Many simple cardinal invariants

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Abstract: For $g < f$ in ω^ω we define $\mathbf{c}(f,g)$ be the least number of uniform trees with g-splitting needed to cover a uniform tree with f-splitting. We show that we can simultaneously force $\aleph_1$ many different values for different functions (f,g). In the language of [*Blass*]: There may be $\aleph_1$ many distinct uniform $\mathbf{\Pi^0_1}$ characteristics.

0. Introduction

[*Blass*] defined a classification of certain cardinal invariants of the continuum, based on the Borel hierarchy. For example, to every Π^0_1 formula $\varphi(x,y) = \forall n R(x{\restriction}n, y{\restriction}n)$ (R recursive) the cardinal

$$\kappa_\varphi := \min\{\mathcal{B} \subseteq {}^\omega\omega : \forall x \in {}^\omega\omega \exists y \in \mathcal{B} : \varphi(x,y)\}$$

is the "uniform Π^0_1 characteristic" associated to φ.

Blass proved structure theorems on simple cardinal invariants, e.g., that there is a smallest $\mathbf{\Pi^0_1}$ characteristic (namely, $\mathbf{Cov}(\mathcal{M})$, the smallest number of first category sets needed to cover the reals), and also that the Π^0_2-characteristics can behave quite chaotically. He asked whether the known uniform Π^0_1 characteristics ($\mathbf{c}$, $\mathbf{d}$, $\mathbf{r}$, $\mathbf{Cov}(\mathcal{M})$) are the only ones or (since that is very unlikely) whether there could be a reasonable classification of the uniform Π^0_1 characteristics — say, a small list that contains all these invariants.

In this paper we give a strong negative answer to this question: For two $\mathbf{\Pi^0_1}$ formulas φ_1, φ_2 we say that φ_1 and φ_2 define "potentially nonequal characteristics" if $\kappa_{\varphi_1} \neq \kappa_{\varphi_2}$ is consistent. We say that φ_1 and φ_2 define "actually different characteristics", if $\kappa_{\varphi_1} \neq \varphi_2$.

We will find a family of $\mathbf{\Pi^0_1}$-formulas indexed by a real parameter (f,g), and we will show not only that there is a perfect set of parameters which defines pairwise potentially nonequal $\mathbf{\Pi^0_1}$-characteristics, but we produce a single universe in which (at least) $\aleph_1$ many cardinals appear as $\mathbf{\Pi^0_1}$-characteristics. (In fact it

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is also possible to produce a universe where there is a perfect set of parameters defining pairwise actually different $\boldsymbol{\Pi}^0_1$-characteristics. See [*Shelah 448a*]).

If we want more than countably many cardinals, we obviously have to use the boldface pointclass. But the proof also produces many lightface uniform Π^0_1 characteristics.

For more information on cardinal invariants, see [*Blass*], [*van Douwen*], [*Vaughan*].

From another point of view, this paper is part of the program of finding consistency techniques for a large continuum, i.e., we want $2^{\aleph_0} > \aleph_2$ and have many values for cardinal invariants. We use a countable support product of forcing notions with an axiom A structure.

We will use invariants that were implicitly introduced in [*Shelah 326*, §2], where it was proved that $\mathbf{c}(f, g)$ and $\mathbf{c}(f', g')$ (see below) may be distinct.

0.1 Definition: If $f \in {}^\omega\omega$, we say that $\bar{B} = \langle B_k : k \in \omega \rangle$ is an f-slalom if for all k, $|B_k| = f(k)$. We write $h \in \bar{B}$ for $h \in \prod_n B_n$, i.e., $\forall n\, h(n) \in B_n$. (See figure 1) This is a Π^0_1-formula in the variables h and $\bar{B}$. Some authors call the set $\{h : h \in \bar{B}\}$ a "belt", or "uniform tree".

For example, $\prod\limits_n f(n)$ is an f-slalom, because we identify the number $f(n)$ with the set of predecessors, $\{0, \dots, f(n) - 1\}$.

Figure 1: A slalom

0.2 Definition: Assume $f, g \in {}^\omega\omega$. Assume that $\mathcal{B}$ is a family of g-slaloms, and $\bar{A} = \langle A_k : k \in \omega \rangle$ is an f-slalom.

We say that $\mathcal{B}$ **covers** $\bar{A}$ iff:

$$(\star) \qquad \text{for all } s \in \bar{A} \text{ there is } \bar{B} \in \mathcal{B} \text{ such that} \quad s \in \bar{B}$$

0.3 Definition: Assume $f, g \in {}^\omega\omega$. Then we define the cardinal invariant $\mathbf{c}(f,g)$ to be the minimal number of g-slaloms needed to cover an f-slalom.
(Clearly this makes sense only if $\forall k\, f(k), g(k) > 0$, so we will assume that from now on.)
This is a uniform Π^0_1-characteristic. (Strictly speaking, we are not working in ${}^\omega\omega$, but rather in ${}^\omega\big([\omega]^{<\omega}\big)$, but a trivial coding translates $\mathbf{c}(f,g)$ into a "uniform Π^0_1 characteristic" as defined above.)
Some relations between these cardinal invariants are provable in ZFC: For example, if $g < g' < f' < f$, then $\mathbf{c}(f',g') \le \mathbf{c}(f,g)$. Also, $\mathbf{c}(f^2,g^2) \le \mathbf{c}(f,g)$.
We will show that if (f,g) is sufficiently different from (f',g'), then the values of $\mathbf{c}(f,g)$ and $\mathbf{c}(f',g')$ are quite independent, and moreover: if $\langle (f_i,g_i) : i < \omega_1 \rangle$ are pairwise sufficiently different, then almost any assignment of the form $\mathbf{c}(f_i,g_i) = \kappa_i$ will be consistent.

Similar results are possible for the "dual" version of $\mathbf{c}(f,g)$: $\mathbf{c}^d(f,g) :=$ the smallest family of g-slaloms $\bar B$ such that for every h bounded by f there are infinitely many k with $h(k) \in B_k$, and for the "tree" version (a g-tree is a tree where every node in level k has $g(k)$ many successors). See [*Shelah 448a*].

We thank Tomek Bartoszynski for pointing out the following known results about the cardinal characteristics $\mathbf{c}(f,g)$:
For example, lemma 1.11 follows from Theorem 3.17 in [*Comfort-Negrepontis*]: Taking $\kappa = \alpha = \omega$, $\beta = n$, and letting $\mathcal{S} \subseteq n^\omega$ be a family of ω-large oscillation, then no family of $n{-}1$-slaloms of size $< 2^{\aleph_0}$ can cover $\mathcal{S}$. Indeed, whenever F is a function on $\mathcal{S}$ such that for each $s \in \mathcal{S}$, $F(s)$ is a $n{-}1$-slalom covering s, then F has to be finite-to-one and in fact at most $n{-}1$-to-one.
Also, since $\mathbf{c}(f, f{-}1)$ is the size of the smallest family of functions below f which does not admit an "infinitely equal" function, i.e.,

$$\mathbf{c}(f,f{-}1) = \min\{|G| : G \subseteq \prod_n f(n)\ \&\ \forall h \in \prod_n f(n)\, \exists g \in G\, \forall^\infty n\ f(n) \ne g(n)\}$$

by [*Miller*] we have that the minimal value of $\mathbf{c}(f,f{-}1)$ is the smallest size of a set of reals which does not have strong measure zero.

