

Every null-additive set is meager-additive[†]

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§1. The basic definitions and the main theorem.

1. Definition. (1) We define addition on ${}^\omega 2$ as addition modulo 2 on each component, i.e., if $x, y, z \in {}^\omega 2$ and $x + y = z$ then for every n we have $z(n) = x(n) + y(n) \pmod{2}$.

(2) For $A, B \subseteq {}^\omega 2$ and $x \in {}^\omega 2$ we set $x + A =^{\text{df}} \{x + y : y \in A\}$, and we define $A + B$ similarly.

(3) We denote the Lebesgue measure on ${}^\omega 2$ with μ . We say that $X \subseteq {}^\omega 2$ is *null-additive* if for every $A \subseteq {}^\omega 2$ which is null, i.e. $\mu(A) = 0$, $X + A$ is null too.

(4) We say that $X \subseteq {}^\omega 2$ is *meager-additive* if for every $A \subseteq {}^\omega 2$ which is meager also $X + A$ is meager.

2. Theorem. Every null-additive set is meager-additive.

3. Outline and discussion. Theorem 2 answers a question of Palikowski. It will be proved in §2. In §3 we shall present direct characterizations of the null-additive sets, and in §4 we shall do the same for the meager-additive sets.

It is obvious that every countable set is both null-additive and meager-additive. Are there uncountable null-additive sets, and even null-additive sets of cardinality 2^{\aleph_0} ? It will be shown in §5 that if the continuum hypothesis holds then there is such a set. Haim Judah has shown that there is a model of ZFC in which all the null-additive sets are countable, but there are in it uncountable meager-additive sets. This is the model obtained by adding to L more than \aleph_1 Cohen reals. In this model the Borel conjecture holds, and therefore every null-additive set is strongly meager and hence countable. On the other hand, in this model the uncountable set of all constructible reals is meager-additive.

§2 The proof of Theorem 2.

4. Notation. (1) we shall use variables as follow: i, j, k, l, m, n for natural numbers, f, g, h for functions from ω to ω , $\eta, \zeta, \nu, \sigma, \tau$ for finite sequences of 0's and 1's, x, y, z for members of ${}^\omega 2$, A, B, X, Y for subsets of ${}^\omega 2$, and S, T for trees.

(2) ${}^{\omega > 2} = \bigcup_{n < \omega} {}^n 2$. We shall denote subsets of ${}^{\omega > 2}$ with U, V . For $\eta \in {}^{\omega > 2}$, $U \subseteq {}^{\omega > 2}$ and $x \in {}^\omega 2$ we shall write $\eta + x$ for $\eta + x \upharpoonright \text{length}(\eta)$, and $U + x$ for $\{\eta + x : \eta \in U\}$.

(3) For $\eta, \nu \in {}^{\omega > 2}$ we write $\eta \trianglelefteq \nu$ if ν is an extension of η .

(4) A tree is a nonempty subset of ${}^{\omega > 2}$ such that

(a) If $\eta \trianglelefteq \nu$ and $\nu \in T$ then also $\eta \in T$, and

(b) If $\eta \in T$ and $n > \text{length}(\eta)$ then there is a ν of length n such that $\eta \trianglelefteq \nu$ and $\nu \in T$.

(5) For a tree T $\text{Lim} T = \{x \in {}^\omega 2 : \text{for every } n < \omega \ x \upharpoonright n \in T\}$.

(6) A tree T is said to be *nowhere dense* if for every $\eta \in T$ there is a $\tau \in {}^{\omega > 2}$ such that

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$\eta \trianglelefteq \tau$ and $\tau \notin T$. A set $B \subseteq {}^\omega 2$ is said to be nowhere dense if $B \subseteq \text{Lim } T$ for some nowhere dense tree T .

(7) For every $x, y \in {}^\omega 2$ we write $x \equiv y$ if $x(n) = y(n)$ for all but finitely many $n < \omega$. For $A \subseteq {}^\omega 2$ $A^{\text{fin}} = \text{df } \{y \in {}^\omega 2 : y \equiv x \text{ for some } x \in A\}$.

(8) $U^{[\nu]} = \text{df } \{\tau \in U : \tau \trianglelefteq \nu \text{ or } \nu \trianglelefteq \tau\}$ (read: U through ν).

(9) $U^{(\nu)} = \text{df } \{\tau \in {}^{\omega > 2} : \nu \frown \tau \in U\}$ (read: U above ν), and for $\eta \in {}^{\omega > 2}$

$\eta^{(k)} = \text{df } \langle \eta(k+i) : i < \text{length } \eta - k \rangle$.

(10) For $\nu, \eta \in {}^{\omega > 2} \cup {}^\omega 2$ we write $\nu \sim_n \eta$ if $\text{length}(\nu) = \text{length}(\eta)$ and $\nu(i) = \eta(i)$ for every $n \leq i < \text{length}(\nu)$. For $S \subseteq {}^{\omega > 2} \cup {}^\omega 2$ we define $S^{\sim n} = \{\nu : \nu \sim_n \eta \text{ for some } \eta \in S\}$.

5. Outline of the proof. Let $X \subseteq {}^\omega 2$ be null-additive. It clearly suffices to prove that for every $A \subseteq {}^\omega 2$ which is nowhere dense $X + A$ is meager. Given a nowhere dense tree S we shall prove in Lemma 6 a condition which is sufficient for a tree T to be such that $T + S$ is nowhere dense. Then we shall split X to a union $X = \bigcup_{i=1}^{\infty} X_i$ such that for each i $X_i \subseteq \text{Lim } T_i$ where T_i is a tree which satisfies that condition. Thus for a nowhere dense S each set $X_i + \text{Lim } S \subseteq \text{Lim}(T_i + S)$ is nowhere dense, hence $X + \text{Lim } S \subseteq \bigcup_{i=1}^{\infty} \text{Lim}(T_i + S)$ is meager.

6. Lemma. Let T be a tree such that

(a) T is nowhere dense.

(b) $f = f_T$ is the function from ω to ω given by

$f(n) = \min\{m : \text{for every } \eta \in {}^n 2 \text{ there is a } \tau \in {}^m 2 \text{ such that } \eta \trianglelefteq \tau \text{ and } \tau \notin T\}$.

Thus for every sequence η of length n there is a witness of length $\leq f(n)$ that T is nowhere dense. Obviously for every $n < \omega$ $f(n) > n$, and if $n < m$ then $f(n) \leq f(m)$.

Let g be a function from ω to ω and $\bar{n} = \langle n_i : i < \omega \rangle$, $\bar{n}' = \langle n'_i : i < \omega \rangle$ increasing sequences of natural numbers such that

(c) $f^{g(i)}(n_i) \leq n'_i < n_{i+1}$ for every $i < \omega$, where f^m denotes the m -th iteration of f .

Then for every tree S which satisfies

(d) S is of width (\bar{n}', g) , i.e., for every $i < \omega$ $|{}^{n'_i} 2 \cap S| \leq g(i)$,

$T + S$ is nowhere dense.

Proof. Let $\eta \in {}^{n_i} 2$. We shall show the existence of an $\eta' \in {}^{n'_i} 2$ such that $\eta \trianglelefteq \eta'$ and $\eta' \notin T + S$.

By (c) there is a sequence $m_0, \dots, m_{g(i)}$ such that $m_0 = n_i$, $f(m_k) \leq m_{k+1}$ for $0 \leq k \leq g(i)$ and $m_{g(i)} = n'_i$. Let $\langle \tau_k : k < k_i \rangle$ enumerate the set ${}^{n'_i} 2 \cap S$. $k_i \leq g(i)$ by (d). We define $\eta_k \in {}^{m_k} 2$ for $0 \leq k \leq k_i$ by recursion as follows. $\eta_0 = \eta$. Given $\eta_k \in {}^{m_k} 2$, for $k < k_i$, we shall define $\eta_{k+1} \in {}^{m_{k+1}} 2$ so that for no extension $\eta' \in {}^{n'_i} 2$ of η_{k+1} we shall have $\eta' + \tau_k \in T$. We have $\eta_k + \tau_k \upharpoonright m_k \in {}^{m_k} 2$ and by the definition of f and by the choice of the m_k 's $\eta_k + \tau_k \upharpoonright m_k$ has an extension $\nu \in {}^{m_{k+1}} 2$ such that $\nu \notin T$. If we take $\eta_{k+1} = \nu + \tau_k \upharpoonright m_{k+1}$ then $\eta_k + \tau_k \upharpoonright m_k \trianglelefteq \nu$ implies $\eta_k \trianglelefteq \eta_{k+1}$, $\eta_{k+1} \in {}^{m_{k+1}} 2$ and $\eta_{k+1} + \tau_k \upharpoonright m_{k+1} = \nu \notin T$, and therefore for every $\eta' \in {}^{n'_i} 2$ such that $\eta_k \trianglelefteq \eta'$ we have $\eta' + \tau_k \notin T$. Let $\eta' = \eta_{k_i}$, and assume that $\eta' \in T + S$. Then, for some $k < k_i \leq g(i)$ $\eta' + \tau_k \in T$, contradicting our choice of $\eta_{k+1} = \eta' \upharpoonright m_{k+1}$. Thus $\eta' \notin T + S$.

7. Lemma. If S, T_i , $i \in \omega$ are trees and $\text{Lim } S \subseteq \bigcup_{i \in \omega} \text{Lim } T_i$ then for some $\eta \in S$ and $j \in \omega$ $S^{[j]} \subseteq T_j$.

