setminus \mathfrak{c}_\alpha^\delta) : \alpha \in a_\delta \rangle$ is $<$ -increasing (see [Sh345b, 2.5] (“good point”)) and $\mathfrak{c}_\alpha^\delta \in J$ and S is stationary in V , so the assumption of 4.3 holds in V^1 whenever $L[A_0] \subseteq V^1 \subseteq V$; hence for $A \subseteq \lambda^+$, in $K[A_0, A]$ the conclusion of 4.2 holds as we are assuming $(*)_\lambda$.

Note: if $A \subseteq \lambda$, in $K[A]$, $\lambda^{<\lambda} = \lambda$ hence if $\alpha < \lambda^+$, $A \subseteq \alpha$ then $K[A] \models “\lambda^{<\lambda} < (\lambda^+)^V”$.

Choose by induction on $\alpha < \lambda^+$ a set $A_\alpha \subseteq [\lambda\alpha, \lambda(\alpha + 1))$ such that: A_0 is as above and for $\alpha > 0$: if $\langle \lambda_x : x \in \mathcal{Y}/e \rangle$, J exemplify the conclusion of 4.2 in $K \left[\bigcup_{\beta < \alpha} A_\beta \right]$, and $\langle f_i : i < \lambda \rangle$ exemplify the $\lambda = \text{tcf} \left(\prod_{x \in \mathcal{Y}/e} \lambda_x / J \right)$, without loss of generality J canonical (all in $K \left[\bigcup_{\beta < \alpha} A_\beta \right]$, canonical means: the normal ideal generated by $\{x : \lambda_x \in \mathfrak{b}_{<\lambda}[\{\lambda_y : y \in \mathcal{Y}/e\}]\}$), then in $K \left[\bigcup_{\beta \leq \alpha} A_\beta \right]$ we can find f , $\bigwedge_{\alpha < \lambda} f <_J \langle \lambda_x : x \in \mathcal{Y}/e \rangle$, $\bigwedge_\alpha f \not<_J f_\alpha$ (as they cannot exemplify the conclusion of 4.5 in V — otherwise we have finished).

Let $A = \bigcup_{\alpha < \lambda^+} A_\alpha$.

Now in $K[A]$ there are e , $\langle \lambda_x : \lambda \in \mathcal{Y}/e \rangle$, $\langle f_i : i < \lambda \rangle$ (and J) exemplifying the conclusion of 4.2 (by $(*)$ and [Sh513, 6.12(3)]). By 4.5 below, for some $\delta < \lambda^+$, e , $\langle \lambda_x : x \in \mathcal{Y}/e \rangle$, $\langle \mathfrak{b}_\sigma[\{\lambda_x : x \in \mathcal{Y}/e\}] : \sigma \in \text{pcf}\{\lambda_x : x \in \mathcal{Y}/e\} \rangle$, $f_\alpha (\alpha < \lambda)$ all belongs to $K \left[\bigcup_{\gamma < \delta} A_\gamma \right]$, and in $K \left[\bigcup_{\gamma \leq \delta} A_\gamma \right]$ we get a contradiction.

If $(*)_\lambda$ holds for every λ we are done. If not, let λ_0 be minimal such that $(*)_{\lambda_0}$ fails; so if $\lambda < \lambda_0$ the conclusion holds, and if $\lambda > \lambda_0$ then let $A \subseteq \lambda_0^+$ be such that in $K[A]$ there is no Ramsey, hence ([DoJ]) for $\mu \geq \lambda_0^+$ in V , $\text{cov}(\mu, \theta, \theta, 2) \leq \mu$, so the assumptions of 4.3 fail. Similarly $\mu > \theta$, $\text{cf}(\mu) = \aleph_1$, $\text{pp}_{\Gamma(\theta, \aleph_1)}(\mu) > \lambda_0^+$ bring a contradiction. ■_{4.3}

4.4 Conclusion: Hypothesis [Sh420, 6.1(C)] in any $K[A]$. (1) Assume $\mu > \text{cf } \mu = \aleph_1$, $\mu_0 < \mu$, $\sigma \geq |\{\lambda : \mu_0 < \lambda < \mu, \lambda \text{ inaccessible}\}| < \mu$. Then

$$\sigma^{+4} > |\{\lambda : \mu_0 < \lambda < \text{pp}_{\Gamma(\sigma, \aleph_1)}(\mu) \text{ and } \lambda \text{ is inaccessible}\}|.$$

(2) The parallel of [Sh400, 4.3].

Proof: See [Sh410, 3.5] and use 4.2(3). ■

By [DoJe]

4.5 THEOREM: If λ is regular ($> \aleph_1$) $A \subseteq \lambda$, $Z \in K[A]$ a bounded subset of λ then for some $\alpha < \lambda$, $Z \in \bigcup_{\alpha < \lambda} K[A \cap \alpha]$.

We shall return to this elsewhere.

5. Densities of Box Products

5.1 Definition: $d_{<\kappa}(\lambda, \theta)$ is the density of the topological space ${}^\lambda\theta$ where the topology is generated by the following family of clopen sets:

$$\{[f]: f \in {}^a\theta \text{ for some } a \subseteq \lambda, |a| < \kappa\}$$

where

$$[f] = \{g \in {}^\lambda\theta: g \subseteq f\}.$$

So

$$d_{<\kappa}(\lambda, \theta) = \text{Min} \{ |F|: F \subseteq {}^\lambda\theta \text{ and if } a \in \mathcal{S}_{<\kappa}(\lambda) \text{ and } g \in {}^a\theta \text{ then } (\exists f \in F) g \subseteq f \}.$$

If $\theta = 2$ we may omit it, if $\kappa = \aleph_0$ we may omit it (i.e. $d(\lambda, \theta) = d_{<\aleph_0}(\lambda, \theta)$). Always we assume $\lambda \geq \aleph_0$, $\kappa \geq \aleph_0$, $\theta > 1$ and $\lambda^+ \geq \kappa$. We write $d_\kappa(\lambda, \theta)$ for $d_{<\kappa^+}(\lambda, \theta)$.

5.1A Discussion: Note: for $\kappa = \aleph_0$ this is the Tichonov product, for higher κ those are called box products and d has obvious monotonicity properties.

$d(2^{\aleph_0}) = \aleph_0$ by the classical Hewitt–Marczewski–Pondiczery theorem [H], [Ma], [P]. This has been generalized by Engelking–Karlłowicz [EK] and by Comfort–Negreponitis [CN1], [CN2] to show, for example, that $d_{<\kappa}(2^\alpha, \alpha) = \alpha$ if and only if $\alpha = \alpha^{<\kappa}$ ([CN1] (Theorem 3.1)). Cater–Erdős–Galvin [CEG] show that every non-degenerate space X satisfies $\text{cf}(d_{<\kappa}(\lambda, X)) \geq \text{cf}(\kappa)$ when $\kappa \leq \lambda^+$, and they note (in our notation) that “ $d_{<\kappa}(\lambda)$ is usually (if not always) equal to the well-known upper bound $(\log \lambda)^{<\kappa}$ ”. It is known (cf. [CEG], [CR]) that $\text{SCH} \Rightarrow d_{<\aleph_1}(\lambda) = (\log \lambda)^{\aleph_0}$, but it is not known whether $d_{<\aleph_1}(\lambda) = (\log \lambda)^{\aleph_0}$ is a theorem of ZFC.

The point in those theorems is the upper bound, as, of course, $d_{<\kappa}(\mu, \theta) > \chi$ if $\mu > 2^\chi$ & $\theta > 2$ [why? because if $F = \{f_i: i < \chi\}$ exemplify $d_{<\kappa}(\mu, \theta) \leq \chi$, the number of possible sequences $\langle \text{Min}\{1, f_i(\zeta)\}: i < \chi \rangle$ (where $\zeta < \mu$) is $\leq 2^\chi$, so

for some $\zeta \neq \xi$ they are equal and we get contradiction by g , $g(\zeta) = 0$, $g(\xi) = 1$, $\text{Dom } g = \{\zeta, \xi\}$.

Also trivial is: for κ limit, $d_{<\kappa}(\lambda, \theta) = \kappa + \sup_{\sigma < \kappa} d_{<\sigma}(\lambda, \theta)$, so we only use κ regular; $d_{<\kappa}(\lambda, \theta) \geq \sigma^\theta$ for $\sigma < \kappa$.

Also if $\text{cf}(\lambda) < \kappa$, λ strong limit then $d_{<\kappa}(\lambda) > \lambda$. The general case (say $2^{<\mu} < \lambda < 2^\mu$, $\text{cf } \mu \leq \theta$) is similar; we ignore it in order to make the discussion simpler.

So the main problem is:

5.2 PROBLEM: *Assume λ is strong limit singular, $\lambda > \kappa > \text{cf}(\lambda)$, what is $d_{<\kappa}(\lambda)$? Is it always 2^λ ? Is it always $> \lambda^+$ when $2^\lambda > \lambda^+$?*

In [Sh93] this question was raised (later and independently) for model theoretic reasons. I thank Comfort for asking me about it in the Fall of '90.

5.3 LEMMA: *Suppose λ is singular strong limit, $\text{cf}(\lambda) = \text{cf}(\delta^*) \leq \delta^* < \text{cf}(\kappa) \leq \kappa < \lambda$, $2 \leq \theta < \lambda$, $\lambda \leq \chi < 2^\lambda$ and $\langle \lambda_\alpha, \mu_\alpha, \chi_\alpha, \chi_\alpha^* : \alpha < \delta^* \rangle$ is such that:*

$$\chi_\alpha = \theta^{\mu_\alpha}, \chi_\alpha^* = \text{cov}(\chi_\alpha, \lambda_\alpha, \lambda_\alpha, 2),$$

$$\alpha < \beta \Rightarrow \mu_\alpha < \mu_\beta,$$

$$\lambda = \bigcup_{\alpha < \delta^*} \mu_\alpha = \text{tlim}_{\alpha < \delta^*} \lambda_\alpha, \theta < \mu_\alpha,$$

$$d_{<\kappa}(\mu_\alpha, \theta) \geq \lambda_\alpha \text{ (this holds e.g. if } (\forall \lambda' < \lambda_\alpha)[2^{\lambda'} < \mu_\alpha]),$$

$$A_\alpha = [\mu_\alpha, \mu_\alpha + \mu_\alpha],$$

$$G_\alpha = \{g: g \text{ a partial function from some } a \in \mathcal{S}_{<\kappa}(A_\alpha) \text{ to } \theta\},$$

$$\text{for } g \in G_\alpha,$$

$$[g] = \{f \in X_\alpha: g \subseteq f\} \text{ where } X_\alpha =: {}^{(A_\alpha)}\theta, \text{ so } |X_\alpha| = \chi_\alpha,$$

$$h_\alpha \text{ is a function from } \mathcal{S}_{<\lambda_\alpha}({}^{(A_\alpha)}\theta) \text{ to } G_\alpha \text{ such that } h_\alpha(a) \text{ "exemplifies"}$$

$$\text{that } a \text{ is not dense in } {}^{(A_\alpha)}\theta, \text{ i.e. } [f \in a \ \& \ g = h_\alpha(a) \Rightarrow g \not\subseteq f].$$

Then $(F) \Rightarrow (E) \Rightarrow (D) \Leftrightarrow (C) \Rightarrow (B) \Leftrightarrow (A)$; and $(E)^\sigma$ decrease with σ and $(E)^\sigma \Rightarrow (G)$ when $\chi_\alpha^* = \chi_\alpha$; and if every λ_α is regular $(G) \Rightarrow (F)$ and if in addition $\bigwedge_{\alpha < \delta^*} \chi_\alpha^* = \chi_\alpha$ then $(G) \Leftrightarrow (F) \Leftrightarrow (E)$, and if $\{\alpha < \delta^*: \sigma \leq \lambda_\alpha\} \neq \emptyset \pmod J$ and $\sigma < \lambda$ then $(E) \Leftrightarrow (E)^\sigma$ (fixing J), where

(A) $d_{<\kappa}(\lambda, \theta) > \chi$;

(B) if $x_\zeta \in \prod_{\alpha < \delta^*} X_\alpha$ for $\zeta < \chi$ then there is $\bar{g} \in \prod_{\alpha < \delta^*} G_\alpha$ such that: for every $\zeta < \chi$, $\{\alpha < \delta^*: x_\zeta(\alpha) \notin [g_\zeta]\} \neq \emptyset$;

(C) if $x_\zeta \in \prod_{\alpha < \delta^*} X_\alpha$ for $\zeta < \chi$ then for some $w_\alpha \in \mathcal{S}_{<\lambda_\alpha}(X_\alpha)$ ($\alpha < \delta^*$) for every $\zeta < \chi$, $\{\alpha < \delta^*: x_\zeta(\alpha) \in w_\alpha\} \neq \emptyset$;

- (D) for every $x_\zeta \in \prod_{\alpha < \delta^*} \chi_\alpha$ for $\zeta < \chi$ there is $\bar{w} \in \prod_{\alpha < \delta^*} \mathcal{S}_{< \lambda_\alpha}(\chi_\alpha)$ such that:
 for each $\zeta < \chi$, $\bigvee_{\alpha < \delta^*} x_\zeta(\alpha) \in w_\alpha$;
- (E) $^\sigma$ for some ideal J on δ^* extending $J_{\delta^*}^{bd}$ for every $x_\zeta \in \prod_{\alpha < \delta^*} \chi_\alpha$ (for $\zeta < \chi$)
 there are $\epsilon(*) < \sigma$ and $\bar{w}^\epsilon \in \prod_{\alpha < \delta^*} \mathcal{S}_{< \lambda_\alpha}(\chi_\alpha)$ for $\epsilon < \epsilon(*)$ such that for
 each ζ we have $\bigvee_\epsilon \{\alpha < \delta^* : x_\zeta(\alpha) \notin w_\alpha^\epsilon\} = \emptyset \pmod J$.
 If $\sigma = 2$ we may omit it;
- (F) for some non-trivial ideal J on δ^* extending $J_{\delta^*}^{bd}$ we have

$$\prod_{\alpha < \delta^*} (\mathcal{S}_{< \lambda_\alpha}(\chi_\alpha), \subseteq) / J \text{ is } \chi^+ \text{-directed};$$

- (G) for some non-trivial ideal J on δ^* extending $J_{\delta^*}^{bd}$, for any $\langle \mathcal{P}_\alpha : \alpha < \delta^* \rangle$, \mathcal{P}_α
 a λ_α -directed partial order of cardinality $\leq \chi_\alpha^*$, we have: $\prod_{\alpha < \delta^*} \mathcal{P}_\alpha / J$ is
 χ^+ -directed.

5.3A Remark:

- (1) Note that the desired conclusion is 5.2(A).
- (2) The interesting case of 5.3 is when $\{\mu_\alpha : \alpha < \delta^*\}$ does not contain a club of λ .
- (3) Note that with notational changes we can arrange “ λ is the disjoint union of $A_\alpha(\alpha < \delta^*)$, hence $\lambda_\theta = \prod_{\alpha < \delta^*} X_\alpha$ ”.

Proof: Check. Clearly (E) $^\sigma$ decreases with σ , i.e. if $\sigma_1 < \sigma_2$ then (E) $^{\sigma_1} \Rightarrow$ (E) $^{\sigma_2}$.

(E) \Rightarrow (D): Just for J varying on non-trivial ideals, we have monotonicity in J ;
 and for $J = \{\emptyset\}$ we get (D).

(D) \Leftrightarrow (C): (C) is a translation of (D).

(C) \Rightarrow (B): If $x_\zeta \in \prod_{\alpha < \delta^*} X_\alpha$ for $\zeta < \chi$, let $\langle w_\alpha : \alpha < \delta^* \rangle$ be as in (C); for each
 α we know that w_α is not a dense subset of X_α (as $d_{< \kappa}(\mu_\alpha, \theta) \geq \lambda_\alpha > |w_\alpha|$) so
 there is $g_\alpha \in G_\alpha$ for which $[g_\alpha] \cap w_\alpha = \emptyset$, so $\bar{g} =: \langle g_\alpha : \alpha < \delta^* \rangle$ is as required in
 (B).

(B) \Leftrightarrow (A): They say the same (see 5.3A(3)).

(F) \Rightarrow (E): Note that (E) just says that in $\prod_{\alpha < \delta^*} (\mathcal{S}_{< \lambda_\alpha}(\chi_\alpha), \subseteq)$, any subset of
 $\{f : f \in \prod_{\alpha < \delta^*} \mathcal{S}_{< \lambda_\alpha}(\chi_\alpha), \text{ such that each } f(\alpha) \text{ is a singleton}\}$ has a \leq_J -upper
 bounded. In this form it is clearly a specific case of (F).

(E)^σ ⇒ (G) WHEN $\chi_\alpha = \chi_\alpha^*$: where $\{\alpha < \delta^* : \sigma \leq \lambda_\alpha\} \neq \emptyset \text{ mod } J$: Easy too.

Next assume every λ_α is regular, J an ideal on δ^* .

(G) ⇒ (F): (F) is a particular case of (G), because $(\mathcal{S}_{<\lambda_\alpha}(\chi_\alpha) \sqsubseteq)$ is λ_α -directed as λ_α is regular and $\mathcal{S}_{<\lambda_\alpha}(\chi_\alpha)$ can be replaced by any cofinal subset and there is one of cardinality χ_α^* by its definition.