Also, note that if r is a random real over V in $\prod_n f(n)$, and if $\sum_{n=1}^\infty 1/f(n) = \infty$, then $\prod_n (1 - 1/f(n)) = 0$, so r cannot be covered by any $f{-}1$-slalom from V.
Conversely, if $\sum_{n=1}^\infty 1/f(n) < \infty$, then for any function $h \in \prod_n f(n) \cap V$ there is a condition forcing that h is covered by the $f{-}1$-slalom $(\{0,\ldots,f(k)-1\} - \{r(k)\} : k \in \omega)$.
Thus, if we add κ many random reals with the measure algebra, a easy density argument shows that in the resulting model we have

$$\mathbf{c}(f,f{-}1) = \begin{cases} \kappa = 2^{\aleph_0} & \text{if } \sum_{n=1}^\infty 1/f(n) = \infty \\ \aleph_1 & \text{otherwise (use any } \aleph_1 \text{ many of the random reals)} \end{cases}$$

That already shows that we can have at least two distinct values of $\mathbf{c}(f,g)$ and $\mathbf{c}(f',g')$.

Contents of the paper: In section 1 we prove results in ZFC of the form

"If (f, g) is in relation ... to (f', g'), then $\mathbf{c}(f, g) \leq \mathbf{c}(f', g')$"

In section 2 we define a forcing notion $Q_{f,g}$ that increases $\mathbf{c}(f, g)$. (I.e., in $V^{Q_{f,g}}$, the g-slaloms from V do not cover $\prod_n f(n)$.) Informally speaking, elements of $Q_{f,g}$ are perfect trees in which the size of the splitting is bounded by f, sometimes $= 1$, but often (i.e., on every branch), much bigger than g.

In section 3 we show that, assuming $\{(f_\xi, g_\xi) : \xi < \omega_1\}$ are sufficiently "independent", a countable support product$\prod_{\xi<\omega_1} Q_\xi^{\kappa_\xi}$ of such forcing notions will force $\forall \xi\, \mathbf{c}(f_\xi, g_\xi) = \kappa_\xi$.

We use the symbol ☺ to denote the end of a proof, and we write 😐 when we leave a proof to the reader.

1. Results in ZFC

1.1 Notation: Operations and relations on functions are understood to be pointwise, e.g., f/g, g^ε, $g < f$, etc. $\lfloor x \rfloor$ is the greatest integer $\leq x$. $\lim f$ is $\lim_{k\to\infty} f(k)$.

We write $f \leq^* g$ for $\exists n \,\forall k \geq n\; f(k) \leq g(k)$.

First we state some obvious facts:

1.2 Fact:

(1) $f \leq g$ iff $\mathbf{c}(f, g) = 1$.

(2) $f \leq^* g$ iff $\mathbf{c}(f, g)$ finite.

(3) If $A := \{k : g(k) < f(k)\}$ is infinite then $\mathbf{c}(f{\restriction}A, g{\restriction}A) = \mathbf{c}(f, g)$.

(4) If π is a permutation of ω, then $\mathbf{c}(f \circ \pi, g \circ \pi) = \mathbf{c}(f, g)$. 😐 1.2

(Strictly speaking, we define $\mathbf{c}(f, g)$ only for functions f, g defined on all of ω, so (3) should be formally rephrased as $\mathbf{c}(f \circ h, g \circ h) = \mathbf{c}(f, g)$, where h is a 1-1 enumeration of A)

1.3 Convention: We will concentrate on the case where $\mathbf{c}(f, g)$ is infinite, so we will wlog assume that $g < f$. By (4), we may also wlog assume that g is nondecreasing.

In these cases we will have that $\mathbf{c}(f, g)$ is infinite, and moreover an easy diagonal argument shows the following fact:

1.4 Fact:

$\mathbf{c}(f, g)$ is uncountable. 😐 1.4

Furthermore, we have the following properties:

1.5 Fact:

(1) (Mononónicity) If $f \leq^* f'$, $g \geq^* g'$, then $\mathbf{c}(f, g) \leq \mathbf{c}(f', g')$.

(2) (Multiplicativity) $\mathbf{c}(f \cdot f', g \cdot g') \leq \mathbf{c}(f, g) \cdot \mathbf{c}(f', g')$.

(3) (Transitivity) $\mathbf{c}(f,h) \le \mathbf{c}(f,g) \cdot \mathbf{c}(g,h)$.

(4) (Invariance) $\mathbf{c}(f,g) = \mathbf{c}(f^-,g^-)$ (where f^- is the function defined by $f^-(n) = f(n+1)$.

(5) (Monotonicity II) If $A \subseteq \omega$ is infinite, then $\mathbf{c}(f{\restriction}A, g{\restriction}A) \le \mathbf{c}(f,g)$. ☺ 1.5

1.6 Remark: (2) implies in particular $\mathbf{c}(f^n,g^n) \le \mathbf{c}(f,g)$. See 3.4 for an example of $\mathbf{c}(f^2,g^2) < \mathbf{c}(f,g)$.

The following inequalities need a little more work.

1.7 Lemma:

(1) $\mathbf{c}(f \cdot \lfloor f/g \rfloor, f) = \mathbf{c}(f,g)$.

(2) $\mathbf{c}(f \cdot \lfloor f/g \rfloor, g) = \mathbf{c}(f,g)$.

(3) $\mathbf{c}(f \cdot \lfloor f/g \rfloor^m, g) = \mathbf{c}(f,g)$ for all $m \in \omega$.

Proof: (2) follows from (1) using transitivity, and (3) follows from (2) by induction, so we only have to prove (1).

Proof of (1): By monotonicity we only have to show $\le$. So let $(N,\in)$ be a reasonably closed model of a large fragment of ZFC (say, $(N,\in) < (H(\chi^+),\in)$, where $\chi = 2^{\mathbf{c}}$) of size $\mathbf{c}(f,g)$ such $\prod_n f(n)$ is covered by the set of all g-slaloms from N.

Define h by $h(k) := f(k) \cdot \lfloor f(k)/g(k) \rfloor$. We can find a family $\langle B_k^i : i < f(k), k \in \omega \rangle$ in N such that for all k, $\{0,\ldots,h(k)-1\} = \bigcup_{i<f(k)} B_k^i$, where $|B_k^i| \le f(k)/g(k)$. We have to show that the set of f-slaloms from N covers $\prod_k h(k)$.

So let x be a function satisfying $\forall k\, x(k) \in \bigcup_{i<f(k)} B_k^i$. We can define a function $y \in \prod_n f(n)$ such that for all k, $x(k) \in B_k^{y(k)}$. So there is some g-slalom $\bar C \in N$ such that for all k, $y(k) \in C_k$.