Proof. Suppose that this is not the case, i.e., for every $\eta \in S$ and $i < \omega$ there is a ζ

such that $\zeta \in S^{[\eta]}$ and $\zeta \notin T_i$. Once there is such a ζ we can assume that $\eta \trianglelefteq \zeta$ and $\text{length } \zeta > \text{length } \eta$. We define now, by induction on i , η_i and k_i so that $k_i = \text{length } \eta_i$, $k_0 = 0$, $\eta_0 = \langle \rangle$, $\eta_i \trianglelefteq \eta_{i+1}$, $k_i < k_{i+1}$, $\eta_{i+1} \in S$ and $\eta_{i+1} \notin T_i$. Let $y = \bigcup_{i \in \omega} \eta_i$. Since $\eta_i \in S$ for every $i \in \omega$ $y \in \text{Lim } S \subseteq \bigcup_{i \in \omega} \text{Lim } T_i$, hence for some $j \in \omega$ $y \in \text{Lim } T_j$. However, $y \upharpoonright k_{j+1} = \eta_{j+1} \notin T_j$, contradicting $y \in \text{Lim } T_j$.

8. Lemma. Let S and T be trees such that $\text{Lim } S \subseteq (\text{Lim } T)^{\text{fin}}$. Then there are $k < \omega$, $\eta, \nu \in {}^k 2$, $\eta \in S$ such that $S^{(\eta)} \subseteq T^{(\nu)}$.

Proof. For $n < \omega$, $\sigma_1, \sigma_2 \in {}^n 2$ and $\sigma_2 \in T$ we define $T_{\sigma_1, \sigma_2} =^{\text{df}} \{\tau : \tau \trianglelefteq \sigma_1\} \cup \{\sigma_1 \hat{\ } \tau : \sigma_2 \hat{\ } \tau \in T\}$ (This is the tree $T^{[\sigma_2]}$ with “ σ_2 replaced by σ_1 ”). Clearly

$$(1) \quad (\text{Lim } T)^{\text{fin}} = \bigcup_{n < \omega, \sigma_1, \sigma_2 \in {}^n 2, \sigma_2 \in T} \text{Lim } T_{\sigma_1, \sigma_2}$$

Since there are only countably many T_{σ_1, σ_2} 's in (1) there are by Lemma 7 a $\zeta \in S$ and $j < \omega$ such that $S^{[\zeta]} \subseteq T_{\sigma_1, \sigma_2}$. Clearly there is an η with $\zeta \trianglelefteq \eta$ and a ν with $\text{length } \nu = \text{length } \eta$ such that $S^{(\eta)} \subseteq T^{(\nu)}$. (If $\zeta \trianglelefteq \sigma_1$ then $\eta = \sigma_1$ and $\nu = \sigma_2$, else $\sigma_1 \trianglelefteq \zeta$ and then $\eta = \zeta$ and $\nu = \sigma_2 \hat{\ } \zeta \upharpoonright [\text{length } \zeta, \text{length } \sigma_2)$).

9. Lemma. Let X be a null-additive set. Let T be a tree such that $\mu(\text{Lim } T) > 0$. There is a tree T^* such that $\mu(\text{Lim } T^*) > 0$, for every $\eta \in T^*$ $\mu(\text{Lim } (T^{*[\eta]})) > 0$, and $((\omega 2 \setminus (\text{Lim } T)^{\text{fin}}) + X) \cap \text{Lim } T^* = \emptyset$, and then $X = \bigcup_{\eta \in T^*, \text{length } \zeta = \text{length } \eta} Y_{\eta, \zeta}$ where $Y_{\eta, \zeta} = \{x \in X : \zeta \hat{\ } x^{(\text{length } \zeta)} + T^{*[\eta]} \subseteq T\}$.

Proof. Since $\mu(\text{Lim } T) > 0$ then, as easily seen, $\mu((\text{Lim } T)^{\text{fin}}) = 1$, hence $\mu(\omega 2 \setminus (\text{Lim } T)^{\text{fin}}) = 0$. Since X is null-additive also $\mu(X + (\omega 2 \setminus (\text{Lim } T)^{\text{fin}})) = 0$. Hence there is a tree T^* such that $\mu(\text{Lim } T^*) > 0$ and $(X + (\omega 2 \setminus (\text{Lim } T)^{\text{fin}})) \cap \text{Lim } T^* = \emptyset$. Without loss of generality we can assume that T^* has been pruned so that for $\eta \in T^*$ $\mu(\text{Lim } T^{*[\eta]}) > 0$.

Let $x \in X$ then $\omega 2 \setminus (x + (\text{Lim } T)^{\text{fin}}) = x + (\omega 2 \setminus (\text{Lim } T)^{\text{fin}}) \subseteq X + (\omega 2 \setminus (\text{Lim } T)^{\text{fin}})$. Hence $(\omega 2 \setminus (x + (\text{Lim } T)^{\text{fin}})) \cap \text{Lim } T^* \subseteq (X + (\omega 2 \setminus (\text{Lim } T)^{\text{fin}})) \cap \text{Lim } T^* = \emptyset$, i.e., $\text{Lim } T^* \subseteq x + (\text{Lim } T)^{\text{fin}}$, and therefore $\text{Lim } (x + T^*) = x + \text{Lim } T^* \subseteq (\text{Lim } T)^{\text{fin}}$. By Lemma 8 there are $\eta \in T^*$ and $\nu \in {}^{\text{length } \eta} 2$ such that $x^{(\text{length } \eta)} + T^{*[\eta]} \subseteq T^{(\nu)}$. Let $\zeta = \eta + \nu$, then $\zeta \hat{\ } \eta = \nu$ and therefore $\zeta \hat{\ } x^{(\text{length } \eta)} + T^{*[\eta]} \subseteq T^{(\nu)} \subseteq T$, hence $x \in Y_{\eta, \zeta}$.

10. Lemma. Let X be null-additive, and let $\bar{n} = \langle n_i : i < \omega \rangle$, $\bar{n}' = \langle n'_i : i < \omega \rangle$ be such that for every $i < \omega$ $n_i < n'_i$ and $n'_i + i \cdot 2^{n'_i} \leq n_{i+1}$, then we can represent X as $\bigcup_{m < \omega} X_m$ such that for each m , for some real $a_m \in (0, 1)$ and S_m of width (\bar{n}', g_{a_m}) we have $X_m \subseteq \text{Lim } (S_m)$, where for every real $a \in (0, 1)$ g_a is the function on ω given by $g_a(0) = 1$, $g_a(i) = \max(1, \text{int}(\log_2(a)/\log_2(1 - 2^{-i})))$, where for a real d $\text{int}(d)$ is the integral part of d .

Proof. Since $n'_i + i \cdot 2^{n'_i} < n_{i+1}$ we can fix for each $0 < i < \omega$ a sequence $\langle u_{i, \tau} : \tau \in {}^{n'_i} 2 \rangle$ of pairwise disjoint subsets of the interval $[n'_i, n_{i+1})$ having i members each. Let $B \subseteq \omega 2$ be given by

$$B = \{y \in \omega 2 : (\forall j > 0)(\exists k \in u_{j, y \upharpoonright n'_j}) y(k) = 1\}.$$

B is clearly a closed subset of $\omega 2$ hence for $T = \{y \upharpoonright n : y \in B \wedge n \in \omega\}$ $B = \text{Lim } (T)$.

The properties of T in which we are interested are

(B0) $T \supseteq {}^{n_1}2$.

(B1) For each $\eta \in T \cap {}^{n'_i}2$ $|T^{[\eta]} \cap {}^{n_{i+1}}2| = 2^{(n_{i+1}-n'_i)}(1-2^{-i})$.

(B2) If $\eta, \nu_0, \dots, \nu_{k-1} \in {}^{n'_i}2$, $\nu_0^+, \dots, \nu_{k-1}^+ \in {}^{n_{i+1}}2$, $\eta + \nu_l \in T$, $\nu_l \leq \nu_l^+$ for $l < k$ and ν_0, \dots, ν_{k-1} is with no repetitions then $|\{\eta^+ : \eta \leq \eta^+ \in {}^{n_{i+1}}2, (\forall l < k)(\eta^+ + \nu_l^+ \in T)\}| \leq 2^{n_{i+1}-n'_i} (1-2^{-i})^k$.

(B3) For every $\eta \in {}^{n'_i}2$ we have : $\eta \upharpoonright n_i \in T$ implies $\eta \in T$.

These properties can be established by an obvious counting argument.

By (B0), (B1) and (B3) we have

$$\begin{aligned} \mu(\text{Lim } T) &= \mu \left(\bigcap_{i=1}^{\infty} \{x \in {}^{\omega}2 : x \upharpoonright n_i \in T\} \right) \\ &= \mu(\{x \in {}^{\omega}2 : x \upharpoonright n_1 \in T\}) \cdot \prod_{i=1}^{\infty} \frac{\mu(\{x \in {}^{\omega}2 : x \upharpoonright n_{i+1} \in T\})}{\mu(\{x \in {}^{\omega}2 : x \upharpoonright n_i \in T\})} \\ &= 1 \cdot \prod_{i=1}^{\infty} \frac{|T \cap {}^{n_{i+1}}2|/2^{n_{i+1}}}{|T \cap {}^{n_i}2|/2^{n_i}} = \prod_{i=1}^{\infty} (1-2^{-i}) > 0 \end{aligned}$$

For the T which we constructed let T^* and $Y_{\eta, \zeta}$ be as in Lemma 9. For $\rho \in {}^{\text{length } \eta}2$ let $Y_{\eta, \zeta, \rho} = \{y \in Y_{\eta, \zeta} : y \upharpoonright \text{length } \eta = \rho\}$. Clearly

$$(2) \quad X = \bigcup_{\eta \in T^*, \text{ length } \eta = \text{length } \zeta = \text{length } \rho} Y_{\eta, \zeta, \rho}$$

Since there are only countably many $Y_{\eta, \zeta, \rho}$'s they can be taken to be the X_m 's we are looking for, provided we show that every such $Y_{\eta, \zeta, \rho}$ is a subset of $\text{Lim}(S)$ for some tree S of width $\langle \bar{n}', g_a \rangle$ for some real $0 < a < 1$. We shall see that this is indeed the case if we take $S = \{y \upharpoonright m : y \in Y_{\eta, \zeta, \rho}, m < \omega\}$ and $a = \mu(T^{*[\eta]})$. $a > 0$ by what we assumed about T^* . As, obviously, $Y_{\eta, \zeta, \rho} \subseteq \text{Lim}(S)$ all we have to do is to show that S is of width $\langle \bar{n}', g_a \rangle$.

