Baire Property and Axiom of Choice

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1 Introduction

In 1979 Shelah proved that in order to obtain a model in which every set of reals has Baire property, a large cardinal assumption is not necessary. The model he constructed satisfied $\omega_1^L = \omega_1$. Therefore Woodin asked if we can get a model for "ZF + DC(ω_1) + each set of reals has Baire property". Recall here that DC(ω_1) is the following sentence:

> if $\mathcal{R}$ is a relation such that $(\forall X)(\exists Y)(\mathcal{R}(X, Y))$ then there is a sequence $< Z_\alpha : \alpha < \omega_1 >$ such that
>
> $$(\forall \alpha < \omega_1)(\mathcal{R}(< Z_\beta : \beta < \alpha > \, , \, Z_\alpha)).$$

Note that DC(ω_1) implies the following version of choice:

> if $\mathcal{R} \subseteq \omega_1 \times \mathbf{R}$
> then there exists a choice function $f : \omega_1 \longrightarrow \mathbf{R}$ such that $\mathcal{R}(\alpha, f(\alpha))$ for each $\alpha < \omega_1$.

In [JS1] we studied the consistency strength of "ZFC + variants of MA + suitable sets of reals have Baire property". We showed that Baire property for Σ^1_3-sets of reals plus MA(σ-centered) implied that ω_1 is a Mahlo cardinal in L.

The natural question that arises at this point is:

> Do we need large cardinals to construct a model in which all projective sets of reals have Baire property and the union of any ω_1 meager sets is meager?

Note that if unions of ω_1 many null sets are null then every Σ^1_2-set of reals is Lebesgue measurable. Consequently if each projective sets of reals has Baire property and any union of ω_1 null sets is null then ω_1 is inaccessible in L.

The aim of the present paper is to prove the following two theorems:

Theorem 1.1 *If ZF is consistent then the following theory is consistent:*

> *ZF + DC(ω_1) + "Every set of reals has Baire property"*

Theorem 1.2 *If ZF is consistent then the following theory is consistent:*

> *ZFC + "Every projective set of reals has Baire property" + "Any union of ω_1 meager sets is meager"*

Our notation is standard and derived from [Jec]. There is one exception, however. We write $p \leq q$ to say that q is a stronger condition then p. $\emptyset$ denotes the smallest element of a forcing notion.

2 Basic definitions and facts

In this section we recall some definitions and results from [She]. They will be applied in the next section.

The basic tool in the construction of models in which definable sets have Baire property is the amalgamation. To define this operation we need the following definition.

Recall that $\mathbf{P} \lessdot \mathbf{P}'$ means $\mathbf{P} \subseteq \mathbf{P}'$ and each maximal antichain in $\mathbf{P}$ is a maximal antichain in $\mathbf{P}'$. For a forcing notion $\mathbf{P}$ let $\Gamma_{\mathbf{P}}$ be a $\mathbf{P}$-name for the generic subset of $\mathbf{P}$.

Definition 2.1 *Suppose that $\mathbf{P} \lessdot BA(\mathbf{Q})$. Then $(\mathbf{Q}\!:\!\mathbf{P})$ is the $\mathbf{P}$-name of a forcing notion which is a subset of $\mathbf{Q}$,*

$$(\mathbf{Q}\!:\!\mathbf{P}) = \{q \in \mathbf{Q} : q \textit{ is compatible with every } p \in \Gamma_{\mathbf{P}}\}.$$

Thus $p \Vdash q \in (\mathbf{Q}\!:\!\mathbf{P})$ if and only if every $p' \in \mathbf{P}$, $p' \geq p$ is compatible with q. Recall that if $\mathbf{P} \lessdot \mathrm{BA}(\mathbf{Q})$ then forcing notions $\mathbf{Q}$ and $\mathbf{P} * (\mathbf{Q} : \mathbf{P})$ are equivalent.

Definition 2.2 *Let $\mathbf{P}^0, \mathbf{P}^1$ and $\mathbf{P}^2$ be forcing notions. Suppose that $f_1 : \mathbf{P}^0 \xrightarrow{1-1} BA(\mathbf{P}^1)$, $f_2 : \mathbf{P}^0 \xrightarrow{1-1} BA(\mathbf{P}^2)$ are complete embeddings (i.e. they preserve order and $f_i[\mathbf{P}^0] \lessdot BA(\mathbf{P}^i)$). We define* the amalgamation of $\mathbf{P}^1$ and $\mathbf{P}^2$ over f_1, f_2 *by* $\mathbf{P}^1 \times_{f_1,f_2} \mathbf{P}^2 =$

$$\{(p_1,p_2)\in\mathbf{P}^1\times\mathbf{P}^2 : (\exists p\in\mathbf{P}^0)(p \Vdash \text{“}p_1\in(\mathbf{P}^1\!:\!f_1[\mathbf{P}^0]) \ \&\ p_2\in(\mathbf{P}^2\!:\!f_2[\mathbf{P}^0])\text{”})\}$$

$\mathbf{P}^1 \times_{f_1,f_2} \mathbf{P}^2$ is ordered in the natural way: $(p_1,p_2) \leq (p_1',p_2')$ if and only if $p_1 \leq p_1'$, $p_2 \leq p_2'$.

Note that $\mathbf{P}^1, \mathbf{P}^2$ can be completely embedded into the amalgamation $\mathbf{P}^1 \times_{f_1,f_2} \mathbf{P}^2$ by $p_1 \in \mathbf{P}^1 \mapsto (p_1, \emptyset)$ and $p_2 \in \mathbf{P}^2 \mapsto (\emptyset, p_2)$. Thus we think of $\mathbf{P}^1 \times_{f_1,f_2} \mathbf{P}^2$ as an forcing notion extending both $\mathbf{P}^1$ and $\mathbf{P}^2$.