The rest should be clear. ■_{5.3}

5.4 CLAIM: Assume λ is strong limit, $\theta < \lambda_0$, $\langle \lambda_\alpha : \alpha < \delta^* \rangle$, $\langle \chi_\alpha^* : \alpha < \delta^* \rangle$ are (strictly) increasing with limit λ , $\delta^* < \kappa \leq \text{cf}(\lambda) < \lambda$, $\lambda < \chi < 2^\lambda$ and $\lambda_\alpha \leq \chi_\alpha^*$, λ_α regular for each $\alpha < \delta^*$. Then (G) of 5.3 holds (hence $d_{<\kappa}(\lambda, \theta) > \chi$) in any of the following cases:

- (a) for some μ_α strong limit, $\text{cf}(\mu_\alpha) < \kappa$, $2^{\mu_\alpha} = \mu_\alpha^+$, $\lambda_\alpha = \mu_\alpha^+$, $\chi_\alpha^* = \mu_\alpha^+$ and $\prod_{\alpha < \delta^*} \mu_\alpha^+ / J$ is χ^+ -directed,
- (b) $k < \omega$ and for every α , $\chi_\alpha^* \leq \lambda_\alpha^{+k}$ and for some ideal J on δ^* , for $\ell \leq k$, $\prod \lambda_\alpha^{+\ell} / J$ is χ^+ -directed, and $d_{<\kappa}(\chi_\alpha^*, \theta) \geq \lambda_\alpha$,
- (c) for some $\gamma < \text{cf}(\lambda)$ for every $\alpha < \delta^*$, $\chi_\alpha^* \leq \lambda_\alpha^{+\gamma}$ and for some ideal J on δ^* for every $\zeta < \gamma$, $\prod_{\alpha < \delta^*} \lambda_\alpha^{+(\zeta+1)} / J$ is χ^+ -directed, and $d_{<\kappa}(\chi_\alpha^*, \theta) \geq \lambda_\alpha$,
- (d) for some ideal J on δ^* extending $J_{\delta^*}^{bd}$ for every regular $\lambda'_\alpha \in [\lambda_\alpha, \chi_\alpha^*]$ satisfying $\text{tlim}_J(\text{cf } \lambda'_\alpha) = \lambda$, we have $\prod_{\alpha < \delta^*} \lambda'_\alpha / J$ is χ^+ -directed and $d_{<\kappa}(\chi_\alpha^*, \theta) \geq \lambda_\alpha$.

Proof: Clearly (a) ⇒ (b) ⇒ (c) ⇒ (d).

Now the statements follow from the following observations 5.4A–5.7.

5.4A Observation: Assume that for $\alpha < \delta$, \mathcal{P}_α is a (non-empty) λ_α -directed partial order of cardinality χ_α , $|\delta|^+ < \lambda_\alpha = \text{cf}(\lambda_\alpha) \leq \chi_\alpha$, J an ideal on δ , $\theta^* = \text{Min}\{\theta : \text{for some } A \text{ and } \bar{f} : \bar{f} = \langle f_i : i < \theta \rangle, f_i \in \prod_{\alpha < \delta} \mathcal{P}_\alpha \text{ is } <_{J+A}\text{-increasing, } A \subseteq \delta, \delta \setminus A \notin J \text{ but for no } g \in \prod_{\alpha < \delta} \mathcal{P}_\alpha, \bigwedge_{i < \theta} \{\alpha : \mathcal{P}_\alpha \models f_i(\alpha) \leq g(\alpha)\} \neq \emptyset \text{ mod } (J + A)\}$. Then $\prod_{\alpha < \delta} \mathcal{P}_\alpha / J$ is θ^* -directed.

Proof: Without loss of generality no \mathcal{P}_α has a maximal element. If the conclusion of 5.4A fails, let F be a subset of $\prod_{\alpha < \delta} \mathcal{P}_\alpha$ with no $<_J$ -upper bound, of minimal cardinality. Let $\theta = |F|$, so let $F = \{f_i : i < \theta\}$; by the choice of F without loss of generality $\alpha < \beta \Rightarrow f_\alpha <_J f_\beta$ hence θ is necessarily regular. If $\{\alpha < \delta : \lambda_\alpha \leq \theta\} \in J$ we can find an upper bound: $g(\alpha)$ is a \mathcal{P}_α -upper bound of $\{f_i(\alpha) : i < \theta\}$ when $\lambda_\alpha > \theta$, and arbitrarily otherwise. So without loss of generality $\bigwedge_\alpha \lambda_\alpha \leq \theta$. Now, remember $|\delta|^+ < \lambda_\alpha$, and so $|\delta|^+ < \theta$. By [Sh420, §1] we can find $\bar{C} = \langle C_i : i < \theta \rangle$,

$C_i \subset i, j \in C_i \Rightarrow C_j = j \cap C_i, \text{otp}(C_i) \leq |\delta|^+$ and $S =: \{i < \lambda: \text{cf}(i) = |\delta|^+, \delta = \text{sup}(C_i)\}$ stationary: so wlog $j \in C_i \Rightarrow \bigwedge_{\alpha < \delta} \mathcal{P}_\alpha \models f_j(\alpha) < f_i(\alpha)$. Now we repeat the proof from [Sh282, 14]; better see [Sh345a, 2.6] or here 6.1.* ■_{5.4A}

5.5 *Observation:* In 5.4A, if A, \bar{f} exemplify $\theta^* = \theta$ then

$$\theta^* \geq \min_{J+A} \{ \text{pre}(\bar{\chi}, \bar{\lambda}): A \subseteq \delta \text{ and } \delta \setminus A \notin J \}$$

where

5.6 *Definition:* For ideal I on δ and $\bar{\chi} = \langle \chi_\alpha: \alpha < \delta \rangle, \bar{\lambda} = \langle \lambda_\alpha: \alpha < \delta \rangle, \lambda_\alpha = \text{cf}(\lambda_\alpha) \leq \chi_\alpha$ we let $\text{pre}_I(\bar{\chi}, \bar{\lambda}) =: \text{Min} \{ |\mathcal{P}|: \mathcal{P} \text{ is a family of sequences of the form } \langle B_\alpha: \alpha < \delta \rangle, B_\alpha \subseteq \chi_\alpha, |B_\alpha| < \lambda_\alpha \text{ such that for every } g \in \prod_{\alpha < \delta} \chi_\alpha \text{ for some } \bar{B} \in \mathcal{P}, \{ \alpha < \delta: g(\alpha) \in B_\alpha \} \neq \emptyset \text{ mod } I \}$.

Proof: Check.

5.6A *Remark:* We use other parts of 5.3.

5.7 *Observation:* Let I be an ideal on $\delta^*, \chi_\alpha \geq \lambda_\alpha > \delta^*$.

- (1) Define $\mathcal{J}[I] = \{ I + A: A \subseteq \delta, \delta \setminus A \notin I \}$.
- (2) If $I_1 \subseteq I_2, \lambda_\alpha^1 \geq \lambda_\alpha^2, \chi_\alpha^1 \leq \chi_\alpha^2$ for $\alpha < \delta$ then $\text{pre}_{I_1}(\bar{\chi}^1, \bar{\lambda}^1) \leq \text{pre}_{I_2}(\bar{\chi}^2, \bar{\lambda}^2)$.
- (3) If δ^* is the disjoint union of $A_1, A_2, A_\ell \notin I$ and $I_\ell =: I + A_\ell$ then $\text{pre}_I(\bar{\chi}, \bar{\lambda}) = \text{Min} \{ \text{pre}_{I_1}(\bar{\chi}, \bar{\lambda}), \text{pre}_{I_2}(\bar{\chi}, \bar{\lambda}) \}$.
- (4) $\text{pre}_I(\bar{\chi}^+, \bar{\lambda}) \leq \text{pre}_I(\bar{\chi}, \bar{\lambda}) + \text{sup} \{ \text{tcf}(\prod \chi_\alpha^+ / I + A): A \subseteq \delta, \delta \setminus A \notin I \}$.**
 Moreover $\text{pre}_I(\bar{\chi}^+, \bar{\lambda}) \leq \text{Min} \{ \text{pre}_{I+A}(\bar{\chi}, \bar{\lambda}) + \text{tcf}(\prod_{\alpha < \delta} \chi_\alpha^+ / (I + A)): A \subseteq \delta, \delta \setminus A \notin I \text{ (and the tcf is well defined)} \}$.
- (5) If each χ_α is a limit cardinal, $\text{cf} \chi_\alpha > \delta^*$, then $\text{sup}_{J \in \mathcal{J}[I]} \text{pre}_J(\bar{\chi}, \bar{\lambda}) = \text{sup}_{\bar{\chi}' < \bar{\chi}} \text{sup}_{J \in \mathcal{J}[I]} \text{pre}_J(\bar{\chi}', \bar{\lambda}) + \text{sup}_{J \in \mathcal{J}[I]} \text{tcf}(\prod \chi_\alpha / I)$.
- (6) $2^{|\delta^*|} + \text{sup}_{J \in \mathcal{J}[I]} \text{sup} \{ \text{tcf}(\prod_{\alpha < \delta} \chi'_\alpha / J): \lambda_\alpha \leq \chi'_\alpha = \text{cf}(\chi'_\alpha) \leq \chi_\alpha \text{ and the true cofinality is well defined} \} \leq 2^{|\delta^*|} + \text{sup}_{J \in \mathcal{J}[I]} \text{pre}_J(\bar{\chi}, \bar{\lambda}) \leq 2^{|\delta^*|} + \text{sup}_{J \in \mathcal{J}[I]} \text{sup} \{ \text{tcf}(\prod_{\alpha < \delta} \chi'_\alpha / J): |\delta^*| < \text{cf}(\chi'_\alpha) \text{ and } \lambda_\alpha \leq \chi'_\alpha \leq \chi_\alpha \}$.
- (7) In part (6), if I is a precipitous ideal then the first inequality is equality.

Proof: Straightforward.

* In the main case here, $\bigwedge_\alpha 2^{|\delta^*|} < \lambda_\alpha$ and then trying all the possible A 's, using their g 's, the proof is very simple.

** Of course, $\bar{\chi}^+ = \langle \chi_\alpha^+: \alpha < \delta \rangle$.

5.9 Observation: In several of the models of set theory in which we know “ λ strong, singular, limit, $2^\lambda > \lambda^+$ ” our sufficient conditions for $d_{\text{cf } \lambda}(\lambda, 2) = 2^\lambda$ usually hold by the sufficient condition 5.4(a) (simplest: if *GCH* holds below λ , cf $\lambda = \aleph_0$).

Remark: We could prove this consistency by looking more at the consistency proofs, adding many Cohen subsets to λ in preliminary forcing; but the present way looks more informative.**

6. Odds and Ends

6.1 LEMMA: Suppose $\text{cf}(\delta) > \kappa^+$, I an ideal on κ , $f_\alpha \in {}^\kappa\text{Ord}$ for $\alpha < \delta$ is \leq_I -increasing. Then there are $J_\alpha, \bar{s}, f'_\alpha(\alpha < \delta)$ such that:

- (A) $\bar{s} = \langle s_i : i < \kappa \rangle$, each s_i a set of $\leq \kappa$ ordinals,
- (B) $\bigwedge_{i < \kappa} \bigwedge_{\alpha < \delta} \bigvee_{\beta \in s_i} f_\alpha(i) \leq \beta$,
- (C) $f'_\alpha \in \prod_{i < \kappa} s_i$ is defined by $f'_\alpha(i) = \text{Min}[s_i \setminus f_\alpha(i)]$,
- (D) $\text{cf}[f'_\alpha(i)] \leq \kappa$ (e.g. $f'_\alpha(i)$ is a successor ordinal) implies $f'_\alpha(i) = f_\alpha(i)$,

such that:

- (E) J_α is an ideal on κ extending I (for $\alpha < \lambda$), decreasing with α (in fact for some $a_{\alpha, \beta} \subseteq \kappa$ (for $\alpha < \beta < \kappa$), $a_{\alpha, \beta}/I$ decreases with β , increases with α and J_α is the ideal generated by $I \cup \{a_{\alpha, \beta} : \alpha < \beta < \lambda\}$) so possibly $J_\alpha = \mathcal{P}(\kappa)$ and possibly $J_\alpha = I$,
- (F) if D is an ultrafilter on κ disjoint to J_α then f'_α/D is a $<_D$ -l.u.b of $\langle f_\beta/D : \beta < \delta \rangle$ and $\{i < \kappa : \text{cf}[f'_\alpha(i)] > \kappa\} \in D$,
- (G) if D is an ultrafilter on κ disjoint to I but for every α not disjoint to J_α then \bar{s} exemplifies $\langle f_\alpha : \alpha < \delta \rangle$ is chaotic for D , i.e. for some club E of δ , $\beta < \gamma \in E \Rightarrow f_\beta \leq_D f'_\beta <_D f_\gamma$,
- (H) if $\text{cf}(\delta) > 2^\kappa$ then $\langle f_\alpha : \alpha < \delta \rangle$ has a \leq_I -l.u.b. and even \leq_I -e.u.b,
- (I) if $b_\alpha =: \{i : f'_\alpha(i) \text{ has cofinality } \leq \kappa \text{ (e.g. is a successor)}\} \notin J_\alpha$ then: for every $\beta \in (\alpha, \delta)$ we have $f'_\alpha \upharpoonright b_\alpha = f'_\beta \upharpoonright b_\alpha \text{ mod } J_\alpha$.

Moreover

(F)⁺ if $\kappa \notin J_\alpha$ then f'_α is an $<_{J_\alpha}$ -e.u.b (= exact upper bound) of $\langle f_\beta : \beta < \delta \rangle$.

Proof: Let $S = \{j : j \leq \sup \bigcup_{\alpha < \delta} \text{Rang}(f_\alpha) \text{ has cofinality } \leq \kappa\}$, $\bar{e} = \langle e_j : j \in S \rangle$ be such that $[j = i + 1 \Rightarrow e_j = \{i\}]$, $[j \text{ limit } \& j' \in S \cap e_j \Rightarrow e_{j'} \subseteq e_j]$, $e_j \subseteq j$ $[j \text{ limit } \Rightarrow j = \sup e_j]$ and $|e_j| \leq \kappa$.

** See much more on independence in a paper of Gitik and Shelah.

For a set $a \subseteq \sup \bigcup_{\alpha < \delta} \text{Rang}(f_\alpha)$ let $\bar{e}[a] = a \cup \bigcup_{j \in a \cap S} e_j$ hence $\bar{e}[\bar{e}[a]] = \bar{e}[a]$ and $[a \subseteq b \Rightarrow \bar{e}[a] \subseteq \bar{e}[b]]$ and $|\bar{e}[a]| \leq |a| + \kappa$. We try to choose by induction on $\zeta < \kappa^+$, the following: $\alpha_\zeta, D_\zeta, g_\zeta, \bar{s}_\zeta = \langle s_{\zeta,i}: i < \kappa \rangle, \langle f_{\zeta,\alpha}: \alpha < \delta \rangle$ such that:

- (a) $g_\zeta \in {}^\kappa \text{Ord}$,
- (b) $s_{\zeta,i} = \bar{e}[\{g_\epsilon(i): \epsilon < \zeta\} \cup \{\sup_{\alpha < \delta} f_\alpha(i) + 1\}]$ so it is a set of $\leq \kappa$ ordinals, increasing with ζ , $\sup_{\alpha < \delta} f_\alpha(i) + 1 \in s_{\zeta,i}$,
- (c) $f_{\zeta,\alpha} \in {}^\kappa \text{Ord}$, $f_{\zeta,\alpha}(i) = \text{Min}[s_{\zeta,i} \setminus f_\alpha(i)]$,
- (d) D_ζ is an ultrafilter on κ disjoint to I ,
- (e) for $\alpha < \delta$, $f_\alpha \leq_{D_\zeta} g_\zeta$,
- (f) α_ζ is an ordinal $< \delta$,
- (g) $\alpha_\zeta \leq \alpha < \lambda \Rightarrow g_\zeta <_{D_\zeta} f_{\zeta,\alpha}$.

If we succeed, let $\alpha(*) = \sup_{\zeta < \kappa^+} \alpha_\zeta$, so as $\text{cf}(\delta) > \kappa^+$ clearly $\alpha(*) < \delta$. Now let $i < \kappa$ and look at $\langle f_{\zeta,\alpha(*)}(i): \zeta < \kappa^+ \rangle$; by its definition (see (c)), $f_{\zeta,\alpha(*)}(i)$ is the minimal member of the set $s_{\zeta,i} \setminus f_{\alpha(*)}(i)$. This set increases with ζ , so $f_{\zeta,\alpha(*)}(i)$ decreases with ζ (though not necessarily strictly), hence is eventually constant; so for some $\zeta_i < \kappa^+$ we have $\zeta \in [\zeta_i, \kappa^+) \Rightarrow f_{\zeta,\alpha(*)}(i) = f_{\zeta_i,\alpha(*)}(i)$. Let $\zeta(*) = \sup_{i < \kappa} \zeta_i$, so $\zeta(*) < \kappa^+$, hence

$$(*) \quad \zeta \in [\zeta(*), \kappa^+) \Rightarrow \bigwedge_i f_{\zeta,\alpha(*)}(i) = f_{\zeta(*),\alpha(*)}(i) \Rightarrow f_{\zeta,\alpha(*)} = f_{\zeta(*),\alpha(*)}.$$

We know that $f_{\alpha(*)} \leq_{D_{\zeta(*)}} g_{\zeta(*)} <_{D_{\zeta(*)}} f_{\zeta(*),\alpha(*)}$ hence for some i , $f_{\alpha(*)}(i) \leq g_{\zeta(*)}(i) < f_{\zeta(*),\alpha(*)}(i)$, but $g_{\zeta(*)}(i) \in s_{\zeta(*)+1,i}$ hence $f_{\zeta(*)+1,\alpha(*)}(i) \leq g_{\zeta(*)}(i) < f_{\zeta(*),\alpha(*)}(i)$, contradicting the choice of $\zeta(*)$.

So necessarily for some $\zeta < \kappa^+$ we are stuck, and clearly $s_{\zeta,i}(i < \kappa)$, $f_{\zeta,\alpha}(\alpha < \lambda)$ are well defined.

Let $s_i =: s_{\zeta,i}$ (for $i < \kappa$) and $f'_\alpha = f_{\zeta,\alpha}$ (for $\alpha < \lambda$). Clearly s_i is a set of $\leq \kappa$ ordinals; now clearly:

- (*)₁ $f_\alpha \leq f'_\alpha$
- (*)₂ $\alpha < \beta \Rightarrow f'_\alpha \leq_I f'_\beta$,
- (*)₃ if $b = \{i: f'_\alpha(i) < f'_\beta(i)\} \notin I$, $\alpha < \beta < \delta$ then $f'_\alpha \upharpoonright b <_I f'_\beta \upharpoonright b$.

We let for $\alpha < \delta$

$$J_\alpha = \left\{ b \subseteq \kappa: b \in I \text{ or } b \notin I \quad \text{and for some } \beta \text{ we have: } \alpha < \beta < \delta \quad \text{and} \right. \\ \left. f'_\alpha \upharpoonright (\kappa \setminus b) =_I f'_\beta \upharpoonright (\kappa \setminus b) \right\}.$$

We let for $\alpha < \beta < \delta$, $a_{\alpha,\beta} =: \{i < \kappa: f'_\alpha(i) < f'_\beta(i)\}$. Then

(*)₄ J_α is an ideal on κ extending I , in fact is the ideal generated by $I \cup \{a_{\alpha,\beta} : \beta \in (\alpha, \delta)\}$.