We can choose a set $W \subseteq S \cap {}^{n_{j+1}}2$ such that the function mapping $\eta \in W$ to $\eta \upharpoonright n'_j$ is one to one and onto $S \cap {}^{n'_j}2$

We fix now η, ζ, ρ and denote $Y_{\eta, \zeta, \rho}$ by Y and the length of η, ζ, ρ by n . Let $z \in {}^{\omega}2$ be such that $z \upharpoonright n = \zeta + \rho$ and $z(i) = 0$ for $i \geq n$. Then for every y such that $y \upharpoonright n = \rho$ we have $y + z = \zeta \wedge y^{(n)}$. Therefore, by the definition of Y we have

$$(3) \quad Y = \{y \in {}^{\omega}2 : y \upharpoonright n = \rho, (\zeta \wedge y^{(n)}) + T^{*[\eta]} \subseteq T\} = \{y \in {}^{\omega}2 : y \upharpoonright n = \rho, y + z + T^{*[\eta]} \subseteq T\}$$

for every $y \in Y$ there is a unique $\tau \in W$ such that $\tau \upharpoonright n'_j = y \upharpoonright n'_j$ (τ may be $y \upharpoonright n_{j+1}$).

Clearly $|W| = |S \cap {}^{n'_j}2|$ and we denote $|W|$ with s , so it suffices to prove $s \leq g_a(j)$. If $n'_j \leq n$ then the only member of $S \cap {}^{n'_j}2$ is $\rho \upharpoonright n'_j$ hence $s = 1$, so $s \leq g_a(j)$. We shall now deal with the case where $n'_j > n$. Let $\tau_0, \dots, \tau_{s-1}$ be the members of W . For $m < s$ $\tau_m = y \upharpoonright n_{j+1}$ for some $y \in Y$, hence, by (3), $\tau_m + z + T^{*[\eta]} \subseteq T$ and therefore $(z + T^{*[\eta]}) \cap {}^{n_{j+1}}2 \subseteq \tau_m + T$. Since this holds for every $\tau \in W$ we have

$$(4) \quad z + T^{*[\eta]} \cap {}^{n_{j+1}}2 \subseteq \bigcap_{m < s} \tau_m + T$$

Let us find out the size of $\bigcap_{m < s} (\tau_m + T)$. Let $\sigma \in {}^{n'_j}2$, and we shall ask how many members τ of $\bigcap_{m < s} (\tau_m + T)$ extend σ . Now $\tau \in \tau_m + T$ for each $m < s$ iff $\tau + \tau_m \in T$ for each $m < s$. If for some $m < s$ $\sigma + \tau_m \upharpoonright n'_j \notin T$ then also $\tau + \tau_m \notin T$, hence σ has no extension in $\bigcap_{m < s} (\tau_m + T)$. If for every $m < s$ $\sigma + \tau_m \upharpoonright n'_j \in T$ then by (B2) (where $\eta = \sigma$, $\nu_m = \tau_m \upharpoonright n'_j$ and $\nu_m^+ = \tau_m$), since $\tau_m \upharpoonright n'_j \neq \tau_l \upharpoonright n'_j$ for $m \neq l$, the number of τ 's such that $\sigma \sqsubseteq \tau \in {}^{n_{j+1}}2$ and $\tau + \tau_m \in T$ for every $m < s$ is $2^{n_{j+1}-n'_j} (1 - 2^{-j})^s$. Since there are $2^{n'_j}$ different σ 's in ${}^{n'_j}2$ we have

$$(5) \quad \left| \bigcap_{m < s} (\tau_m + T) \right| \leq 2^{n_{j+1}} \cdot (1 - 2^{-j})^s.$$

On the other hand, since $\mu(T^{*[n]}) = a \cdot T^{*[n]} \cap {}^{n_{j+1}}2$ has at least $a \cdot 2^{n_{j+1}}$ members, and so has $z + T^{*[n]} \cap {}^{n_{j+1}}2$. Comparing (4) with (5) we get $a \cdot 2^{n_{j+1}} \leq 2^{n_{j+1}} (1 - 2^{-j})^s$, i.e., $a \leq (1 - 2^{-j})^s$, $\log_2(a) \leq s \cdot \log_2(1 - 2^{-j})$, $s \leq \log_2(a) / \log_2(1 - 2^{-j})$.

11. Proof of Theorem 2. Let X be null-additive. As mentioned in 5 it suffices to show that for every nowhere dense tree T $X + \text{Lim}(T)$ is meager. Let $f = f_T$ as in Lemma 6. Define by recursion $n_0 = 0$, $n'_i = f^{g_{1/(i+1)}(i)}(n_i) + 1$ and $n_{i+1} = n'_i + i \cdot 2^{n_i} + 1$. By Lemma 10 $X \subseteq \bigcup_{m < \omega} \text{Lim}(S_m)$, where for some $a_m \in (0, 1)$ S_m is of width $\langle \bar{n}', g_{a_m} \rangle$, hence it suffices to show that if S is of width $\langle \bar{n}', g_a \rangle$ for some $a \in (0, 1)$ then $\text{Lim}(S) + \text{Lim}(T) = \text{Lim}(S+T)$ is meager. Let j be such that $\frac{1}{j+1} \leq a$ and let η_1, \dots, η_k be all the members of S of length n'_j . Then $S = \bigcup_{l=1}^k S^{[n_l]}$ and $\text{Lim} S = \bigcup_{l=1}^k \text{Lim}(S^{[n_l]})$. Therefore it suffices to prove that for $1 \leq l \leq k$ $\text{Lim}(S_l) + \text{Lim}(T)$ is meager and this follows once we show that $S_l + T$ is nowhere dense. To prove this we show that the requirements of Lemma 6 hold here for S_l, T . (a) and (b) hold by our choice of T and f . Let g be defined by $g(i) = 1$ for $i < j$ and $g(i) = g_a(i)$ for $i \geq j$. Now we shall see that (c) holds. For $i < j$ we have $n'_i = f^{g_{1/(i+1)}(i)}(n_i) + 1 \geq f(n_i) + 1 = f^{g(i)}(n_i) + 1$, since $f(n) \geq n$ for every n , and for $i \geq j$ we have $n'_i = f^{g_{1/(i+1)}(i)}(n_i) + 1 \geq f^{g_a(i)}(n_i) + 1 = f^{g(i)}(n_i) + 1$, since $a \geq \frac{1}{j+1} \geq \frac{1}{i+1}$ and $g_a(i)$ is a decreasing function of a . Thus for every $i < \omega$ $f^{g(i)}(n_i) \leq f^{g_{1/(i+1)}(i)}(n_i) \leq n'_i$. (d) of Lemma 6 holds since for $i < j$ $|n'_i 2 \cap S_l| = 1 = g(i)$ and for $i \geq j$ $|n'_i 2 \cap S_l| \leq |n'_i 2 \cap S| \leq g_a(i) = g(i)$.

§3 Characterization of the null-additive sets

12. Definition. By a *corset* we mean a non decreasing function f from ω to $\omega \setminus \{0\}$ which converges to infinity (i.e., for every $n < \omega$ $f(m) > n$ for all sufficiently large m). For a corset f , we say that a tree T is *of width f* if for every $n < \omega$ $|T \cap {}^n 2| \leq f(n)$; and we say that T is *almost of width f* if $|T \cap {}^n 2| \leq f(n)$ for all sufficiently large n .

13. Theorem. For every $X \subseteq {}^\omega 2$ the following conditions are equivalent:

- X is null-additive.
- For every corset f there is a tree S of width f such that $X \subseteq \text{Lim}(S)^{\text{fin}}$.
- For every corset f there are trees S_m , $m < \omega$, which are almost of width f such that $X \subseteq \bigcup_{m < \omega} \text{Lim}(S_m)^{\text{fin}}$.
- For every corset f there are trees S_m , $m < \omega$, of width f such that $X \subseteq \bigcup_{m < \omega} \text{Lim}(S_m)$.

Proof. (b) \rightarrow (c) is obvious.

(c)→(d). Let S be a tree almost of width f . Then for some k we have $|T \cap {}^n 2| \leq f(n)$ for all $n \geq k$. By (1) of Lemma 8 $\text{Lim}(S)^{\text{fin}} = \bigcup_{\sigma_1, \sigma_2 \in {}^{k2}, \sigma_2 \in S} \text{Lim}(S_{\sigma_1, \sigma_2})$. Each S_{σ_1, σ_2} is of width f since for $n \leq k$ we have $|S_{\sigma_1, \sigma_2} \cap {}^n 2| = 1$ and for $n > k$ we have $|S_{\sigma_1, \sigma_2} \cap {}^n 2| \leq |S \cap {}^n 2| \leq f(n)$. Therefore, if $X \subseteq \bigcup_{m < \omega} \text{Lim}(S_m)^{\text{fin}}$ as in (c) then each S_m can be replaced by countably many S_{σ_1, σ_2} 's and (d) holds.

