All meager filters may be null

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Abstract

We show that it is consistent with ZFC that all filters which have the Baire property are Lebesgue measurable. We also show that the existence of a Sierpinski set implies that there exists a nonmeasurable filter which has the Baire property.

The goal of this paper is to show yet another example of nonduality between measure and category.

Suppose that $\mathcal{F}$ is a nonprincipal filter on ω. Identify $\mathcal{F}$ with the set of characteristic functions of its elements. Under this convention $\mathcal{F}$ becomes a subset of 2^ω and a question about its topological or measure-theoretical properties makes sense.

It has been proved by Sierpinski that every non-principal filter has either Lebesgue measure zero or is nonmeasurable. Similarly it is either meager or does not have the Baire property.

In [T] Talagrand proved that

Theorem 0.1 *There exists a measurable filter which does not have the Baire property.*■

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In fact we have an even stronger result. In [Ba] it is proved that

Theorem 0.2 *Every measurable filter can be extended to a measurable filter which does not have the Baire property.* ■

We show that the dual result is false.

1 A model where all meager filters are null

In this section we prove the following theorem:

Theorem 1.1 *It is consistent with ZFC that every filter which has the Baire property is measurable.*

Proof We will use the following more general result:

Theorem 1.2 *Let $\mathbf{V} \models GCH$ and suppose that $\mathbf{V}[G]$ is a generic extension extension of $\mathbf{V}$ obtained adding ω_2 Cohen reals. Then in $\mathbf{V}[G]$ for any two sets $A, B \subset 2^\omega$ if $A + B = \{a + b : a \in A, b \in B\}$ is a meager set then either A or B has measure zero.*

Proof Note that we apply this lemma only for the case $A = B$. Therefore to simplify the notation we assume that $A = B$. The proof of the general case is almost the same. We follow [Bu].

We will use the following notation. Let

$$Fn(X, 2) = \{s : \mathrm{dom}(s) \in [X]^{<\omega} \text{ and } \mathrm{range}(s) \subset \{0, 1\}\}$$

be the notion of forcing adding $|X|$-many Cohen reals. For $s \in Fn(X, 2)$ let $[s] = \{f \in 2^X : s \subset f\}$.

Let $\mathbf{V} \models GCH$ be a model of ZFC and let G_{ω_2} be a $Fn(\omega_2, 2)$-generic filter over $\mathbf{V}$. Clearly $c = \bigcup G_{\omega_2}$ is a generic sequence of ω_2 Cohen reals and $\mathbf{V}[c] = \mathbf{V}[G_{\omega_2}]$.

Let $\{F_n : n \in \omega\}$ be a sequence of closed, nowhere dense sets such that $A + A \subseteq \bigcup_{n\in\omega} F_n$. Without loss of generality we can assume that $\{F_n : n \in \omega\} \in \mathbf{V}$.

Let $\{a_\xi : \xi < \omega_2\}$ be an enumeration of all elements of A. For every $\xi < \omega_2$ let $\dot{a}_\xi$ be a name for a_ξ. In other words for every $\xi < \omega_2$ we have a countable set $I_\xi \subset \omega_2$ such that $\dot{a}_\xi$ is a Borel function from 2^{I_ξ} into 2^ω. Moreover a_ξ is the value of of the function $\dot{a}_\xi$ on Cohen real i.e. $\dot{a}_\xi(c|I_\xi) = a_\xi$. In addition we can find a dense G_δ set $H_\xi \subseteq 2^{I_\xi}$ such that $\dot{a}_\xi|H_\xi$ is a continuous function.

For $\alpha, \xi, \eta < \omega_2$ define $\xi \simeq_\alpha \eta$ if

1. I_ξ and I_η are order isomorphic,
2. the order-isomorphism between I_ξ and I_η transfers $\dot{a}_\xi$ onto $\dot{a}_\eta$ and H_ξ onto H_η,

3. $I_\xi \cap \alpha = I_\eta \cap \alpha$.

Notice that for every $\alpha < \omega_2$ the relation $\simeq_\alpha$ is an equivalence relation with ω_1 many equivalence classes.

