version of 92/01/03

Linear liftings for non-complete probability spaces

Maxim R. Burke[1)]

and

Saharon Shelah[2)]

Abstract: We show that it is consistent with ZFC that $L^\infty(Y, \mathcal{B}, \nu)$ has no linear lifting for many non-complete probability spaces $(Y, \mathcal{B}, \nu)$, in particular for $Y = [0,1]^A$, $\mathcal{B} =$ Borel subsets of Y, $\nu =$ usual Radon measure on $\mathcal{B}$.

AMS Subject Classification 1980 (1985 revision):
Primary: 28A51 Secondary: 03E35

[1)] Research supported by UPEI Senate Grant no 602101, by the Research Institute for Mathematical Sciences at Bar-Ilan University, and by the Landau Center for Mathematical Research in Analysis (supported by the Minerva Foundation). The author would like to thank the organizers of the Winter Institute on the Set Theory of the Reals for their hospitality while part of this research was being carried out.

[2)] Partially supported by the Foundation for Basic Research of the Israel Academy of Science. Publication number 437.

§1 Introduction

In **[S 83]** the second author showed that it is consistent that Lebesgue measure on $[0,1]$ has no Borel lifting. This argument was generalized in **[J 89]** and **[BJ 89]** to produce a model where there is no lifting ρ for the usual product measure on $[0,1]^A$ such that for each measurable set E, $\rho(E) = E' \times [0,1]^{A-B}$ where $B \subseteq A$ is countable and $E' \subseteq [0,1]^B$ is projective. In particular $[0,1]^A$ has no Baire lifting. The approach taken there did not shed any light on the question of whether one can produce in ZFC a Borel lifting for $[0,1]^A$ when A is uncountable. In this paper we show that this is not possible. D.H. Fremlin suggested the use of linear liftings for this purpose. The technique is a modification of the one used in **[S 83]**. We assume that the reader is familiar with **[S 83]**. Most definitions which we need are given below. See **[IT 69]** for more details concerning liftings.

1.1 Definitions

1. If $(Y, \mathcal{B}, \nu)$ is any probability space (not necessarily complete) then as usual we say that $f\colon Y \to \mathbf{R}$ is *measurable* if $f^{-1}(a,b) \in \mathcal{B}$ for every rational interval $(a,b) \subseteq \mathbf{R}$.
2. $L^\infty(Y, \mathcal{B}, \nu) = \{f \in \mathbf{R}^Y : f \text{ is bounded and measurable }\}$.
3. $\rho\colon L^\infty(Y, \mathcal{B}, \nu) \to L^\infty(Y, \mathcal{B}, \nu)$ is a *linear lifting* if for all $f, g \in L^\infty(Y, \mathcal{B}, \nu)$ and all $x, y \in \mathbf{R}$,
   (a) $f = g$ a.e. implies $\rho(f) = \rho(g)$ (everywhere).
   (b) $\rho(f) = f$ a.e.
   (c) $\rho(xf + yg) = x\rho(f) + y\rho(g)$.
   (d) $\rho(1) = 1$ where 1 is the constant function with value 1.
   (e) $f \geq 0$ a.e. implies $\rho(f) \geq 0$.
4. $\rho\colon L^\infty(Y, \mathcal{B}, \nu) \to L^\infty(Y, \mathcal{B}, \nu)$ is a *lifting* if ρ is a linear lifting and $\rho(fg) = \rho(f)\rho(g)$ for all $f, g \in L^\infty(Y, \mathcal{B}, \nu)$. In this case ρ corresponds in a canonical way to a lifting for the measure algebra of $(Y, \mathcal{B}, \nu)$. See **[IT 69]**.
5. When ρ is a linear lifting for $L^\infty(Y, \mathcal{B}, \nu)$ and $E \in \mathcal{B}$, we will write $\rho(E)$ instead of $\rho(\chi_E)$, where χ_E is the characteristic function of the set E.
6. For sequences of real numbers, we will use the expressions *increasing* and *decreasing* to mean *strictly* increasing and *strictly* decreasing, respectively.
7. For real numbers $a \neq b$, (a,b) will denote $\{x \in \mathbf{R} : a < x < b\}$ if $a < b$, and $\{x \in \mathbf{R} : b < x < a\}$ if $b < a$.