The amalgamation is applied in constructing of Boolean algebras admitting a lot of automorphisms. The mapping

$$f_2^{-1} \circ f_1^{-1} : f_1[\mathbf{P}^0] \longrightarrow \mathbf{P}^2$$

can be naturally extended to an embedding

$$\phi : \mathbf{P}^1 \longrightarrow \mathbf{P}^1 \times_{f_1,f_2} \mathbf{P}^2.$$

Now. suppose that $\mathcal{B}$ is a complete Boolean algebra such that for sufficiently many pairs $(\mathbf{P}^1, \mathbf{P}^2)$ of complete suborders of $\mathcal{B}$ and for complete embeddings $f_i : \mathbf{P}^0 \longrightarrow \mathbf{P}^i$, $(i = 1,2)$ the algebra $\mathcal{B}$ contains the amalgamation $\mathbf{P}^1 \times_{f_1,f_2} \mathbf{P}^2$. Then $\mathcal{B}$ is *strongly Cohen-homogeneous*:
Suppose τ is a $\mathcal{B}$-name for an ω_1-sequence of ordinals. Then there exists a complete subalgebra $\mathcal{B}'$ of the algebra $\mathcal{B}$ such that

- τ is a $\mathcal{B}'$-name,
- if $\mathcal{B}' \lessdot \mathcal{B}'' \lessdot \mathcal{B}$, $\mathcal{B}' \Vdash$"$(\mathcal{B}'' : \mathcal{B}')$ is the Cohen algebra" and $f : \mathcal{B}'' \longrightarrow \mathcal{B}$ is a complete embedding such that $f|\mathcal{B}' = \mathrm{id}_{\mathcal{B}'}$
  then there exists an automorphism $\phi : \mathcal{B} \stackrel{\text{onto}}{\longrightarrow} \mathcal{B}$ extending f.

For more details on extending homomorphisms see [JuR].

Solovay showed the connection between the strong homogeneity of the algebra $\mathcal{B}$ and the fact that in generic extensions via $\mathcal{B}$ all projective sets of reals have Baire property. Let $\mathbf{S}_1$ be the class of all ω_1-sequences of ordinal numbers.

Theorem 2.3 (Solovay) *Let $\mathcal{B}$ be a strongly Cohen homogeneous complete Boolean algebra satisfying ccc. Suppose that for any $\mathcal{B}$-name τ for an ω_1-sequence of ordinals*

$\mathcal{B} \Vdash$ "the union of all meager Borel sets coded in $\mathbf{V}[\tau]$ is meager".

Then $\mathcal{B} \Vdash$ "any set of reals definable over $\mathbf{S}_1$ has Baire property".

Proof   See theorem 2.3 of [JuR]. ■

The class $\mathbf{HOD}(\mathbf{S}_1)$ consists of all sets hereditarily ordinal definable over $\mathbf{S}_1$.

Theorem 2.4 (Solovay) *Assume that every set of reals ordinal definable over $\mathbf{S}_1$ has Baire property. Then*

$\mathbf{HOD}(\mathbf{S}_1) \models$ " $\mathbf{ZF}$ + $\mathbf{DC}(\omega_1)$ + *every set of reals has Baire property".*

Proof   See [Sol]. ■

In the next section we will built a model in which there exists an algebra $\mathcal{B}$ satisfying the assumptions of theorem 2.3 and such that

$\mathcal{B} \Vdash$"the union of ω_1 meager sets is meager".

To be sure that the algebra $\mathcal{B}$ satisfies ccc we will use the following notion.

Definition 2.5   *A triple $(\mathbf{P}, \mathcal{D}, \{E_n\}_{n\in\omega})$ is* a model of sweetness *if*

1. *$\mathbf{P}$ is a notion of forcing and $\mathcal{D}$ is a dense subset of $\mathbf{P}$,*
2. *E_n are equivalence relations on $\mathcal{D}$ such that*

- *each* E_n *has countably many equivalence classes (the equivalence class of the element* $p \in \mathcal{D}$ *in the relation* E_n *will be denoted by* $[p]_n$*),*
- *equivalence classes of all relations* E_n *are upward directed,*
- *if* $\{p_i : i \leq \omega\} \subseteq \mathcal{D}$*,* $p_i \in [p_\omega]_i$ *for all* i *then for every* $n < \omega$ *there exists* $q \in [p_\omega]_n$ *which is stronger than all* p_i *for* $i \geq n$*,*
- *if* $p, q \in \mathcal{D}$*,* $p \leq q$ *and* $n \in \omega$ *then there exists* $k \in \omega$ *such that*

$$(\forall p' \in [p]_k)(\exists q' \in [q]_n)(p' \leq q').$$

Note that if $(\mathbf{P}, \mathcal{D}, \{E_n\}_{n\in\omega})$ is a model of sweetness then $\mathbf{P}$ is σ-centered.

Definition 2.6 *We say that a model of sweetness* $(\mathbf{P}^2, \mathcal{D}^2, \{E^2_n\}_{n\in\omega})$ extends *a model* $(\mathbf{P}^1, \mathcal{D}^1, \{E^1_n\}_{n\in\omega})$ *(we write* $(\mathbf{P}^1, \mathcal{D}^1, \{E^1_n\}_{n\in\omega}) < (\mathbf{P}^2, \mathcal{D}^2, \{E^2_n\}_{n\in\omega})$*) whenever*

1. $\mathbf{P}^1 \lessdot \mathbf{P}^2$*,* $\mathcal{D}^1 \subseteq \mathcal{D}^2$ *and* $E^1_n = E^2_n | \mathcal{D}^1$ *for each* $n \in \omega$*,*
2. *if* $p \in \mathcal{D}^1$*,* $n \in \omega$ *then* $[p]^2_n \subseteq \mathcal{D}^1$*,*
3. *if* $p \leq q$*,* $p \in \mathcal{D}^2, q \in \mathcal{D}^1$ *then* $p \in \mathcal{D}^1$*.*

Lemma 2.7 a) *The relation* $<$ *is transitive on models of sweetness.*

b) *Suppose that* $(\mathbf{P}^i, \mathcal{D}^i, \{E^i_n\}_{n\in\omega})$ *are models of sweetness such that*

$$(\mathbf{P}^i, \mathcal{D}^i, \{E^i_n\}_{n\in\omega}) < (\mathbf{P}^j, \mathcal{D}^j, \{E^j_n\}_{n\in\omega})$$

for all $i < j < \xi$ $(\xi < \omega_1)$*. Then*

$$\lim_{i<\xi} (\mathbf{P}^i, \mathcal{D}^i, \{E^i_n\}_{n\in\omega}) = (\bigcup_{i<\xi} \mathbf{P}^i, \bigcup_{i<\xi} \mathcal{D}^i, \{\bigcup_{i<\xi} E^i_n\}_{n\in\omega})$$

is a model of sweetness extending all models $(\mathbf{P}^i, \mathcal{D}^i, \{E^i_n\}_{n\in\omega})$*.* $\blacksquare$

The sweetness may be preserved by the amalgamation.