Lemma 1.3 *There exists $\alpha^\star < \omega_2$ such that*

$$\forall \xi, \beta \; \exists \eta \; (\xi \simeq_{\alpha^\star} \eta \;\&\; I_\eta \cap (\beta - \alpha^\star) = \emptyset) \; .$$

PROOF For every $\alpha < \omega_2$ let $\mathcal{E}_\alpha$ be the set $\{[\xi]_\alpha : \xi < \omega_2\}$ of $\simeq_\alpha$-equivalence classes. Let

$$\mathcal{E}^0_\alpha = \{E \in \mathcal{E}_\alpha : \sup_{\eta \in E}(\min(I_\eta - \alpha)) < \omega_2\} \text{ and}$$

$$\mathcal{E}^1_\alpha = \mathcal{E}_\alpha - \mathcal{E}^0_\alpha \; .$$

Let

$$\gamma(\alpha) = \sup_{E \in \mathcal{E}^0_\alpha} (\sup_{\eta \in E}(\min(I_\eta - \alpha))) \; .$$

Note that $\gamma(\alpha) < \omega_2$ since $|\mathcal{E}_\alpha| < \aleph_1$.

Find $\alpha^\star < \omega_2$ such that $\gamma(\alpha) < \alpha^\star$ for all $\alpha < \alpha^\star$ and $\mathrm{cf}(\alpha^\star) = \omega_1$. We claim that $\alpha^\star$ satisfies the statement of the lemma.

Take any $\xi < \omega_2$ and any β. If $\beta < \alpha^\star$ or $I_\xi \subseteq \alpha^\star$, then we can choose $\eta = \xi$. So assume $\beta > \alpha^\star$ and $I_\xi - \alpha^\star \neq \emptyset$. There is $\alpha < \alpha^\star$ such that $I_\xi \cap \alpha = I_\xi \cap \alpha^\star$. Let $E = [\xi]_\alpha$.

CASE 1 $E \in \mathcal{E}^0_\alpha$. Then

$$\sup_{\eta \in E}(\min(I_\eta - \alpha)) \leq \gamma(\alpha) < \alpha^\star$$

which is a contradiction since $\min(I_\xi - \alpha) \geq \alpha^\star$ and $\xi \in E$.

CASE 2 $E \notin \mathcal{E}^0_\alpha$. So

$$\sup_{\eta \in E}(\min(I_\eta - \alpha)) = \omega_2$$

hence there is $\eta \in E$ with $\min(I_\eta - \alpha) \geq \beta$ i.e. $I_\eta \cap (\beta - \alpha) = \emptyset$.

So $I_\xi \cap \alpha^\star = I_\xi \cap \alpha = I_\eta \cap \alpha = I_\eta \cap \alpha^\star$, where the last equality holds because $I_\eta \cap (\alpha^\star - \alpha) \subseteq I_\eta \cap (\beta - \alpha) = \emptyset$. Also $I_\eta \cap (\beta - \alpha^\star) \subseteq I_\eta \cap (\beta - \alpha) = \emptyset$. ■

Let $\alpha^\star$ be the ordinal from the above lemma. Work in $\mathbf{V}' = \mathbf{V}[c|\alpha^\star]$.

For every $\xi < \omega_2$ define

$$D_\xi = \{s \in Fn(\omega_2 - \alpha^\star, 2) : \mathrm{cl}(\dot{a}_\xi([s])) \text{ has measure zero } \} \; .$$

Lemma 1.4 *D_ξ is dense in $Fn(\omega_2 - \alpha^\star, 2)$ for every $\xi < \omega_2$.*

PROOF Notice that it is enough to show that $D_\xi \cap Fn(I_\xi - \alpha^\star, 2)$ is dense in $Fn(I_\xi - \alpha^\star, 2)$ for $\xi < \omega_2$.

Suppose that this fails. Find $\xi < \omega_2$ and $s_0 \in Fn(I_\xi - \alpha^\star, 2)$ such that for all $s \supseteq s_0$ the set $\mathrm{cl}(\dot{a}_\xi([s]))$ has positive measure.

Using the lemma with $\beta > \sup(I_\xi)$ we can find $\eta < \omega_2$ such that $\xi \simeq_{\alpha^\star} \eta$ and $(I_\xi - \alpha^\star) \cap (I_\eta - \alpha^\star) = \emptyset$. Notice that there exists $t_0 \in Fn(I_\eta - \alpha^\star, 2)$ (the image of s_0 under the isomorphism between I_ξ and I_η) such that for every $t \supseteq t_0$ the set $\mathrm{cl}(\dot{a}_\eta([t]))$ has positive measure.