We will prove the following theorem:

1.2 Theorem *The following is consistent with ZFC:*

Let $\Sigma =$ the σ-algebra of Borel subsets of $[0,1]$, $\mu =$ Lebesgue measure on Σ. Then $L^\infty([0,1], \Sigma, \mu)$ has no linear lifting.

1.3 Corollary *The following is consistent with ZFC:*

Suppose that

1. $(Y, \mathcal{B}, \nu)$ *is a probability space (not necessarily complete),*

2. *There is a measurable inverse-measure-preserving function* $\varphi: Y \to [0,1]$,

3. *There is a* Borel disintegration *of* ν, *i.e., there is a family* $\langle \nu_x : x \in [0,1] \rangle$ *of probability measures on* $\mathcal{B}$ *such that for each* $g \in L^\infty(Y, \mathcal{B}, \nu)$, *the function* $x \mapsto \int g\, d\nu_x$ *is Borel measurable and equal a.e. to* $E(g|\varphi^{-1}(\Sigma))$. *(Here* $E(\cdot)$ *is the conditional expectation operator.)*

Then $L^\infty(Y, \mathcal{B}, \nu)$ *has no linear lifting. (In particular* $(Y, \mathcal{B}, \nu)$ *has no lifting.)*

[Proof: If ρ is a linear lifting for $L^\infty(Y, \mathcal{B}, \nu)$, then $\bar\rho$ is a linear lifting for $L^\infty([0,1], \Sigma, \mu)$ where $\bar\rho(f)(x) = \int \rho(f \circ \varphi)\, d\nu_x$. [For a.a. x we have $\bar\rho(f)(x) = E(f \circ \varphi | \varphi^{-1}(\Sigma))(x) = f(x)$.]]

1.4 Examples

1. $Y = [0,1]^A$, $\mathcal{B} =$ Borel subsets of $[0,1]^A$, $\nu =$ usual Radon product measure on $\mathcal{B}$.
2. $Y = \{0,1\}^A$, $\mathcal{B} =$ Borel subsets of $\{0,1\}^A$, $\nu =$ usual Haar measure on $\mathcal{B}$.
3. $(Z, \mathcal{C}, \lambda)$ is any probability space, $Y = [0,1] \times Z$, $\mathcal{B} =$ the σ-algebra generated by the rectangles $E \times F$, $E \in \Sigma$, $F \in \mathcal{C}$, $\nu =$ the usual product measure on $\mathcal{B}$.

Note that the third hypothesis of the corollary is needed. To see this, consider the hyperstonian space $(Y, \mathcal{B}, \nu)$ of [0,1] and the canonical projection $\varphi: Y \to [0,1]$. We know that $(Y, \mathcal{B}, \nu)$ has a lifting (even a continuous lifting). (See **[F 89]**.) However in the model which we will construct, none of the disintegrations of ν will be Borel, so there is no contradiction.

1.5 Problem Is it consistent with ZFC that there is a translation-invariant linear lifting for $L^\infty([0,1), \Sigma, \mu)$? ($\rho$ is *translation invariant* if $\rho(f_a)(x) = \rho(f)(a+x)$, where $f_a(y) = f(a+y)$ (all additions are mod 1), for $a, x, y \in [0,1)$, $f \in L^\infty([0,1), \Sigma, \mu)$.)

§2 Proof of theorem 1.2

Let L^∞ stand for $L^\infty([0,1], \Sigma, \mu)$.

Assume $V = L$. As in **[S 83]** (the technique is explained in **[S 82]**), we use an oracle-cc iteration of length $\aleph_2$, and it will suffice to prove the following lemma.

2.1 Main Lemma *Let* $\overline{M}$ *be an* $\aleph_1$*-oracle and let* ρ *be a linear lifting of* L^∞. *Then there is a forcing notion* P *satisfying the* $\overline{M}$*-cc and a* P*-name* $\dot X$ *of an open set such that for every* $G \subseteq P \times Q$ *generic over* V *(where* Q *is Cohen forcing), there is no Borel function* h *in* $V[G]$ *such that*

(a) $h = \chi_{\dot X[G]}$ *a.e.*

(b) *for every* $g \in (L^\infty)^V$, *if* $g \le \chi_{\dot X[G]}$ *a.e. then* $\rho(g) \le h$.