Define $\bar A = \langle A_k : k \in \omega \rangle$ by $A_k := \bigcup_{i \in C_k} B_k^i$. Then $|A_k| \le |C_k| \cdot |B_k^i| \le g(k) \cdot f(k)/g(k) = f(k)$, so $\bar A$ is an f-slalom in N, and for all k, $x(k) \in A_k$. ☺ 1.7

1.8 Lemma: Assume $f > g > 0$. Assume that $\langle w_i : i \in \omega \rangle$ is a partition of ω into finite sets, and for each i there are $\bar H^i = \langle H_l^i : l \in w_i \rangle$ satisfying (a)–(c). Then $\mathbf{c}(f',g') \le \mathbf{c}(f,g)$.

(a) $\operatorname{dom} H_l^i = f'(i) = \{0,\ldots,f'(i)-1\}$

(b) $\operatorname{rng} H_l^i \subseteq f(l) = \{0,\ldots,f(l)-1\}$

(c) Whenever $\langle u_l : l \in w_i \rangle$ satisfies

$u_l \subseteq f(l)$

$|u_l| \le g(l)$

then $\{n < f'(i) : \forall l \in w_i\, H_l^i(n) \in u_l\}$ has cardinality $\le g'(i)$

Proof: To any g-slalom $\bar B = \langle B_l : l \in \omega \rangle$ we can associate a g'-slalom $\bar B^* = \langle B_i^* : i \in \omega \rangle$ by letting

$$B_i^* := \{n < f'(i) : \forall l \in w_i\ H_l^i(n) \in w_l\}$$

Conversely, to any function $x \in \prod_i f'(i)$ we can define a function x^* in $\prod_n f(n)$ by

$$\text{if } l \in w_i, \text{ then } x^*(l) = H_l^i(x(i))$$

It is easy to check that if x^* is in $\bar{B}$ then x is in $\bar{B}^*$. The result follows. ☺ 1.8

1.9 Corollary: Assume $0 = n_0 < n_1 < \cdots$, and let

$$f'(i) := f(n_i) \cdot f(n_i + 1) \cdots f(n_{i+1} - 1)$$

$$g'(i) := g(n_i) \cdot g(n_i + 1) \cdots g(n_{i+1} - 1)$$

Then $\mathbf{c}(f', g') \leq \mathbf{c}(f, g)$.

Proof: Identify the set of numbers less than $f(n_i) \cdot f(n_i + 1) \cdots f(f_{i+1} - 1)$ with the cartesian product $\prod_{n_i \leq k < n_{i+1}} f(k)$, and let

$$H_l^i : \prod_{n_i \leq k < n_{i+1}} f(k) \ \to \ f(l)$$

be the projection onto the l-coordinate. We leave the verification of 1.8(c) to the reader. ☺ 1.9

1.10 Lemma: If g is constant, $f(k) \geq 2^k$, then $\mathbf{c}(f, g) = \mathbf{c}$.

Proof: Let $\forall k \, g(k) = n$, $f(k) = 2^k$. Assume that $\prod_l {}^l 2$ can be covered by $< \mathbf{c}$ many g-slaloms.
For any $\eta \in {}^\omega 2$, the sequence $\bar{\eta} := \langle \eta{\restriction}l : l \in \omega \rangle$ is in $\prod_l {}^l 2$. But any g-slalom can contain only n many such $\bar{\eta}$, i.e. for any g-slalom $\bar{B} = \langle B_l : l \in \omega \rangle$ we have

$$\left| \{ \eta \in {}^\omega 2 : \forall l \ \eta{\restriction}l \in B_l \} \right| \leq m$$

Since there are continuum many η we need continuum many g-slaloms to cover $\prod_l f(l)$ (or equivalently, $\prod_l {}^l 2$). ☺ 1.10

1.11 Lemma: If f and g are constant with $f > g$, then $\mathbf{c}(f, g) = \mathbf{c}$.

Proof: Using monotonicity wlog we assume that $f(k) = n + 1$, $g(k) = n$ for all k. We will use 1.8. Let $\omega = \bigcup_{i \in \omega} w_i$ be a partition of ω where $|w_i| = n^{2^i}$.
Let $f'(i) = 2^i$, $g'(i) = n$, and let $\langle H_l^i : l \in w_i \rangle$ enumerate all functions from 2^i to n.
We plan to show $\mathbf{c}(f, g) \geq \mathbf{c}(f', g')$ (so $\mathbf{c}(f, g) = \mathbf{c}$ by 1.10). We want to apply 1.8, so fix a sequence $\langle u_l : l \in w_i \rangle$,where $u_l \subseteq f(l)$ and $|u_l| \leq g(l)$.
To show that the hypotheses of 1.8 are satisfied, fix i_0 and let

$$A := \{ x < f'(i_0) : \forall l \in w_{i_0} \, H_l^{i_0}(x) \in u_l \}$$

and assume A has cardinality $> g'(i_0) = n$. So let $x_0, \ldots, x_n$ be distinct elements of A. Let $H : f'(i_0) \to n+1$ be a function satisfying

$$\forall j \leq n \ H(x_j) = j$$

H is one of the functions $\{ H_l^{i_0} : l \in w_{i_0} \}$, say $H = H_{l_0}^{i_0}$. Let $j_0 \notin u_{l_0}$, then also

$$x_{j_0} \notin \{ x < f'(i_0) : H_{l_0}^{i_0}(x) \in u_{l_0} \} \supseteq A,$$

contradicting $x_{j_0} \in A$. ☺ 1.11

1.12 Corollary: If $f > g$, and $\liminf_{k\to\infty} g(k) < \infty$, then $\mathbf{c}(f,g) = \mathbf{c}$.

Proof: This follows from 1.11, using monotonicity and monotonicity II. ☺ 1.12

We can now extend 1.7 as follows:

1.13 Theorem: If for some $\varepsilon > 0$, $g^{1+\varepsilon} \le f$, then for all n, $\mathbf{c}(f^n, g) = \mathbf{c}(f,g)$.

Proof: First we consider a special case: Assume that $g^2 \le f$. Then we get

$$\mathbf{c}(f,g) \le \mathbf{c}(f^2,g) \le \mathbf{c}(f^2,f)\cdot\mathbf{c}(f,g) \le \mathbf{c}(f^2,g^2)\cdot\mathbf{c}(f,g) = \mathbf{c}(f,g)$$

Now we use this result on (f,g), then on (f^2,g), etc, to get

$$\mathbf{c}(f,g) = \mathbf{c}(f^2,g) = \mathbf{c}(f^4,g) = \mathbf{c}(f^8,g) = \cdots$$

and use monotonicity to get the general result under the assumption $g^2 \le f$.
Now we consider the general case $g^{1+\varepsilon} \le f$:
If g does not diverge to infinity, we have already (by 1.12) $\mathbf{c}(f,g) = \mathbf{c}$. Otherwise we can find some $\delta > 0$ such that for almost all k,

$$\frac{f(k)}{g(k)} \ge g(k)^\delta + 1,$$

so

$$\left\lfloor \frac{f(k)}{g(k)} \right\rfloor \ge g(k)^\delta$$

Now choose m such that $m\cdot\delta > 1$. Then $\lfloor f(k)/g(k)\rfloor^m \ge g$. By 1.7, $\mathbf{c}(f\cdot\lfloor f/g\rfloor^m, g) = \mathbf{c}(f,g)$ and so by monotonicity also $\mathbf{c}(f\cdot g, g) = \mathbf{c}(f,g)$. Since $g^2 \le f\cdot g$, we can apply the result from the special case above to get $\mathbf{c}(f,g) = \mathbf{c}(f^n\cdot g^n, g)$ so in particular, $\mathbf{c}(f^n,g) = \mathbf{c}(f,g)$. ☺ 1.13

If f is not much bigger than g, the assumption in 1.7 and 1.13 may be false. For these cases, we can prove the following:

1.14 Lemma:

(1) $\mathbf{c}(2f-g, f) = \mathbf{c}(f,g)$.
(2) $\mathbf{c}(2f-g, g) = \mathbf{c}(f,g)$.
(3) $\mathbf{c}(f+m(f-g), g) = \mathbf{c}(f,g)$ for all $m\in\omega$.