Since s_0 and t_0 have disjoint domains, $s_0 \cup t_0 \in Fn(\omega_2 - \alpha^\star, 2)$. Find $n \in \omega$ and a condition $u \in Fn(\omega_2 - \alpha^\star, 2)$ extending $s_0 \cup t_0$ such that $u \Vdash \dot{a}_\xi(\dot{c}) + \dot{a}_\eta(\dot{c}) \in F_n$. u can be written as $u_1 \cup u_2 \cup u_3$ where $s_0 \subseteq u_1 \in Fn(I_\xi - \alpha^\star, 2)$, $t_0 \subseteq u_2 \in Fn(I_\eta - \alpha^\star, 2)$ and $u_3 \in Fn(\omega_2 - (I_\xi \cup I_\eta \cup \alpha^\star), 2)$. By the assumption the sets $\mathrm{cl}(\dot{a}_\xi([u_1])), \mathrm{cl}(\dot{a}_\eta([u_2]))$ have positive measure. By well-known theorem of Steinhaus the set $\mathrm{cl}(\dot{a}_\xi([u_1])) + \mathrm{cl}(\dot{a}_\eta([u_2]))$ contains an open set (hence also $(\mathrm{cl}(\dot{a}_\xi([u_1])) + \mathrm{cl}(\dot{a}_\eta([u_2]))) - F_n$ contains an open set). Using the fact that $\dot{a}_\xi$ and $\dot{a}_\eta$ are continuous functions we can find $u_1 \subseteq s_1 \in Fn(I_\xi - \alpha^\star, 2)$ and $u_2 \subseteq t_1 \in Fn(I_\eta - \alpha^\star, 2)$ such that $(\mathrm{cl}(\dot{a}_\xi([s_1])) + \mathrm{cl}(\dot{a}_\eta([t_1]))) \cap F_n = \emptyset$. But this is a contradiction since

$$s_1 \cup t_1 \cup u_3 \Vdash \dot{a}_\xi(\dot{c}) + \dot{a}_\eta(\dot{c}) \notin F_n \ . \blacksquare$$

Notice that for $\xi < \omega_2$

$$D_\xi = \{s \in Fn(I_\xi) : \text{ there exists a closed measure zero set } F \in V'$$
$$\text{such that } s \Vdash \dot{a}_\xi(\dot{c}) \in F\} \ .$$

Therefore by the above lemma

$$A \subseteq \bigcup \{F : F \text{ is a closed measure zero set coded in } \mathbf{V}'\} \ .$$

Since $\mathbf{V}$ contains Cohen reals over $\mathbf{V}'$, the union of all closed measure zero sets coded in $\mathbf{V}'$ has measure zero in $\mathbf{V}$. We conclude that A has measure zero. $\blacksquare$

Let $\mathcal{F}$ be a non-principal filter. Denote by $\mathcal{F}^c = \{X \subseteq \omega : \omega - X \in \mathcal{F}\}$. $\mathcal{F}^c$ is an ideal and it is very easy to see that $\mathcal{F}$ is measurable (has the Baire property) iff $\mathcal{F}^c$ is measurable (has the Baire property).

Lemma 1.5 $\mathcal{F} + \mathcal{F} = \mathcal{F}^c$.

PROOF Suppose that $X, Y \in \mathcal{F}$. Then $\{n : X(n) + Y(n) = 0\} \supseteq X^{-1}(1) \cap Y^{-1}(1) \in \mathcal{F}$. In general $\mathcal{F} + \cdots + \mathcal{F}$ is equal to $\mathcal{F}$ or $\mathcal{F}^c$ depending whether there is an even or odd number of $\mathcal{F}$'s.

Let $\mathbf{V} \models GCH$ and suppose that $\mathbf{V}[G]$ is a generic extension of $\mathbf{V}$ obtained by adding ω_2 Cohen reals. By the above lemma if $\mathcal{F}$ is a meager filter then $\mathcal{F}^c = \mathcal{F} + \mathcal{F}$ is meager. So by 1.2 $\mathcal{F}$ has measure zero. $\blacksquare$

2 Filters which are meager and nonmeasurable

Theorem 1.1 shows that in order to construct a filter which is meager and nonmeasurable we need some extra assumptions.