(c) *for every* $g \in (L^\infty)^V$, *if* $\chi_{\dot X[G]} \le g$ *a.e. then* $h \le \rho(g)$.

2.2 Proof of the main lemma

Let $\mathcal{S}$ denote the set of triples

$$\bar{a} = \big(\langle a_{0i} : i < \omega\rangle, \langle a_{1i} : i < \omega\rangle, a_\omega\big)$$

such that the a_{ji} are rational numbers in $(0,1)$ ($j < 2$, $i < \omega$), a_ω is irrational, $\langle a_{0i} : i < \omega\rangle$ is an increasing sequence converging to a_ω and $\langle a_{1i} : i < \omega\rangle$ is a decreasing sequence converging to a_ω.

Define a partial order $P = P(\langle \bar{a}^\alpha : \alpha < \beta\rangle)$ where $\beta \le \omega_1$, $\bar{a}^\alpha \in \mathcal{S}$, and the numbers a^α_ω are pairwise distinct, as follows: $p \in P$ *iff* the following conditions hold:

(a) $p = (U_p, f_p)$, where U_p is an open subset of $(0,1)$, $\mathrm{cl}(U_p)$ has measure $< 1/2$, and $f_p \colon U_p \to \{0,1\}$.

(b) There is a finite sequence of rational numbers $0 = b_0 < b_1 < \ldots < b_n = 1$ such that $U_p = \bigcup_{\ell=0}^{n-1} I_\ell$, $\mathrm{cl}(I_\ell) \subseteq (b_\ell, b_{\ell+1})$.

(c) I_ℓ is either a rational interval, in which case $f_p | I_\ell$ is constant, or there are $\alpha < \beta$ and $n(\ell) < \omega$ such that

$$I_\ell = \bigcup_{j<2} \bigcup_{n(\ell) \le m < \omega} (a^\alpha_{j,2m}, a^\alpha_{j,2m+1})$$

and $f_p | (a^\alpha_{j,4m+2k}, a^\alpha_{j,4m+2k+1})$ is identically equal to k, ($j < 2$, $n(\ell) \le 2m + k$, $m < \omega$, $k < 2$).

The order on P is: $p \le q$ if and only if $U_p \subseteq U_q$, $f_p \subseteq f_q$, and $\mathrm{cl}(U_p) \cap U_q = U_p$.

Let $\dot{X}$ be a P-name for $\bigcup\{(a,b) : (a,b)$ is a rational interval $\subseteq (0,1)$ and for some $p \in G_P$, $(a,b) \subseteq U_p$ and $f_p|(a,b)$ is identically zero$\}$.

As in **[S 83]**, the main lemma will follow if we prove the following claim.

2.3 Main Claim *Let $P_\delta = P(\langle \bar{a}^\alpha : \alpha < \delta\rangle)$, $\delta < \omega_1$ be given, as well as a countable M_δ, $P_\delta \in M_\delta$, a condition $(p^*, r^*) \in P_\delta \times Q$ and a $P_\delta \times Q$-name τ for a code for a member of L^∞. (We shall identify Borel functions and their codes. This should not cause any confusion.) Then we can find $\bar{a}^\delta \in \mathcal{S}$ such that, letting $P_{\delta+1} = P(\langle \bar{a}^\alpha : \alpha \le \delta\rangle)$, the following conditions hold:*

(A) *Every predense subset of P_δ which belongs to M_δ is a predense subset of $P_{\delta+1}$.*

(B) *There is a condition $(p', r') \in P_{\delta+1} \times Q$ such that $(p^*, r^*) \le (p', r')$ and one of the following two conditions holds for some n:*

(B1) $(p', r') \Vdash_{P_{\delta+1}\times Q}$ " $\tau(a^\delta_\omega) \ge 1/2$
and $\rho(\bigcup_{j<2} \bigcup_{n \le m < \omega} (a^\delta_{j,4m+2}, a^\delta_{j,4m+3}))(a^\delta_\omega) \ge 3/4$
and $\dot{X} \cap \bigcup_{j<2} \bigcup_{n \le m < \omega} (a^\delta_{j,4m+2}, a^\delta_{j,4m+3}) = \emptyset$."

or

(B2) $(p', r') \Vdash_{P_{\delta+1}\times Q}$ " $\tau(a^\delta_\omega) \le 1/2$
and $\rho(\bigcup_{j<2} \bigcup_{n \le m < \omega} (a^\delta_{j,4m}, a^\delta_{j,4m+1}))(a^\delta_\omega) \ge 3/4$
and $\bigcup_{j<2} \bigcup_{n \le m < \omega} (a^\delta_{j,4m}, a^\delta_{j,4m+1}) \subseteq \dot{X}$."