Proof: The proof is similar to the proof of 1.7. Again we only have to show (1). Let $(N,\in)$ be a reasonably closed model of a large fragment of ZFC (say, $(N,\in) \prec (H(\chi^+),\in)$, where $\chi = 2^{\mathbf{c}}$) of size $\mathbf{c}(f,g)$ such $\prod_n f(n)$ is covered by the set of all g-slaloms from N.
Define h by $h(k) := f(k) + f(k) - g(k)$. We can find a family $\langle B^i_k : i < f(k), k\in\omega\rangle$ in N such that for all k, $\{0,\ldots,h(k)-1\} = \bigcup_{i<f(k)} B^i_k$, where $|B^i_k| = 2$ for $l < f(k)-g(k)$, and $|B^i_k| = 1$ otherwise. We have to show that the set of f-slaloms from N covers $\prod_k h(k)$.
So let x be a function satisfying $\forall k\, x(k) \in \bigcup_{i<f(k)} B^i_k$. We can define a function $y \in \prod_n f(n)$ such that for all k, $x(k) \in B^{y(k)}_k$. So there is some g-slalom $\bar C \in N$ such that for all k, $y(k)\in C_k$.

Define $\bar{A} = \langle A_k : k \in \omega \rangle$ by $A_k := \bigcup_{i \in C_k} B^i_k$. Thus A_k is the union of $g(k)$ many sets, of which at most $f(k) - g(k)$ are pairs, and the others singletons. Thus $|A_k| \le g(k) + (f(k) - g(k)) = f(k)$, so $\bar{A}$ is an f-slalom in N, and for all k, $x(k) \in A_k$. ☺ 1.14

Similar to the proof of 1.13 we now get:

1.15 Lemma:

(1) If $2g \le f$, then for all n, $\mathbf{c}(nf, g) = \mathbf{c}(f, g)$.

(2) If for some $\varepsilon > 0$, $(1 + \varepsilon) g \le f$, then for all n, $\mathbf{c}(nf, g) = \mathbf{c}(f, g)$. 😐 1.15

2. The forcing notion $Q_{f,g}$

2.1 Definition: We fix sequences $\langle n^-_k : k \in \omega \rangle$ and $\langle n^+_k : k \in \omega \rangle$ that increase very quickly and satisfy $n^-_0 \ll n^+_0 \ll n^-_1 \ll n^+_1 \ll \cdots$. In particular, we demand

(1) For all k $\prod_{j<k} n^-_j \le n^-_k$

(2) $\lim_{k \to \infty} \dfrac{\log n^+_k}{\log n^-_k} = 0$.

(3) $n^-_k \cdot n^+_k < n^-_{k+1}$.

We will only consider functions f, g satisfying $n^-_k \le g(k) < f(k) \le n^+_k$. This is partly justified by 1.9, and it also helps to keep the formulation of the main theorem relatively simple.

2.2 Definition: Let $X \ne \emptyset$ be finite, $c, d \in \omega$. A (c, d)-complete norm on $\mathbf{P}(X)$ is a map

$$\| \ \| : \mathbf{P}(\mathbf{X}) - \{\emptyset\} \to \omega$$

mapping any nonempty $a \subseteq X$ to a number $\|a\|$ such that

whenever $a = a_1 \cup \cdots \cup a_c \subseteq X$, then for some $i_1, \ldots, i_d \in \{1, \ldots, c\}$, $\|a_{i_1} \cup \cdots \cup a_{i_d}\| \ge \|a\| - 1$.
($|a|$ is the cardinality of the set a)

A natural (c, d)-complete norm is given by $\|a\| := \log_{c/d} |a|$. c-complete means $(c, 1)$-complete.

2.3 Definition: We call (f, g, h) **progressive**, if f, g, h are functions in ${}^\omega\omega$, satisfying

(1) For all k, $n^-_k \le g(k) < f(k) \le n^+_k$

(2) For all k, $n^-_k \le h(k)$

(3) $\lim_k \log \frac{f(k)}{g(k)} \Big/ \log h(k) = \infty$.

We call (f, g) progressive, if there is a function h such that (f, g, h) is progressive(or equivalently, if (f, g, n^-) is progressive, where n^- is the function defined by $n^-(k) = n^-_k$).

2.4 Remark: For example, if f and g satisfy (1), then (f,g,g) is progressive iff $\log f/\log g \to \infty$. ☹ 2.4

In 2.6 we will define a forcing notion $Q_{f,g,h}$ for any progressive (f,g,h). First we recall the following notation:

2.5 Notation: ${}^{<\omega}\omega = \bigcup_n {}^n2$ is the set of finite sequences of natural numbers. For $s \in {}^{<\omega}\omega$, $|s|$ is the length of s.

A tree p is a nonempty subset of ${}^{<\omega}\omega$ with the properties

$\forall \eta \in p \, \forall k < |\eta| : \eta{\restriction}k \in p$

$\forall \eta \in p : \mathrm{succ}_p(\eta) \neq \emptyset$, where

$$\mathrm{succ}_p(\eta) := \{\nu \in p : \eta \subset \nu, |\eta| + 1 = |\nu|\}.$$

A branch b of p is a maximal linearly $\subseteq$-ordered subset of p. Every branch b defines a function $\bar b : \omega \to \omega$ by $\bar b = \bigcup b$. We usually identify b and $\bar b$, so we write $b{\restriction}k$ (instead of $(\bigcup b){\restriction}k$) for the kth element of b.

The set of all branches of p is written as $[p]$.

For $\eta \in p$, we let

$$p^{[\eta]} := \{\nu \in p : \nu \subseteq \eta \text{ or } \eta \subseteq \nu\}$$

We let

$$\mathrm{split}(p) := \{\eta \in p : |\mathrm{succ}_p(\eta)| > 1\} \qquad \text{(the splitting nodes of } p\text{)}$$

$$\mathrm{split}_n(p) := \{\eta \in \mathrm{split}(p) : |\{\nu \subset \eta : \nu \in \mathrm{split}(p)\}| = n\} \quad \text{(the } n\text{-th splitting level)}$$

and we define the stem of p to be the unique element of $\mathrm{split}_0(p)$.

2.6 Definition: Assume f, g, h are as in 2.3. Then we define for all k, and for all sets x

$$\|x\|_k := \left\lfloor \frac{\log(|x|/g(k))}{\log h(k)} \right\rfloor$$

and we define the forcing notion $Q_{f,g}$ (or more accurately, $Q_{f,g,h}$) to be the set of all p satisfying

(1) p is a perfect tree.