In [T] Talagrand showed that

Theorem 2.1 *Suppose that the real line is not the union of $< 2^{\aleph_0}$ many measure zero sets. Then there exists a nonmeasurable filter which is meager.* ■

Let κ be a regular uncountable cardinal. Recall that S is a generalized Sierpinski set of size κ if $|S \cap H < \kappa$ for every null set H. It is clear that all $S' \subseteq S$ of size κ are also nonmeasurable.

Theorem 2.2 *Assume that there exists a generalized Sierpinski set. Then there exists a nonmeasurable meager filter.*

Proof Let S be a generalized Sierpinski set of size κ. Build a sequence $\{x_\xi : \xi < \kappa\} \subset S$ and an elementary chain of models $\{M_\xi : \xi < \kappa\}$ of size κ such that

1. $\{x_\xi : \xi < \alpha\} \subset M_\alpha$ for $\alpha < \kappa$,
2. x_β is a random real over M_α for $\beta > \alpha$.

Suppose that M_β, x_β are already constructed for $\beta < \alpha$. Since S is a Sierpinski set

$$\bigcup \{S \cap H : H \text{ is a null set coded in } M_\beta \text{ for } \beta < \alpha\}$$

has size $< \kappa$. Let x_α be any element of S avoiding this set.

Let $X_\xi = x_\xi^{-1}(1)$ for $\xi < \kappa$. Let $\mathcal{F}$ be the filter generated by the family $\{X_\xi : \xi < \kappa\}$. We will show that $\mathcal{F}$ has the required properties.

For $X \subset \omega$ let

$$d(X) = \lim_{n\to\infty} \frac{|X \cap n|}{n}$$

if the above limit exists.

By easy induction we show that for $\xi_1, \ldots, \xi_n < \kappa$ we have $d(X_{\xi_1} \cap \cdots \cap X_{\xi_n}) = 2^{-n}$. This shows that

$$\mathcal{F} \subseteq \{X \subset \omega : \liminf_{n\to\infty} \frac{|X \cap n|}{n} > 0\}$$

which is a meager set. To check that $\mathcal{F}$ is nonmeasurable notice that $\mathcal{F}$ contains the nonmeasurable set $\{x_\xi : \xi < \kappa\}$.■

It is an open problem whether one can construct a meager nonmeasurable filter assuming the existence of a nonmeasurable set of size $\aleph_1$. We only have some partial results.

Let $\mathbf{b}$ be the size of the smallest unbounded family in ω^ω and let **unif** be the size of the smallest nonmeasurable set.

For $X \subseteq \omega$ let $f_X \in \omega^\omega$ be an increasing function enumerating X. For a filter $\mathcal{F}$ let $\mathcal{F}^\star = \{f_X : X \in \mathcal{F}\}$. In [J] it is proved that

Theorem 2.3 *For every filter $\mathcal{F}$,*
$\mathcal{F}$ has the Baire property iff $\mathcal{F}^\star$ is bounded. ■

Theorem 2.4 *Suppose that* **unif** $<$ **b**. *Then there exists a nonmeasurable filter which is meager.*

PROOF Let $X \subseteq 2^\omega$ be a nonmeasurable set of size **unif**. Let M be a model of the same size containing X as a subset. Then $M \cap 2^\omega$ does not have measure zero, so it is nonmeasurable. Consider any filter $\mathcal{F}$ such that $M \models \mathcal{F}$ is an ultrafilter. $\mathcal{F}$ generates a filter in $\mathbf{V}$ and this filter is meager by 2.3 and the fact that it is generated by **unif** $<$ **b** many elements. On the other hand $M \models 2^\omega = \mathcal{F} \cup \mathcal{F}^c$ and we know that $M \cap 2^\omega$ is a nonmeasurable set. Hence $\mathcal{F}$ is nonmeasurable.■

The previous theorem depended on the implication:

If $\mathcal{F}$ has measure zero then $M \cap 2^\omega$ has measure zero.

This implication is not true in general for any set $X \in M$ having outer measure 1 in M as is showed by the following example.

EXAMPLE It is consistent with ZFC that there are models $M \subset \mathbf{V}$ such that only *some* sets which have outer measure 1 in M have measure 0 in V.