(d)→(b). Let f be a corset. We can easily define by recursion a sequence $0 = n_0 < n_1 < \dots$ of natural numbers and a corset f^* such that for all $j < \omega$ and $m \geq n_{j+1}$ we have $(j+1) \cdot 2^{n_j} \cdot f^*(m) \leq f(m)$.

For a given corset f , if X satisfies (d) let S_m^* , $m < \omega$, be as in (d) for the corset f^* . We construct now a set $S \subseteq {}^\omega 2$ by defining $S \cap {}^m 2$ by recursion on m . $S \cap {}^0 2 = \{\langle \rangle\}$. For $n_i < m \leq n_{i+1}$ let

$$S \cap {}^m 2 = \{\eta \in {}^m 2 : \eta \upharpoonright n_i \in S \cap {}^{n_i} 2 \text{ and } \eta \in S_j^{*\sim n_j} \text{ for some } j < i \vee j = 0\}.$$

S can be easily seen to be a tree, and clearly $\text{Lim}(S)^{\text{fin}} \supseteq \bigcup_{n < \omega} \text{Lim}(S_n^*) \supseteq X$. For $m \leq n_1$ easily, for $n_i \leq m < n_{i+1}, i \geq 1$ we have

$$|S \cap {}^m 2| \leq \sum_{j \leq i} |S_j^{*\sim n_j} \cap {}^m 2| = \sum_{j \leq i} 2^{n_j} |S_j^* \cap {}^m 2| \leq (i+1) \cdot 2^{n_j} \cdot f^*(m) \leq f(m),$$

thus S is of width f .

(d)→(a). Assume now that (d) holds for X , and let $A \subseteq {}^\omega 2$, $\mu(A) = 0$; we shall prove that $\mu(X + A) = 0$. First we shall mention two lemmas of measure theory the proof of which is left to the reader.

Lemma A. For every tree T with $\mu(\text{Lim}(T)) = a > 0$ and $\epsilon > 0$ there is an $N \in \omega$ such that for every $n \geq N$ there is a $t \subseteq {}^n 2 \cap T$ such that $|t| \geq 2^n(a - \epsilon)$ and for each $\eta \in t$ $\mu(\text{Lim}(T^{[\eta]})) > 2^{-n}(1 - \epsilon)$.

Using Lemma A one can prove

Lemma B. For every tree T with $\mu(\text{Lim}(T)) > 0$, every $\epsilon > 0$ and every sequence $\langle \epsilon_i : 0 < i < \omega \rangle$ of positive reals there is a subtree T' of T and an increasing sequence $\langle n_i : i < \omega \rangle$ of natural numbers such that $n_0 = 0$, $\mu(\text{Lim}(T')) > \mu(\text{Lim}(T)) - \epsilon$ and

$$(6) \quad \text{for } i > 0 \text{ and every } \eta \in {}^{n_i} 2 \cap T' \quad \mu(\text{Lim}(T'^{[\eta]})) > 2^{-n_i}(1 - \epsilon_i)$$

By basic measure theory $\mu(A^{\text{fin}}) = 0$ so there is a tree T such that $\mu(\text{Lim}(T)) > 0$ and $\text{Lim}(T) \cap A^{\text{fin}} = \emptyset$ hence $\text{Lim}(T)^{\text{fin}} \cap A = \emptyset$. Given $\epsilon < \mu(\text{Lim}(T))$ and $\langle \epsilon_i : i < \omega \rangle$ as in Lemma B we obtain a subtree T' of T as in that lemma with $\mu(\text{Lim}(T')) > 0$. The union of sufficiently many “finite translates” of T' , i.e., trees T'_{σ_1, σ_2} as in (1) of Lemma 8 is a tree T'' satisfying (6) with $\mu(\text{Lim}(T'')) \geq \frac{1}{2}$. $\text{Lim}(T'')^{\text{fin}} = \text{Lim}(T')^{\text{fin}} \subseteq \text{Lim}(T)^{\text{fin}}$ and hence $\text{Lim}(T'') \cap A \subseteq \text{Lim}(T)^{\text{fin}} \cap A = \emptyset$. We take now $\epsilon_i = \frac{1}{4(i+1)^3}$ and take T to be T'' and we get $\mu(\text{Lim}(T)) \geq \frac{1}{2}$ and

$$(7) \quad \text{for } i > 0 \text{ and every } \eta \in {}^{n_i} 2 \cap T \quad \mu(\text{Lim}(T^{[\eta]})) > 2^{-n_i} \left(1 - \frac{1}{4(i+1)^3}\right)$$

Let f be the corset given by $f(n) = i+1$ for $n_i \leq n < n_{i+1}$. By (d) there are trees S_m of width f such that $X \subseteq \bigcup_{m < \omega} \text{Lim}(S_m)$. To show that $\mu(X + A) = 0$ it clearly suffices to show that for every tree S of width f $\mu(\text{Lim}(S) + A) = 0$.

We define

$$T^* = \{\eta \in {}^\omega 2 : \nu + \eta \in T \text{ for every } \nu \in S \text{ of the same length as } \eta\}$$

We do not show that T^* is a tree but obviously if $\zeta \sqsubseteq \eta \in T^*$ then $\zeta \in T^*$, thus $\text{Lim}(T^*)$ is defined. If $\mu(\text{Lim}(T^*)) > 0$ then, by a well-known property of the measure, $\mu(\text{Lim}(T^*)^{\text{fin}}) = 1$, hence in order to prove $\mu(\text{Lim}(S) + A) = 0$ it suffices to prove $(\text{Lim}(S) + A) \cap \text{Lim}(T^*)^{\text{fin}} = \emptyset$. Assume $y \in (\text{Lim}(S) + A) \cap \text{Lim}(T^*)^{\text{fin}}$. Since $y \in \text{Lim}(T^*)^{\text{fin}}$ there is a $y' \in {}^\omega 2$ such that $y'(n) = y(n)$ for all sufficiently big n 's and $y' \in \text{Lim}(T^*)$. Since $y \in \text{Lim}(S) + A$ there is an $x \in \text{Lim}(S)$ such that $y + x \in A$, hence $y + x \notin \text{Lim}(T)^{\text{fin}}$, hence $y' + x \notin \text{Lim}(T)$. Therefore, for some n $y' \upharpoonright n + x \upharpoonright n \notin T$, hence, by the definition of T^* , $y' \upharpoonright n \notin T^*$ contradicting $y' \in \text{Lim}(T^*)$.

We still have to prove that $\mu(\text{Lim}(T^*)) > 0$. We shall prove, by induction on i , that

$$(8) \quad n_i \leq n \leq n_{i+1} \rightarrow |(T \setminus T^*) \cap {}^n 2| \leq 2^n \cdot \sum_{j < i} \frac{1}{4(j+1)^2}.$$

Once we establish (8) we notice that since

$$\begin{aligned} \text{Lim}(T) \setminus \text{Lim}(T^*) &= \bigcup_{n < \omega} \text{Lim}(T) \setminus \{x \in {}^\omega 2 : x \upharpoonright n \in T^*\}, \text{ and the set} \\ \text{Lim}(T) \setminus \{x \in {}^\omega 2 : x \upharpoonright n \in T^*\} &\text{ is increasing with } n \text{ hence } \mu(\text{Lim}(T) \setminus \text{Lim}(T^*)) \\ &= \lim_{n \rightarrow \infty} \mu(\text{Lim}(T) \setminus \{x \in {}^\omega 2 : x \upharpoonright n \in T^*\}) \leq \lim_{n \rightarrow \infty} 2^{-n} |(T \setminus T^*) \cap {}^n 2| \\ &\leq \lim_{n \rightarrow \infty} \sum_{j=0}^n \frac{1}{4(j+1)^2} = \sum_{j=0}^{\infty} \frac{1}{4(j+1)^2} = \frac{\pi^2}{24} < \frac{1}{2} \text{ and since } \mu(\text{Lim}(T)) \geq \frac{1}{2} \\ \mu(\text{Lim}(T^*)) &> 0. \end{aligned}$$

To prove (8), assume now $n_i \leq n \leq n_{i+1}$. By the definition of T^*

$$\begin{aligned} (T \setminus T^*) \cap {}^n 2 &= \{\eta \in T \cap {}^n 2 : (\exists \rho \in S \cap {}^n 2) \rho + \eta \notin T\} \\ &= \{\eta \in T \cap {}^n 2 : (\exists \rho \in S \cap {}^n 2) (\eta \upharpoonright n_i + \rho \upharpoonright n_i \notin T)\} \cup \\ &\quad \bigcup_{\rho \in S \cap {}^n 2} \{\eta \in T \cap {}^n 2 : \eta \upharpoonright n_i + \rho \upharpoonright n_i \in T \wedge \eta + \rho \notin T\} \\ &\subseteq \{\eta \in {}^n 2 : \eta \upharpoonright n_i \in T \setminus T^*\} \cup \bigcup_{\rho \in S \cap {}^n 2} \{\eta \in {}^n 2 : \eta + \rho \in \{\sigma \in {}^n 2 : \sigma \upharpoonright n_i \in T \wedge \sigma \notin T\}\}. \end{aligned}$$