Lemma 2.8 *Suppose that* $(\mathbf{P}^i, \mathcal{D}^i, \{E^i_n\}_{n\in\omega})$ *for* $i = 1, 2$ *are models of sweetness and* $f_i : \mathbf{P}^0 \longrightarrow BA(\mathbf{P}^i)$ *are complete embeddings. Then there exists a model of sweetness* $(\mathbf{P}^1 \times_{f_1,f_2} \mathbf{P}^2, \mathcal{D}^*, \{E^*_n\}_{n\in\omega})$ *based on the amalgamation* $\mathbf{P}^1 \times_{f_1,f_2} \mathbf{P}^2$ *and extending both* $(\mathbf{P}^1, \mathcal{D}^1, \{E^1_n\}_{n\in\omega})$ *and* $(\mathbf{P}^2, \mathcal{D}^2, \{E^2_n\}_{n\in\omega})$*.*

PROOF see lemmas 7.5, 7.12 of [She]. ■

To ensure that our algebra satisfies

$\mathcal{B} \Vdash$ "the union of ω_1 meager sets is meager"

we will use the Hechler order $\mathbf{D}$. Recall that $\mathbf{D}$ consists of all pairs (n, f) such that $n \in \omega$, $f \in \omega^{\omega}$. It is ordered by

$(n, f) \leq (n', f')$ if and only if
$n \leq n'$, $f|n = f'|n$ and $(\forall k \in \omega)(f(k) \leq f'(k))$.

The forcing with $\mathbf{D}$ adds both a dominating real and a Cohen real. Consequently

$\mathbf{D} * \dot{\mathbf{D}} \Vdash$ "the union of all Borel meager sets coded in the ground model is meager".

The iteration with $\mathbf{D}$ preserves sweetness.

Lemma 2.9 *Let $(\mathbf{P}, \mathcal{D}, \{E_n\}_{n\in\omega})$ be a model of sweetness and let $\dot{\mathbf{D}}$ be a $\mathbf{P}$-name for the Hechler forcing. Then there exists a model of sweetness $(\mathbf{P} * \dot{\mathbf{D}}, \mathcal{D}^*, \{E^*_n\}_{n\in\omega})$ based on $\mathbf{P} * \dot{\mathbf{D}}$ and extending the model $(\mathbf{P}, \mathcal{D}, \{E_n\}_{n\in\omega})$.*

PROOF Similar to the proof of lemmas 7.6, 7.11 of [She]. ■

3 The proof of the main result

In this section we present proofs of theorems 1.2 and 1.1.

Definition 3.1 *Let $\mathcal{K}$ be the class consisting of all sequences $\bar{\mathbf{P}} = <(P^i, M^i) : i < \omega_1>$ such that*

1. *M^i is a model of sweetness based on P^i,*
2. *if $i < j < \omega_1$ then $P^i \lessdot P^j$.*

If $\bar{\mathbf{P}} \in \mathcal{K}$ is as above then we put $P^{\omega_1} = \bigcup_{i<\omega_1} P^i$.

Note that if $\bar{\mathbf{P}} \in \mathcal{K}$ then each P^i is σ-centered. Consequently P^{ω_1} satisfies ccc.

We define the relation $\leq$ on $\mathcal{K}$.

Definition 3.2 *Let $\bar{\mathbf{P}}_1, \bar{\mathbf{P}}_2 \in \mathcal{K}$. We say $\bar{\mathbf{P}}_1 \leq \bar{\mathbf{P}}_2$ if $P_1^{\omega_1} \lessdot P_2^{\omega_1}$ and there exists a closed unbounded subset C of ω_1 such that*

(!) *if $i \in C$ then $M_1^i < M_2^i$*

(!!) *if $i \in C$, $q \in P_1^{\omega_1}$, $p \in P_1^i$ and $p \Vdash_{P_1^i} q \in (P_1^{\omega_1} : P_1^i)$ then $p \Vdash_{P_2^i} q \in (P_2^{\omega_1} : P_2^i)$.*

Clearly the relation $\leq$ is transitive and reflexive.

Lemma 3.3 *Suppose that $\bar{\mathbf{P}}_m \in \mathcal{K}$ for $m < \omega$ are such that $m_1 < m_2 < \omega$ implies $\bar{\mathbf{P}}_{m_1} \leq \bar{\mathbf{P}}_{m_2}$ (and let C_{m_1,m_2} witness it). Let $C = \bigcap_{m_1<m_2<\omega} C_{m_1,m_2}$. Put*

$$P_\omega^i = \bigcup_{m<\omega} P_m^{\cap(C\setminus i)}, \ M_\omega^i = \lim_{m<\omega} M_m^{\cap(C\setminus i)}.$$

Then $\bar{\mathbf{P}}_\omega = < (P_\omega^i, M_\omega^i) : i < \omega_1 > \in \mathcal{K}$ and $\bar{\mathbf{P}}_m \leq \bar{\mathbf{P}}_\omega$ for each $m < \omega$.

Proof First note that C is a closed unbounded subset of ω_1. Since $C \subseteq \bigcap_{m<\omega} C_{m,m+1}$ we may apply lemma 2.7 b) to conclude that each M_ω^i is a model of sweetness based on P_ω^i.

Claim: If $i < j < \omega_1$ then $P_\omega^i \lessdot P_\omega^j$.