(2) $\forall \eta \in p \, \forall i \in \mathrm{dom}(\eta) \; \eta(i) < f(i)$.

(3) $\forall \eta \in \mathrm{split}_n(p) \; \|\mathrm{succ}_p(\nu)\|_{|\nu|} \geq n$.

We let $p \leq q$ ("q extends p") iff $q \subseteq p$.

2.7 Remark: If we define

$$p \sqsubseteq_k q \text{ iff } p \leq q \text{ and } \mathrm{split}_k(p) \subseteq q$$

then $Q_{f,g,h}$ satisfies axiom A, and is in fact strongly ${}^{\omega}\omega$-bounding, i.e., for name of an ordinal, $\underset{\sim}{\alpha}$, for any p and for any n there is a finite set A and a condition $q \sqsupseteq_n p$, $q \Vdash \underset{\sim}{\alpha} \in A$. However, it will be more convenient to use the relation $\leq_n$ that is based on *levels* rather than *splitting levels.*

2.8 Definition: For p, $q \in Q$, $n \in \omega$ we define

$$p \leq_n q \text{ iff } p \leq q \text{ and } p \cap {}^{\leq n}\omega \subseteq q$$

2.9 Notation: We will usually write $\|\eta\|_p$ instead of $\|\mathrm{succ}_p(\eta)\|_{|\eta|}$.

2.10 Remark: This forcing is similar to the forcing in [*Shelah 326*], but note the following important difference: Whereas in [*Shelah 326*] all nodes above the stem have to be splitting points, we allow many nodes to have only one successor, as long as there "many" nodes with high norm.

2.11 Remark:

(1) The norm $\|\cdot\|_k$ is $h(k)$-complete (hence also n_k^--complete).
(2) If $c/d \leq h(k)$, then the norm is (c, d)-complete.
(3) If $\|a\|_k > 0$, then $|a| > g(k)$.
(4) $\|f(k)\|_k \to \infty$ (so $Q_{f,g,h}$ is nonempty). ☺ 2.11

We will see in the next section that this forcing (and any countable support product of such forcings) is proper and ${}^\omega\omega$-bounding. For the moment, we only show why this forcing is useful in connection with $\mathbf{c}(f, g)$:

2.12 Fact: Any generic filter $G \subseteq Q_{f,g}$ defines a "generic branch"

$$r := \bigcup_{p \in G} \mathrm{stem}(p)$$

that avoids all g-slaloms from V.

Proof: Let $\bar{B} = \langle B_k : k \in \omega \rangle$ be a g-slalom in V, and let $p \in Q_{f,g}$ be a condition. Let $\eta \in p$ be a node satisfying $\|\eta\|_p > 0$. Let $k := |\eta|$. Then $|\mathrm{succ}_p(\eta)| > g(k)$ by 2.11(3), so there is $i \notin B_k$, $\eta^\frown i \in p$. So $p^{[\eta^\frown i]} \Vdash r(k) = i \notin B_k$. ☺ 2.12

3. The construction

In this section we will prove the following theorem:

3.1 Theorem (CH): Assume that $(f_\xi, g_\xi : \xi < \omega_1)$ is a sequence of progressive functions, witnessed by functions h_ξ (see 2.3).

Let $(\kappa_\xi : \xi < \omega_1)$ be a sequence of cardinals satisfying $\kappa_\xi^\omega = \kappa_\xi$ such that whenever $\kappa_\xi < \kappa_\zeta$, then

$$\lim_{k\to\infty} \min\left(\frac{f_\zeta(k)}{g_\xi(k)}, \frac{f_\xi(k)}{g_\xi(k)} \Big/ h_\zeta(k)\right) = 0$$

(or informally: either $f_\zeta \ll g_\xi$, or $f_\xi/g_\xi \ll h_\zeta$, or a combination of these two condition holds)

Then there is a proper forcing notion P not collapsing cardinals nor changing cofinalities such that

$$\Vdash_P \forall\xi : \mathbf{c}(f_\xi, g_\xi) = \kappa_\xi$$

For the proof we use a countable support product of the forcing notions Q_{f_ξ,g_ξ,h_ξ} described in the previous section.

3.2 Remark: The theorem is of course also true (with the same proof) if we have countably or finitely many functions to deal with.

If we are only interested in 2 cardinal invariants $\mathbf{c}(f',g')$, $\mathbf{c}(f,g)$, then we can phrase the theorem without the auxiliary functions h as follows: If (f,g) and (f',g') are progressive, and satisfy

$$\min\left(\frac{f'}{g}, \frac{\log(f/g)}{\log(f'/g')}\right) \to 0$$

then $\mathbf{c}(f,g) < \mathbf{c}(f',g')$ is consistent.

In particular, this shows that our result is quite sharp: For example, if for some function d we have $\lim d = \infty$, $f' = f^d$, $g' = g^d$ (and (f,g), (f',g') are progressive with the same n_k^-, n_k^+), then $\mathbf{c}(f,g) < \mathbf{c}(f',g')$ is consistent. On the other hand, $\mathbf{c}(f^n,g^n) \le \mathbf{c}(f,g)$ for every fixed n.

Proof: Choose h' such that $\log h' \approx 2\log(f/g)$ whenever $\frac{f'}{g} \ge \frac{\log(f/g)}{\log(f'/g')}$. (f',g',h') is progressive, and the assumptions of the theorem are satisfied. (Recall that (f,g) is progressive, hence $\log f/g \gg \log n^-$, so h' will satisfy $h'(k) \ge n_k^-$). ☺ 3.2

A similar simplified formulation of 3.1 is possible when we deal with only countably many functions.

3.3 Example: There is a family $\langle (f_\xi, g_\xi, g_\xi : \xi < \mathbf{c}\rangle$ of continuum many progressive functions such that for any $\zeta \ne \xi$, $\min\left(\frac{f_\xi}{g_\zeta}, \frac{f_\zeta}{g_\xi}\right) \to 0$. [In particular, under CH we may choose any family $(\kappa_\xi : \xi < \omega_1)$ of cardinals satisfying $\kappa_\xi^\omega = \kappa_\xi$ and get an extension where $\mathbf{c}(f_\xi, g_\xi) = \kappa_\xi$.]

Proof: Let $\ell_k := \left\lfloor \frac{1}{2}\sqrt{\log\frac{\log n_k^+}{\log n_k^-}} \right\rfloor$. (Here, "log" can be the logarithm to any (fixed) base, say 2.) Then $\lim_{k\to\infty} \ell_k = \infty$, and by invariance (1.5(4)) we may assume $\ell_k \ge 1$ for all k.

Let $T \subseteq 2^{<\omega}$ be a perfect tree such that for all k we have $|T \cap 2^k| = \ell_k$, say, $T \cap 2^k = \{s_1(k), \ldots, s_{\ell_k}(k)\}$.