[The proof of the Main Lemma is a bookkeeping argument using the Main Claim. P is obtained, in the notation of the Main Claim, as $P = \bigcup_{\delta<\omega_1} P_\delta$, and the bookkeeping is needed to ensure that all triples (p^*, r^*, τ) are considered in the construction, where $(p^*, r^*) \in P \times Q$ and τ is a $P \times Q$-name for a code of a Borel function. If there were an h contradicting the Main Lemma, then there would be a $P \times Q$-name τ for h and a condition $(p^*, r^*) \in P \times Q$ forcing that τ satisfies (a), (b), (c) of the Main Lemma. But then condition (B) of the Main Claim gives a contradiction. For more details of such oracle-cc arguments see pp. 114ff of [**S 82**].]

2.4 Proof of main claim 2.3

Choose a sufficiently large regular λ and choose a countable $N \prec H_\lambda$ such that $\rho, P_\delta, \langle \bar{a}^\alpha : \alpha < \delta \rangle, \tau, M_\delta \in N$. Choose a random real over N, $a^\delta_\omega \in (0,1) - \mathrm{cl}(U_{p^*})$. Note that for any rational interval $(a,b) \subseteq (0,1)$ we have $\rho((a,b))(a^\delta_\omega) = \chi_{(a,b)}(a^\delta_\omega)$.

Let $u_0 = \rho((0, a^\delta_\omega))(a^\delta_\omega)$, $u_1 = \rho((a^\delta_\omega, 1))(a^\delta_\omega)$. Then $u_0 + u_1 = 1$.

Note that for any number x, if $0 \le x < a^\delta_\omega$, then $\rho((x, a^\delta_\omega))(a^\delta_\omega) = u_0$. [Otherwise, for any rational number b such that $x < b < a^\delta_\omega$, we have $\rho((0,b))(a^\delta_\omega) > 0$, contradicting the choice of a^δ_ω.] A similar statement holds for u_1. Putting these together we see that $\rho((x,y))(a^\delta_\omega) = 1$ for any numbers x and y such that $0 \le x < a^\delta_\omega < y \le 1$.

Choose an increasing sequence of rational numbers $\langle b_{0n} : n < \omega \rangle \in N[a^\delta_\omega]$ converging to a^δ_ω, and choose a decreasing sequence of rational numbers $\langle b_{1n} : n < \omega \rangle \in N[a^\delta_\omega]$ also converging to a^δ_ω. In $N[a^\delta_\omega]$ define the partial order R for adding a Mathias real as follows:

$$R = \{(s, A) : s \text{ is a finite subset of } \omega,\ A \subseteq \omega,\ \max(s) < \min(A)\},$$

ordered by $(s, A) \ge (t, B)$ iff t is an initial segment of s, $A \subseteq B$, $s - t \subseteq B$.

For sets $A \subseteq \omega$, let us identify A with its enumerating function, so that we may write $A = \{A(i) : i < |A|\}$. We need the following special case of the known fact that an infinite subset of a Mathias real is a Mathias real. (See [**M 77:** Theorem 2.0]; the special case which we need here is a fairly routine exercise.)

2.5 Fact *If $X \subseteq \omega$ is R-generic over $N[a^\delta_\omega]$, and $g \in \omega^\omega \cap N[a^\delta_\omega]$ is increasing, then $Y = \{X(g(n)) : n < \omega\}$ is also R-generic over $N[a^\delta_\omega]$.* ∎

Let f^* be the enumerating function of a set which is R-generic over $N[a^\delta_\omega]$.

In $N[a^\delta_\omega][f^*]$, define for increasing functions $f \in \omega^\omega$,

$$A^k_m(f) = \bigcup_{j<2} \bigcup_{k \le \ell < \omega} (b_{j, f(4\ell+m)}, b_{j, f(4\ell+m+1)}).$$

Define $f^*_3(\ell) = f^*(3\ell)$ for $\ell < \omega$.