Therefore $|(T \setminus T^*) \cap {}^n 2| \leq 2^{n-n_i} |(T \setminus T^*) \cap {}^{n_i} 2| + |S \cap {}^n 2| |\{\sigma \in {}^n 2 : \sigma \upharpoonright n_i \in T \wedge \sigma \notin T\}|$. For $i > 0$ we have, by the induction hypothesis $|T \setminus T^* \cap {}^{n_i} 2| \leq 2^{n_i} \sum_{j < i} \frac{1}{4(j+1)^2}$. For $i = 0$ we have $(T \setminus T^*) \cap {}^{n_0} 2 = \emptyset$ since $n_0 = 0$ and $\emptyset \in T^*$. $|S \cap {}^n 2| \leq f(n) = i$ and $|\{\sigma \in {}^n 2 : \sigma \upharpoonright n_i \in T \wedge \sigma \notin T\}| \leq \frac{2^n}{4(i+1)^3}$, by (7). Thus

$$|(T \setminus T^*) \cap {}^n 2| \leq 2^{n-n_i} \cdot 2^{n_i} \sum_{j < i} \frac{1}{4(j+1)^2} + (i+1) \cdot \frac{2^n}{4(i+1)^3} \leq 2^n \sum_{j < i+1} \frac{1}{4(j+1)^2} \text{ which is what we had to show.}$$

(a)→(c). Most of the proof follows that of Lemma 10. We need also the following Lemma 14, which will be proved later. Let f be a corset.

14. Lemma. There is an infinite sequence $0 = n_0 < n_1 < n_2 < \dots$ and a tree T such that for every $i \in \omega$ $f(n_{i+1}) > (i+1) \cdot 2^{i+1} + 1$ and

(B1) For each $\eta \in T \cap {}^{n_i} 2$ we have $|T^{[\eta]} \cap {}^{n_{i+1}} 2| = 2^{(n_{i+1}-n_i)} \cdot (1 - 2^{-(i+1)})$.

(B2) If $\eta, \nu_0, \dots, \nu_{k-1} \in {}^{n_i} 2$, $\nu_0^+, \dots, \nu_{k-1}^+ \in {}^{n_{i+1}} 2$, $\nu_j^+ \neq \nu_l^+$ for $j < l < k$, $\eta + \nu_l \in T$, $\nu_l \sqsubseteq \nu_l^+$ for $l < k$ then

$$|\{\eta^+ : \eta \sqsubseteq \eta^+ \in {}^{n_{i+1}} 2, (\forall l < k)(\eta^+ + \nu_l^+ \in T)\}| \leq 2^{n_{i+1}-n_i} (1 - 2^{-(i+1)})^{k-1}.$$

Let $\langle n_i : i \in \omega \rangle$ and T be as in Lemma 14. As in the proof of Lemma 10 we get $\mu(\text{Lim } T) > 0$. Let T^* and $Y_{\eta, \zeta}$ be as in Lemma 9 and let $Y_{\eta, \zeta, \rho}$, S and z be as in the proof of Lemma 10. All we have to do is to show that S is almost of width f . Let us fix η , ζ and ρ . We shall now see that

$$(9) \quad \begin{aligned} & \text{If } \eta' \in T^{*[\eta]} \cap {}^{n_i}2 \text{ then} \\ & |\{\eta^+ : \eta' \sqsubseteq \eta^+ \in T^* \cap {}^{n_{i+1}}2\}| / 2^{(n_{i+1}-n_i)} \leq (1 - 2^{-(i+1)})^{|S \cap {}^{n_i}2| - 1} \end{aligned}$$

Let $\eta^+ \in T^{*[\eta]} \cap {}^{n_{i+1}}2$, then, by the definition of S (see (3)), if $\rho^+ \in S \cap {}^{n_{i+1}}2$ then $\rho^+ + \eta^+ + z \in T$. Thus

$$\{\eta^+ : \eta' \sqsubseteq \eta^+ \in T^* \cap {}^{n_{i+1}}2\} \subseteq \{\eta^+ : \eta' \sqsubseteq \eta^+ \in {}^{n_{i+1}}2, (\forall \rho^+ \in S) \rho^+ + \eta^+ + z \in T\}.$$

Let us take in (B2) $\eta = \eta'$, $k = |S \cap {}^{n_i}2|$, $\{\tau_l : l < k\} = S \cap {}^{n_i}2$, $\{\tau_l^+ : l < k\} \subseteq S \cap {}^{n_{i+1}}2$, and for $l < k$ $\tau_l^+ \upharpoonright n_i = \tau_l$, $\nu_l = \tau_l + z$, $\nu_l^+ = \tau_l^+ + z$, hence $\nu_l = \nu_l^+ \upharpoonright n_i$ for $l < k$. Since for $l < k$ $\nu_l^+ + z = \tau_l^+ \in S \cap {}^{n_{i+1}}2$ we have

$$\begin{aligned} \{\eta^+ : \eta' \sqsubseteq \eta^+ \in {}^{n_{i+1}}2, (\forall \rho^+ \in S) (\rho^+ + \eta^+ + z \in T)\} \\ \subseteq \{\eta^+ : \eta' \sqsubseteq \eta^+ \in {}^{n_{i+1}}2, (\forall l < k) (\nu_l^+ + \eta^+ \in T)\}, \end{aligned}$$

therefore by (B2)

$$|\{\eta^+ : \eta' \sqsubseteq \eta^+ \in {}^{n_{i+1}}2, (\forall \rho^+ \in S) (\rho^+ + \eta^+ + z \in T)\}| \leq 2^{n_{i+1}-n_i} (1 - 2^{-(i+1)})^{|S \cap {}^{n_i}2| - 1},$$

which establishes (9).

(9) tells us how T^* grows from the level n_i to the level n_{i+1} and therefore

$$|T^* \cap {}^{n_i}2| \cdot 2^{-n_i} \leq \prod_{j < i} (1 - 2^{-(j+1)})^{|S \cap {}^{n_j}2| - 1}.$$

Let $c_0 = \mu(\text{Lim } T^*)$. We know that $c_0 > 0$ and we can assume $c_0 < 1$. Then

$-\infty < \log c_0 \leq \log(|T^* \cap {}^{n_i}2| \cdot 2^{-n_i}) \leq \sum_{j < i} (\log(1 - 2^{-(j+1)}) \cdot (|S \cap {}^{n_j}2| - 1))$. Since $\log(1 - x) \leq -\frac{1}{2}x$ we get $\sum_{j < i} 2^{-(j+2)} \cdot (|S \cap {}^{n_{i+1}}2| - 1) \leq \log \frac{1}{c_0}$. We shall denote $4 \log \frac{1}{c_0}$ by c , so $\sum_{j < i} 2^{-j} \cdot (|S \cap {}^{n_j}2| - 1) \leq c$, and for every j $2^{-j} (|S \cap {}^{n_j}2| - 1) \leq c$, hence $|S \cap {}^{n_j}2| \leq c \cdot 2^j + 1$. For $j > c$ we have, by our choice of the n_i 's, $f(n_j) > j \cdot 2^j + 1 > c \cdot 2^j + 1 \geq |S \cap {}^{n_j}2|$, hence S is almost of width f .

Lemma 14 follows immediately from the following Lemma.

15. Lemma. For every $n \in \omega$ and $0 < p < 1$ there is an $N > n$ such that for every $n' \geq N$ and $t \subseteq {}^{n'}2$ there is a $t' \subseteq {}^{n'}2$ which satisfies the following (i)–(iii).

(i) For each $\zeta \in t'$ $\zeta \upharpoonright n \in t$.

(ii) For each $\eta \in t$ $|t'^{[\eta]}| \geq 2^{n'-n} \cdot p$.

(iii) If $0 < k \leq 2^n$, $\eta, \nu_0, \dots, \nu_{k-1} \in {}^{n'}2$, $\nu_0^+, \dots, \nu_{k-1}^+ \in {}^{n'}2$, $\nu_j^+ \neq \nu_l^+$ for $j < l < k$, $\eta + \nu_l \in t$, $\nu_l = \nu_l^+ \upharpoonright n$ for $l < k$ then $|\{\eta^+ : \eta \sqsubseteq \eta^+ \in {}^{n'}2, (\forall l < k) \eta^+ + \nu_l^+ \in t'\}| \leq 2^{n'-n} p^{k-1}$.

Proof. We shall prove the lemma by the probabilistic method. Let $n' > n$ and let $A = \{\eta^+ \in {}^{n'}2 : \eta^+ \upharpoonright n \in t\}$. We construct a subset A^* of A as follows. We take a coin which yields heads with probability p . For each $\eta^+ \in A$ we toss this coin and we put η^+ in A^* iff the coin shows heads. We shall see that if we take $t' = A^*$ then, for sufficiently large n' , the probability that (ii) holds has a positive lower bound which does not depend on n' while the probability that (iii) holds is arbitrarily close to 1. Hence there is an N and a t' as claimed by the lemma. We prove first two lemmas.