As $\langle f'_\alpha : \alpha < \delta \rangle$ is \leq_I -increasing (i.e. (*)₁):

(*)₅ J_α decreases with α , in fact $a_{\alpha,\beta}/I$ increases with β , decreases with α ,

(*)₆ if D is an ultrafilter on κ disjoint to J_α , then f'_α/D is a $<_D$ -lub of $\{f_\beta/D : \beta < \delta\}$.

[Why? We know that $\beta \in (\alpha, \delta) \Rightarrow a_{\alpha,\beta} = \emptyset \text{ mod } D$, so $f_\beta \leq f'_\beta =_D f'_\alpha$ for $\beta \in (\alpha, \delta)$, so f'_α/D is an \leq_D -upper bound. If it is not a least upper bound then for some $g \in {}^\kappa \text{Ord}$, $\bigwedge_\beta f_\beta \leq_D g <_D f'_\alpha$ and we can get a contradiction to the choice of ζ, \bar{s}, f'_β as: (D, g) could serve as D_ζ, g_ζ .]

(*)₇ If D is an ultrafilter on κ disjoint to I but not to J_α (for every $\alpha < \lambda$) then \bar{s} exemplifies $\langle f_\alpha : \alpha < \delta \rangle$ is chaotic for D .

[Why? For every $\alpha < \delta$ for some $\beta \in (\alpha, \delta)$ we have $a_{\alpha,\beta} \in D$, i.e. $\{i < \kappa : f'_\alpha(i) < f'_\beta(i)\} \in D$, so $\langle f'_\alpha/D : \alpha < \delta \rangle$ is not eventually constant, so if $\alpha < \beta$, $f'_\alpha <_D f'_\beta$ then $f'_\alpha <_D f_\beta$ (by (*)₃) and $f_\beta \leq_D f'_\beta$ (by (c)) as required.]

(*)₈ if $\kappa \notin J_\alpha$ then f'_α is an \leq_{J_α} -e.u.b. of $\langle f_\beta : \beta < \delta \rangle$.

[Why? By (*)₆, f'_α is a \leq_{J_α} -upper bound of $\langle f_\beta : \beta < \delta \rangle$; so assume that it is not a \leq_{J_α} -e.u.b. of $\langle f_\beta : \beta < \delta \rangle$, hence there is a function g with domain κ , such that $g(i) < \text{Max}\{1, f'_\alpha(i)\}$, but for no $\beta < \delta$ do we have

$$C_\beta =: \{i < \kappa : g(i) < \text{Max}\{1, f_\beta(i)\}\} = \kappa \text{ mod } J_\alpha.$$

Clearly $\langle C_\beta : \beta < \delta \rangle$ is increasing modulo J_α so there is an ultrafilter D on κ disjoint to $J_\alpha \cup \{C_\beta : \beta < \delta\}$. So $f_\beta \leq_D g \leq_D f'_\alpha$, so we get a contradiction to (*)₆ except when $g =_D f'_\alpha$ and then $f'_\alpha =_D O_\kappa$ (as $g(i) < 1 \vee g(i) < f'_\alpha(i)$). If we can demand $b^* = \{i : f'_\alpha(i) = 0\} \notin D$ we are done, but easily $b^* \setminus C_\beta \in J_\alpha$ so we finish.]

(*)₉ If $\text{cf}[f'_\alpha(i)] \leq \kappa$ then $f'_\alpha(i) = f_\alpha(i)$.

[Why? By the definition of $s_\zeta = \bar{e}[\dots]$ and the choice of \bar{e} , and $f'_\alpha(i)$.]

(*)₁₀ Clause (I) of the conclusion holds.

[Why? As $f_\alpha \leq_{J_\alpha} f_\beta \leq_{J_\alpha} f'_\alpha$ and $f_\alpha \upharpoonright b =_{J_\alpha} f'_\alpha \upharpoonright b$ by (*)₉.]

The reader can check the rest. ■_{6.1}

6.1A Example: We show that l.u.b and e.u.b are not the same. Let I be an ideal on κ , $\kappa^+ < \lambda = \text{cf}(\lambda)$, $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ be a sequence of subsets of κ , (strictly) increasing modulo I , $\kappa \setminus a_\alpha \notin I$ but there is no $b \in \mathcal{P}(\kappa) \setminus I$ such that

$\bigwedge_\alpha b \cap a_\alpha \in I$. [Does this occur? E.g. for $I = \mathcal{S}_{<\aleph_0}(\omega)$, the existence of such \bar{a} is known to be consistent; e.g. MA & $\kappa = \aleph_0$ & $\lambda = 2^{\aleph_0}$. Moreover, for any κ and $\kappa^+ < \lambda = \text{cf } \lambda \leq 2^\kappa$ we can find $a_\alpha \subseteq \kappa$ for $\alpha < \lambda$ such that, e.g., any Boolean combination of the a_α 's has cardinality κ (less needed). Let I_0 be the ideal on κ generated by $\mathcal{S}_{<\kappa}(\kappa) \cup \{a_\alpha \setminus a_\beta : \alpha < \beta < \lambda\}$, and let I be maximal in $\{J : J \text{ an ideal on } \kappa, I_0 \subseteq J \text{ and } [\alpha < \beta < \lambda \Rightarrow a_\beta \setminus a_\alpha \notin J]\}$. So if G.C.H. fails, we have examples.] For $\alpha < \lambda$, we let $f_\alpha : \kappa \rightarrow \text{Ord}$ be:

$$f_\alpha(i) = \begin{cases} \alpha & \text{if } \alpha \in \kappa \setminus a_i, \\ \lambda + \alpha & \text{if } \alpha \in a_i. \end{cases}$$

Now the constant function $f \in {}^\kappa \text{Ord}$, $f(i) = \lambda + \lambda$ is a l.u.b of $\langle f_\alpha : \alpha < \lambda \rangle$ but not an e.u.b. (both mod J) (not e.u.b. is exemplified by $g \in {}^\kappa \text{Ord}$ which is constantly λ).

6.2 CLAIM: Suppose $\mu > \kappa = \text{cf } \mu$, $\mu = \text{tlim}_J \lambda_i$, $\delta < \mu$, $\lambda_i = \text{cf}(\lambda_i) > \delta$ for $i < \delta$, J a σ -complete ideal on δ and $\lambda = \text{tcf}(\prod_{i < \delta} \lambda_i / J)$, and $\langle f_\alpha : \alpha < \lambda \rangle$ exemplifies this.

Then we have

(*) if $\langle u_\beta : \beta < \lambda \rangle$ is a sequence of pairwise disjoint non-empty subsets of λ , each of cardinality $\leq \sigma$ (not $< \sigma!$) and $\alpha^* < \mu$, then we can find $B \subseteq \lambda$ such that:

- (a) $\text{otp}(B) = \alpha^*$,
- (b) if $\beta \in B$, $\gamma \in B$ and $\beta < \gamma$ then $\sup u_\beta < \min u_\gamma$,
- (c) we can find $s_\zeta \in J$ for $\zeta \in \bigcup_{i \in B} u_i$ such that: if $\zeta \in \bigcup_{\beta \in B} u_\beta$, $\xi \in \bigcup_{\beta \in B} u_\beta$, $\zeta < \xi$ and $i \in \delta \setminus s_\zeta \setminus s_\xi$, then $f_\zeta(i) < f_\xi(i)$.

Proof: For each regular $\theta, \theta^+ < \mu$, there is a stationary $S_\theta \subseteq \{\delta < \lambda : \text{cf}(\delta) = \theta < \delta\}$ which is in $I[\lambda]$ (see [Sh420, 1.5]) which is equivalent (see [Sh420, 1.2(1)]) to:

- (*) there is $\bar{C}^\theta = \langle C_\alpha^\theta : i < \lambda \rangle$,
- (α) C_α^θ a subset of α , with no accumulation points (in C_α^θ),
- (β) $[\alpha \in \text{nacc}(C_\beta^\theta) \Rightarrow C_\alpha^\theta = C_\beta^\theta \cap \alpha]$,
- (γ) for some club E_θ^0 of λ ,

$$[\delta \in S_\theta \cap E_\theta^0 \Rightarrow \text{cf}(\delta) = \theta < \delta \ \& \ \delta = \sup C_\delta^\theta \ \& \ \text{otp}(C_\delta^\theta) = \theta].$$

Without loss of generality $S_\theta \subseteq E_\theta^0$, and $\bigwedge_{\alpha < \delta} \text{otp}(C_\delta^\theta) \leq \theta$. By [Sh365, 2.3, Def. 1.3] for some club E_θ of λ , $\langle \text{gl}(C_\alpha^\theta, E_\theta) : \alpha \in S_\theta \rangle$ guess clubs (i.e. for every

club $E \subseteq E_\theta$ of λ , for stationarily many $\zeta \in S_\theta$, $gl(C_\zeta^\theta, E_\theta) \subseteq E$ (remember $gl(C_\delta^\theta, E_\theta) = \{\sup(\gamma \cap E_\theta) : \gamma \in C_\delta^\theta; \gamma > \text{Min}(E_\theta)\}$). Let $C_\alpha^{\theta,*} = \{\gamma \in C_\alpha^\theta : \gamma = \text{Min}(C_\alpha^\theta \setminus \sup(\gamma \cap E_\theta))\}$, they have all the properties of the C_α^θ 's and guess clubs in a weak sense: for every club E of λ for some $\alpha \in S_\theta \cap E$, if $\gamma_1 < \gamma_2$ are successive members of E then $|(\gamma_1, \gamma_2] \cap C_\alpha^{\theta,*}| \leq 1$; moreover, the function $\gamma \mapsto \sup(E \cap \gamma)$ is one to one on $C_\zeta^{\theta,*}$.

Now we define by induction on $\zeta < \lambda$, an ordinal α_ζ and functions $g_\theta^\zeta \in \prod_{i < \delta} \lambda_i$ (for each $\theta \in \Theta = \{\theta : \theta < \mu, \theta \text{ regular uncountable}\}$).

For given ζ , let $\alpha_\zeta < \lambda$ be minimal such that:

$$\begin{aligned} \xi < \zeta &\Rightarrow \alpha_\xi < \alpha_\zeta, \\ \xi < \zeta \ \&\ \theta \in \Theta &\Rightarrow g_\theta^\xi < f_{\alpha_\zeta} \text{ mod } J. \end{aligned}$$

Now α_ζ exists as $\langle f_\alpha : \alpha < \lambda \rangle$ is $<_J$ -increasing cofinal in $\prod_{i < \lambda_i} / J$. Now for each $\theta \in \Theta$ we define g_θ^ζ as follows:

for $i < \delta^*$, $g_\theta^\zeta(i)$ is $\sup \left[\{g_\theta^\xi(i) + 1 : \xi \in C_\zeta^\theta\} \cup \{f_{\alpha_\zeta}(i) + 1\} \right]$ if this number is $< \lambda_i$, and $f_{\alpha_\zeta}(i)$ otherwise.

Having made the definition we prove the assertion. We are given $\langle u_\beta : \beta < \lambda \rangle$, a sequence of pairwise disjoint non-empty subsets of λ , each of cardinality $< \sigma$ and $\alpha^* < \mu$. We should find B as promised; let $\theta = (|\alpha^*| + |\delta|)^+$ so $\theta < \mu$ is regular $> |\delta|$. Let $E = \{\delta \in E_\theta : \text{for every } \zeta : [\zeta < \delta \Leftrightarrow \sup u_\zeta < \delta \Leftrightarrow u_\zeta \subseteq \delta \Leftrightarrow \alpha_\zeta < \delta]\}$. Choose $\alpha \in S_\theta \cap \text{acc}(E)$ such that $gl(C_\alpha^\theta, E_\theta) \subseteq E$; hence letting $C_\alpha^{\theta,*} = \{\gamma_i : i < \theta\}$ (increasing) we know $\bigwedge_i (\gamma_i, \gamma_{i+1}) \cap E \neq \emptyset$. Now $B = \{\gamma_{5i+3} : i < \alpha^*\}$ are as required. For $\alpha \in \bigcup_{\zeta < \alpha^*} u_{5\zeta+3}$ let $s_\alpha = s_\alpha^0 \cup s_\alpha^1$. For $\alpha \in u_{5\zeta+3}$, $\zeta < \alpha^*$, let $s_\alpha^0 = \{i < \delta : g_\theta^{5\zeta+1}(i) < f_\alpha(i) < g_\theta^{5\zeta+4}(i)\}$, for each $\zeta < \alpha^*$; let $\langle \alpha_\epsilon : \epsilon < |u_{5\zeta+3}| \rangle$ enumerate $u_{5\zeta+3}$ and

$$s_{\alpha_\epsilon}^1 = \{i : \text{for every } \xi < \epsilon, f_{\alpha_\xi}(i) < f_{\alpha_\epsilon}(i) \Leftrightarrow \alpha_\xi < \alpha_\epsilon \Leftrightarrow f_{\alpha_\xi}(i) \leq f_{\alpha_\epsilon}(i)\}. \quad \blacksquare_{6.2}$$

6.2A Remark: In 6.2: (1) We can avoid guessing clubs.

(2) Assume $\sigma < \theta_1 < \theta_2 < \mu$ are regular and there is $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \theta_1\}$ from $I[\lambda]$ such that for every $\zeta < \lambda$ (or at least a club) of cofinality θ_2 , $S \cap \zeta$ is stationary and $\langle f_\alpha : \alpha < \lambda \rangle$ obey suitable \bar{C}^θ (see [Sh345a, §2]). Then for some $A \subseteq \lambda$ unbounded, for every $\langle u_\beta : \beta < \theta_2 \rangle$ sequence of pairwise disjoint non-empty subsets of A , each of cardinality $< \sigma$ with $[\min u_\beta, \sup u_\beta]$ pairwise disjoint we have: for every $B_0 \subseteq A$ of order type θ_2 , for some $B \subseteq B_0$, $|B| = \theta_1$, (c) of (*) of 6.2 holds.

(3) In (*) of 6.2, “ $\alpha^* < \mu$ ” can be replaced by “ $\alpha^* < \mu^+$ ” (prove by induction on α^*).

6.3 OBSERVATION: Assume $\lambda < \lambda^{<\lambda}$, $\mu = \text{Min}\{\mu: 2^\mu > \lambda\}$. Then there are δ , χ and \mathcal{T} , satisfying the condition (*) below for $\chi = 2^\mu$ or at least arbitrarily large regular $\chi \leq 2^\mu$.

(*) \mathcal{T} a tree with δ levels, (where $\delta \leq \mu$) with a set X of $\geq \chi$ δ -branches, and for $\alpha < \delta$, $\bigcup_{\beta < \alpha} |\mathcal{T}_\beta| < \lambda$.

Proof of Observation: So let $\chi \leq 2^\mu$ be regular, $\chi > \lambda$.

CASE 1: $\bigwedge_{\alpha < \mu} 2^{|\alpha|} < \lambda$. Then $\mathcal{T} = {}^{\mu>}2$, $\mathcal{T}_\alpha = {}^\alpha 2$ are O.K. (the set of branches ${}^\mu 2$ has cardinality 2^μ).

CASE 2: *Not Case 1.* So for some $\theta < \mu$, $2^\theta \geq \lambda$, but by the choice of μ , $2^\theta \leq \lambda$, so $2^\theta = \lambda$, $\theta < \mu$ and so $\theta \leq \alpha < \mu \Rightarrow 2^{|\alpha|} = 2^\theta$. Note $|{}^{\mu>}2| = \lambda$ as $\mu \leq \lambda$.

SUBCASE 2A: $\text{cf}(\lambda) \neq \text{cf}(\mu)$. Let ${}^{\mu>}2 = \bigcup_{j < \lambda} B_j$, B_j increasing with j , $|B_j| < \lambda$. For each $\eta \in {}^\mu 2$, (as $\text{cf}(\lambda) \neq \text{cf}(\mu)$) for some $j_\eta < \lambda$,

$$\mu = \sup \{ \zeta < \mu: \eta \upharpoonright \zeta \in B_{j_\eta} \}.$$

So as $\text{cf}(\chi) > \mu$, for some ordinal $j^* < \lambda$ we have

$$\{ \eta \in {}^\mu 2: j_\eta \leq j^* \} \text{ has cardinality } \geq \chi.$$

As $\text{cf}(\lambda) \neq \text{cf}(\mu)$ and $\mu \leq \lambda$ (by its definition) clearly $\mu < \lambda$, hence $|B_{j^*}| \times \mu < \lambda$. Let

$$\mathcal{T} = \{ \eta \upharpoonright \epsilon: \epsilon < \ell g(\eta) \text{ and } \eta \in B_{j^*} \}.$$

It is as required.

SUBCASE 2B: *Not 2A so* $\text{cf}(\lambda) = \text{cf}(\mu)$. As $(\forall \sigma)[\theta \leq \sigma < \mu \Rightarrow \lambda = 2^\sigma \Rightarrow \text{cf}(\lambda) = \text{cf}(2^\sigma) > \sigma]$, clearly $\text{cf}(\lambda) \geq \mu$ so μ is regular. If $\lambda = \mu$ we get $\lambda = \lambda^{<\lambda}$ contradicting an assumption.

So $\lambda > \mu$, so λ singular. So if $\alpha < \mu$, $\mu < \sigma_i = \text{cf}(\sigma_i) < \lambda$ for $i < \alpha$ then (see [Sh-g, 345a, 1.3(10)]) $\max \text{pcf}\{\sigma_i: i < \alpha\} \leq \prod_{i < \alpha} \sigma_i \leq \lambda^{|\alpha|} \leq (2^\theta)^{|\alpha|} \leq 2^{<\mu} = \lambda$, but as λ is singular and $\max \text{pcf}\{\sigma_i: i < \alpha\}$ is regular (see [Sh345a, 1.9]), clearly the inequality is strict, i.e. $\max \text{pcf}\{\sigma_i: i < \alpha\} < \lambda$. So let $\langle \sigma_i: i < \mu \rangle$ be a strictly increasing sequence of regulars in (μ, λ) with limit λ , and by [Sh355, 3.4] there

is $T \subseteq \prod_{i < \mu} \sigma_i$, $|\{\nu \upharpoonright i : \nu \in T\}| \leq \max \text{pcf}\{\lambda_j : j < i\} < \lambda$, and number of μ -branches $> \lambda$. In fact we can get any regular cardinal in $(\lambda, \text{pp}^+(\lambda))$ in the same way. Let $\lambda^* = \min\{\lambda' : \mu < \lambda' \leq \lambda, \text{cf}(\lambda') = \mu \text{ and } \text{pp}(\lambda') > \lambda\}$, so (by [Sh355, 2.3]), also λ^* has those properties and $\text{pp}(\lambda^*) \geq \text{pp}(\lambda)$. So if $\text{pp}^+(\lambda^*) = (2^\mu)^+$ or $\text{pp}(\lambda^*) = 2^\mu$ is singular, we are done. So assume this fails.