Then $\{A^0_m(f^*_3) : m < 4\}$ is a partition of $(b_{0,f^*(0)}, b_{1,f^*(0)})$. For some $m < 4$ we have

$$(*) \qquad \rho(A^0_m(f^*_3))(a^\delta_\omega) \le 1/4.$$

2.6 Claim *For any $\bar{m} < 4$ and $k < \omega$, we can find an increasing function $g \in N[a^\delta_\omega] \cap \omega^\omega$ such that $g(i) = i$ for all $i < k$ and $\rho(A^0_{\bar{m}}(f^* \circ g))(a^\delta_\omega) \ge 3/4$.*

Proof of claim Let $g(i) = i$ for $i < 4k+\bar{m}+1$ and define $g(4\ell+\bar{m}+1+j) = 12\ell+3m+j$ for $\ell \geq k$ and $j < 4$. We leave it for the reader to check, using $(*)$, that g has the desired property. (The reader might find it helpful, for seeing the role of g, to mark off the first few elements of its range on a line.) ■

Let us provisionally let $\bar{a}^\delta = (\langle b_{0,f^*(\ell)} : \ell < \omega\rangle, \langle b_{1,f^*(\ell)} : \ell < \omega\rangle, a^\delta_\omega)$.

2.7 Proof of condition (A) of main claim 2.3

Let $J \subseteq P_\delta$ be predense, $J \in M_\delta$. We must show that J is predense in $P_{\delta+1}$. Let $p \in P_{\delta+1}$, $p \notin P_\delta$. By the definition of $P_{\delta+1}$, there are $q \in P_\delta$ and rational numbers c_0, c_1 and $\ell(0) \in \omega$ such that

$$0 < b_{0,f^*(4\ell(0))-1} < c_0 < b_{0,f^*(4\ell(0))} < a^\delta_\omega < b_{1,f^*(4\ell(0))} < c_1 < b_{1,f^*(4\ell(0))-1} < 1,$$

$\mathrm{cl}(U_q) \cap [c_0, c_1] = \emptyset$, $U_p = U_q \cup A_0^{\ell(0)}(f^*) \cup A_2^{\ell(0)}(f^*)$, $f_p = f_q \cup 0_{A_0^{\ell(0)}(f^*)} \cup 1_{A_2^{\ell(0)}(f^*)}$. (For $i = 0, 1$, i_A denotes the function with domain A and constant value i.)

The proof of the following fact is exactly as in **[S 83]**.

2.8 Fact *If $r \in P_\delta$, $J \subseteq P_\delta$ is dense, $(c_o, c_1) \subseteq (0,1)$ and $(c_0, c_1) \cap U_r = \emptyset$, then*

$$\mu\big((c_0, c_1) \cap \bigcap\{\mathrm{cl}(U_{r_1}) : r_1 \in J, r_1 \geq r\}\big) = 0. \quad ■$$

Let $J_1 = \{r \in P_\delta : \exists q_1 \in J\ q_1 \leq r\}$. For every $k > f^*(4\ell(0))$ let
$T_k = \{t \in P_\delta : U_t$ is the union of finitely many intervals whose endpoints are from $\{b_{j,\ell} : j < 2, f^*(4\ell(0)) \leq \ell \leq k\}$ and $\mu(U_q \cup U_t) < 1/2\}$.

So T_k is finite and for each $t \in T_k$, $q \leq q \cup t \in P_\delta$ and $a^\delta_\omega \notin \mathrm{cl}(U_t)$. In N, define for each $k > f^*(4\ell(0))$ and $t \in T_k$,

$$J_t = (b_{0,k}, b_{1,k}) \cap \bigcap\{\mathrm{cl}(U_{r_1}) : r_1 \in J_1,\ r_1 \geq q \cup t\}.$$

By fact 2.8, J_t has measure zero, and hence $a^\delta_\omega \notin J_t$. Thus there is an $r_t \in J_1$, such that $r_t \geq q \cup t$ and $a^\delta_\omega \notin \mathrm{cl}(U_{r_t})$. Define $g{:}\bigcup\{T_k : k > f^*(4\ell(0))\} \to \omega$ and $G{:}\omega \to \omega$ such that $[b_{0,g(t)}, b_{1,g(t)}] \cap \mathrm{cl}(U_{r_t}) = \emptyset$, $b_{1,g(t)} - b_{0,g(t)} < (1/2) - \mu(U_{r_t})$, $G(k) = \max\{g(t) : t \in T_k\}$.