Indeed, let $i < j$. We may assume that $i, j \in C$ (recall that $P_\omega^i = P_\omega^{\cap(C\setminus i)}$). Note that $P_m^i \lessdot P_\omega^i$ and $P_m^i \lessdot P_m^j$ for each $m \in \omega$. Let $\mathcal{A} \subseteq P_\omega^i$ be a maximal antichain. Clearly it is an antichain in P_ω^j but we have to prove that it is maximal. Let $q \in P_\omega^j$. Then $q \in P_m^j$ for some $m < \omega$. Let

$$Z = \{r \in P_m^i : (\exists p_r \in \mathcal{A})(r \Vdash_{P_m^i} p_r \in (P_\omega^i : P_m^i))\}$$

Clearly Z is dense in P_m^i. Hence we find $r \in Z$ such that $r \Vdash_{P_m^i} q \in (P_m^j : P_m^i)$. Let $p_r \in \mathcal{A}$ witness $r \in Z$. Take k such that $p_r \in P_k^i$, $m < k < \omega$. Consider $\bar{\mathbf{P}}_m$ and $\bar{\mathbf{P}}_k$. Since $i, j \in C \subseteq C_{m,k}$ we may apply condition **(!!)** to conclude that

$$r \Vdash_{P_k^i} q \in (P_k^j : P_k^i).$$

By the choice of p_r we have

$$r \Vdash_{P^i_m} p_r \in (P^i_k : P^i_m).$$

Thus p_r and r are compatible and any $p' \in P^i_k$, $p' \geq r, p_r$ is compatible with q. Consequently q and p_r are compatible. The claim is proved.

It follows from the above claim that $\bar{\mathbf{P}}_\omega \in \mathcal{K}$.

Claim: The club C witness that $\bar{\mathbf{P}}_m \leq \bar{\mathbf{P}}_\omega$ for each $m < \omega$.

Indeed, first note that

$$P^{\omega_1}_\omega = \bigcup_{i<\omega_1} P^i_\omega = \bigcup_{i<\omega_1} \bigcup_{m<\omega} P^{\cap(C\setminus i)}_m = \bigcup_{m<\omega} P^{\omega_1}_m.$$

Since $P^{\omega_1}_{m_1} \lessdot P^{\omega_1}_{m_2}$ for each $m_1 < m_2$ we see that $P^{\omega_1}_m \lessdot P^{\omega_1}_\omega$. It follows from the definition of M^i_ω and lemma 2.7 that if $i \in C$ then $M^i_m < M^i_\omega$. Thus we have to check condition **(!!)** only. Suppose $i \in C$, $q \in P^{\omega_1}_m$, $p \in P^i_m$ and $p \Vdash_{P^i_m} q \in (P^{\omega_1}_m : P^i_m)$. Assume $p \not\Vdash_{P^i_\omega} q \in (P^{\omega_1}_\omega : P^i_\omega)$. Then we find $r \in P^i_\omega$ such that $r \geq p$ and r is incompatible with q. Let $k > m$ be such that $r \in P^i_k$. Since $i \in C_{m.k}$ we have $p \Vdash_{P^i_k} q \in (P^{\omega_1}_k : P^i_k)$ (by condition **(!!)** for $\bar{\mathbf{P}}_m, \bar{\mathbf{P}}_k$). But $r \Vdash_{P^i_k} q \notin (P^{\omega_1}_k : P^i_k)$ - a contradiction. $\blacksquare$

Lemma 3.4 *Assume that*

- $\bar{\mathbf{P}}_\xi \in \mathcal{K}$ *for* $\xi < \omega_1$,
- *if* $\xi < \zeta < \omega_1$ *then* $\bar{\mathbf{P}}_\xi \leq \bar{\mathbf{P}}_\zeta$ *is witnessed by the club* $C_{\xi,\zeta} \subseteq \omega_1$,
- *if* $\delta < \omega_1$ *is a limit ordinal and* $i \in \bigcap_{\xi<\zeta<\delta} C_{\xi,\zeta}$ *then* $M^i_\delta = \lim_{\xi<\delta} M^i_\xi$.

Let

$$C = \{\delta < \omega_1 : \delta \text{ is limit } \& \ (\forall \xi < \zeta < \delta)(\delta \in C_{\xi,\zeta})\}$$

and let $C(i) = \cap(C\setminus i)$ *for* $i < \omega_1$. *Put* $P^i_{\omega_1} = P^{C(i)}_{C(i)}$, $M^i_{\omega_1} = M^{C(i)}_{C(i)}$.
Then $\bar{\mathbf{P}}_{\omega_1} \in \mathcal{K}$ *and* $(\forall \xi < \omega_1)(\bar{\mathbf{P}}_\xi \leq \bar{\mathbf{P}}_{\omega_1})$.

Proof First note that the set $\{\delta < \omega_1 : (\forall \xi < \zeta < \delta)(\delta \in C_{\xi,\zeta})\}$ is the diagonal intersection of clubs $\bigcap_{\xi<\zeta} C_{\xi,\zeta}$ (for $\zeta < \omega_1$). Hence C is closed and unbounded and $\bar{\mathbf{P}}_{\omega_1}$ is well defined.

Claim: If $i < j < \omega_1$ then $P^i_{\omega_1} \lessdot P^j_{\omega_1}$.

Indeed, suppose $i < j < \omega_1$. Then $P^i_{\omega_1} = P^{C(i)}_{C(i)}$, $P^j_{\omega_1} = P^{C(j)}_{C(j)}$ and we may assume that $C(i) < C(j)$. By 3.1 2) we have that $P^{C(i)}_{C(i)} \lessdot P^{C(j)}_{C(i)}$. Since C consists of limit ordinals only and $C(j) \in \bigcap_{\xi<\zeta<C(j)} C_{\xi,\zeta}$ we get $P^{C(j)}_{C(j)} = \bigcup_{\xi<C(j)} P^{C(j)}_{\xi}$ (and it is a direct limit). Since $C(i) < C(j)$ we conclude $P^{C(j)}_{C(i)} \lessdot P^{C(j)}_{C(j)}$ and consequently $P^{C(i)}_{C(i)} \lessdot P^{C(j)}_{C(j)}$. The claim is proved.

Since each $M^i_{\omega_1}$ is a model of sweetness based on $P^i_{\omega_1}$ we have proved that $\bar{\mathbf{P}}_{\omega_1} \in \mathcal{K}$. Let $\xi < \omega_1$.