If $\mu > \aleph_0$, then (as in 3.4) $\alpha < 2^\mu \Rightarrow \text{cov}(\alpha, \mu^+, \mu^+, \mu) < 2^\mu$ and we can finish as in subcase 2A (as in 3.4; actually $\text{cov}(2^{<\mu}, \mu^+, \mu^+, \mu) < 2^\mu$ suffices which holds by the previous sentence and [Sh355, 5.4]). If $\mu = \aleph_0$ all is easy. ■_{6.3}

6.4 CLAIM: Assume $\mathfrak{b}_k \subseteq \mathfrak{b}_{k+1} \subseteq \dots$ for $k < \omega$, $\mathfrak{a} = \bigcup_{k < \omega} \mathfrak{b}_k$ (and $|\mathfrak{a}| < \text{Min } \mathfrak{a}$) and $\lambda \in \text{pcf } \mathfrak{a} \setminus \bigcup_{k < \omega} \text{pcf}(\mathfrak{b}_k)$.

- (1) Then we can find finite $\mathfrak{d}_k \subseteq \text{pcf}(\mathfrak{b}_k \setminus \mathfrak{b}_{k-1})$ (stipulating $\mathfrak{b}_{-1} = \emptyset$) such that $\lambda \in \text{pcf } \bigcup_{k < \omega} \mathfrak{d}_k$.
- (2) Moreover, we can demand $\mathfrak{d}_k \subseteq (\text{pcf } \mathfrak{b}_k) \setminus (\text{pcf}(\mathfrak{b}_{k-1}))$.

Proof: We start to repeat the proof of [Sh371, 1.5] for $\kappa = \omega$. But there we apply [Sh371, 1.4] to $\langle \mathfrak{b}_\zeta : \zeta < \kappa \rangle$ and get $\langle \langle \mathfrak{c}_{\zeta, \ell} : \ell \leq n_\zeta \rangle : \zeta < \kappa \rangle$ and let $\lambda_{\zeta, \ell} = \max \text{pcf}(\mathfrak{c}_{\zeta, \ell})$. Here we apply the same claim ([Sh371, 1.4]) to $\langle \mathfrak{b}_k \setminus \mathfrak{b}_{k-1} : k < \omega \rangle$ to get part (1). As for part (2), in the proof of [Sh371, 1.5] we let $\delta = |\mathfrak{a}|^+ + \aleph_2$ choose $\langle N_i : i < \delta \rangle$, but now we have to adapt the proof of [Sh371, 1.4] (applied to \mathfrak{a} , $\langle \mathfrak{b}_k : k < \omega \rangle$, $\langle N_i : i < \delta \rangle$); we have gotten there, toward the end, $\alpha < \delta$ such that $E_\alpha \subseteq E$. Let $E_\alpha = \{i_k : k < \omega\}$, $i_k < i_{k+1}$. But now instead of applying [Sh371, 1.3] to each \mathfrak{b}_ℓ separately, we try to choose $\langle \mathfrak{c}_{\zeta, \ell} : \ell \leq n(\zeta) \rangle$ by induction on $\zeta < \omega$. For $\zeta = 0$ we apply [Sh371, 1.3]. For $\zeta > 0$, we apply [Sh371, 1.3] to \mathfrak{b}_ζ but there defining by induction on ℓ $\mathfrak{c}_\ell = \mathfrak{c}_{\zeta, \ell} \subseteq \mathfrak{a}$ such that $\max(\text{pcf}(\mathfrak{a} \setminus \mathfrak{c}_{\zeta, 0} \setminus \dots \setminus \mathfrak{c}_{\zeta, \ell-1}) \cap \text{pcf } \mathfrak{b}_\zeta)$ is strictly decreasing with ℓ . We use:

6.4A Observation: If $|\mathfrak{a}_i| < \text{Min}(\mathfrak{a}_i)$ for $i < i^*$, then $\mathfrak{c} = \bigcap_{i < i^*} \text{pcf}(\mathfrak{a}_i)$ has a last element or is empty.

Proof: Wlog $\langle |\mathfrak{a}_i| : i < i^* \rangle$ is nondecreasing. By [Sh345b, 1.12]

$$(*)_1 \quad \mathfrak{d} \subseteq \mathfrak{c} \ \& \ |\mathfrak{d}| < \text{Min } \mathfrak{d} \Rightarrow \text{pcf}(\mathfrak{d}) \subseteq \mathfrak{c}.$$

By [Sh371, 2.6]

if $\lambda \in \text{pcf}(\mathfrak{d})$, $\mathfrak{d} \subseteq \text{pcf}(\mathfrak{c})$, $|\mathfrak{d}| < \text{Min}(\mathfrak{d})$ then
for some $\mathfrak{e} \subseteq \mathfrak{d}$ we have $|\mathfrak{e}| \leq \text{Min } |\mathfrak{a}_0|$, $\lambda \in \text{pcf}(\mathfrak{e})$.

Now choose by induction on $\zeta < |\mathfrak{a}_0|^+$, $\theta_\zeta \in \mathfrak{c}$, satisfying $\theta_\zeta > \max \text{pcf}\{\theta_\epsilon: \epsilon < \zeta\}$. If we are stuck in ζ , $\max \text{pcf}\{\theta_\epsilon: \epsilon < \zeta\}$ is the desired maximum by $(*)_1$. If we succeed $\theta = \max \text{pcf}\{\theta_\epsilon: \epsilon < |\mathfrak{a}_0|^+\}$ is in $\text{pcf}\{\theta_\epsilon: \epsilon < \zeta\}$ for some $\zeta < |\mathfrak{a}_0|^+$ by $(*)_2$; easy contradiction. ■6.4A

■6.4

6.5 Conclusion: Assume $\aleph_0 = \text{cf}(\mu) \leq \kappa \leq \mu_0 < \mu$, $[\mu' \in (\mu_0, \mu) \ \& \ \text{cf}(\mu') \leq \kappa \Rightarrow \text{pp}_\kappa(\mu') < \lambda]$ and $\text{pp}_\kappa^+(\mu) > \lambda = \text{cf}(\lambda) > \mu$. Then we can find λ_n for $n < \omega$, $\mu_0 < \lambda_n < \lambda_{n+1} < \mu$, $\mu = \bigcup_{n < \omega} \lambda_n$ and $\lambda = \text{tcf} \prod_{n < \omega} \lambda_n / J$ for some ideal J on ω (extending J_ω^{bd}).

Proof: Let $\mathfrak{a} \subseteq (\mu, \mu) \cap \text{Reg}$, $|\mathfrak{a}| \leq \kappa$, $\lambda \in \text{pcf}(\mathfrak{a})$. Without loss of generality $\lambda = \max \text{pcf} \mathfrak{a}$, let $\mu = \bigcup_{n < \omega} \mu_n^0$, $\mu_0 \leq \mu_n^0 < \mu_{n+1}^0 < \mu$, let $\mu_n^1 = \mu_n^0 + \sup\{\text{pp}_\kappa(\mu'): \mu_0 < \mu' \leq \mu_n^0 \text{ and } \text{cf}(\mu') \leq \kappa\}$, by [Sh355, 2.3] $\mu_n^1 < \mu$, $\mu_n^1 = \mu_n^0 + \sup\{\text{pp}_\kappa(\mu'): \mu_0 < \mu' < \mu_n^1 \text{ and } \text{cf}(\mu') \leq \kappa\}$ and obviously $\mu_n^1 \leq \mu_{n+1}^1$; by replacing by a subsequence without loss of generality $\mu_n^1 < \mu_{n+1}^1$. Now let $\mathfrak{b}_n = \mathfrak{a} \cap \mu_n^1$ and apply the previous claim: to $\mathfrak{b}_k =: \mathfrak{a} \cap (\mu_n^1)^+$, note:

$$\max \text{pcf}(\mathfrak{b}_k) \leq \mu_k^1 < \text{Min}(\mathfrak{b}_{k+1} \setminus \mathfrak{b}_k). \quad \blacksquare_{6.5}$$

6.6 CLAIM:

- (1) Assume $\aleph_0 < \text{cf}(\mu) = \kappa < \mu_0 < \mu$, $2^\kappa < \mu$ and $[\mu_0 \leq \mu' < \mu \ \& \ \text{cf}(\mu') \leq \kappa \Rightarrow \text{pp}_\kappa \mu' < \mu]$. If $\mu < \lambda = \text{cf}(\lambda) < \text{pp}^+(\mu)$ then there is a tree \mathcal{T} with κ levels, each level of cardinality $< \mu$, \mathcal{T} has exactly λ κ -branches.
- (2) Suppose $\langle \lambda_i: i < \kappa \rangle$ is a strictly increasing sequence of regular cardinals, $2^\kappa < \lambda_0$, $\mathfrak{a} =: \{\lambda_i: i < \kappa\}$, $\lambda = \max \text{pcf} \mathfrak{a}$, $\lambda_j > \max \text{pcf}\{\lambda_i: i < j\}$ for each $j < \kappa$ (or at least $\sum_{i < \kappa} \lambda_i > \max \text{pcf}\{\lambda_i: i < j\}$) and $\mathfrak{a} \notin J$ where $J = \{\mathfrak{b} \subseteq \mathfrak{a}: \mathfrak{b} \text{ is the union of countably many members of } J_{< \lambda}[\mathfrak{a}]\}$ (so $J \supseteq J_\mathfrak{a}^{bd}$, $\text{cf} \kappa > \aleph_0$). Then the conclusion of (1) holds with $\mu = \sum_{i < \kappa} \lambda_i$.

Proof: (1) By (2) and [Sh371, §1] (or can use the conclusion of [Sh-g, AG 5.7]).

(2) For each $\mathfrak{b} \subseteq \mathfrak{a}$ define the function $g_\mathfrak{b}: \kappa \rightarrow \text{Reg}$ by

$$g_\mathfrak{b}(i) = \max \text{pcf}[\mathfrak{b} \cap \{\lambda_j: j < i\}].$$

Clearly $[\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \Rightarrow g_{\mathfrak{b}_1} \leq g_{\mathfrak{b}_2}]$. As $\text{cf}(\kappa) > \aleph_0$, J \aleph_1 -complete, there is $\mathfrak{b} \subseteq \mathfrak{a}$, $\mathfrak{b} \notin J$ such that:

$$\mathfrak{c} \subseteq \mathfrak{b} \ \& \ \mathfrak{c} \notin J \Rightarrow \neg g_\mathfrak{c} <_J g_\mathfrak{b}.$$

Let $\lambda_i^* = \max \text{pcf}(\mathfrak{b} \cap \{\lambda_j: j < i\})$. For each i let $\mathfrak{b}_i = \mathfrak{b} \cap \{\lambda_j: j < i\}$ and $\langle\langle f_{\lambda,\alpha}^{\mathfrak{b}}: \alpha < \lambda \rangle\rangle: \lambda \in \text{pcf}(\mathfrak{b})$ be as in [Sh371, §1]. Let

$$\mathcal{T}_i^0 = \left\{ \text{Max}_{\ell=1,n} f_{\lambda_\ell,\alpha_\ell}^{\mathfrak{b}} \upharpoonright \mathfrak{b}_i: \lambda_\ell \in \text{pcf}(\mathfrak{b}_i), \alpha_\ell < \lambda_\ell, n < \omega \right\}.$$

Let $\mathcal{T}_i = \{f \in T_i^0: \text{for every } j < i, f \upharpoonright \mathfrak{b}_j \in \mathcal{T}_j^0 \text{ moreover for some } f' \in \prod_{j < \kappa} \lambda_j, \text{ for every } j, f' \upharpoonright j \in \mathcal{T}_i^0 \text{ and } f \subseteq f'\}$, and $\mathcal{T} = \bigcup_{i < \kappa} \mathcal{T}_i$, clearly it is a tree, \mathcal{T}_i its i th level (or empty), $|\mathcal{T}_i| \leq \lambda_i^*$. By [Sh371, 1.3, 1.4] for every $g \in \prod \mathfrak{b}$ for some $f \in \prod \mathfrak{b}$, $\bigwedge_{i < \kappa} f \upharpoonright \mathfrak{b}_i \in \mathcal{T}_i^0$ hence $\bigwedge_{i < \kappa} f \upharpoonright \mathfrak{b}_i \in \mathcal{T}_i$. So $|\mathcal{T}_i| = \lambda_i^*$, and \mathcal{T} has $\geq \lambda$ κ -branches. By the observation below we can finish (apply it essentially to $F = \{\eta: \text{for some } f \in \prod \mathfrak{b} \text{ for } i < \kappa \text{ we have } \eta(i) = f \upharpoonright \mathfrak{b}_i \text{ and for every } i < \kappa, f \upharpoonright \mathfrak{b}_i \in \mathcal{T}_i^0\}$), then find $A \subseteq \kappa$, $\kappa \setminus A \in J$ and $g^* \in \prod_{i < \kappa} (\lambda_i + 1)$ such that $Y' =: \{f \in F: f \upharpoonright A < g^* \upharpoonright A\}$ has cardinality λ and then the tree will be \mathcal{T}' where $\mathcal{T}'_i =: \{f \upharpoonright \mathfrak{b}_i: f \in Y'\}$ and $\mathcal{T}' = \bigcup_{i < \kappa} \mathcal{T}'_i$. (So actually this proves that if we have such a tree with $\geq \theta$ ($\text{cf}(\theta) > 2^\kappa$) κ -branches then there is one with exactly θ κ -branches.)

6.6A OBSERVATION: (1) If $F \subseteq \prod_{i < \kappa} \lambda_i$, J an \aleph_1 -complete ideal on κ , and $[f \neq g \in F \Rightarrow f \neq_J g]$ and $|F| \geq \theta$, $\text{cf} \theta > 2^\kappa$, then for some $g^* \in \prod_{i < \kappa} (\lambda_i + 1)$ we have:

- (a) $Y = \{f \in F: f <_J g^*\}$ has cardinality θ ,
- (b) for $f' <_J g^*$, we have $|\{f \in F: f \leq_J f'\}| < \theta$,
- (c) there* are $f_\alpha \in Y$ for $\alpha < \theta$ such that: $f_\alpha <_J g^*$, $[\alpha < \beta < \theta \Rightarrow \neg f_\beta <_J f_\alpha]$.

Proof: Let $Z =: \{g: g \in \prod_{i < \kappa} (\lambda_i + 1) \text{ and } Y_g =: \{f \in F: f \leq_J g\} \text{ has cardinality } \geq \theta\}$. Clearly $\langle \lambda_i: i < \kappa \rangle \in Z$ so there is $g^* \in Z$ such that: $[g' \in Z \Rightarrow \neg g' <_J g^*]$; so (b) holds. Let $Y = \{f \in F: f <_J g^*\}$, easily $Y \subseteq Y_{g^*}$ and $|Y_{g^*} \setminus Y| \leq 2^\kappa$ hence $|Y| \geq \theta$, also clearly $[f_1 \neq f_2 \in F \ \& \ f_1 \leq_J f_2 \Rightarrow f_1 <_J f_2]$; if (a) fails, necessarily (by (b)) $|Y| > \theta$. For each $f \in Y$ let $Y_f = \{h \in Y: h \leq_D f\}$, so $|Y_f| < \theta$ hence by the Hajnal free subset theorem for some $Z' \subseteq Z$, $|Z'| = \lambda^+$, and $f_1 \neq f_2 \in Z' \Rightarrow f_1 \notin Y_{f_2}$ so $[f_1 \neq f_2 \in Z' \Rightarrow \neg f_1 <_J f_2]$. But there is no such Z' of cardinality $> 2^\kappa$ ([Sh111, 2.2, p. 264]) so (a) holds. As for (c): choose $f_\alpha \in F$ by induction on α , such that $f_\alpha \in Y \setminus \bigcup_{\beta < \alpha} Y_{f_\beta}$; it exists by cardinality considerations and $\langle f_\alpha: \alpha < \theta \rangle$ is as required (in (c)). ■6.6A

■6.6

* Or straightening clause (i) see the proof of 6.6B

6.6B OBSERVATION: Let $\kappa < \lambda$ be regular uncountable, $2^\kappa < \mu_i < \lambda$ (for $i < \kappa$), μ_i increasing in i . The following are equivalent:

- (A) there is $F \subseteq {}^\kappa\lambda$ such that:
 - (i) $|F| = \lambda$,
 - (ii) $|\{f \upharpoonright i: f \in F\}| \leq \mu_i$,
 - (iii) $[f \neq g \in F \Rightarrow f \neq_{J_{\kappa}^{bd}} g]$;
- (B) there be a sequence $\langle \lambda_i: i < \kappa \rangle$ such that:
 - (i) $2^\kappa < \lambda_i = \text{cf}(\lambda_i) \leq \mu_i$,
 - (ii) $\max \text{pcf}\{\lambda_i: i < \kappa\} = \lambda$,
 - (iii) for $j < \kappa$, $\mu_j \geq \max \text{pcf}\{\lambda_i: i < j\}$;
- (C) there is an increasing sequence $\langle \mathfrak{a}_i: i < \kappa \rangle$ such that $\lambda \in \text{pcf} \bigcup_{i < \kappa} \mathfrak{a}_i$, $\text{pcf} \mathfrak{a}_i \subseteq \mu_i$ (so $\text{Min}(\bigcup_{i < \kappa} \mathfrak{a}_i) > |\bigcup_{i < \kappa} \mathfrak{a}_i|$).

Proof:

(B) \Rightarrow (A): By [Sh355, 3.4].

(A) \Rightarrow (B): If $(\forall \theta)[\theta \geq 2^\kappa \Rightarrow \theta^\kappa \leq \theta^+]$ we can directly prove (B) if for a club of $i < \kappa$, $\mu_i > \bigcup_{j < i} \mu_j$, and contradict (A) if this fails. Otherwise every normal filter D on κ is nice (see [Sh386, §1]). Let F exemplify (A).