For any $x \in [T]$ (i.e., $x \in 2^\omega$, $\forall k\, x{\restriction}k \in T$) we now define functions f_x, g_x, h_x by:

If $x{\restriction}k = s_i(k)$, then

$$f_x(k) = \left(n_k^-\right)^{\ell_k^{2i}}$$
$$h_x(k) = g_x(k) = \left(n_k^-\right)^{\ell_k^{2i-1}}$$

We leave the verification that (f_x, g_x, h_x) is indeed progressive to the reader. [Recall 2.4, and also note that $\log\log f_x(k) \le 2\ell_k \log \ell_k + \log\log n_k^- < \log\log n_k^+$. Finally, note that if $x \ne y$, then for almost all k we have $\min\left(\frac{f_x(k)}{g_y(k)}, \frac{f_y(k)}{h_x(k)}\right) \ll \frac{1}{n_k^-}$.] ☺ 3.3

3.4 Example: It is consistent to have $\mathbf{c}(f^2,g^2) < \mathbf{c}(f,g)$ (for certain f, g).

Proof: Let $\ell_k := \left\lfloor \frac{1}{6}\log\frac{n_k^+}{n_k^-} \right\rfloor$. Assume $\ell_k > 0$ for all k. Then, letting

$$f(k) := \left(n_k^-\right)^{3\ell_k}$$
$$g(k) := \left(n_k^-\right)^{2\ell_k}$$
$$h(k) := n_k^-$$

We have that (f, g, h) and (f^2, g^2, h) are progressive, and $\lim \frac{f}{g^2} = 0$, so we can apply the theorem. ☺ 3.4

3.5 Definition:

Let κ be a disjoint union $\kappa = \bigcup_{\xi<\omega_1} A_\xi$, where $|A_\xi| = \kappa_\xi$.

For $\alpha < \kappa$, let Q_α be the forcing Q_{f_ξ, g_ξ, h_ξ}, if $\alpha \in A_\xi$, and let $P = \prod_{\alpha<\kappa} Q_\alpha$ be the **countable support product** of the forcing notions Q_α, i.e., elements of P are countable functions p with $\mathrm{dom}(p) \subseteq \kappa$, and $\forall \alpha \in \mathrm{dom}(p)\ p(\alpha) \in Q_\alpha$.

For $A \subseteq \kappa$, we write $P \restriction A := \{p \restriction A : p \in P\}$. Clearly $P \restriction A \lessdot P$ for any A. In particular, $Q_\alpha \lessdot P$.

We write $\underset{\sim}{r}_\alpha$ for the Q_α-name (or P-name) for the generic branch introduced by a generic filter on Q_α.

We say that q **strictly extends** p, if $q \geq p$ and $\mathrm{dom}(q) = \mathrm{dom}(p)$.

3.6 Facts: Assume CH. Then

(1) each Q_α is proper and ${}^\omega\omega$-bounding.

(2) P is proper and ${}^\omega\omega$-bounding.

(3) P satisfies the $\aleph_2$-cc.

(4) Neither cardinals nor cofinalities are changed by forcing with P.

Proof of (1), (2): See below (3.23, 3.24)

Proof of (3): A straightforward Δ-system argument, using CH.

(4) follows from (2) and (3). ☺ 3.6

We plan to show that $\Vdash_P \mathbf{c}_\xi = \kappa_\xi$ for all $\xi < \omega_1$.

3.7 Definition: If $p \in P$, $k \in \omega$, we let the level k of p be

$$\begin{aligned}\mathrm{Level}_k(p) := \big\{\bar\eta : {} & \mathrm{dom}(\bar\eta) = \mathrm{dom}(p), \\ & \forall \alpha \in \mathrm{dom}(\bar\eta) : |\bar\eta(\alpha)| = k,\ \bar\eta(\alpha) \in p(\alpha) \ \big\}\end{aligned}$$

We define the set of active ordinals at level k as

$$\mathrm{active}_k(p) := \{\alpha \in \mathrm{dom}(p) : |\mathrm{stem}(p(\alpha))| \leq k\}$$

3.8 Remark: Sometimes we identify the set $\mathrm{Level}_k(p)$ with the set

$$\begin{aligned}&\{\bar\eta : \mathrm{dom}(\bar\eta) = \mathrm{active}_k(p), \forall \alpha \in \mathrm{dom}(\bar\eta) : |\bar\eta(\alpha)| = k\} \\ &= \{\bar\eta \restriction \mathrm{active}_k(p) : \bar\eta \in \mathrm{Level}_k(p)\}\end{aligned}$$

3.9 Definition: We say that the kth level is a splitting level of p (or "k is a splitting level of p") iff

$$\exists \alpha \in \mathrm{dom}(p)\, \exists \eta \in \mathrm{split}(p(\alpha)) : |\eta| = k$$

3.10 Definition: If $\bar\eta \in \mathrm{Level}_k(p)$, $\bar\eta' \in \mathrm{Level}_{k'}(p)$, $k < k'$, then we say that $\bar\eta'$ extends $\bar\eta$ iff for all $\alpha \in \mathrm{dom}(\bar\eta)$, $\bar\eta'(\alpha)$ extends (i.e., $\supseteq$) $\bar\eta(\alpha)$.

3.11 Definition: For $p, q \in P$, $k \in \omega$, we let

$$p \leq_k q \text{ iff } p \leq q \text{ and } \forall \alpha \in \mathrm{dom}(p) : p(\alpha) \leq_k q(\alpha) \text{ and } \mathrm{active}_k(p) = \mathrm{active}_k(q)$$

That is, we allow $\mathrm{dom}(q)$ to be bigger than $\mathrm{dom}(p)$, but for all new $\alpha \in \mathrm{dom}(q) - \mathrm{dom}(p)$ we require that $|\mathrm{stem}(q(\alpha))| > k$.

3.12 Definition: Let $A \subseteq P$. A set $D \subseteq P$ is

$$\begin{aligned} \text{dense in } A, &\text{ if } \forall p \in A\, \exists q \in D : p \leq q \\ \text{strictly dense in } A, &\text{ if } \forall p \in A\, \exists q \in D : p \leq q \text{ and } \mathrm{dom}(p) = \mathrm{dom}(q) \\ \text{open in } A, &\text{ if } \forall p \in D\, \forall q \in A \text{: } (p \leq q \text{ implies } q \in D) \\ \text{almost open in } A, &\text{ if } \forall p \in D\, \forall q \in A \text{: } (p \leq q \text{ and } \mathrm{dom}(p) = \mathrm{dom}(q) \text{ implies } q \in D) \end{aligned}$$

These definitions can also be relativized to conditions above a given condition p_0. If we omit A we mean $A = P$.

3.13 Definition: If $\bar\eta \in \mathrm{Level}_k(p)$, we let $q = p^{[\bar\eta]}$ be the condition defined by $\mathrm{dom}(q) = \mathrm{dom}(p)$, and

$$\forall \alpha \in \mathrm{dom}(q)\ q(\alpha) = p(\alpha)^{[\bar\eta(\alpha)]}$$

3.14 Definition: If $p \Vdash \underset{\sim}{x} \in V$, and $\bar\eta \in \mathrm{Level}_k(p)$, we say that $\bar\eta$ decides $\underset{\sim}{x}$ (or more accurately, $p^{[\bar\eta]}$ decides $\underset{\sim}{x}$) if for some $y \in V$, $p^{[\bar\eta]} \Vdash \underset{\sim}{x} = \check{y}$.