Claim: $P^{\omega_1}_{\xi} \lessdot P^{\omega_1}_{\omega_1}$

First note that

$$P^{\omega_1}_{\omega_1} = \bigcup_{i<\omega_1} P^i_{\omega_1} = \bigcup_{i<\omega_1} P^{C(i)}_{C(i)} = \bigcup_{\zeta,i<\omega_1} P^i_{\zeta} = \bigcup_{\zeta<\omega_1} P^{\omega_1}_{\zeta}.$$

Since $\zeta_1 < \zeta_2 < \omega_1$ implies $\bar{\mathbf{P}}_{\zeta_1} \leq \bar{\mathbf{P}}_{\zeta_2}$ we have $P^{\omega_1}_{\zeta_1} \lessdot P^{\omega_1}_{\zeta_2}$ for $\zeta_1 < \zeta_2 < \omega_1$. Consequently $P^{\omega_1}_{\xi} \lessdot P^{\omega_1}_{\omega_1}$.

Claim: If $i \in C\setminus(\xi+1)$ then $M^i_{\xi} < M^i_{\omega_1}$.

If $i \in C\setminus(\xi+1)$ then $C(i) = i > \xi$. Moreover it follows from our assumptions that $M^i_i = \lim_{\zeta<i} M^i_{\zeta}$. By lemma 2.7 we get $M^i_{\xi} < M^i_i = M^{C(i)}_{C(i)} = M^i_{\omega_1}$.

Claim: Suppose $i \in C\setminus(\xi+1)$, $q \in P^{\omega_1}_{\xi}$, $p \in P^i_{\xi}$ and $p \Vdash_{P^i_{\xi}} q \in (P^{\omega_1}_{\xi} : P^i_{\xi})$. Then $p \Vdash_{P^i_{\omega_1}} q \in (P^{\omega_1}_{\omega_1} : P^i_{\omega_1})$.

Assume not. Then we have $r \in P^i_{\omega_1} = P^i_i$, $r \geq p$ such that r and q are incompatible. There is $\zeta \in (\xi, i)$ such that $r \in P^i_{\zeta}$. Thus $p \nVdash_{P^i_{\zeta}} q \in (P^{\omega_1}_{\zeta} : P^i_{\zeta})$. Since $i \in C_{\xi,\zeta}$ we get a contradiction with condition **(!!)** for $\bar{\mathbf{P}}_{\xi} \leq \bar{\mathbf{P}}_{\zeta}$.

We have proved that the club $C\setminus(\xi+1)$ witness $\bar{\mathbf{P}}_{\xi} \leq \bar{\mathbf{P}}_{\omega_1}$. ■

Suppose $\bar{\mathbf{P}} = \langle (P^i, M^i) : i < \omega_1 \rangle \in \mathcal{K}$. Let

$$P^i_D = \{(p,\tau) \in P^{\omega_1} * \dot{\mathbf{D}} : p \in P^i \ \&\ \tau \text{ is a } P^i\text{-name } \}.$$

Note that P^i_D is isomorphic to $P^i * \dot{\mathbf{D}}$. Let M^i_D be the canonical model of sweetness based on P^i_D and extending the model M^i (see lemma 2.9). Let

$$\bar{\mathbf{P}}_D = \langle (P^i_D, M^i_D) : i < \omega_1 \rangle .$$

Lemma 3.5 $\bar{\mathbf{P}}_D \in \mathcal{K}$, $\bar{\mathbf{P}} \leq \bar{\mathbf{P}}_D$ *and* $P^{\omega_1}_D = P^{\omega_1} * \dot{\mathbf{D}}$.

Proof The last assertion is a consequence of the fact that P^{ω_1} is a ccc notion of forcing. It follows from properties of Souslin forcing (cf [JS2]) that $P^i_D \lessdot P^j_D$ provided $i < j$. Consequently $\bar{\mathbf{P}}_D \in \mathcal{K}$. To show $\bar{\mathbf{P}} \leq \bar{\mathbf{P}}_D$ note that $M^i < M^i_D$ for all $i < \omega_1$ and $P^{\omega_1} \lessdot P^{\omega_1}_D$. Suppose now that $i < \omega_1$, $p \in P^i$, $q \in P^{\omega_1}$ and $p \Vdash_{P^i} q \in (P^{\omega_1} : P^i)$. Assume that $p \not\Vdash_{P^i_D} q \in (P^{\omega_1}_D : P^i_D)$. Then we find a condition $r = (r_0, \tau) \in P^i_D$ above p which is inconsistent with q. Note that q may be a member of $P^i * \dot{\mathbf{D}} \lessdot P^j$ (for some $j > i$) but we consider it as an element of P^{ω_1}, while r is an element of $P^{\omega_1} * \dot{\mathbf{D}}$. Consequently incompatibility of q and r means that q and r_0 are not compatible. But $r_0 \in P^i$ lies above p - a contradiction. ■

Lemma 3.6 *Suppose that $\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{C}_0$ are complete Boolean algebras such that*

(1) $\mathcal{B} \lessdot \mathcal{D} \lessdot \mathcal{C}$, $\mathcal{C}_0 \lessdot \mathcal{C}$

Let $\mathcal{B}_0 = \mathcal{B} \cap \mathcal{C}_0$, $\mathcal{D}_0 = \mathcal{D} \cap \mathcal{C}_0$ (note that $\mathcal{B}_0 \lessdot \mathcal{D}_0 \lessdot \mathcal{C}_0$). We assume that

(2) $\mathcal{B} \Vdash$ *"$(\mathcal{D} : \mathcal{B})$ is a subset of $(\mathcal{C}_0 : \mathcal{B})$"*

(3) *if $b \in \mathcal{B}, b_0 \in \mathcal{B}_0$ and $b_0 \Vdash_{\mathcal{B}_0} b \in (\mathcal{B} : \mathcal{B}_0)$ then $b_0 \Vdash_{\mathcal{C}_0} b \in (\mathcal{C} : \mathcal{C}_0)$.*

Then

(3*) *if $d \in \mathcal{D}, d_0 \in \mathcal{D}_0$ and $d_0 \Vdash_{\mathcal{D}_0} d \in (\mathcal{D} : \mathcal{D}_0)$ then $d_0 \Vdash_{\mathcal{C}_0} d \in (\mathcal{C} : \mathcal{C}_0)$.*

Proof

Claim: Suppose $c \in \mathcal{C}_0, d_0 \in \mathcal{D}_0$ and $d_0 \Vdash_{\mathcal{D}_0} c \in (\mathcal{C}_0 : \mathcal{D}_0)$. Then $d_0 \Vdash_{\mathcal{D}} c \in (\mathcal{C} : \mathcal{D})$.