Let $K = \{(D, g): D \text{ a normal filter on } \kappa, g \in {}^\kappa(\lambda + 1), \lambda = |\{f \in F: f <_D g\}|\}$. Clearly K is not empty (let g be constantly λ) so by [Sh386] we can find $(D, g) \in K$ such that:

(*)₁ if $A \subseteq \kappa$, $A \neq \emptyset \text{ mod } D$, $g_1 <_{D+A} g$ then $\lambda > |\{f \in F: f <_{D+A} g_1\}|$.

Let $F^* = \{f \in F: f <_D g\}$, so (as in the proof of 6.6) $|F^*| = \lambda$.

We claim:

(*)₂ if $h \in F^*$ then $\{f \in F^*: \neg h \leq_D f\}$ has cardinality $< \lambda$.

[Why? Otherwise for some $h \in F^*$, $F' =: \{f \in F^*: \neg h \leq_D f\}$ has cardinality λ , for $A \subseteq \kappa$ let $F'_A = \{f \in F^*: f \upharpoonright A \leq h \upharpoonright A\}$ so $F' = \bigcup \{F'_A: A \subseteq \kappa, A \neq \emptyset \text{ mod } D\}$, hence for some $A \subseteq \kappa$, $A \neq \emptyset \text{ mod } D$ and $|F'_A| = \lambda$; now $(D + A, h)$ contradicts (*)₁].

By (*)₂ we can choose by induction on $\alpha < \lambda$, a function $f_\alpha \in F^*$ such that $\bigwedge_{\beta < \alpha} f_\beta <_D f_\alpha$. By [Sh355, 1.2A(3)] $\langle f_\alpha: \alpha < \lambda \rangle$ has an e.u.b. f^* . Let $\lambda_i = \text{cf}(f^*(i))$, clearly $\{i < \kappa: \lambda_i \leq 2^\kappa\} = \emptyset \text{ mod } D$, so without loss of generality $\bigwedge_{i < \kappa} \text{cf}(f^*(i)) > 2^\kappa$ so λ_i is regular $\in (2^\kappa, \lambda]$, and $\lambda = \text{tcf}(\prod_{i < \kappa} \lambda_i / D)$. Let $J_i = \{A \subseteq i: \max \text{pcf}\{\lambda_j: j < i\} \leq \mu_i\}$; so (remembering (ii) of (A)) we can find $h_i \in \prod_{j < i} f^*(j)$ such that:

(*)₃ if $\{j: j < i\} \notin J_i$, then for every $f \in F$, $f \upharpoonright i <_{J_i} h_i$.

Let $h \in \prod_{i < \kappa} f^*(i)$ be defined by: $h(i) = \sup \{h_j(i): j \in (i, \kappa) \text{ and } \{j: j < i\} \notin J_i\}$. As $\bigwedge_i \text{cf}[f^*(i)] > 2^\kappa$, clearly $h < f^*$ hence by the choice of f^* for some $\alpha(*) < \lambda$ we have: $h <_D f_{\alpha(*)}$ and let $A =: \{i < \kappa: h(i) < f_{\alpha(*)}\}$, so $A \in D$. Define λ'_i as follows: λ'_i is λ_i if $i \in A$, and is $(2^\kappa)^+$ if $i \in \kappa \setminus A$. Now $\langle \lambda'_i: i < \kappa \rangle$ is as required in (B).

(B) \Rightarrow (C): Straightforward.

(C) \Rightarrow (B): By [Sh371, §1]. ■_{6.6B}

6.6C CLAIM: If $F \subseteq {}^\kappa\text{Ord}$, $2^\kappa < \theta = \text{cf}(\theta) \leq |F|$ then we can find $g^* \in {}^\kappa\text{Ord}$ and a proper ideal I on κ and $A \subseteq \kappa$, $A \in I$ such that:

- (a) $\prod_{i < \kappa} g^*(i)/I$ has true cofinality θ , and for each $i \in \kappa \setminus A$ we have $\text{cf}[g^*(i)] > 2^\kappa$,
- (b) for every $g \in {}^\kappa\text{Ord}$ satisfying $g \upharpoonright A = g^* \upharpoonright A$, $g \upharpoonright (\kappa \setminus A) < g^* \upharpoonright (\kappa \setminus A)$ we can find $f \in F$ such that: $f \upharpoonright A = g^* \upharpoonright A$, $g \upharpoonright (\kappa \setminus A) < f \upharpoonright (\kappa \setminus A) < g^* \upharpoonright (\kappa \setminus A)$.

Proof: As in [Sh410, 3.7 proof of (A) \Rightarrow (B)]. (In short let $f_\alpha \in F$ for $\alpha < \theta$ be distinct, χ large enough, $\langle N_i: i < (2^\kappa)^+ \rangle$ as there, $\delta_i =: \sup(\theta \cap N_i)$, $g_i \in {}^\kappa\text{Ord}$, $g_i(\zeta) =: \text{Min}[N \cap \text{Ord} \setminus f_{\delta_i}(\zeta)]$, $A \subseteq \kappa$ and $S \subseteq \{i < (2^\kappa)^+ : \text{cf}(i) = \kappa^+\}$ stationary, $[i \in S \Rightarrow g_i = g^*]$, $[\zeta < \alpha \ \& \ i \in S \Rightarrow [f_{\delta_i}(\zeta) = g^*(\zeta) \equiv \zeta \in A]]$ and for some $i(*) < (2^\kappa)^+$, $g^* \in N_{i(*)}$, so $[\zeta \in \kappa \setminus A \Rightarrow \text{cf } g^*(\zeta) > 2^\kappa]$.) ■_{6.6C}

6.6D CLAIM: Suppose D is a filter on $\theta = \text{cf}(\theta)$, σ -complete, $\theta > |\alpha|^\kappa$ for $\alpha < \sigma$, and for each $\alpha < \theta$, $\bar{\beta} = \langle \beta_\epsilon^\alpha: \epsilon < \kappa \rangle$ is a sequence of ordinals. Then for every $X \subseteq \theta$, $X \neq \emptyset \text{ mod } D$ there is $\langle \beta_\epsilon^*: \epsilon < \kappa \rangle$ (a sequence of ordinals) and $w \subseteq \kappa$ such that:

- (a) $\epsilon \in \kappa \setminus w \Rightarrow \sigma \leq \text{cf}(\beta_\epsilon^*) \leq \theta$,
- (b) if $\beta'_\epsilon \leq \beta_\epsilon^*$ and $[\epsilon \in w \equiv \beta'_\epsilon = \beta_\epsilon^*]$, then $\{\alpha \in X: \text{ for every } \epsilon < \kappa \text{ we have } \beta'_\epsilon \leq \beta_\epsilon^\alpha \leq \beta_\epsilon^* \text{ and } [\epsilon \in w \equiv \beta_\epsilon^\alpha = \beta_\epsilon^*]\} \neq \emptyset \text{ mod } D$.

Proof: Essentially by the same proof as 6.6C (replacing δ_i by $\text{Min}\{\alpha \in X: \text{ for every } Y \in N_i \cap D \text{ we have } \alpha \in Y\}$). See more [Sh513, §6]. ■_{6.6D}

6.6E Remark: We can rephrase the conclusion as:

- (a) $B =: \{\alpha \in X: \text{ if } \epsilon \in w \text{ then } \beta_\epsilon^\alpha = \beta_\epsilon^*, \text{ and: if } \epsilon \in \kappa \setminus w \text{ then } \beta_\epsilon^\alpha \text{ is } < \beta_\epsilon^* \text{ but } > \sup\{\beta_\zeta^*: \zeta < \epsilon, \beta_\zeta^\alpha < \beta_\epsilon^*\}\} \neq \emptyset \text{ mod } D$.
- (b) If $\beta'_\epsilon < \beta_\epsilon$ for $\epsilon \in \kappa \setminus w$ then $\{\alpha \in B: \text{ if } \epsilon \in \kappa \setminus w \text{ then } \beta_\epsilon^\alpha > \beta'_\epsilon\} \neq \emptyset \text{ mod } D$.

(c) $\epsilon \in \kappa \setminus w \Rightarrow \text{cf}(\beta'_\epsilon)$ is $\leq \theta$ but $\geq \sigma$.

6.6F Remark: (1) If $|\mathfrak{a}| < \min(\mathfrak{a})$, $F \subseteq \Pi \mathfrak{a}$, $|F| = \theta = \text{cf } \theta \notin \text{pcf}(\mathfrak{a})$ and even $\theta > \sigma = \sup(\theta^+ \cap \text{pcf}(\mathfrak{a}))$ then for some $g \in \Pi \mathfrak{a}$, the set $\{f \in F: f < g\}$ is unbounded in θ (or use a σ -complete D as in 6.6E). (This is as $\Pi \mathfrak{a}/J_{<\theta}[\mathfrak{a}]$ is $\min(\text{pcf}(\mathfrak{a}) \setminus \theta)$ -directed as the ideal $J_{<\theta}[\mathfrak{a}]$ is generated by $\leq \sigma$ sets; this is discussed in [Sh513, §6].)

6.6G Remark: It is useful to note that 6.6D is useful to use [Sh462, §4, 5.14]: e.g. for if $n < \omega$, $\theta_0 < \theta_1 < \dots < \theta_n$, satisfying $(*)$ below, for any $\beta'_\epsilon \leq \beta^*_\epsilon$ satisfying $[\epsilon \in w \equiv \beta'_\epsilon < \beta^*_\epsilon]$ we can find $\alpha < \gamma$ in X such that:

$$i \in w \equiv \beta^\alpha_\epsilon = \beta^*_\epsilon,$$

$$\{\epsilon, \zeta\} \subseteq \kappa \setminus w \ \& \ \{\text{cf}(\beta^*_\epsilon), \text{cf}(\beta^*_\zeta)\} \subseteq [\theta_l, \theta_{l+1}) \ \& \ l \text{ even} \Rightarrow \beta^\alpha_\epsilon < \beta^\gamma_\zeta,$$

$$\{\epsilon, \zeta\} \subseteq \kappa \setminus w \ \& \ \{\text{cf}(\beta^*_\zeta), \text{cf}(\beta^*_\epsilon)\} \subseteq [\theta_l, \theta_{l+1}) \ \& \ l \text{ odd} \Rightarrow \beta^\gamma_\epsilon < \beta^\alpha_\zeta$$

where

$(*)$ (a) $\epsilon \in \kappa \setminus w \Rightarrow \text{cf}(\beta^*_\epsilon) \in [\theta_0, \theta_n)$, and

(b) $\max \text{pcf}[\{\text{cf}(\beta^*_\epsilon): \epsilon \in \kappa \setminus w\} \cap \theta_l] \leq \theta_l$ (which holds if $\theta_l = \sigma_l^+$, $\sigma_l^\kappa = \sigma_l$ for $l \in \{1, \dots, n\}$).

6.7 CLAIM: For any \mathfrak{a} , $|\mathfrak{a}| < \text{Min}(\mathfrak{a})$, we can find $\bar{\mathfrak{b}} = \langle \mathfrak{b}_\lambda: \lambda \in \mathfrak{a} \rangle$ such that:

(α) $\bar{\mathfrak{b}}$ is a generating sequence, i.e.

$$\lambda \in \mathfrak{a} \Rightarrow J_{\leq \lambda}[\mathfrak{a}] = J_{< \lambda}[\mathfrak{a}] + \mathfrak{b}_\lambda,$$

(β) $\bar{\mathfrak{b}}$ is smooth, i.e. for $\theta < \lambda$ in \mathfrak{a} ,

$$\theta \in \mathfrak{b}_\lambda \Rightarrow \mathfrak{b}_\theta \subseteq \mathfrak{b}_\lambda,$$

(γ) $\bar{\mathfrak{b}}$ is closed, i.e. for $\lambda \in \text{pcf}(\mathfrak{a})$ we have $\mathfrak{b}_\lambda = \mathfrak{a} \cap \text{pcf}(\mathfrak{b}_\lambda)$.

Proof: Let $\langle \mathfrak{b}_\theta[\mathfrak{a}]: \theta \in \text{pcf } \mathfrak{a} \rangle$ be as in [Sh371, 2.6]. For $\lambda \in \mathfrak{a}$, let $\bar{f}^{\mathfrak{a}, \lambda} = \langle f_\alpha^{\mathfrak{a}, \lambda}: \alpha < \mathfrak{a} \rangle$ be a $<_{J_\lambda[\mathfrak{a}]}$ -increasing cofinal sequence of members of $\prod \mathfrak{a}$, satisfying:

$(*)_1$ if $\delta < \lambda$, $|\mathfrak{a}| < \text{cf}(\delta) < \text{Min } \mathfrak{a}$ and $\theta \in \mathfrak{a}$ then:

$$f_\delta^{\mathfrak{a}, \lambda}(\theta) = \text{Min} \left\{ \bigcup_{\alpha \in C} f_\alpha^{\mathfrak{a}, \lambda}(\theta): C \text{ a club of } \delta \right\}$$

[exists by [Sh345a, Def. 3.3(2)^b + Fact 3.4(1)]]].

Let $\chi = \beth_{\omega}(\sup \mathfrak{a})^+$, $|\mathfrak{a}| < \kappa = \text{cf } \kappa < \text{Min } \mathfrak{a}$ (without loss of generality there is such κ) and $\bar{N} = \langle N_i : i < \kappa \rangle$ be an increasing continuous sequence of elementary submodels of $(H(\chi), \in, <_{\chi}^*)$, $N_i \cap \kappa$ an ordinal, $\bar{N} \upharpoonright (i+1) \in N_{i+1}$, $\|N_i\| < \kappa$, and \mathfrak{a} , $\langle \bar{f}^{\mathfrak{a},\lambda} : \lambda \in \mathfrak{a} \rangle$ belong to N_0 . Let $N_{\kappa} = \bigcup_{i < \kappa} N_i$. For every $\lambda \in \mathfrak{a}$, for some club E_{λ} of κ ,

$$(*) \theta \in \mathfrak{a} \Rightarrow f_{\sup(N_{\kappa} \cap \lambda)}^{\mathfrak{a},\lambda}(\theta) = \bigcup_{\alpha \in E_{\lambda}} f_{\sup(N_{\alpha} \cap \lambda)}^{\mathfrak{a},\lambda}(\theta).$$

Let $E = \bigcap_{\lambda \in \mathfrak{a}} E_{\lambda}$, so E is a club of κ . For any $i < j < \kappa$ let

$$\mathfrak{b}_{\lambda}^{i,j} = \left\{ \theta \in \mathfrak{a} : \sup(N_i \cap \theta) < f_{\sup(N_j \cap \lambda)}^{\mathfrak{a},\lambda}(\theta) \right\}.$$

As in the proof of [Sh371, 1.3], possibly shrinking E , we have:

(*)₂ for $i < j$ from* E and $\lambda \in \mathfrak{a}$, we have:

- (α) $J_{\leq \lambda}[\mathfrak{a}] = J_{< \lambda}[\mathfrak{a}] + \mathfrak{b}_{\lambda}^{i,j}$ (hence $\mathfrak{b}_{\lambda}^{i,j} = \mathfrak{b}_{\lambda}[\mathfrak{a}] \bmod J_{< \lambda}[\mathfrak{a}]$),
- (β) $\mathfrak{b}_{\lambda}^{i,j} \subseteq \lambda^+ \cap \mathfrak{a}$,
- (γ) $\langle \mathfrak{b}_{\lambda}^{i,j} : \lambda \in \mathfrak{a} \rangle \in N_{j+1}$,
- (δ) $f_{\sup(N_{\kappa} \cap \lambda)}^{\mathfrak{a},\lambda} \upharpoonright \mathfrak{b}_{\lambda}^{i,j} = \langle (\theta, \sup(N_{\kappa} \cap \theta)) : \theta \in \mathfrak{b}_{\lambda}^{i,j} \rangle$,
- (ϵ) $f_{\sup(N_{\kappa} \cap \lambda)}^{\mathfrak{a},\lambda} \leq \langle (\theta, \sup(N_{\kappa} \cap \theta)) : \theta \in \mathfrak{a} \rangle$.

We now define by induction on $\epsilon < |\mathfrak{a}|^+$, for $\lambda \in \mathfrak{a}$ (and $i < j < \kappa$), the set $\mathfrak{b}_{\lambda}^{i,j,\epsilon}$:

$$\begin{aligned} \mathfrak{b}_{\lambda}^{i,j,0} &= \mathfrak{b}_{\lambda}^{i,j} \\ \mathfrak{b}_{\lambda}^{i,j,\epsilon+1} &= \mathfrak{b}_{\lambda}^{i,j,\epsilon} \cup \bigcup \left\{ \mathfrak{b}_{\theta}^{i,j,\epsilon} : \theta \in \mathfrak{b}_{\lambda}^{i,j,\epsilon} \right\} \cup \left\{ \theta \in \mathfrak{a} : \theta \in \text{pcf } \mathfrak{b}^{i,j,\epsilon} \right\}, \\ \mathfrak{b}_{\lambda}^{i,j,\epsilon} &= \bigcup_{\zeta < \epsilon} \mathfrak{b}_{\lambda}^{i,j,\zeta} \text{ for } \epsilon < |\mathfrak{a}|^+ \text{ limit.} \end{aligned}$$

Clearly for $\lambda \in \mathfrak{a}$, $\langle \mathfrak{b}_{\lambda}^{i,j,\epsilon} : \epsilon < |\mathfrak{a}|^+ \rangle$ belongs to N_{j+1} and is a non-decreasing sequence of subsets of \mathfrak{a} , hence for some $\epsilon(i, j, \lambda) < |\mathfrak{a}|^+$,

$$\left[\epsilon \in (\epsilon(i, j, \lambda), |\mathfrak{a}|^+) \Rightarrow \mathfrak{b}_{\lambda}^{i,j,\epsilon} = \mathfrak{b}_{\lambda}^{i,j,\epsilon(i,j,\lambda)} \right].$$

So letting $\epsilon(i, j) = \sup_{\lambda \in \mathfrak{a}} \epsilon(i, j, \lambda) < |\mathfrak{a}|^+$ we have:

$$(*)_3 \epsilon(i, j) \leq \epsilon < |\mathfrak{a}|^+ \Rightarrow \bigwedge_{\lambda \in \mathfrak{a}} \mathfrak{b}_{\lambda}^{i,j,\epsilon(i,j)} = \mathfrak{b}_{\lambda}^{i,j,\epsilon}.$$

Which of the properties required from $\langle \mathfrak{b}_{\lambda} : \lambda \in \mathfrak{a} \rangle$ are satisfied by $\langle \mathfrak{b}_{\lambda}^{i,j,\epsilon(i,j)} : \lambda \in \mathfrak{a} \rangle$? Note (β), (γ) hold by the inductive definition of $\mathfrak{b}_{\lambda}^{i,j,\epsilon}$ (and the choice of $\epsilon(i, j)$), as for property (α), one half, $J_{\leq \lambda}[\mathfrak{a}] \subseteq J_{< \lambda}[\mathfrak{a}] + \mathfrak{b}_{\lambda}^{i,j,\epsilon(i,j)}$ hold by (*)₂(α) (and $\mathfrak{b}_{\lambda}^{i,j} = \mathfrak{b}_{\lambda}^{i,j,0} \subseteq \mathfrak{b}_{\lambda}^{i,j,\epsilon(i,j)}$), so it is enough to prove (for $\lambda \in \mathfrak{a}$) :

* Actually for any $i < j < \kappa$ clauses (β), (γ), (δ) hold.

$$(*)_4 \mathfrak{b}_\lambda^{i,j,\epsilon(i,j)} \in J_{\leq \lambda}[\mathfrak{a}].$$

For this end we define by induction on $\epsilon < |\mathfrak{a}|^+$ functions $f_\alpha^{a,\lambda,\epsilon}$ with domain $\mathfrak{b}_\lambda^{i,j,\epsilon}$ for every $\alpha < \lambda \in \mathfrak{a}$, such that $\zeta < \epsilon \Rightarrow f_\alpha^{a,\lambda,\zeta} \subseteq f_\alpha^{a,\lambda,\epsilon}$, so the domain increases with ϵ .