Lemma 16. For $k, \eta, \nu_0, \dots, \nu_{k-1}, \nu_0^+, \dots, \nu_{k-1}^+$ as in Lemma 15 there are reals $c_1, c_2 > 0$

which depend only on p , n and k such that

$$\Pr \left(\left| \{ \eta^+ : \eta \sqsubseteq \eta^+ \in {}^{n'}2, \bigwedge_{l < k} \eta^+ + \nu_l^+ \in A^* \} \right| \geq p^{k-1} 2^{n'-n} \right) < c_1 e^{-c_2 \cdot 2^{n'}}$$

Proof. We denote $2^{n'-n}$ with m . We set $({}^{n'}2)^{[\eta]} = \{ \eta_j^+ : j < m \}$. Let G be the graph on m given by

$$iGj \text{ iff } \{ \eta_i^+ + \nu_l^+ : l < k \} \cap \{ \eta_j^+ + \nu_l^+ : l < k \} \neq \emptyset$$

Obviously each $i < m$ has at most k^2 neighbors in G hence, by a well known theorem, m can be decomposed into $k^2 + 1$ pairwise disjoint sets B_0, \dots, B_{k^2} such that for every $i \leq k^2$ if $j, l \in B_i$ and $j \neq l$ then jGl does not hold. Let $d < \frac{1}{2} \min \{ p^{l-1} - p^l : l \leq 2^n \} = \frac{1}{2} p^{2^n-1} (1-p) > 0$.

$$\begin{aligned} & \Pr \left(\left| \{ j < m : \bigwedge_{l < k} \eta_j^+ + \nu_l^+ \in A^* \} \right| \geq m \cdot p^{k-1} \right) \\ (10) \quad & \leq \Pr \left(\left| \{ j < m : \bigwedge_{l < k} \eta_j^+ + \nu_l^+ \in A^* \} \right| > m(p^k + d) \right) \quad \text{since } p^k + d < p^{k-1} \end{aligned}$$

Assume that

$$\begin{aligned} & \text{for every } i \leq k^2 \text{ such that } |B_i| \geq \frac{dm}{2k^2 + 2} \\ (11) \quad & \text{we have } \left| \{ j \in B_i : \bigwedge_{l < k} \eta_j^+ + \nu_l^+ \in A^* \} \right| \leq |B_i| \left(p^k + \frac{d}{2} \right) \end{aligned}$$

then

$$\left\{ j < m : \bigwedge_{l < k} \eta_j^+ + \nu_l^+ \in A^* \right\} \subseteq \bigcup_{i \leq k^2, |B_i| \geq \frac{dm}{2k^2 + 2}} \left\{ j \in B_i : \bigwedge_{l < k} \eta_j^+ + \nu_l^+ \in A^* \right\} \cup \bigcup_{i \leq k^2, |B_i| < \frac{dm}{2k^2 + 2}} B_i$$

hence

$$\begin{aligned} & \left| \{ j < m : \bigwedge_{l < k} \eta_j^+ + \nu_l^+ \in A^* \} \right| \\ & \leq \sum_{i \leq k^2, |B_i| \geq \frac{dm}{2k^2 + 2}} \left| \{ j \in B_i : \bigwedge_{l < k} \eta_j^+ + \nu_l^+ \in A^* \} \right| + \sum_{i \leq k^2, |B_i| < \frac{dm}{2k^2 + 2}} |B_i| \\ & \leq \sum_{i \leq k^2, |B_i| \geq \frac{dm}{2k^2 + 2}} |B_i| \left(p^k + \frac{d}{2} \right) + \sum_{i \leq k^2, |B_i| < \frac{dm}{2k^2 + 2}} |B_i|, \quad \text{by (11)} \\ & \leq m \left(p^k + \frac{d}{2} \right) + (k^2 + 1) \frac{dm}{2k^2 + 2} = m(p^k + d) \end{aligned}$$

Therefore the event $|\{j < m : \bigwedge_{l < k} \eta_j^+ + \nu_l^+ \in A^*\}| > m(p^k + d)$ is incompatible with (11), so we continue the inequality (10) by

$$(12) \quad \begin{aligned} &\leq \Pr \left(\bigvee_{i \leq k^2, |B_i| \geq \frac{dm}{2k^2+2}} \left(|\{j \in B_i : \bigwedge_{l < k} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|(p^k + \frac{d}{2}) \right) \right) \\ &\leq \sum_{i \leq k^2, |B_i| \geq \frac{dm}{2k^2+2}} \Pr \left(|\{j \in B_i : \bigwedge_{l < k} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|(p^k + \frac{d}{2}) \right) \end{aligned}$$

For a fixed $j < m$ the events $\eta_j^+ + \nu_l^+ \in A^*$ for different l 's are independent hence $\Pr \left(\bigwedge_{l < k} \eta_j^+ + \nu_l^+ \in A^* \right) = p^k$. For a fixed i the events $\bigwedge_{l < k} \eta_j^+ + \nu_l^+ \in A^*$ for different j 's in B_i are independent since, by the definition of the B_i 's, if $j_1, j_2 \in B_i$ and $j_1 \neq j_2$ then $\eta_{j_1}^+ + \nu_{l_1}^+ \neq \eta_{j_2}^+ + \nu_{l_2}^+$. We have here $|B_i|$ independent events, each with probability p^k . By a formula of probability theory (see, e. g., the formula $\Pr[X > a] < e^{-2a^2/n}$ in Spencer [2], p. 29)

$$\Pr \left(|\{j \in B_i : \bigwedge_{k < l} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|p^k + \epsilon \right) < e^{-\frac{2\epsilon^2}{|B_i|}}$$

and taking $\epsilon = \frac{1}{2}|B_i|d$ we get

$$\Pr \left(|\{j \in B_i : \bigwedge_{k < l} \eta_j^+ + \nu_l^+ \in A^*\}| > |B_i|(p^k + \frac{d}{2}) \right) < e^{-\frac{d^2|B_i|}{2}}$$

Continuing (12) we get

$$\leq \sum_{i \leq k^2, |B_i| \geq \frac{dm}{2k^2+2}} e^{-\frac{d^2|B_i|}{2}} \leq \sum_{i \leq k^2, |B_i| \geq \frac{dm}{2k^2+2}} e^{-\frac{d^2}{2} \frac{dm}{2k^2+2}} \leq (k^2 + 1)e^{-\frac{d^3 2^{n'} - n}{4k^2+4}}$$

Combining this with the inequalities (10) and (12) we get

$$\begin{aligned} &\Pr \left(|\{\eta^+ : \eta \leq \eta^+ \in n'2, \bigwedge_{l < k} \eta^+ + \nu_l^+ \in A^*\}| \geq p^{k-1}2^{n'-n} \right) \\ &< (k^2 + 1)e^{-\frac{d^3 2^{n'} - n}{4k^2+4}} = (k^2 + 1)e^{-\frac{d^3 2^{-n} 2^{n'}}{4k^2+4}} \end{aligned}$$

Since $d = \frac{1}{2}p^{2^n-1}(1-p)$ this proves Lemma 16.

17. Lemma. There are c_3, c_4 which depend only on p and n such that

$$(13) \quad \begin{aligned} &\Pr \left(\bigvee_{k, \eta, \nu_0, \dots, \nu_{k-1}, \nu_0^+, \dots, \nu_{k-1}^+} \left(|\{\eta^+ : \eta \leq \eta^+ \in n'2, (\forall l < k) \eta^+ + \nu_l^+ \in A^*\}| \geq 2^{n'-n} p^{k-1} \right) \right) \\ &\leq c_3(2^{n'})^{2^n} e^{-c_4 2^{n'}} \end{aligned}$$

where $k, \eta, \nu_0, \dots, \nu_{k-1}, \nu_0^+, \dots, \nu_{k-1}^+$ are as in (iii) of Lemma 15.

Proof. By our requirements on $k, \eta, \nu_0, \dots, \nu_{k-1}, \nu_0^+, \dots, \nu_{k-1}^+$ there are at most 2^n possible k 's and η 's and $(2^{n'})^{2^n}$ sequences $\langle \nu_0^+, \dots, \nu_{k-1}^+ \rangle$, while ν_0, \dots, ν_{k-1} are determined by $\nu_0^+, \dots, \nu_{k-1}^+$ and n . Therefore we get, by Lemma 16,

$$\begin{aligned} & \Pr\left(\bigvee_{k, \eta, \nu_0, \dots, \nu_{k-1}, \nu_0^+, \dots, \nu_{k-1}^+} (|\{\eta^+ : \eta \leq \eta^+ \in {}^{n'}2, (\forall l < k) \eta^+ + \nu_l^+ \in A^*\}| \geq 2^{n'-n} p^{k-1})\right) \\ & \leq \sum_{k, \eta, \nu_0, \dots, \nu_{k-1}, \nu_0^+, \dots, \nu_{k-1}^+} \Pr\left(|\{\eta^+ : \eta \leq \eta^+ \in {}^{n'}2, (\forall l < k) \eta^+ + \nu_l^+ \in A^*\}| \geq 2^{n'-n} p^{k-1}\right) \\ & \leq 2^n \cdot 2^n \cdot (2^{n'})^{(2^n)} \cdot c_1 e^{-c_2 2^{n'}} \end{aligned}$$

Proof of Lemma 15 (continued). For each $\eta^+ \in {}^{n'}2$ such that $\eta \leq \eta^+ \eta^+ \in A^*$ if the coin shows heads and different tosses are independent $|A^{*[\eta]}|$ is a binomial random variable with expectation $2^{n'-n}p$. By the central limit theorem of probability theory (see, e. g., Feller [1, Ch. 7]) the limit, as $n' \rightarrow \infty$, of $\Pr(|A^{*[\eta]}| \geq 2^{n'-n}p)$ is $\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{2}$, hence there is an N such that for every $n' \geq N$ $\Pr(|A^{*[\eta]}| \geq 2^{n'-n}p) \geq \frac{1}{3}$. For different $\eta \in t$ the random variables $|A^{*[\eta]}|$ are independent, hence

$$(14) \quad \Pr\left(\bigwedge_{\eta \in t} (|A^{*[\eta]}| \geq 2^{n'-n}p)\right) \geq \frac{1}{3^{|t|}} \geq \frac{1}{3^{2^n}}$$

The right-hand side of (13) clearly vanishes as $n \rightarrow \infty$, let us take N to be such that for $n' \geq N$ the right-hand side of (13) is $< 3^{-2^n}$. Therefore we have, by (13) and (14),

$$(15) \quad \begin{aligned} & \Pr\left(\bigvee_{k, \eta, \nu_0, \dots, \nu_{k-1}, \nu_0^+, \dots, \nu_{k-1}^+} (|\{\eta^+ : \eta \leq \eta^+ \in {}^{n'}2, (\forall l < k) \eta^+ + \nu_l^+ \in A^*\}| < 2^{n'-n} p^{k-1}) \right. \\ & \quad \left. \wedge \bigwedge_{\eta \in t} (|A^{*[\eta]}| \geq 2^{n'-n} p)\right) > 0 \end{aligned}$$

By (15) there is a t' as required by the lemma.