Since f^* is R-generic over $N[a^\delta_\omega]$, for all but finitely many $\ell < \omega$ we have

$$f^*(4\ell+2) \geq G(f^*(4\ell+1)).$$

Choose such an $\ell \geq \ell(0)$. Let $k = f^*(4\ell+1)$, $t = (U_t, f_t)$, where

$$U_t = U_p \cap \big([b_{0,f^*(4\ell(0))}, b_{0,k}] \cup [b_{1,k}, b_{1,f^*(4\ell(0))}]\big),$$

$f_t = f_p | U_t$. Then $t \in T_k$ and we have $r_t \in J_1$, $r_t \geq q \cup t$. Also, $[b_{0,G(k)}, b_{1,G(k)}] \cap \mathrm{cl}(U_{r_t}) = \emptyset$ and hence $[b_{0,f^*(4\ell+2)}, b_{1,f^*(4\ell+2)}] \cap \mathrm{cl}(U_{r_t}) = \emptyset$. Thus p and r_t are compatible, and this proves part (A) of main claim 2.3.

2.9 Proof of condition (B) of main claim 2.3

Let

$$p_1^* = (U_p \cup A_0^k(f^*) \cup A_2^k(f^*), f_{p^*} \cup 0_{A_0^k(f^*)} \cup 1_{A_2^k(f^*)})$$

where k is large enough so that $p_1^* \in P_{\delta+1}$. So $p_1^* \in N[a_\omega^\delta][f^*]$ and $(p_1^*, r^*) \geq (p^*, r^*)$. In $N[a_\omega^\delta][f^*]$, choose $(p', r') \geq (p_1^*, r^*)$ deciding whether $\tau(a_\omega^\delta) \geq 1/2$ or $\tau(a_\omega^\delta) \leq 1/2$, say the first. We will get (p', r') so that condition (B1) of main claim 2.3 is satisfied. The other case is handled similarly. For some $(t, B) \in R \cap N[a_\omega^\delta]$ we have $f^*(n) = t(n)$ for all $n < |t|$, $f^*(n) \in B$ for all $n \geq |t|$, and

$$N[a_\omega^\delta] \models (t, B) \Vdash_R \text{“}(p', r') \Vdash_{P_{\delta+1} \times Q} \tau(a_\omega^\delta) \geq 1/2\text{”}.$$

By claim 2.6 and fact 2.5 above, we can replace f^* by another R-generic real, maintaining $f^*(n) = t(n)$ for $n < |t|$ and $f^*(n) \in B$ for $n \geq |t|$, so that $\rho(A_2^0(f^*))(a_\omega^\delta) \geq 3/4$. (B1) is now satisfied. This completes the proof of main claim 2.3 and of theorem 1.2. ■

Acknowledgment We thank the referee for several suggestions which helped improve the readability of the paper.

References:

[BJ 91] Burke M.R., Just W., *Liftings for Haar measure on* $\{0,1\}^\kappa$, Israel J. Math, ?? (1991) ???–???.

[J 89] Just W., *A modification of Shelah's oracle-cc with applications*, to appear in Trans. Amer. Math. Soc.

[F 89] Fremlin D.H., *Measure algebras*, in Handbook of Boolean Algebra, ed. J.D. Monk, North-Holland, 1989, 877–980.

[IT 69] Ionescu Tulcea A., Ionescu Tulcea C., *Topics in the theory of liftings*, Springer, 1969.

[M 77] Mathias A.R.D., *Happy families*, Ann. Math. Logic, 12 (1977) 59–111.

[S 82] Shelah S., *Proper Forcing*, Lecture Notes in Math 940, Springer-Verlag, 1982.

[S 83] Shelah S., *Lifting problem of the measure algebra*, Israel J. Math., 45 (1983) 90–96.

Authors' addresses:

1) Department of Mathematics and Computer Science
University of Prince Edward Island
Charlottetown, P.E.I. C1A 4P3
Canada

2) Institute of Mathematics
The Hebrew University of Jerusalem
Jerusalem, Israel