We let $f_\alpha^{a,\lambda,0} = f_\alpha^{a,\lambda} \upharpoonright \mathfrak{b}_\lambda^{i,j}$, $f_\alpha^{a,\lambda,\zeta} = \bigcup_{\zeta < \epsilon} f_\alpha^{a,\lambda,\zeta}$ for $\epsilon < |\mathfrak{a}|^+$ limit, and $f_\alpha^{a,\lambda,\epsilon+1}$ is defined by defining each $f_\alpha^{a,\lambda,\epsilon+1}(\theta)$ as follows:

CASE 1: If $\theta \in \mathfrak{b}_\lambda^{i,j,\epsilon}$ then $f_\alpha^{a,\lambda,\epsilon}(\theta)$.

CASE 2: If $\mu \in \mathfrak{b}_\lambda^{i,j,\epsilon}$, $\theta \in \mathfrak{b}_\mu^{i,j,\epsilon}$ and not Case 1 and μ minimal under those conditions, then $f_\beta^{a,\mu,\epsilon}(\theta)$ where we choose $\beta = f_\alpha^{a,\lambda,\epsilon}(\mu)$.

CASE 3: If $\theta \in \mathfrak{a} \cap \text{pcf}(\mathfrak{b}_\lambda^{i,j,\epsilon})$ and not Case 1 or 2, then

$$\text{Min} \{ \gamma < \theta : f_\alpha^{a,\lambda,\epsilon} \upharpoonright \mathfrak{b}_\theta[\mathfrak{a}] \leq_{J_{< \theta}[\mathfrak{a}]} f_\gamma^{a,\theta,\epsilon} \}.$$

Now $\langle \langle \mathfrak{b}_\lambda^{i,j,\epsilon} : \lambda \in \mathfrak{a} \rangle : \epsilon < |\mathfrak{a}|^+ \rangle$ can be computed from \mathfrak{a} and $\langle \mathfrak{b}_\lambda^{i,j} : \lambda \in \mathfrak{a} \rangle$. But the latter belong* to N_{j+1} , so the former belongs to N_{j+1} , so as also $\langle \langle f_\alpha^{a,\lambda} : \alpha < \lambda \rangle : \lambda \in \text{pcf } \mathfrak{a} \rangle$ belongs to N_{j+1} we clearly get that

$$\langle \langle \langle f_\alpha^{a,\lambda,\epsilon} : \epsilon < |\mathfrak{a}|^+ \rangle : \alpha < \lambda \rangle : \lambda \in \mathfrak{a} \rangle$$

belongs to N_{j+1} . Next we prove by induction on ϵ that, for $\lambda \in \mathfrak{a}$, we have:

$$\otimes_1 \quad \theta \in \mathfrak{b}_\lambda^{i,j,\epsilon} \ \& \ \lambda \in \mathfrak{a} \Rightarrow f_{\sup(N_\kappa \cap \theta)}^{a,\lambda,\epsilon}(\theta) = \sup(N_\kappa \cap \theta).$$

For $\epsilon = 0$ this is by $(*)_2(\delta)$. For ϵ limit, by the induction hypothesis and the definition of $f_\alpha^{a,\lambda,\epsilon}$. For $\epsilon + 1$, we check $f_{\sup(N_\kappa \cap \lambda)}^{a,\lambda,\epsilon+1}(\theta)$ according to the case in its definition; for Case 1 use the induction hypothesis applied to $f_{\sup(N_\kappa \cap \lambda)}^{a,\lambda,\epsilon}$. For Case 2 (with μ), by the induction hypothesis applied to $f_{\sup(N_\kappa \cap \mu)}^{a,\mu,\epsilon}$. Lastly, for Case 3 (with θ) we should note:

- (i) $\mathfrak{b}_\lambda^{i,j,\epsilon} \cap \mathfrak{b}_\theta[\mathfrak{a}] \notin J_{< \theta}[\mathfrak{a}]$ (by the case's assumption and $(*)_2(\alpha)$ above),
- (ii) $f_{\sup(N_\kappa \cap \lambda)}^{a,\lambda,\epsilon} \upharpoonright (\mathfrak{b}_\lambda^{i,j,\epsilon} \cap \mathfrak{b}_\theta^{i,j,\epsilon}) \subseteq f_{\sup(N_\kappa \cap \theta)}^{a,\theta,\epsilon}$ (by the induction hypothesis for ϵ , used concerning λ and θ) hence (by the definition in case 3 and (i) + (ii)),
- (iii) $f_{\sup(N_\kappa \cap \lambda)}^{a,\lambda,\epsilon+1}(\theta) \leq \sup(N_\kappa \cap \theta)$.

* As $\langle \mathfrak{b}_\lambda^{i,j,\epsilon} : \lambda \in \mathfrak{a} \rangle : \epsilon < |\mathfrak{a}|^+$ is eventually constant, also each member of the sequence belongs to N_{j+1} .

Now if $\gamma < \sup(N_\kappa \cap \theta)$ then for some $\gamma(1)$, $\gamma < \gamma(1) \in N_\kappa \cap \theta$, so letting $\mathfrak{b} =: \mathfrak{b}_\lambda^{i,j,\epsilon} \cap \mathfrak{b}_\theta[\mathfrak{a}] \cap \mathfrak{b}_\theta^{i,j,\epsilon}$, it belongs to $J_{\leq \theta}[\mathfrak{a}] \setminus J_{< \theta}[\mathfrak{a}]$, we have

$$f_\gamma^{\mathfrak{a},\theta} \upharpoonright \mathfrak{b} <_{J_{< \theta}[\mathfrak{a}]} f_{\gamma(1)}^{\mathfrak{a},\theta} \upharpoonright \mathfrak{b} \leq f_{\sup(N_\kappa \cap \theta)}^{\mathfrak{a},\theta,\epsilon}$$

hence $f_{\sup(N_\kappa \cap \lambda)}^{\mathfrak{a},\lambda,\epsilon+1}(\theta) > \gamma$; as this holds for every $\gamma < \sup(N_\kappa \cap \theta)$ we have obtained

$$(iv) \quad f_{\sup(N_\kappa \cap \lambda)}^{\mathfrak{a},\lambda,\epsilon+1}(\theta) \geq \sup(N_\kappa \cap \theta);$$

together we have finished proving the inductive step for $\epsilon + 1$, hence we have proved \otimes_1 .

This is enough for proving $\mathfrak{b}_\lambda^{i,j,\epsilon} \in J_{\leq \lambda}[\mathfrak{a}]$: Why? If it fails, as $\mathfrak{b}_\lambda^{i,j,\epsilon} \in N_{j+1}$ and $\langle f_\alpha^{\mathfrak{a},\lambda,\epsilon} : \alpha < \lambda \rangle$ belongs to N_{j+1} , there is $g \in \prod \mathfrak{b}_\lambda^{i,j,\epsilon}$ s.t.

$$(*) \quad \alpha < \lambda \Rightarrow f_\alpha^{\mathfrak{a},\lambda,\epsilon} \upharpoonright \mathfrak{b}_\lambda^{i,j,\epsilon} < g \text{ mod } J_{\leq \lambda}[\mathfrak{a}].$$

Wlog $g \in N_{j+1}$; by $(*)$, $f_{\sup(N_\kappa \cap \lambda)}^{\mathfrak{a},\lambda,\epsilon} < g \text{ mod } J_{\leq \lambda}[\mathfrak{a}]$. But $g < \langle \sup(N_\kappa \cap \theta) : \theta \in \mathfrak{b}_\lambda^{i,j,\epsilon} \rangle$. Together this contradicts \oplus_1 !

This ends the proof of 6.7. ■_{6.7}

6.7A CLAIM: Assume $|\mathfrak{a}| < \kappa = \text{cf}(\kappa) < \text{Min}(\mathfrak{a})$, σ an infinite ordinal, $|\sigma|^+ < \kappa$. Let \bar{f} , $\bar{N} = \langle N_i : i < \kappa \rangle$, N_κ be as in the proof of 6.7. Then we can find $\bar{i} = \langle i_\alpha : \alpha \leq \sigma \rangle$, $\bar{\mathfrak{a}} = \langle \mathfrak{a}_\alpha : \alpha < \sigma \rangle$ and $\langle \langle \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_\beta \rangle : \beta < \sigma \rangle$ such that:

- (a) \bar{i} is a strictly increasing continuous sequence of ordinals $< \kappa$,
- (b) for $\beta < \sigma$ we have $\langle i_\alpha : \alpha \leq \beta \rangle \in N_{i_{\beta+1}}$ (hence* $\langle N_{i_\alpha} : \alpha \leq \beta \rangle \in N_{i_{\beta+1}}$) and $\langle \mathfrak{b}_\lambda^\gamma[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_\gamma \text{ and } \gamma \leq \beta \rangle \in N_{i_{\beta+1}}$,
- (c) $\mathfrak{a}_\beta = N_{i_\beta} \cap \text{pcf}(\mathfrak{a})$, so \mathfrak{a}_β is increasing continuous in β , $\mathfrak{a} \subseteq \mathfrak{a}_\beta \subseteq \text{pcf } \mathfrak{a}$, $|\mathfrak{a}_\beta| < \kappa$,
- (d) $\mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}] \subseteq \mathfrak{a}_\beta$ (for $\lambda \in \mathfrak{a}_\beta$),
- (e) $J_{\leq \lambda}[\mathfrak{a}_\beta] = J_{< \lambda}[\mathfrak{a}_\beta] + \mathfrak{b}_\lambda^\beta[\mathfrak{a}]$ (so $\lambda \in \mathfrak{b}_\lambda[\mathfrak{a}]$ and $\mathfrak{b}_\lambda[\mathfrak{a}] \subseteq \lambda^+$),
- (f) if $\mu < \lambda$ are in \mathfrak{a}_β and $\mu \in \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$ then $\mathfrak{b}_\mu^\beta[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$ (i.e. smoothness),
- (g) $\mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}] = \mathfrak{a}_\beta \cap \text{pcf } \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$ (i.e. closedness),
- (h) if $\mathfrak{c} \subseteq \mathfrak{a}_\beta$, $\beta < \sigma$, $\mathfrak{c} \in N_{i_{\beta+1}}$ then for some finite $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+1} \cap \text{pcf}(\mathfrak{c})$, we have $\mathfrak{c} \subseteq \bigcup_{\mu \in \mathfrak{d}} \mathfrak{b}_\mu^{\beta+1}[\bar{\mathfrak{a}}]$; more generally,**
- (h)⁺ if $\mathfrak{c} \subseteq \mathfrak{a}_\beta$, $\beta < \sigma$, $\mathfrak{c} \in N_{i_{\beta+1}}$, $\theta = \text{cf}(\theta) \in N_{i_{\beta+1}}$, then for some $\mathfrak{d} \in N_{i_{\beta+1}}$, $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+1} \cap \text{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ we have $\mathfrak{c} \subseteq \bigcup_{\mu \in \mathfrak{d}} \mathfrak{b}_\mu^{\beta+1}[\bar{\mathfrak{a}}]$ and $|\mathfrak{d}| < \theta$,

* We can get $\bar{i} \upharpoonright (\beta + 1) \in N_{i_{\beta+1}}$ if κ successor of regular and \bar{C} a square later.

** If in (h)⁺, $\theta = \aleph_0$, we get (h).

- (i) $\mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$ increases with β .

This will be proved below.

6.7B CLAIM: In 6.7A we can also have:

- (1) if we let $\mathfrak{b}_\lambda[\bar{\mathfrak{a}}] = \mathfrak{b}_\lambda^\sigma[\bar{\mathfrak{a}}] = \bigcup_{\beta < \sigma} \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$, $\mathfrak{a}_\sigma = \bigcup_{\beta < \sigma} \mathfrak{a}_\beta$ then also for $\beta = \sigma$ we have (b) (use $N_{i_{\beta+1}}$), (c), (d), (f), (i).
- (2) If $\sigma = \text{cf}(\sigma) > |\mathfrak{a}|$ then for $\beta = \sigma$ also (e), (g).
- (3) If $\text{cf}(\sigma) > |\mathfrak{a}|$, $\mathfrak{c} \in N_{i_\sigma}$, $\mathfrak{c} \subseteq \mathfrak{a}_\sigma$ (hence $|\mathfrak{c}| < \text{Min}(\mathfrak{c})$ and $\mathfrak{c} \subseteq \mathfrak{a}_\sigma$), then for some finite $\mathfrak{d} \subseteq (\text{pcf } \mathfrak{c}) \cap \mathfrak{a}_\sigma$ we have $\mathfrak{c} \subseteq \bigcup_{\mu \in \mathfrak{d}} \mathfrak{b}_\mu[\bar{\mathfrak{a}}]$. Similarly for θ -complete, $\theta < \text{cf}(\sigma)$ (i.e. we have clauses (h), (h)⁺ for $\beta = \sigma$).
- (4) We can have continuity in $\delta \leq \sigma$ when $\text{cf}(\delta) > |\mathfrak{a}|$, i.e. $\mathfrak{b}_\lambda^\delta = \bigcup_{\beta < \delta} \mathfrak{b}_\lambda^\beta$.

6.7C Remark:

- (1) If we want to use length κ , use \bar{N} as produced in [Sh420, 2.6] so $\sigma = \kappa$.
- (2) Concerning 6.7B, in 6.7C(1) for a club E of $\sigma = \kappa$, we have $\alpha \in E \Rightarrow \mathfrak{b}_\lambda^\alpha[\bar{\mathfrak{a}}] = \mathfrak{b}_\lambda[\bar{\mathfrak{a}}] \cap \mathfrak{a}_\alpha$.
- (3) We can also use 6.7 (6.7A, 6.7B) to give an alternative proof of part of the localization theorems similar to the one given in the Spring '89 lectures.
For example:
 - (3A) If $|\mathfrak{a}| < \theta = \text{cf } \theta < \text{Min}(\mathfrak{a})$, for no $\lambda_i \in \text{pcf } \mathfrak{a}$ ($i < \theta$) $\alpha < \theta$, do we have $\bigwedge_{\alpha < \theta} [\lambda_\alpha > \max \text{pcf} \{ \lambda_i : i < \alpha \}]$.
 - (3B) if $|\mathfrak{a}| < \text{Min}(\mathfrak{a})$, $|\mathfrak{b}| < \text{Min } \mathfrak{b}$, $\mathfrak{b} \subseteq \text{pcf}(\mathfrak{a})$, $\lambda \in \text{pcf}(\mathfrak{a})$, then for some $\mathfrak{c} \subseteq \mathfrak{b}$ we have $|\mathfrak{c}| \leq |\mathfrak{a}|$ and $\lambda \in \text{pcf}(\mathfrak{c})$.