First we simplify the form of our conditions such that all levels are finite.

3.15 Fact: The set of all conditions p satisfying

$\boxed{\text{I}}$ $\forall k\, |\mathrm{active}_k(p)| < \omega$, and moreover:

$\boxed{\text{II}}$ For any splitting level k there is exactly one pair (η, α) such that $|\mathrm{succ}_{p(\alpha)}(\eta)| > 1$.

is dense in P. ☹ 3.15

3.16 Fact: If p is in the dense set given by (I) and (II), then the size of level k is $\leq n^-_{k-1} \cdot n^+_{k-1} < n^-_k$.

Proof: By induction. ☹ 3.16

From now on we will only work in the dense set of conditions satisfying (I) and (II).

3.17 Notation: For p satisfying (I)–(II), we let $k_l = k_l(p)$ be the lth splitting level. Let $\eta_l = \eta_l(p)$ and $\alpha_l = \alpha_l(p)$ be such that $|\eta_l(p)| = k_l(p)$, $\eta_l(p) \in \mathrm{split}(p(\alpha_l))$. We let $\zeta_l = \zeta_l(p)$ be such that $\alpha_l \in A_{\zeta_l}$.

We write $\|p\|_{k_l}$ for $\|\eta_l\|_{p(\alpha_l)}$, i.e., for $\|\mathrm{succ}_{p(\alpha_l)}(\eta_l)\|_{\zeta_l, k_l}$. (See figure 2)

3.18 Definition: If p is a condition, $l \in \omega$, $\alpha^* := \alpha_l(p)$, $\eta^* := \eta_l(p)$, $\nu^* \in \mathrm{succ}_{p(\alpha^*)}(\eta^*)$, we can define a stronger condition q by letting $q(\alpha) = p(\alpha)$ for all $\alpha \neq \alpha^*$, and

$$q(\alpha^*) := \{\eta \in p(\alpha^*) : \text{If } \eta^* \subset \eta, \text{ then } \nu^* \subseteq \eta\}$$

In this case, we say that q was obtained from p by "pruning the splitting node η^*."

To simplify the notation in the fusion arguments below, we will use the following game:

Figure 2: A condition satisfying (I) and (II)

3.19 Definition: For any condition $p \in P$, $G(P,p)$ is the following two person game with perfect information:

There are two players, the *spendthrift* and the *accountant*. A play in $G(P,p)$ last ω many moves (starting with move number 1) The *accountant* moves first. We let $p_0 := p$, $i_0 := 0$.

In the n-th move, the *accountant* plays a pair (η^n, α^n) with $\eta^n \in p_{n-1}(\alpha^n)$, $|\eta^n| = i_{n-1}$, and a number b_n. Player *spendthrift* responds by playing a condition p_n and a finite sequence ν^n (letting $i_n := |\nu^n| + 1$) satisfying the following: (See Figure 3)

(1) $p_n \geq_{i_{n-1}} p_{n-1}$.

(2) $\nu^n \in p_n(\alpha^n)$

(3) $\|\nu^n\|_{p_n(\alpha^n)} > b_n$.

(4) $\nu^n \supset \eta^n$.

(5) For all $\alpha \in \mathrm{dom}(p_n) - \mathrm{dom}(p_{n-1})$, $|\mathrm{stem}(p_n(\alpha))| > |\nu^n|$.

(6) $|\mathrm{Level}_{|\nu^n|}(p_n)| = |\mathrm{Level}_{|\eta^n|}(p_n)| = |\mathrm{Level}_{|\eta^n|}(p_{n-1})|$

(Remember that all conditions p_n have to be in the dense set given by (I) and (II)) Player *accountant* wins iff after ω many moves there is a condition q such that for all n, $p_n \leq q$, or equivalently, if the function q with domain $\bigcup_n \mathrm{dom}(p_n)$, defined by

$$q(\alpha) = \bigcup_{\alpha \in \mathrm{dom}(p_n)} p_n(\alpha)$$

is a condition. Note that we have $\eta_l(q) = \nu^l$, $\alpha_l(q) = \alpha^l$, since the only splitting points are the ones chosen by *spendthrift*.

3.20 Fact: Player *accountant* has a winning strategy in $G(P,p)$.

Figure 3: stage n

Proof: We leave the proof to the reader, after pointing out that a finitary bookkeeping will ensure that the limit of the conditions p_n is in fact a condition.

In particular, this shows that *spendthrift* has no winning strategy. Below we will define various strategies for the *spendthrift*, and use only the fact that there is a play in which the *accountant* wins. ☺ $_{3.20}$

The game gives us the following lemma:

3.21 Lemma: Assume that p is a condition satisfying (I)–(II). For each l let $\emptyset \neq F_{\eta_l} \subseteq \mathrm{succ}_{p(\alpha_l)}(\eta_l)$ be a set of norm $\|F_{\eta_l}\|_{k_l} \geq \left\|\mathrm{succ}_{p(\alpha_l)}(\eta_l)\right\| / 2$.

Then there is a condition $q \geq p$, $\mathrm{dom}(q) = \mathrm{dom}(p)$ such that for all l:

$$(*) \qquad\qquad \text{If } \eta_l(p) \in q(\alpha_l(p)), \text{ then } \mathrm{succ}_{q(\alpha_l(p))}(\eta_l(p)) \subseteq F_{\eta_l}$$

Proof: The condition q can be constructed by playing the game. In the n-th move, *spendthrift* first finds a $\eta^n \supset \nu^n$ satisfying $\eta^n(i) \in F_{\eta_i}$ whenever this is applicable, and $\left\|\mathrm{succ}_{p_{n-1}}(\eta^n)\right\| > 2b_n$. Then *spendthrift* obtains p_n by pruning (see 3.18) all splitting nodes of p_{n-1} whose height is between $|\eta^n|$ and $|\nu^n|$ and further thinning out the successors of η^n to satisfy $\mathrm{succ}_{p_n}(\eta^n) = F_{\eta^n}$. (Note that $F_{\eta^n} \subseteq \mathrm{succ}_{p_{n-1}}(\eta^n) = \mathrm{succ}_{p_0}(\eta^n)$.)

In the resulting condition q the only splitting nodes will be the nodes η^n, so $(*)$ will be satisfied. ☺ $_{3.21}$

(Note that in general $\eta_l(q) \neq \eta_l(p)$, and indeed $k_l(q) \neq k_l(p)$, since many splitting levels of p are not splitting levels in q anymore.)

3.22 Lemma: Assume $\underset{\sim}{\tau}$ is a P-name of a function from ω to ω, or even from ω into ordinals. Then the set of conditions satisfying (I)–(III) is dense and almost open.

$\boxed{\text{III}}$ Whenever k is a splitting level, then every $\bar\eta$ in level $k+1$ decides $\underset{\sim}{\tau}\restriction k$.

Proof of (III): We will use the game from 3.19. We will define a strategy for the *spendthrift* ensuring that the condition q the *accountant* produces at the end will satisfy (III).