§4 Characterization of the meager-additive sets

18. Theorem. For every $X \subseteq {}^\omega 2$ the following conditions are equivalent:

- a. X is meager additive.
- b. For every sequence $n_0 < n_1 < n_2 < \dots$ of natural numbers there is a sequence $i_0 < i_1 < \dots$ of natural numbers and a $y \in {}^\omega 2$ such that for every $x \in X$ and for every sufficiently big $k < \omega$ there is an $l \in [i_k, i_{k+1})$ such that $x \upharpoonright [n_l, n_{l+1}) = y \upharpoonright [n_l, n_{l+1})$.

Proof. Throughout this proof, if $x \in {}^\omega 2 \cup {}^{>\omega} 2$, $k, l \in \omega$ and $k < l$ then $x \upharpoonright [k, l)$ will denote the sequence $\xi \in {}^{l-k} 2$ such that $\xi(i) = x(k+i)$ for all $i < l-k$.

(b)→(a). In order to prove (a) it clearly suffices to show that $X + \text{Lim } T$ is meager for every nowhere dense tree T .

For a nowhere dense tree T let $\langle n_i : i < \omega \rangle$ be an ascending sequence of natural numbers such that $n_0 = 0$ and for every $i \in \omega$ there is a sequence $\nu_i \in {}^{n_{i+1}-n_i}2$ such that for every $\tau \in {}^{n_i}2$ $\tau \cap \nu_i \notin T$. Let $\langle i_j : j < \omega \rangle$ and y be as in (b), then, by (b), $X = \bigcup_{k \in \omega} X_k$ where $X_k = \{x \in X : (\forall m \geq k)(\exists l \in [i_m, i_{m+1})) x \upharpoonright [n_l, n_{l+1}) = y \upharpoonright [n_l, n_{l+1})\}$. It clearly suffices to prove that $X_k + \text{Lim } T$ is nowhere dense.

Let $\tau \in {}^{n_{i_m}}2$ for some $m \geq k$; we shall show that τ has an extension which is not in $X_k + \text{Lim } T$. Let $\nu = \nu_{i_m} \cap \nu_{i_{m+1}} \cap \dots \cap \nu_{i_{m+1}-1}$ and let $\rho = y \upharpoonright [n_{i_m}, n_{i_{m+1}}) + \nu$. We show that no extension z of $\tau \cap \rho$ is in $X_k + \text{Lim } T$. Suppose $\tau \cap \rho \trianglelefteq z \in X_k + \text{Lim } T$ then $z = x + w$, $x \in X_k$ $w \in \text{Lim } T$. Therefore $\tau = \tau_1 + \tau_2$ and $\rho = \rho_1 + \rho_2$ such that $\tau_1 \cap \rho_1 \trianglelefteq x$ and $\tau_2 \cap \rho_2 \trianglelefteq w$, hence $\tau_2 \cap \rho_2 \in T$. Let $\xi \in {}^{n_{i_m}}2$ be such that $\xi(j) = 0$ for every $j < n_{i_m}$, and let $\rho' = \xi \cap \rho$, $\rho'_1 = \xi \cap \rho_1$, $\rho'_2 = \xi \cap \rho_2$. Clearly $\rho' = \rho'_1 + \rho'_2$. Since $x \in X_k$ there is, by (b), an $l \in [i_m, i_{m+1})$ such that $x \upharpoonright [n_l, n_{l+1}) = y \upharpoonright [n_l, n_{l+1})$. Since $\tau_1 \cap \rho_1 \trianglelefteq x$ we have $\rho'_1 \upharpoonright [n_{i_m}, n_{i_{m+1}}) = x \upharpoonright [n_{i_m}, n_{i_{m+1}})$ and hence $\rho_1 \upharpoonright [n_l, n_{l+1}) = x \upharpoonright [n_l, n_{l+1}) = y \upharpoonright [n_l, n_{l+1})$. Therefore, by the definition of ρ and ν

$$\begin{aligned} y \upharpoonright [n_l, n_{l+1}) + \rho'_2 \upharpoonright [n_l, n_{l+1}) &= \rho'_1 \upharpoonright [n_l, n_{l+1}) + \rho'_2 \upharpoonright [n_l, n_{l+1}) = \rho' \upharpoonright [n_l, n_{l+1}) \\ &= y \upharpoonright [n_l, n_{l+1}) + \nu_l \end{aligned}$$

hence $\rho'_2 \upharpoonright [n_l, n_{l+1}) = \nu_l$. By the definition of ν_l $\tau_2 \cap \rho_2 \notin T$, contradicting $\tau_2 \cap \rho_2 \in T$.

(a)→(b). Let X be meager-additive. Let $\langle n_i : i < \omega \rangle$ be an ascending sequence of natural numbers. Let $B = \{x \in {}^\omega 2 : \forall j(\exists k \in [n_j, n_{j+1})) x(k) \neq 0\}$ and $T = \{x \upharpoonright n : x \in B, n \in \omega\}$. Clearly $B = \text{Lim } T$ is nowhere dense, so $X + \text{Lim } T$ is meager, hence there are nowhere dense trees S_n , $n \in \omega$ such that for every n $S_n \subseteq S_{n+1}$ and $X + \text{Lim } T \subseteq \bigcup_{n \in \omega} S_n$. We define now the sequences $\langle i_l : l < \omega \rangle$, which is an ascending sequence of natural numbers, and $\langle \nu_l : l < \omega \rangle$ by recursion as follows. $i_0 = 0$. Given i_l let ν_l and i_{l+1} be such that $\nu_l \in {}^{n_{i_{l+1}}-n_{i_l}}2$ and for every $\rho \in {}^{n_{i_l}}2$ $\rho \cap \nu_l \notin S_l$; there are such ν_l and i_{l+1} since S_l is nowhere dense. Let $y \in {}^\omega 2$ be given by $y \upharpoonright [n_{i_l}, n_{i_{l+1}}) = \nu_l$ for every $l < \omega$. We shall now prove that $\langle i_l : l < \omega \rangle$ and y are as required by (b).

Let $x \in X$, so $\text{Lim}(x + T) = x + \text{Lim } T \subseteq X + \text{Lim } T \subseteq \bigcup_{n \in \omega} S_n$. Therefore, by Lemma 7 (where we take $x + T$ for S) there is an $\eta \in T$ and $n \in \omega$ such that $x + T^{[\eta]} \subseteq S_n$. Let k be such that $k \geq n$ and $i_k \geq \text{length } \eta$. By $x + T^{[\eta]} \subseteq S_n$ we have $x \upharpoonright n_{i_{k+1}} + (T^{[\eta]} \cap {}^{n_{i_{k+1}}}2) \subseteq S_n \subseteq S_k$. Thus for every $\rho \in T^{[\eta]} \cap {}^{n_{i_{k+1}}}2$ $x \upharpoonright n_{i_{k+1}} + \rho \in S_k$, hence, by the definition of ν_k and y , $x \upharpoonright [n_{i_k}, n_{i_{k+1}}) + \rho \upharpoonright [n_{i_k}, n_{i_{k+1}}) \neq \nu_k = y \upharpoonright [n_{i_k}, n_{i_{k+1}})$ and therefore $x \upharpoonright [n_{i_k}, n_{i_{k+1}}) - y \upharpoonright [n_{i_k}, n_{i_{k+1}}) \neq \rho \upharpoonright [n_{i_k}, n_{i_{k+1}})$, i.e., $x \upharpoonright [n_{i_k}, n_{i_{k+1}}) - y \upharpoonright [n_{i_k}, n_{i_{k+1}}) \notin \{\rho \upharpoonright [n_{i_k}, n_{i_{k+1}}) : \rho \in T^{[\eta]}\}$. Since $i_k > \text{length } \eta$ this can happen, by the definition of T , only if for some $i_k \leq j < i_{k+1}$ $x \upharpoonright [n_j, n_{j+1}) - y \upharpoonright [n_j, n_{j+1})$ is identically zero, and this is what we had to prove.

§5. An uncountable null-additive set.