Proof of (3A) from 6.7C(3): Without loss of generality $\text{Min } \mathfrak{a} > \theta^{+3}$, let $\kappa = \theta^{+2}$, let \bar{N} , N_κ , $\bar{\mathfrak{a}}$, \mathfrak{b} (as a function), $\langle i_\alpha : \alpha \leq \sigma =: |\mathfrak{a}|^+ \rangle$ be as in 6.7A but also $\langle \lambda_i : i < \theta \rangle \in N_0$. So for $j < \theta$, $\mathfrak{c}_j =: \{ \lambda_i : i < j \} \in N_0$ (and $\mathfrak{c}_j \subseteq \mathfrak{a}_0$) hence (by clause (h) of 6.7A), for some finite $\mathfrak{d}_j \subseteq \mathfrak{a}_1 \cap \text{pcf } \mathfrak{c}_j = N_{i_1} \cap \text{pcf } \mathfrak{a} \cap \text{pcf } \mathfrak{c}_j$ we have $\mathfrak{c}_j \subseteq \bigcup_{\lambda \in \mathfrak{d}_j} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}]$. Assume $j(1) < j(2) < \theta$. Now if $\mu \in \mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_{j(1)}} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}]$ then for some $\mu_0 \in \mathfrak{d}_{j(1)}$ we have $\mu \in \mathfrak{b}_{\mu_0}^1[\bar{\mathfrak{a}}]$; now $\mu_0 \in \mathfrak{d}_{j(1)} \subseteq \text{pcf}(\mathfrak{c}_{j(1)}) \subseteq \text{pcf}(\mathfrak{c}_{j(2)}) \subseteq \text{pcf} \left(\bigcup_{\lambda \in \mathfrak{d}_{j(2)}} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}] \right) = \bigcup_{\lambda \in \mathfrak{d}_{j(2)}} \text{pcf}(\mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}])$ hence (by clause (g) of 6.7A as $\mu_0 \in \mathfrak{d}_{j(0)} \subseteq N_1$) for some $\mu_1 \in \mathfrak{d}_{j(2)}$, $\mu_0 \in \mathfrak{b}_{\mu_1}^1[\bar{\mathfrak{a}}]$. So by clause (f) of 6.7A we have $\mathfrak{b}_{\mu_0}^1[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}_{\mu_1}^1[\bar{\mathfrak{a}}]$ so remembering $\mu \in \mathfrak{b}_{\mu_0}^1[\bar{\mathfrak{a}}]$, we have $\mu \in \mathfrak{b}_{\mu_1}^1[\bar{\mathfrak{a}}]$. Remembering μ was any member of $\mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_{j(1)}} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}]$, we have $\mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_{j(1)}} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}] \subseteq \mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_{j(2)}} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}]$ (holds without “ $\mathfrak{a} \cap$ ” but not used). So $\langle \mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_j} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}] : j < \theta \rangle$ is a non-decreasing sequence of subsets of \mathfrak{a} , but $\text{cf}(\theta) > |\mathfrak{a}|$, so the sequence is

eventually constant, say for $j \geq j(*)$. But

$$\begin{aligned} \max \text{pcf} \left(\mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_j} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}] \right) &\leq \max \text{pcf} \left(\bigcup_{\lambda \in \mathfrak{d}_j} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}] \right) \\ &= \max_{\lambda \in \mathfrak{d}_j} (\max \text{pcf}(\mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}])) \\ &= \max_{\lambda \in \mathfrak{d}_j} \lambda \leq \max \text{pcf} \{ \lambda_i : i < j \} < \lambda_j \\ &= \max \text{pcf} \left(\mathfrak{a} \cap \bigcup_{\lambda \in \mathfrak{d}_{j+1}} \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}] \right) \end{aligned}$$

(last equality as $\mathfrak{b}_{\lambda_j}[\mathfrak{a}] \subseteq \mathfrak{b}_\lambda^1[\bar{\mathfrak{a}}] \bmod J_{<\lambda}[\mathfrak{a}_1]$). Contradiction. ■_{6.7C}

Proof of 6.7C(3B) (like [Sh371, §3]): Included for completeness. If this fails choose a counterexample $(\mathfrak{a}, \mathfrak{b}, \lambda)$ with $|\mathfrak{b}|$ minimal, and among those with $\max \text{pcf}(\mathfrak{b})$ minimal and among those with $\bigcup \{ \mu^+ : \mu \in \lambda \cap \text{pcf}(\mathfrak{b}) \}$ minimal. So $\max \text{pcf}(\mathfrak{b}) = \lambda$, and $\mu = \sup[\lambda \cap \text{pcf}(\mathfrak{a})]$ is not in $\text{pcf}(\mathfrak{b})$ or $\mu = \lambda$. Try to choose by induction on $i < |\mathfrak{a}|^+$, $\lambda_i \in \lambda \cap \text{pcf}(\mathfrak{b})$, $\lambda_i > \max \text{pcf} \{ \lambda_j : j < i \}$, by 6.7C(3A), we will be stuck at some i , and by the previous sentence (and choice of $(\mathfrak{a}, \mathfrak{b}, \lambda)$, i is limit, so $\text{pcf}(\{ \lambda_j : j < i \}) \not\subseteq \lambda$ but it is $\subseteq \text{pcf}(\mathfrak{b}) \subseteq \lambda^+$, so $\lambda = \max \text{pcf} \{ \lambda_j : j < i \}$. For each j , by the minimality condition for some $\mathfrak{b}_j \subseteq \mathfrak{b}$, we have $|\mathfrak{b}_j| \leq |\mathfrak{a}|$, $\lambda_j \in \text{pcf}(\mathfrak{b}_j)$. So $\lambda \in \text{pcf} \{ \lambda_j : j < i \} \subseteq \text{pcf}(\bigcup_{j < i} \mathfrak{b}_j)$ but $\bigcup_{j < i} \mathfrak{b}_j$ is a subset of \mathfrak{b} of cardinality $\leq |i| \times |\mathfrak{a}| = |\mathfrak{a}|$.

6.7D Proof of 6.7A: Let $\langle \langle f_\alpha^{\mathfrak{a}, \lambda} : \alpha < \lambda \rangle : \lambda \in \text{pcf } \mathfrak{a} \rangle$ be chosen as in the proof of 6.7. For $\zeta < \kappa$ we define $\mathfrak{a}^\zeta =: N_\zeta \cap \text{pcf } \mathfrak{a}$; we also define ${}^\zeta \bar{f}$ as $\langle \langle f_\alpha^{\mathfrak{a}^\zeta, \lambda} : \alpha < \lambda \rangle : \lambda \in \text{pcf } \mathfrak{a} \rangle$ where $f_\alpha^{\mathfrak{a}^\zeta, \lambda} \in \prod \mathfrak{a}^\zeta$ is defined as follows:

- (a) if $\theta \in \mathfrak{a}$, $f_\alpha^{\mathfrak{a}^\zeta, \lambda}(\theta) = f_\alpha^{\mathfrak{a}, \lambda}(\theta)$,
- (b) if $\theta \in \mathfrak{a}^\zeta \setminus \mathfrak{a}$ and $\text{cf}(\alpha) \notin (|\mathfrak{a}^\zeta|, \text{Min } \mathfrak{a})$, then

$$f_\alpha^{\mathfrak{a}^\zeta, \lambda}(\theta) = \text{Min} \{ \gamma < \theta : f_\alpha^{\mathfrak{a}, \lambda} \upharpoonright \mathfrak{b}_\theta[\mathfrak{a}] \leq_{J_{<\theta}[\mathfrak{b}_\theta[\mathfrak{a}]]} f_\gamma^{\mathfrak{a}, \theta} \upharpoonright \mathfrak{b}_\theta[\mathfrak{a}] \},$$

- (c) if $\theta \in \mathfrak{a}^\zeta \setminus \mathfrak{a}$ and $\text{cf}(\alpha) \in (|\mathfrak{a}^\zeta|, \text{Min } \mathfrak{a})$, define $f_\alpha^{\mathfrak{a}^\zeta, \lambda}(\theta)$ so as to satisfy $(*)_1$ in the proof of 6.7.

Now ${}^\zeta \bar{f}$ is legitimate except that we have only

$$\beta < \gamma < \lambda \in \text{pcf } \mathfrak{a} \Rightarrow f_\beta^{\mathfrak{a}^\zeta, \lambda} \leq f_\gamma^{\mathfrak{a}^\zeta, \lambda} \bmod J_{<\lambda}[\mathfrak{a}^\zeta]$$

(instead of strict inequality) and $\bigwedge_{\beta < \lambda} \bigvee_{\gamma < \lambda} [f_{\beta}^{\mathfrak{a}^{\zeta}, \lambda} < f_{\gamma}^{\mathfrak{a}^{\zeta}, \lambda} \bmod J_{< \lambda}[\mathfrak{a}^{\zeta}]]$, but this suffices. (The first statement is actually proved in [Sh371, 3.2A], the second in [Sh371, 3.2B]; by it also ${}^{\zeta}\bar{f}$ is cofinal in the required sense.)

For every $\zeta < \kappa$ we can apply 6.7 with $(N_{\zeta} \cap \text{pcf } \mathfrak{a})$, ${}^{\zeta}\bar{f}$ and $\langle N_{\zeta+1+i} : i < \kappa \rangle$ here standing for \mathfrak{a} , \bar{f} , \bar{N} there. In the proof of 6.7 get a club E_{ζ} of κ (so any $i < j$ from E_{ζ} are O.K.). Now we can define for $\zeta < \kappa$ and $i < j$ in E_{ζ} , ${}^{\zeta}\mathfrak{b}_{\lambda}^{i,j}$ and $\langle {}^{\zeta}\mathfrak{b}_{\lambda}^{i,j,\epsilon} : \epsilon < |\mathfrak{a}^{\zeta}|^+ \rangle$, $\langle \epsilon^{\zeta}(i, j, \lambda) : \lambda \in \mathfrak{a}^{\zeta} \rangle$, $\epsilon^{\zeta}(i, j)$, as well as in the proof of 6.7. Let:

$$E = \{i < \kappa : i \text{ is a limit ordinal } (\forall j < i)(j+j < i \& j \times j < i) \text{ and } \bigwedge_{j < i} i \in E_j \}.$$

So by [Sh420, §1] we can find $\bar{C} = \langle C_{\delta} : \delta \in S \rangle$, $S \subseteq \{\delta < \kappa : \text{cf } \delta = \text{cf } \sigma\}$ stationary, C_{δ} a club of δ , $\text{otp } C_{\delta} = \omega^2 \sigma$ such that:

- (1) for each $\alpha < \lambda$, $\{C_{\delta} \cap \alpha : \alpha \in \text{nacc}(C_{\delta})\}$ has cardinality $< \kappa^*$ and
- (2) for every club E' of θ for stationarily many $\delta \in S$, $C_{\delta} \subseteq E'$.

Without loss of generality $\bar{C} \in N_0$. For some δ^* , $C_{\delta^*} \subseteq E$, and let $\{j_{\zeta} : \zeta \leq \omega^2 \sigma\}$ enumerate $C_{\delta^*} \cup \{\delta^*\}$. So $\langle j_{\zeta} : \zeta \leq \omega^2 \sigma \rangle$ is a strictly increasing continuous sequence of ordinals from $E \subseteq \kappa$ such that $\langle j_{\epsilon} : \epsilon \leq \zeta \rangle \in N_{j_{\zeta+1}}$. Let $j(\zeta) = j_{\zeta}$, $i(\zeta) = i_{\zeta} =: j_{\omega^2(1+\zeta)}$, $\mathfrak{a}_{\zeta} = N_{i_{\zeta}} \cap \text{pcf } \mathfrak{a}$, and $\bar{\mathfrak{a}} =: \langle \mathfrak{a}_{\zeta} : \zeta < \sigma \rangle$, $\mathfrak{b}_{\lambda}^{\zeta}[\bar{\mathfrak{a}}] =: i(\zeta) \mathfrak{b}_{\lambda}^{j(\omega^2 \zeta+1), j(\omega^2 \zeta+2), \epsilon^{\zeta}(j(\omega^2 \zeta+1), j(\omega^2 \zeta+2))}$. Most of the requirements follow immediately, as

- (*) for each $\zeta < \sigma$, we have \mathfrak{a}_{ζ} , $\langle \mathfrak{b}_{\lambda}^{\zeta}[\bar{\mathfrak{a}}] : \lambda \in \mathfrak{a}_{\zeta} \rangle$ are as in 6.7 and belong to $N_{i_{\beta+3}} \subseteq N_{i_{\beta+1}}$.

We are left (for proving 6.7A) with proving (h)⁺ and (i) (remember (h) is a special case of (h)⁺ choosing $\theta = \aleph_0$).

For proving clause (i) note that for $\zeta < \xi < \kappa$, $f_{\alpha}^{\mathfrak{a}^{\zeta}, \lambda} \subseteq f_{\alpha}^{\mathfrak{a}^{\xi}, \lambda}$ hence ${}^{\zeta}\mathfrak{b}_{\lambda}^{i,j} \subseteq {}^{\xi}\mathfrak{b}_{\lambda}^{i,j}$. Now we can prove by induction on ϵ that ${}^{\zeta}\mathfrak{b}_{\lambda}^{i,j,\epsilon} \subseteq {}^{\xi}\mathfrak{b}_{\lambda}^{i,j,\epsilon}$ for every $\lambda \in \mathfrak{a}_{\zeta}$ (check the definition after (*)₂ in the proof of 6.7) and the conclusion follows.

Instead of proving (h)⁺ we prove an apparently weaker version (h)' below, and then note that $\bar{i}' = \langle i_{\omega^2 \zeta} : \zeta < \sigma \rangle$, $\bar{\mathfrak{a}}' = \langle \mathfrak{a}_{\omega^2 \zeta} : \zeta < \sigma \rangle$, $\langle N_{i(\omega^2 \zeta)} : \zeta < \sigma \rangle$, $\langle \mathfrak{b}_{\lambda}^{\omega^2 \zeta}[\bar{\mathfrak{a}}'] : \zeta < \sigma, \lambda \in \mathfrak{a}'_{\zeta} = \mathfrak{a}_{\omega^2 \zeta} \rangle$ will exemplify the conclusion** where

- (h)' if $\mathfrak{c} \subseteq \mathfrak{a}_{\beta}$, $\beta < \sigma$, $\mathfrak{c} \in N_{i_{\beta+1}}$, $\theta = \text{cf}(\theta) \in N_{i_{\beta+1}}$ then for some $\mathfrak{d} \in N_{i_{\beta+\omega+1+1}}$, $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+\omega} \cap \text{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ we have $\mathfrak{c} \subseteq \bigcup_{\mu \in \mathfrak{d}} \mathfrak{b}_{\mu}^{\beta+\omega}[\bar{\mathfrak{a}}]$ and $|\mathfrak{d}| < \theta$.

* If κ is successor of regular, then we can get $[\gamma \in C_{\alpha} \cap C_{\beta} \Rightarrow C_{\alpha} \cap \gamma = C_{\beta} \cap \gamma]$.

** Assuming $\sigma > \aleph_0$ hence, $\omega^2 \sigma = \sigma$ for notational simplicity.

Proof of (h)': So let $\theta, \beta, \mathfrak{c}$ be given; let $\langle \mathfrak{b}_\mu[\mathfrak{a}] : \mu \in \text{pcf } \mathfrak{c} \rangle (\in N_{i_{\beta+1}})$ be a generating sequence. We define by induction on $n < \omega$, $A_n, \langle \mathfrak{c}_\eta, \lambda_\eta : \eta \in A_n \rangle$ such that:

- (a) $A_0 = \{\langle \rangle\}$, $\mathfrak{c}_{\langle \rangle} = \mathfrak{c}$, $\lambda_{\langle \rangle} = \max \text{pcf } \mathfrak{c}$,
- (b) $A_n \subseteq {}^n\theta$, $|A_n| < \theta$,
- (c) if $\eta \in A_{n+1}$ then $\eta \upharpoonright n \in A_n$, $\mathfrak{c}_\eta \subseteq \mathfrak{c}_{\eta \upharpoonright n}$, $\lambda_\eta < \lambda_{\eta \upharpoonright n}$ and $\lambda_\eta = \max \text{pcf}(\mathfrak{c}_\eta)$,
- (d) $A_n, \langle \mathfrak{c}_\eta, \lambda_\eta : \eta \in A_n \rangle$ belongs to $N_{i_{\beta+1+n}}$ hence $\lambda_\eta \in N_{i_{\beta+1+n}}$,
- (e) if $\eta \in A_n$ and $\lambda_\eta \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{c}_\eta)$ and $\mathfrak{c}_\eta \not\subseteq \mathfrak{b}_{\lambda_\eta}^{\beta+1+n}[\bar{\mathfrak{a}}]$ then $(\forall \nu)[\nu \in A_{n+1} \ \& \ \eta \subseteq \nu \Leftrightarrow \nu = \eta \hat{\ } \langle 0 \rangle]$ and $\mathfrak{c}_{\eta \hat{\ } \langle 0 \rangle} = \mathfrak{c}_\eta \setminus \mathfrak{b}_{\lambda_\eta}^{\beta+1+n}[\bar{\mathfrak{a}}]$ (so $\lambda_{\eta \hat{\ } \langle 0 \rangle} = \max \text{pcf } \mathfrak{c}_{\eta \hat{\ } \langle 0 \rangle} < \lambda_\eta = \max \text{pcf } \mathfrak{c}_\eta$),
- (f) if $\eta \in A_n$ and $\lambda_\eta \notin \text{pcf}_{\theta\text{-complete}}(\mathfrak{c}_\eta)$ then

$$\mathfrak{c}_\eta = \bigcup \{ \mathfrak{b}_{\lambda_{\eta \hat{\ } \langle i \rangle}}[\mathfrak{c}] : i < i_n < \theta, \eta \hat{\ } \langle i \rangle \in A_{n+1} \},$$

and if $\nu = \eta \hat{\ } \langle i \rangle \in A_{n+1}$ then $\mathfrak{c}_\nu = \mathfrak{b}_{\lambda_\nu}[\mathfrak{c}]$,

- (g) if $\eta \in A_n$, and $\lambda_\eta \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{c}_\eta)$ but $\mathfrak{c}_\eta \subseteq \mathfrak{b}_{\lambda_\eta}^{\beta+1-n}[\bar{\mathfrak{a}}]$, then $\neg(\exists \nu)[\eta \triangleleft \nu \in A_{n+1}]$.

There is no problem to carry the definition (we use 6.7F(1) below*, the point is that $\mathfrak{c} \in N_{i_{\beta+1+n}}$ implies $\langle \mathfrak{b}_\lambda[\mathfrak{c}] : \lambda \in \text{pcf}_\theta[\mathfrak{c}] \rangle \in N_{i_{\beta+1+n}}$ and as there is \mathfrak{d} as in 6.7F(1), there is one in $N_{i_{\beta+1+n+1}}$ so $\mathfrak{d} \subseteq \mathfrak{a}_{\beta+1+n+1}$). Now let

$$\mathfrak{d}_n =: \left\{ \lambda_\eta : \eta \in A_n \text{ and } \lambda_\eta \in \underset{\theta\text{-complete}}{\text{pcf}}(\mathfrak{c}_\eta) \text{ and } \mathfrak{c}_\eta \subseteq \mathfrak{b}_{\lambda_\eta}^{\beta+1+n}[\bar{\mathfrak{a}}] \right\}$$

and $\mathfrak{d} =: \bigcup_{n < \omega} \mathfrak{d}_n$; we shall show that it is as required.