We have to prove that each $d \geq d_0, d \in \mathcal{D}$ is compatible with c. Let $d \geq d_0, d \in \mathcal{D}$. By (2) we find $b \in \mathcal{B}$ and $d_1 \in \mathcal{D}_0$ such that

$$b \Vdash_{\mathcal{B}} \text{"}d \in (\mathcal{D} : \mathcal{B}) \ \&\ d \equiv_{(\mathcal{D}:\mathcal{B})} d_1\text{"}$$

(the last means that $b \cdot d = b \cdot d_1$). Thus $b \cdot d_1 \cdot d_0 = b \cdot d \cdot d_0 = b \cdot d_0 \neq \mathbf{0}$. We find $b_0 \in \mathcal{B}_0$ such that $b_0 \Vdash_{\mathcal{B}_0} b \in (\mathcal{B} : \mathcal{B}_0)$ and $b_0 \cdot d_1 \cdot d_0 \neq \mathbf{0}$ (it is enough to take b_0 such that $b_0 \Vdash_{\mathcal{B}_0} b \cdot d_1 \cdot d_0 \in (\mathcal{D} : \mathcal{B}_0)$). Note that then $b_0 \Vdash_{\mathcal{C}_0} b \in (\mathcal{C} : \mathcal{C}_0)$ (by (3)). Since $b_0 \cdot d_1 \cdot d_0 \in \mathcal{D}_0$ and it is stronger than d_0 we get $b_0 \cdot d_1 \cdot d_0 \cdot c \neq \mathbf{0}$. The last condition is stronger than b_0 and belongs to $\mathcal{C}_0$. Hence $b \cdot b_0 \cdot d_1 \cdot d_0 \cdot c \neq \mathbf{0}$.

Finally note that $b \cdot b_0 \cdot d_1 \cdot d_0 \cdot c \geq b \cdot d_1 = b \cdot d \geq d$ so d and c are compatible. The claim is proved.
Now suppose that $d \in \mathcal{D}, d_0 \in \mathcal{D}_0$ and $d_0 \Vdash_{\mathcal{D}_0} d \in (\mathcal{D} : \mathcal{D}_0)$. Let $c \in \mathcal{C}_0, c \geq d_0$. Take $d^* \in \mathcal{D}_0$ such that $d^* \geq d_0$ and $d^* \Vdash_{\mathcal{D}_0} c \in (\mathcal{C}_0 : \mathcal{D}_0)$. By the claim we have $d^* \Vdash_{\mathcal{D}} c \in (\mathcal{C} : \mathcal{D})$. Since $d^* \geq d_0$ we have $d^* \cdot d \neq \mathbf{0}, d^* \cdot d \in \mathcal{D}$ and consequently $d^* \cdot d \cdot c \neq \mathbf{0}$. Hence d and c are compatible and we are done. ■

Suppose that $\bar{\mathbf{P}}_0, \bar{\mathbf{P}}_1, \bar{\mathbf{P}}_2, \bar{\mathbf{P}}_3 \in \mathcal{K}$ and the club $C \subseteq \omega_1$ witness that both $\bar{\mathbf{P}}_0 \leq \bar{\mathbf{P}}_1$ and $\bar{\mathbf{P}}_2 \leq \bar{\mathbf{P}}_3$. Assume that $\mathbf{Q}_0, \mathbf{Q}_2$ are complete Boolean algebras such that for some $i_0 < \omega_1$

- $\mathrm{BA}(P_0^{\omega_1}) \lessdot \mathbf{Q}_0 \lessdot \mathrm{BA}(P_1^{\omega_1})$, $\mathrm{BA}(P_2^{\omega_1}) \lessdot \mathbf{Q}_2 \lessdot \mathrm{BA}(P_3^{\omega_1})$
- $\mathrm{BA}(P_0^{\omega_1}) \Vdash (\mathbf{Q}_0 : \mathrm{BA}(P_0^{\omega_1})) \subseteq (\mathrm{BA}(P_1^{i_0}) : \mathrm{BA}(P_0^{\omega_1}))$
  $\mathrm{BA}(P_2^{\omega_1}) \Vdash (\mathbf{Q}_2 : \mathrm{BA}(P_2^{\omega_1})) \subseteq (\mathrm{BA}(P_3^{i_0}) : \mathrm{BA}(P_2^{\omega_1}))$

Let $f : \mathbf{Q}_0 \longrightarrow \mathbf{Q}_2$ be an isomorphism such that $f[\mathbf{Q}_0 \cap \mathrm{BA}(P_1^i)] = \mathbf{Q}_2 \cap \mathrm{BA}(P_3^i)$ for all $i \in C \setminus i_0$. For $i \in C \setminus i_0$ put

$$P^i = \{(p_1, p_2) \in P_1^{\omega_1} \times_{\mathrm{id}, f} P_3^{\omega_1} : p_1 \in P_1^i \ \& \ p_2 \in P_3^i\},$$

where id stands for the identity on $\mathbf{Q}_0$. It follows from lemma 3.6 that P^i is isomorphic to $P_1^i \times_{f_1, f_3} P_3^i$, where $f_3 = f | \mathbf{Q}_0 \cap \mathrm{BA}(P_1^i)$ and f_1 is the identity on $\mathbf{Q}_0 \cap \mathrm{BA}(P_1^i)$. Therefore we have the canonical model of sweetness M^i based on P^i and extending both models M_1^i and M_2^i (compare lemma 2.8). Let

$$\bar{\mathbf{P}}_1 \times_f \bar{\mathbf{P}}_3 = < (P^i, M^i) : i < \omega_1 > .$$

Note that $\bigcup_{i < \omega_1} P^i = P_1^{\omega_1} \times_{\mathrm{id}, f} P_3^{\omega_1}$.

Lemma 3.7 $\bar{\mathbf{P}}_1 \times_f \bar{\mathbf{P}}_3 \in \mathcal{K}$ *and* $\bar{\mathbf{P}}_1, \bar{\mathbf{P}}_3 \leq \bar{\mathbf{P}}_1 \times_f \bar{\mathbf{P}}_3$.