19. Theorem. If the continuum hypothesis holds then there is an uncountable null-additive set.

Proof. Let $\langle f_\alpha : \alpha < \omega_1 \rangle$ be a sequence containing all corsets and let $\langle T_\alpha : \alpha < \omega_1 \rangle$ be a sequence containing all perfect trees. Let E be the set of all limit ordinals $\delta < \omega_1$ such that for every $\alpha, \beta < \delta$ and $n < \omega$ there is a $\gamma < \delta$ such that

$$T_\gamma \subseteq T_\alpha, \quad T_\gamma \cap {}^n 2 = T_\alpha \cap {}^n 2 \quad \text{and for all } m \quad |T_\gamma \cap {}^m 2| \leq \max(|T_\alpha \cap {}^m 2|, f_\beta(m))$$

Clearly E is closed. For every $\alpha, \beta < \omega_1$ there is a perfect tree T such that $T \subseteq T_\alpha$, $T \cap {}^n 2 = T_\alpha \cap {}^n 2$ and for all $m < \omega$ $|T \cap {}^m 2| \leq \max(|T_\alpha \cap {}^m 2|, f_\beta(m))$. This tree T is T_γ for some $\gamma < \omega_1$. By a simple closure argument this implies that E is unbounded.

We need now the following lemma which will be proved later.

20. Lemma. There is an increasing and continuous sequence $\langle \delta_\zeta : \zeta < \omega_1 \rangle$ of ordinals in E such that for every $\zeta < \omega_1$, $k < \omega$ and $\alpha < \delta_\zeta$ there is an ordinal γ which is good for (ζ, α, k) , where by γ is good for (ζ, α, k) we mean that

- (16) (i) $\gamma < \delta_{\zeta+1}$
(ii) $T_\gamma \subseteq T_\alpha$, $T_\gamma \cap {}^k 2 = T_\alpha \cap {}^k 2$
(iii) for all $\xi \leq \zeta$ such that $\delta_\xi > \alpha$ and for every $\epsilon < \delta_\zeta$, there is a $\beta < \delta_\xi$ such that $T_\gamma \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ

For $\xi < \omega_1$ let γ_ξ be the γ which is good for $(\xi, 0, 0)$. We choose $\eta_\xi \in \text{Lim } T_{\gamma_\xi} \setminus \{\eta_\beta : \beta < \xi\}$, and let $X = \{\eta_\xi : \xi < \omega_1\}$. X is clearly uncountable. We shall prove that X is null-additive by proving that X satisfies condition (c) of Theorem 13. For a given corset f $f = f_\epsilon$ for some $\epsilon < \omega_1$. Let $\xi < \omega_1$ be such that $\delta_\xi > \epsilon$. Let $Z = \{\beta < \delta_{\xi+1} : T_\beta \text{ is almost of width } f_\epsilon\}$. We shall see that

$X \subseteq \{\eta_\zeta : \zeta \leq \xi\} \cup \bigcup_{\beta \in Z} \text{Lim } T_\beta$. Since Z and ξ are countable condition (c) of Theorem 13 holds.

Let $\zeta > \xi$, it suffices to prove that $\eta_\zeta \in \text{Lim } T_\beta$ for some $\beta \in Z$. $\epsilon < \delta_\xi$ and since γ_ζ is good for $\alpha = k = 0$ hence there is a $\beta < \delta_\xi$ such that $T_{\gamma_\zeta} \subseteq T_\beta$, and T_β is of width f_ϵ . Thus $\beta \in Z$ and $\eta_\zeta \in \text{Lim } T_{\gamma_\zeta} \subseteq \text{Lim } T_\beta$.

Proof of Lemma 20. We define $\langle \delta_\zeta : \zeta < \omega_1 \rangle$ as follows. δ_0 is the least member of E . For a limit ordinal ζ $\delta_\zeta = \bigcup_{\xi < \zeta} \delta_\xi$. Since $\delta_\xi \in E$ for $\xi < \zeta$ also $\delta_\zeta \in E$. We shall now define $\delta_{\zeta+1}$. We shall assume, as an induction hypothesis, that for each $\xi < \zeta$ the lemma holds. For each $\alpha < \delta_\zeta$ and $k < \omega$ we shall find a $\gamma(\alpha, k)$ which is good for (ζ, α, k) and we shall choose $\delta_{\zeta+1}$ to be the least member of E greater than all these $\gamma(\alpha, k)$'s.

First we shall show that what the lemma claims holds for the case where ζ is a successor or 0. Whenever we shall write $\zeta - 1$ we shall assume that ζ is a successor. Let $\alpha < \delta_\zeta$ and $k < \omega$ be given, and let $\{\epsilon_n : n < \omega\} = \{\epsilon : \epsilon < \delta_\zeta\}$. We define sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle k_n : n < \omega \rangle$ so that

- (a) $k_0 = k$. If $\zeta = 0$ or $\alpha < \delta_{\zeta-1}$ then $\alpha_0 = \alpha$. If $\alpha \geq \delta_{\zeta-1}$ then α_0 is an ordinal which is good for $(\zeta - 1, \alpha, k)$. In any case $\alpha_0 < \delta_\zeta$, $T_{\alpha_0} \subseteq T_\alpha$ and $T_{\alpha_0} \cap {}^k 2 = T_\alpha \cap {}^k 2$.
(b) $\alpha_{n+1} < \delta_\zeta$.
(c) $T_{\alpha_{n+1}} \subseteq T_{\alpha_n}$.
(d) $T_{\alpha_{n+1}} \cap {}^{k_n} 2 = T_{\alpha_n} \cap {}^{k_n} 2$.
(e) $T_{\alpha_{n+1}}$ is almost of width f_{ϵ_n} .

(f) $k_{n+1} > k_n$ and every $\eta \in T_{\alpha_{n+1}} \cap {}^{k_n}2$ has at least two extensions in $T_{\alpha_{n+1}} \cap {}^{k_{n+1}}2$. There are indeed such sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle k_n : n < \omega \rangle$. (a) determines k_0 and α_0 ; if $\alpha < \delta_{\zeta-1}$ then there is an α_0 as in (a) by the induction hypothesis. δ_ζ in E and let us take $\alpha_n, \epsilon_n, k_n, \alpha_{n+1}$ for α, β, n, γ in the definition of E , then $\delta_\zeta \in E$ says that there is an α_{n+1} which satisfies (b)–(e). Since $T_{\alpha_{n+1}}$ is perfect there is a k_{n+1} as in (f).

Let $T = \bigcap_{n \in \omega} T_{\alpha_n}$. By (c),(d),(f) T is a perfect tree, hence it is T_γ for some $\gamma < \omega_1$. Since T , and therefore also γ , depend on α and k we denote γ with $\gamma(\alpha, k)$. As is easily seen $T_{\gamma(\alpha, k)} \subseteq T_\alpha$, $T_{\gamma(\alpha, k)} \cap {}^k 2 = T_\alpha \cap {}^k 2$, and for every $\epsilon < \delta_\zeta$ $T_{\gamma(\alpha, k)} \subseteq T_{\alpha_{l+1}} \subseteq T_\alpha$, where l is such that $\epsilon = \epsilon_l$. This means that (iii) of (16) holds for $\xi = \zeta$. We shall have to show that (iii) holds for $\xi < \zeta$ and to deal with the case where ζ is a limit ordinal.

If ζ is a limit ordinal let $\langle \zeta_n : n < \omega \rangle$ be an increasing sequence such that $\delta_{\zeta_0} > \alpha$ and $\bigcup_{n < \omega} \zeta_n = \zeta$. We construct the sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle k_n : n < \omega \rangle$ as in the case where ζ is a successor, except that (a),(b),(e) are replaced by

(a') $k_0 = k$, $\alpha_0 = \alpha$.

(b') $\alpha_n < \delta_{\zeta_n}$.

(e') α_{n+1} is good for (ζ_n, α, k) .

By the induction hypothesis that the lemma holds for the ζ_n 's there are indeed such sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle k_n : n < \omega \rangle$. Let $T = \bigcap_{n < \omega} T_{\alpha_n}$. As above, T is a perfect tree and $T = T_{\gamma(\alpha, k)}$, $T_{\gamma(\alpha, k)} \subseteq T_\alpha$ and $T_{\gamma(\alpha, k)} \cap {}^k 2 = T_\alpha \cap {}^k 2$.

We shall now see that for both cases of ζ with which we are dealing (iii) holds for $\xi < \zeta$. If ζ is a successor then $\xi \leq \zeta - 1$ and since α_o is, by (a), good for $(\zeta - 1, \alpha, k)$ there is a $\beta < \delta_\zeta$ such that $T_{\alpha_0} \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ . Note that if $\alpha < \delta_{\zeta-1}$ then, by the induction hypothesis, we have a $\gamma < \delta_\zeta$ which is good for (ζ, α, k) , and if $\zeta = 0$ then (iii) holds vacuously, hence we may assume that $\zeta > 0$ and $\alpha \in [\delta_{\zeta-1}, \delta_\zeta)$. Since $T_{\gamma(\alpha, k)} \subseteq T_{\alpha_0}$ β is as required by (iii). If ζ is a limit ordinal then $\xi \leq \zeta_n$ for some $n < \omega$. Since α_{n+1} is good for ζ_n then there is a $\beta < \delta_\xi$ such that $T_{\alpha_{n+1}} \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ . Since $T_{\gamma(\alpha, k)} \subseteq T_{\alpha_{n+1}}$ β is as required by (iii).

The only case left is that where ζ is a limit ordinal and $\xi = \zeta$ in (iii). Since $\alpha, \epsilon < \zeta$ also $\alpha, \epsilon < \zeta_n$ for some $n < \omega$. α_{n+1} is good for ζ_n hence there is a $\beta < \delta_{\zeta_n}$ such that $T_{\alpha_{n+1}} \subseteq T_\beta \subseteq T_\alpha$ and T_β is almost of width f_ϵ . Since $T_{\gamma(\alpha, k)} \subseteq T_{\alpha_{n+1}}$ and $\zeta_n < \zeta$ β is as required by (iii).

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