The main point is $\mathfrak{c} \subseteq \bigcup_{\lambda \in \mathfrak{d}} \mathfrak{b}_\lambda^{\beta+\omega}[\bar{\mathfrak{a}}]$; note that

$$\left[\lambda_\eta \in \mathfrak{d}, \eta \in A_n \Rightarrow \mathfrak{b}_{\lambda_\eta}^{\beta+1+n}[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}_{\lambda_\eta}^{\beta+\omega}[\bar{\mathfrak{a}}] \right]$$

hence it suffices to show $\mathfrak{c} \subseteq \bigcup_{n < \omega} \bigcup_{\lambda \in \mathfrak{d}_n} \mathfrak{b}_\lambda^{\beta+1+n}[\bar{\mathfrak{a}}]$, so assume $\theta \in \mathfrak{c} \setminus \bigcup_{n < \omega} \bigcup_{\lambda \in \mathfrak{d}_n} \mathfrak{b}_\lambda^{\beta+1+n}[\bar{\mathfrak{a}}]$, and we choose by induction on n , $\eta_n \in A_n$ such that $\eta_0 = \langle \rangle$, $\eta_{n+1} \upharpoonright n = \eta_n$ and $\theta \in \mathfrak{c}_{\eta_n}$; by clauses (e) + (f) above this is possible and $\langle \max \text{pcf } \mathfrak{c}_{\eta_n} : n < \omega \rangle$ is strictly decreasing, contradiction.

The minor point is $|\mathfrak{d}| < \theta$; if $\theta > \aleph_0$ note that $\bigwedge_n |A_n| < \theta$ and $\theta = \text{cf}(\theta)$ so $|\mathfrak{d}| \leq |\bigcup_n A_n| < \theta + \aleph_1 = \theta$.

* No vicious circle; 6.7F(1) does not depend on 6.7B.

If $\theta = \aleph_0$ (i.e. clause (h)) we should have $\bigcup_n A_n$ finite; the proof is as above noting the clause (f) is vacuous now. So $\bigwedge_n |A_n| = 1$ and $\bigvee_n A_n = \emptyset$, so $\bigcup_n A_n$ is finite. Another minor point is $\mathfrak{d} \in N_{i_{\beta+\omega+1}}$; this holds as the construction is unique from $\langle N_j: j < i_{\beta+\omega} \rangle, \langle i_j: j \leq \beta + \omega \rangle, \langle \langle \mathfrak{a}_{i(\zeta)}, \langle \mathfrak{b}_\lambda^\zeta: \lambda \in \mathfrak{a}_{i(\zeta)} \rangle \rangle: \zeta \leq \beta + \omega \rangle$; no “outside” information is used so $\langle \langle A_n, \langle \langle c_\eta, \lambda_\eta \rangle: \eta \in A_n \rangle \rangle: n < \omega \rangle \in N_{i_{\beta+\omega+1}}$, so (using a choice function) really $\mathfrak{d} \in N_{i_{\beta+\omega+1}}$. ■6.7A

6.7E Proof of 6.7B: Let $\mathfrak{b}_\lambda[\bar{\mathfrak{a}}] = \mathfrak{b}_\lambda^\sigma = \bigcup_{\beta < \sigma} \mathfrak{b}_\lambda^\beta[\mathfrak{a}_\beta]$ and $\mathfrak{a}_\sigma = \bigcup_{\zeta < \sigma} \mathfrak{a}_\zeta$. Part (1) is straightforward. For part (2), for clause (g), for $\beta = \sigma$, the inclusion “ \subseteq ” is straightforward; so assume $\mu \in \mathfrak{a}_\beta \cap \text{pcf } \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$. Then by 6.7A(c) for some $\beta_0 < \beta$, we have $\mu \in \mathfrak{a}_{\beta_0}$, and by 6.7C(3B) (which depends on 6.7A only) for some $\beta_1 < \beta$, $\mu \in \text{pcf } \mathfrak{b}_\lambda^{\beta_1}[\bar{\mathfrak{a}}]$; by monotonicity wlog $\beta_0 = \beta_1$, by clause (g) of 6.7A applied to β_0 , $\mu \in \mathfrak{b}_\lambda^{\beta_0}[\bar{\mathfrak{a}}]$. Hence by clause (i) of 6.7A, $\mu \in \mathfrak{b}_\lambda^\beta[\bar{\mathfrak{a}}]$, thus proving the other inclusion.

The proof of clause (e) (for 6.7B(2)) is similar, and also 6.7B(3). For 6.7(B)(4) for $\delta < \sigma$, $\text{cf}(\delta) > |\mathfrak{a}|$ redefine $\mathfrak{b}_\lambda^\delta[\bar{\mathfrak{a}}]$ as $\bigcup_{\beta < \delta} \mathfrak{b}_\lambda^{\beta+1}[\mathfrak{a}]$. ■6.7B

6.7F CLAIM: *Let θ be regular.*

- (0) If $\alpha < \theta$, $\text{pcf}_{\theta\text{-complete}}(\bigcup_{i < \alpha} \mathfrak{a}_i) = \bigcup_{i < \alpha} \text{pcf}_{\theta\text{-complete}}(\mathfrak{a}_i)$.
- (1) If $\langle \mathfrak{b}_\theta[\mathfrak{a}]: \theta \in \text{pcf } \mathfrak{a} \rangle$ is a generating sequence for \mathfrak{a} , $\mathfrak{c} \subseteq \mathfrak{a}$, then for some $\mathfrak{d} \subseteq \text{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ we have: $|\mathfrak{d}| < \theta$ and $\mathfrak{c} \subseteq \bigcup_{\theta \in \mathfrak{a}} \mathfrak{b}_\theta[\mathfrak{a}]$.
- (2) If $|\mathfrak{a} \cup \mathfrak{c}| < \text{Min } \mathfrak{a}$, $\mathfrak{c} \subseteq \text{pcf}_{\theta\text{-complete}}(\mathfrak{a})$, $\lambda \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ then $\lambda \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{a})$.
- (3) In (2) we can weaken $|\mathfrak{a} \cup \mathfrak{c}| < \text{Min } \mathfrak{a}$ to $|\mathfrak{a}| < \text{Min } \mathfrak{a}$, $|\mathfrak{c}| < \text{Min } \mathfrak{c}$.
- (4) We cannot find $\lambda_\alpha \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{a})$ for $\alpha < |\mathfrak{a}|^+$ such that $\lambda_i > \sup \text{pcf}_{\theta\text{-complete}}(\{\lambda_j: j < i\})$.
- (5) Assume $\theta \leq |\mathfrak{a}|$, $\mathfrak{c} \subseteq \text{pcf}_{\theta\text{-complete}} \mathfrak{a}$ (and $|\mathfrak{c}| < \text{Min } \mathfrak{c}$; of course $|\mathfrak{a}| < \text{Min } \mathfrak{a}$). If $\lambda \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{c})$ then for some $\mathfrak{d} \subseteq \mathfrak{c}$ we have $|\mathfrak{d}| \leq |\mathfrak{a}|$ and $\lambda \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{d})$.

Proof: (0) and (1): Check.

(2) See [Sh345b, 1.10–1.12].

(3) Similarly.

(4) If $\theta = \aleph_0$ we already know it (e.g. 6.7C(3A)), so assume $\theta > \aleph_0$ and, without loss of generality, θ is regular $\leq |\mathfrak{a}|$. We use 6.7A with $\{\theta, \langle \lambda_i: i < |\mathfrak{a}|^+ \rangle\} \in N_0$, $\sigma = |\mathfrak{a}|^+$, $\kappa = |\mathfrak{a}|^{+3}$ where, without loss of generality, $\kappa < \text{Min}(\mathfrak{a})$. For each $\alpha < |\mathfrak{a}|^+$ by (h)⁺ of 6.7A there is $\mathfrak{d}_\alpha \in N_{i_1}$, $\mathfrak{d}_\alpha \subseteq \text{pcf}_{\theta\text{-complete}}(\{\lambda_i: i < \alpha\})$,

$|\mathfrak{d}_\alpha| < \theta$ such that $\{\lambda_i: i < \alpha\} \subseteq \bigcup_{\theta \in \mathfrak{d}_\alpha} \mathfrak{b}_\theta^1[\bar{\mathfrak{a}}]$; hence by clause (g) of 6.7A and 6.7F(0) we have $\mathfrak{a}_1 \cap \text{pcf}_{\theta\text{-complete}}(\{\lambda_i: i < \alpha\}) \subseteq \bigcup_{\theta \in \mathfrak{d}_\alpha} \mathfrak{b}_\theta^1[\bar{\mathfrak{a}}]$. So for $\alpha < \beta < |\mathfrak{a}|^+$, $\mathfrak{d}_\alpha \subseteq \mathfrak{a}_1 \cap \text{pcf}_{\theta\text{-complete}}\{\lambda_i: i < \alpha\} \subseteq \mathfrak{a}_1 \cap \text{pcf}_{\theta\text{-complete}}\{\lambda_i: i < \beta\} \subseteq \bigcup_{\theta \in \mathfrak{d}_\beta} \mathfrak{b}_\theta^1[\bar{\mathfrak{a}}]$. As the sequence is smooth (i.e. clause (f) of 6.7A) clearly $\alpha < \beta \Rightarrow \bigcup_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \subseteq \bigcup_{\mu \in \mathfrak{d}_\beta} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}]$.

So $\langle \bigcup_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}: \alpha < |\mathfrak{a}|^+ \rangle$ is a non-decreasing sequence of subsets of \mathfrak{a} of length $|\mathfrak{a}|^+$, hence for some $\alpha(*) < |\mathfrak{a}|^+$ we have:

$$(*)_1 \quad \alpha(*) \leq \alpha < |\mathfrak{a}|^+ \Rightarrow \bigcup_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a} = \bigcup_{\mu \in \mathfrak{d}_{\alpha(*)}} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}.$$

If $\tau \in \mathfrak{a}_1 \cap \text{pcf}_{\theta\text{-complete}}(\{\lambda_i: i < \alpha\})$ then $\tau \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{a})$ (by 6.7F(2),(3)), and $\tau \in \mathfrak{b}_{\mu_\tau}^1[\bar{\mathfrak{a}}]$ for some $\mu_\tau \in \mathfrak{d}_\alpha$ so $\mathfrak{b}_\tau^1[\bar{\mathfrak{a}}] \subseteq \mathfrak{b}_{\mu_\tau}^1[\bar{\mathfrak{a}}]$, also $\tau \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{b}_\tau^1[\bar{\mathfrak{a}}] \cap \mathfrak{a})$ (by clause (e) of 6.7A), hence

$$\begin{aligned} \tau \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{b}_\tau^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}) &\subseteq \text{pcf}_{\theta\text{-complete}}(\mathfrak{b}_{\mu_\tau}^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}) \\ &\subseteq \text{pcf}_{\theta\text{-complete}}\left(\bigcup_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}\right). \end{aligned}$$

So $\mathfrak{a}_1 \cap \text{pcf}_{\theta\text{-complete}}(\{\lambda_i: i < \alpha\}) \subseteq \text{pcf}_{\theta\text{-complete}}\left(\bigcup_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}\right)$. But for each $\alpha < |\mathfrak{a}|^+$ we have $\lambda_\alpha > \sup \text{pcf}_{\theta\text{-complete}}(\{\lambda_i: i < \alpha\})$, whereas $\mathfrak{d}_\alpha \subseteq \text{pcf}_{\sigma\text{-complete}}\{\lambda_i: i < \alpha\}$, hence $\lambda_\alpha > \sup \mathfrak{d}_\alpha$ hence

$$(*)_2 \quad \lambda_\alpha > \sup_{\mu \in \mathfrak{d}_\alpha} \max \text{pcf} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \geq \sup \text{pcf}_{\theta\text{-complete}}\left(\bigcup_{\mu \in \mathfrak{d}_\alpha} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}\right).$$

On the other hand,

$$(*)_3 \quad \lambda_\alpha \in \text{pcf}_{\theta\text{-complete}}\{\lambda_i: i < \alpha + 1\} \subseteq \text{pcf}_{\theta\text{-complete}}\left(\bigcup_{\mu \in \mathfrak{d}_{\alpha+1}} \mathfrak{b}_\mu^1[\bar{\mathfrak{a}}] \cap \mathfrak{a}\right).$$

For $\alpha = \alpha(*)$ we get contradiction by $(*)_1 + (*)_2 + (*)_3$.

(5) Assume $\mathfrak{a}, \mathfrak{c}, \lambda$ form a counterexample with λ minimal. Without loss of generality $|\mathfrak{a}|^{+3} < \text{Min}(\mathfrak{a})$ and $\lambda = \max \text{pcf} \mathfrak{a}$ and $\lambda = \max \text{pcf} \mathfrak{c}$ (just let $\mathfrak{a}' =: \mathfrak{b}_\lambda[\mathfrak{a}]$, $\mathfrak{c}' =: \mathfrak{c} \cap \text{pcf}_\theta[\mathfrak{a}']$; if $\lambda \notin \text{pcf}_{\theta\text{-complete}}(\mathfrak{c}')$ then necessarily $\lambda \in \text{pcf}(\mathfrak{c} \setminus \mathfrak{c}')$ (by 6.7F(0)) and similarly $\mathfrak{c} \setminus \mathfrak{c}' \subseteq \text{pcf}_{\theta\text{-complete}}(\mathfrak{a} \setminus \mathfrak{a}')$ hence by 6.7F(2),(3) $\lambda \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{a} \setminus \mathfrak{a}')$, contradiction).

Also without loss of generality $\lambda \notin \mathfrak{c}$. Let $\kappa, \sigma, \bar{N}, \langle i_\alpha = i(\alpha): \alpha \leq \sigma \rangle$, $\bar{\mathfrak{a}} = \langle \mathfrak{a}_i: i \leq \sigma \rangle$ be as in 6.7A with $\mathfrak{a} \in N_0, \mathfrak{c} \in N_0, \lambda \in N_0, \sigma = |\mathfrak{a}|^+, \kappa = |\mathfrak{a}|^{+3} < \text{Min} \mathfrak{a}$. We choose by induction on $\epsilon < |\mathfrak{a}|^+, \lambda_\epsilon, \mathfrak{d}_\epsilon$ such that:

- (a) $\lambda_\epsilon \in \mathfrak{a}_{\omega^2\epsilon + \omega + 3}, \mathfrak{d}_\epsilon \in N_{i(\omega^2\epsilon + \omega + 1)}$,
- (b) $\lambda_\epsilon \in \mathfrak{c}$,
- (c) $\mathfrak{d}_\epsilon \subseteq \mathfrak{a}_{\omega^2\epsilon + \omega + 1} \cap \text{pcf}_{\theta\text{-complete}}(\{\lambda_\zeta: \zeta < \epsilon\})$,

- (d) $|\mathfrak{d}_\epsilon| < \theta$,
- (e) $\{\lambda_\zeta: \zeta < \epsilon\} \subseteq \bigcup_{\theta \in \mathfrak{d}_\epsilon} \mathfrak{b}_\theta^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$,
- (f) $\lambda_\epsilon \notin \text{pcf}_{\theta\text{-complete}}\left(\bigcup_{\theta \in \mathfrak{d}_\epsilon} \mathfrak{b}_\theta^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]\right)$.

For every $\epsilon < |\mathfrak{a}|^+$ we first choose \mathfrak{d}_ϵ as the $<_\chi^*$ -first element satisfying (c) + (d) + (e) and then if possible λ_ϵ as the $<_\chi^*$ -first element satisfying (b) + (f). It is easy to check the requirements and in fact $\langle \lambda_\zeta: \zeta < \epsilon \rangle \in N_{\omega^2\epsilon+1}$, $\langle \mathfrak{d}_\zeta: \zeta < \epsilon \rangle \in N_{\omega^2\epsilon+1}$ (so clause (a) will hold). But why can we choose at all? Now $\lambda \notin \text{pcf}_{\theta\text{-complete}}\{\lambda_\zeta: \zeta < \epsilon\}$ as \mathfrak{a} , \mathfrak{c} , λ form a counterexample with λ minimal and $\epsilon < |\mathfrak{a}|^+$ (by 6.7F(3)). As $\lambda = \max \text{pcf } \mathfrak{a}$ necessarily $\text{pcf}_{\theta\text{-complete}}(\{\lambda_\zeta: \zeta < \epsilon\}) \subseteq \lambda$ hence $\mathfrak{d}_\epsilon \subseteq \lambda$ (by clause (c)). By part (0) of the claim (and clause (a)) we know:

$$\begin{aligned} \text{pcf}_{\theta\text{-complete}} \left[\bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}] \right] &= \bigcup_{\mu \in \mathfrak{d}_\epsilon} \text{pcf}_{\theta\text{-complete}} \left[\mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}] \right] \\ &\subseteq \bigcup_{\mu \in \mathfrak{d}_\epsilon} (\mu + 1) \subseteq \lambda \end{aligned}$$

(note $\mu = \max \text{pcf } \mathfrak{b}_\mu^\beta[\bar{\mathfrak{a}}]$). So $\lambda \notin \text{pcf}_{\theta\text{-complete}}\left(\bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]\right)$ hence by part (0) of the claim $\mathfrak{c} \not\subseteq \bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ so λ_ϵ exists. Now \mathfrak{d}_ϵ exists by 6.7A clause (h)⁺.

Now clearly $\left\langle \mathfrak{a} \cap \bigcup_{\mu \in \mathfrak{d}_\epsilon} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]: \epsilon < |\mathfrak{a}|^+ \right\rangle$ is non-decreasing (as in the earlier proof) hence eventually constant, say for $\epsilon \geq \epsilon(*)$ (where $\epsilon(*) < |\mathfrak{a}|^+$).

But

- (α) $\lambda_\epsilon \in \bigcup_{\mu \in \mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ [clause (e) in the choice of $\lambda_\epsilon, \mathfrak{d}_\epsilon$],
- (β) $\mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}] \subseteq \bigcup_{\mu \in \mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]$ [by clause (f) of 6.7A and (α) alone],
- (γ) $\lambda_\epsilon \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{a})$ [as $\lambda_\epsilon \in \mathfrak{c}$ and a hypothesis],
- (δ) $\lambda_\epsilon \in \text{pcf}_{\theta\text{-complete}}(\mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}])$ [by (γ) above and clause (e) of 6.7A],
- (ϵ) $\lambda_\epsilon \notin \text{pcf}(\mathfrak{a} \setminus \mathfrak{b}_{\lambda_\epsilon}^{\omega^2\epsilon+\omega+1})$,
- (ζ) $\lambda_\epsilon \in \text{pcf}_{\theta\text{-complete}}\left(\mathfrak{a} \cap \bigcup_{\mu \in \mathfrak{d}_{\epsilon+1}} \mathfrak{b}_\mu^{\omega^2\epsilon+\omega+1}[\bar{\mathfrak{a}}]\right)$ [by (δ) + (ϵ) + (β)].

But for $\epsilon = \epsilon(*)$, the statement (ζ) contradicts the choice of $\epsilon(*)$ and clause (f) above. ■6.7F

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