PROOF To prove $\bar{\mathbf{P}}_1 \times_f \bar{\mathbf{P}}_3 \in \mathcal{K}$ we have to show the following
CLAIM: $P^i \lessdot P^j$ for each $i < j < \omega_1$, $i, j \in C \setminus i_0$.
Let $\mathcal{A} \subseteq P^i$ be a maximal antichain and let $(p_1, p_2) \in P^j$. Let $q \in \mathbf{Q}_0$ be such that

$$q \Vdash \text{“} p_1 \in (P_1^{\omega_1} : \mathbf{Q}_0) \ \& \ p_2 \in (P_3^{\omega_1} : f[\mathbf{Q}_0]) \text{”}.$$

Take $r_1 \in P_1^i$ such that $r_1 \Vdash_{P_1^i}$ “$p_1, q \in (P_1^{\omega_1} : P_1^i)$” (note that q and p_1 are compatible). Next find $q' \in \mathbf{Q}_0$ such that $q' \geq q$ and $q' \Vdash r_1 \in (P_1^{\omega_1} : \mathbf{Q}_0)$

(recall that r_1 and q are compatible). Since p_2 and $f(q')$ are compatible we find $r_2 \in P_3^i$ such that $r_2 \Vdash_{P_3^i}$ "$p_2, f(q') \in (P_3^{\omega_1} : P_3^i)$". Consider the pair (r_1, r_2). There is $q'' \in \mathbf{Q}_0, q'' \geq q'$ such that $q'' \Vdash r_2 \in (P_3^{\omega_1} : f[\mathbf{Q}_0])$. Then

$$q'' \Vdash \text{“} r_1 \in (P_1^{\omega_1} : \mathbf{Q}_0) \ \& \ r_2 \in (P_3^{\omega_1} : f[\mathbf{Q}_0]) \text{”}$$

and consequently $(r_1, r_2) \in P^i$. Since (r_1, r_2) has to be compatible with some element of $\mathcal{A}$ we are done.

Claim: Suppose $q \in P_1^{\omega_1}$, $i \in C \setminus i_0$, $p \in P_1^i$ are such that $p \Vdash_{P_1^i} q \in (P_1^{\omega_1} : P_1^i)$. Then $p \Vdash_{P^i} q \in (P^{\omega_1} : P^i)$.

Suppose $r \in P^i$ is stronger than p. Let $r = (r_1, r_2)$ and let $r_0 \in \mathbf{Q}_0$ witness $r \in P_1^{\omega_1} \times_{\mathrm{id}, f} P_3^{\omega_1}$. We may get $r_0 \in \mathbf{Q}_0 \cap \mathrm{BA}(P_1^i)$. Remember that really we have $p \simeq (p, \emptyset)$, $q \simeq (q, \emptyset)$. Since $r_0, r_1 \in \mathrm{BA}(P_1^i)$ are compatible and $r_1 \geq p$ we find $r_1^* \in P_1^{\omega_1}$ above r_0, r_1 and q. Then $(r_1^*, r_2) \in P^{\omega_1}$ and it is a condition stronger than both (r_1, r_2) and $(q, \emptyset)$. The claim is proved.

Since $M_1^i < M^i$ for each $i \in C \setminus i_0$ it follows from the above claim that $\bar{\mathbf{P}}_1 \leq \bar{\mathbf{P}}_1 \times_f \bar{\mathbf{P}}_3$ (and $C \setminus i_0$ is a witness for it). Similarly one can prove $\bar{\mathbf{P}}_3 \leq \bar{\mathbf{P}}_1 \times_f \bar{\mathbf{P}}_3$. ■

Lemma 3.8 *Suppose $\bar{\mathbf{P}}_0, \bar{\mathbf{P}}_1 \in \mathcal{K}$, $\bar{\mathbf{P}}_0 \leq \bar{\mathbf{P}}_1$. Let $\mathbf{Q}_0, \mathbf{Q}_1$ be complete Boolean algebras such that (for $k = 0, 1$):*

- *$BA(P_0^{\omega_1}) \lessdot \mathbf{Q}_k \lessdot BA(P_1^{\omega_1})$*
- *$BA(P_0^{\omega_1}) \Vdash$ "$(\mathbf{Q}_k : BA(P_0^{\omega_1}))$ is the Cohen algebra"*

Let $f : \mathbf{Q}_0 \longrightarrow \mathbf{Q}_1$ be an isomorphism such that $f | BA(P_0^{\omega_1}) = id$.
Then there exist $\bar{p} \in \mathcal{K}$ and an automorphism $\phi : P^{\omega_1} \overset{onto}{\longrightarrow} P^{\omega_1}$ such that $\bar{\mathbf{P}}_1 \leq \bar{\mathbf{P}}$ and $f \subseteq \phi$.

Proof We may apply lemma 3.7 to get that $\bar{\mathbf{P}}_2 = \bar{\mathbf{P}}_1 \times_f \bar{\mathbf{P}}_1 \in \mathcal{K}$. The amalgamation over f produces an extension of f — there is $f_1 : P_1^{\omega_1} \longrightarrow P_2^{\omega_1}$ such that $f \subseteq f_1$ (we identify $p \in P_1^{\omega_1}$ with $(\emptyset, p) \in P_2^{\omega_1}$). Moreover $\bar{\mathbf{P}}_1, \bar{\mathbf{P}}_2, f_1$ satisfy assumptions of lemma 3.7 and thus $\bar{\mathbf{P}}_3 = \bar{\mathbf{P}}_2 \times_{f_1} \bar{\mathbf{P}}_2 \in \mathcal{K}$. If we identify $p \in P_2^{\omega_1}$ with $(p, \emptyset) \in \bar{\mathbf{P}}_3$ we get a partial isomorphism f_2 such that $f_1 \subseteq f_2$ and $\mathrm{rng}(f_2) = P_2^{\omega_1}$. And so on, we build $\bar{\mathbf{P}}_m \in \mathcal{K}$ and partial isomorphisms f_m such that $\bar{\mathbf{P}}_m \leq \bar{\mathbf{P}}_{m+1}$, $f_m \subseteq f_{m+1}$ and either $P_m^{\omega_1} \subseteq$

$\mathrm{dom}(f_m)$ or $P_m^{\omega_1} \subseteq \mathrm{rng}(f_m)$. Next we apply lemma 3.3 to conclude that $\bar{\mathbf{P}}_\omega \in \mathcal{K}$ and $f_\omega = \bigcup_{m\in\omega} f_m : P_\omega^{\omega_1} \overset{\text{onto}}{\longrightarrow} P_\omega^{\omega_1}$ is the desired automorphism. ■

Definition 3.9 *We define the following notion of forcing*

- $\mathbf{R} = \{\bar{\mathbf{P}} \in \mathcal{K} : \bar{p} \in \mathcal{H}(\omega_2)\}$
- $\leq_{\mathbf{R}}$ *is the relation* $\leq$ *of 3.2.*

A notion of forcing $\mathbf{P}$ is (ω_1+1)-strategically closed if the second player has a winning strategy in the following game of the length ω_1+1.