Let $\mathbf{V} = \mathbf{L}[c][\langle r_\xi : \xi < \omega_1 \rangle]$ where c is a Cohen real over $\mathbf{L}$ and $\langle r_\xi : \xi < \omega_1 \rangle$ is a sequence of random reals over $\mathbf{L}[c]$ (added side by side). Let $M = \mathbf{L}[\langle r_\xi : \xi < \omega_1 \rangle]$. Consider the set $X = \mathbf{L} \cap 2^\omega$. It is known that X is a nonmeasurable set in M but X has measure 0 in $\mathbf{V}$. On the other hand the set $\{r_\xi : \xi < \omega_1\}$ is nonmeasurable in $\mathbf{V}$.■

We conclude the paper with a canonical example of a filter which does not generate an ultrafilter. In other words we have the following:

Theorem 2.5 *Let M be a model for ZFC and let r be a real which does not belong to M. Then there exists a filter $\mathcal{F}$ such that $M \models \mathcal{F}$ is an ultrafilter but*

$$M[r] \models \{X \subseteq \omega : \exists Y \in \mathcal{F}\ Y \subseteq X\} \textit{ is not an ultrafilter }.$$

PROOF Let $\{k_n : n \in \omega\}$ be a fast increasing sequence of natural numbers. Let T be a tree on $2^{<\omega}$ such that:

1. For $s \in T$ we have $|s| = k_n$ iff $s^\frown 0 \in T$ and $s^\frown 1 \in T$,
2. let $\{s_1, \ldots, s_{2^n}\}$ be the list of $T \cap 2^{k_n}$ in lexicographical order. Then for every $w \subseteq \mathcal{P}(2^n) - \{\emptyset, 2^n\}$ there exists $m \in [k_n + 1, k_{n+1})$ such that $s_l(m) = 0$ iff $l \in w$,
3. there is no $m \in \omega$ such that for all $s \in T \cap 2^{m+1}$ we have $s(m) = 0$ or for all $s \in T \cap 2^{m+1}$ we have $s(m) = 1$.

Let $S \subseteq T$ be a subtree of T. Define

$$A^0_S = \{m : \forall s \in S \cap 2^{m+1} \; s(m) = 0\} \text{ and}$$

$$A^1_S = \{m : \forall s \in S \cap 2^{m+1} \; s(m) = 1\} \; .$$

Let $\mathcal{J}$ be the ideal generated by sets $\{A^0_S, A^1_S : S \text{ is a perfect subtree of } T\}$.

One can easily verify that all finite subsets of ω belong to $\mathcal{J}$.

Lemma 2.6 *$\mathcal{J}$ is a proper ideal.*

Proof Let $S_1, \ldots, S_m$ be perfect subtrees of T. Find n sufficiently big so that $|S_j \cap 2^{k_n}| > m$ for $j \leq m$. Let $s_1, \ldots, s_{2^n}$ be the list of $T \cap 2^{k_n}$ in lexicographical ordering. Let $w_1, \ldots, w_m$ be such that $S_j \cap 2^{k_n} = \{s_i : i \in w_j\}$ for $j \leq m$. Let $w = \{\min(w_1), \ldots, \min(w_m)\}$. Then for all j, $w_j \not\subseteq w$ and $w_j \cap w \neq \emptyset$. By the definition of T there is $k < k_n$ such that $w = \{l : s_l(k) = 0\}$. By the property of w for every $j \leq m$ there exist $s^0, s^1 \in S_j \cap 2^{k_n}$ such that $s^0(k) = 0$ and $s^1(k) = 1$. Therefore $k \notin A^0_{S_1} \cup A^1_{S_1} \cup \cdots \cup A^0_{S_m} \cup A^1_{S_m}$.

Let $\mathcal{F}$ be any ultrafilter in M extending the filter $\{\omega - X : X \in \mathcal{J}\}$. Let r be a real which does not belong to M. Without loss of generality we can assume that r is a branch through T.

Assume that $\mathcal{F}$ generates an ultrafilter and let $X_r = \{n : r(n) = 1\}$. We can assume that there exists an element $X \in \mathcal{F}$ such that $X \subseteq X_r$. Let $S = \{s \in T : \forall k \in X \; (|s| > k \; \rightarrow s(k) = 1)\}$. Clearly r is a branch through S. But in that case S contains a perfect subtree $S_1 \subseteq S$ (since it contains a new branch). Therefore $X \subseteq A^1_{S_1} \in \mathcal{J}$. Contradiction. ■

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