For $i=0$ Player I gives $p_0 \in \mathbf{P}$;
Player I gives in the i-th move a dense subset D_i of $\mathbf{P}$;
Player II gives $p_{i+1} \geq p_i$, $p_{i+1} \in D_i$, for a limit i Player II gives
p_i above all p_j (for $j<i$).

Player II looses if he is not able to give the respective element of $\mathbf{P}$ for some $i \leq \omega_1$.

Note that (ω_1+1)-strategically closed notions of forcings do not add new ω_1-sequences of elements of the ground model.

Proposition 3.10 *The forcing notion* $\mathbf{R}$ *is* ω_1*-closed and* (ω_1+1)*-strategically closed. Consequently forcing with* $\mathbf{R}$ *does not collapse* ω_1 *and* ω_2*.*

Proof For the first assertion use lemma 3.3. The second follows from 3.3 and 3.4. ■

Note that $|\mathbf{R}| = 2^{\omega_1}$. Thus if we assume that $2^{\omega_1} = \omega_2$ then forcing with $\mathbf{R}$ does not collapse cardinals.

Suppose $\mathbf{V} \models$GCH.

Let $G \subseteq \mathbf{R}$ be a generic over $\mathbf{V}$. Let $\mathbf{P} = \bigcup\{P^{\omega_1} : \bar{\mathbf{P}} \in G\}$.

Proposition 3.11 *1.* $\mathbf{P}$ *is a ccc notion of forcing.*

2. *If* τ *is a* $\mathbf{P}$*-name for an* ω_1*-sequence of ordinals then* $\mathbf{P} \Vdash$ *"the union of all Borel meager sets coded in* $\mathbf{V}[\tau]$ *is meager".*

3. The Boolean algebra BA($\mathbf{P}$) is strongly Cohen-homogeneous.

4. $\mathbf{P} \Vdash$ "any union of ω_1 meager sets is meager".

PROOF 1. Work in $\mathbf{V}$. Suppose that $\dot{\mathcal{A}}$ is a $\mathbf{R}$-name for an ω_1-sequence of pairwise incompatible elements of $\mathbf{P}$. Let $\bar{\mathbf{P}} \in \mathbf{R}$. By proposition 3.10 there is $\bar{\mathbf{P}}_1 \geq \bar{\mathbf{P}}$ which decides all values of $\dot{\mathcal{A}}$. We may assume that all these elements belong to $P_1^{\omega_1}$. A contradiction.

2. Let τ be a $\mathbf{P}$-name for an ω_1-sequence of ordinals. Then τ is actually an ω_1-sequence of (countable) antichains in $\mathbf{P}$. Therefore $\tau \in \mathbf{V}$ and it is a $P_0^{\omega_1}$-name for some $\bar{\mathbf{P}}_0 \in G$. By density arguments we have that $(\bar{\mathbf{P}}_D)_D \in G$ for some $\bar{\mathbf{P}} \geq \bar{\mathbf{P}}_0$ (compare lemma 2.9). Hence

> $\mathbf{P} \Vdash$"the union of all Borel meager sets coded in $\mathbf{V}[G][\tau]$ is meager"

3. Work in $\mathbf{V}[G]$. Let τ be a $\mathbf{P}$-name for an ω_1-sequence of ordinals. As in 2. we find $\bar{\mathbf{P}}_0 \in G$ such that τ is a $P_0^{\omega_1}$-name. Suppose now that

- $\mathrm{BA}(P_0^{\omega_1}) \lessdot \mathcal{B} \lessdot \mathrm{BA}(\mathbf{P})$,
- $\mathrm{BA}(P_0^{\omega_1}) \Vdash$ "$(\mathcal{B}{:}\mathrm{BA}(P_0^{\omega_1}))$ is the Cohen algebra",
- $f : \mathcal{B} \longrightarrow \mathrm{BA}(\mathbf{P})$ is a complete embedding such that $f|\mathrm{BA}(P_0^{\omega_1}) = \mathrm{id}$.

Note that $\mathcal{B}$ and f are determined by countably many elements. Each element of BA($\mathbf{P}$) is a countable union of elements of $\mathbf{P}$. Consequently $\mathcal{B}, f \in \mathbf{V}$ and there is $\bar{\mathbf{P}}_1 \in G$ such that $\mathcal{B}, \mathrm{rng}(f) \subseteq \mathrm{BA}(P_1^{\omega_1})$, $\bar{\mathbf{P}}_0 \leq \bar{\mathbf{P}}_1$. By density argument and lemma 3.8 we find $\bar{\mathbf{P}}_2 \in G$ and f_2 such that $\bar{\mathbf{P}}_1 \leq \bar{\mathbf{P}}_2$ and f_2 is an automorphism of $\mathrm{BA}(P_2^{\omega_1})$ extending f. Similarly, if $\bar{\mathbf{P}}_4 \in G$, $\bar{\mathbf{P}}_3 \leq \bar{\mathbf{P}}_4$ and f_3 is an automorphism of $\mathrm{BA}(P_3^{\omega_1})$ then there are $\bar{\mathbf{P}}_5 \in G$, f_5 such that f_5 is an automorphism of $\mathrm{BA}(P_5^{\omega_1})$ extending f_3.

It follows from the above that, in $V[G]$, we can extend f to an automorphism of BA($\mathbf{P}$).

4. Similar arguments as in 1. and 2. ■

Theorems 1.2 and 1.1 follow directly from the above proposition and theorems 2.3 and 2.4.

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