In the n-th move, *spendthrift* finds a condition $r_n \geq_{i_{n-1}} p_{n-1}$ such that for every $\bar\eta \in \mathrm{Level}_{i_{n-1}}(r_n)$ the condition $(p_n)^{[\bar\eta]}$ decides $\underset{\sim}{\tau}\restriction i_{n-1}+10$. Then *spendthrift* finds $\eta^n \in r_n(\alpha^n)$ satisfying the rules and obtains p_n with $\eta^n \in p_n(\alpha^n)$ from r_n by pruning all splitting levels between i_{n-1} and $|\eta_n|$. ☺ 3.22

Since all levels of q are finite, it is thus possible to find a finite sequence $\bar B = \langle B_k : k\in\omega\rangle$ in the ground model that will cover $\underset{\sim}{\tau}$. (I.e. $q \Vdash \underset{\sim}{\tau}(k)\in B_k$). The rest of this section will be devoted to finding "small" such sets B_k.

3.23 Corollary: P is ${}^\omega\omega$-bounding and does not collapse ω_1. ☺ 3.23

3.24 Remark: Although it does not literally follow from 3.22, the reader will have no difficulty in showing that P is actually α-proper for any $\alpha<\omega_1$. ☺ Indeed, using the partial orders $\sqsubseteq_n$ from 2.7, it is possible to carry out straightforward fusion arguments, without using the game 3.19 at all. However, the orderings $\leq_n$ are more easy to handle, since in induction steps we only have to take care of a single η^n, instead of a front.

3.25 Fact: $\Vdash_P \forall\tau\in{}^\omega\omega\,\exists B\subseteq\kappa$, B countable, $B\in V$, and $\tau\in V[G\restriction B]$.

Proof: Let p be any condition and let $\underset{\sim}{\tau}$ be a name for a real. There is a stronger condition q satisfying (I), (II) and (III). Let $B := \mathrm{dom}(q)$. Clearly $q\Vdash \underset{\sim}{\tau}\in V[G\restriction B]$. ☺ 3.25

3.26 Corollary: If $\lambda = |A|^\omega$, then $\Vdash_{P\restriction A} 2^{\aleph_0}\leq\lambda$.

Proof: For each countable subset $B\subseteq A$, $\Vdash_{P\restriction B} CH$. Since every real in $V[G]$ is in some such $V[G\restriction B]$, the result follows. ☺ 3.26

3.27 Fact and Notation: If p satisfies (II), then

(1) If $\bar\eta(\alpha_l)=\eta_l$, and $\nu\in\mathrm{succ}_{p(\alpha_l)}(\eta_l)$, then the requirement

$$\bar\eta^{+\nu}(\alpha_l)=\nu$$

uniquely defines an extension $\bar\eta^{+\nu}$ of $\bar\eta$ in $\mathrm{Level}_{k_l+1}(p)$.

(2) If $\bar\eta(\alpha_l)\neq\eta_l$, $\bar\eta$ has a unique extension $\bar\eta^{+}\in\mathrm{Level}_{k_l+1}(p)$. To simplify the notation in 3.33 below, we also define for this case, for any $\nu\in\mathrm{succ}_{p(\alpha_l)}(\eta_l)$, $\bar\eta^{+\nu}:=\bar\eta^{+}$.

3.28 Fact: The set of conditions satisfying (IV) is strictly dense (but not almost open) in the set of conditions satisfying (I)–(II).

$\boxed{\text{IV}}$ For all l:

$$|\mathrm{Level}_{k_l}(p)| < \min\left(\frac{\|p\|_{k_l}}{2}, n^-_{k_l}\right)$$

For the proof, note that $|\mathrm{Level}_{k_l}(p)| = |\mathrm{Level}_{k_{l-1}+1}(p)|$. ☺ 3.28

3.29 Lemma: Assume $\underset{\sim}{\tau}$ is a P-name of a function $\in {}^\omega\omega$, and $\Vdash_P \forall k\, \underset{\sim}{\tau}(k) < n_k^+$. Then the set of conditions satisfying (V) is strictly dense and almost open in the set give by (I), (II), (III). where

$\boxed{\text{V}}$ Whenever k is a splitting level, then every $\bar\eta$ in level k decides $\underset{\sim}{\tau}\restriction k$.

Proof: Fix p satisfying (I), (II), (III), (IV).

Let $k_l := k_l(p)$, etc. Let $m_l := |\text{Level}_{k_l}|$.

Proof: We will use 3.21. For each $l \in \omega$, $F_{\eta_l} \subseteq \text{succ}_{p(\alpha_l)}(\eta_l)$ will be defined as follows: Let $m_l := |\text{Level}_{k_l}(p)|$, and let $\bar\eta^0, \ldots, \bar\eta^{m-1}$ enumerate $\text{Level}_k(p)$. Find a sequence

$$\text{succ}_{p(\alpha_l)}(\eta_l) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^m \qquad \forall i \left\|F^{i+1}\right\|_k \geq \left\|F^i\right\|_k - 1$$

such that for all i there exists x^i such that for all $\nu \in F^{i+1}$ we have $p^{[(\bar\eta^i)^{+\nu}]} \Vdash \underset{\sim}{\tau}\restriction k = x$. It is possible to find such F^{i+1} since $\|\cdot\|_k$ is n_k^--complete, and there are only $n_0^+ \cdot n_1^+ \cdots n_{k-1}^+ < n_k^-$ many possible values of $\underset{\sim}{\tau}\restriction k$.

Finally, let $F_{\eta_l} := F^m$. Applying 3.21 will yield the desired result. ☺ 3.29

3.30 Remark: Note that (V) in particular implies

$\boxed{\text{Va}}$ Whenever k is not a splitting level, then every $\bar\eta$ in level k decides $\underset{\sim}{\tau}(k)$.

3.31 Proof that $\Vdash_P \mathbf{c}(f_\xi, g_\xi) \geq \kappa_\xi$: (This proof is essentially the same as 2.12.)

Recall that $\underset{\sim}{r}_\alpha$ is the generic real added by the forcing Q_α. Working in $V[G]$, let $\mathcal{B}$ be a family of less than κ_ξ many g_ξ-slaloms. We will show that they cannot cover $\prod f_\xi$, by finding an α such that $\underset{\sim}{r}_\alpha$ is forced not to be covered.

There exists a set $A \in V$ of size $< \kappa_\xi$ such that $\mathcal{B} \subseteq V[G\restriction A]$. Since $|A| < \kappa_\xi$ there is $\alpha \in A_\xi - A$.

Assume that $\bar B$ is a g_ξ-slalom in $V[G\restriction A]$ covering r_α. So in V there are a $P\restriction A$-name $\underset{\sim}{\bar B}$ and a condition p such that

$$\Vdash_{P\restriction A} \underset{\sim}{\bar B} \text{ is a } g\text{-slalom}$$

and

$$p \Vdash_P \underset{\sim}{\bar B} \text{ covers } r_\alpha$$

We can find a node η in $p(\alpha)$ with $\text{succ}_{p(\alpha)}(\eta)$ having more than $g(|\eta|)$ elements. Increase $p\restriction A$ to decide $\underset{\sim}{B}_{|\eta|}$, then increase $p(\alpha)$ to make r_α avoid this set. ☺ 3.31

3.32 Fact: Fix ξ^*. Then the set of conditions p satisfying

$\boxed{\text{VI}}$ For all l: If $\kappa_{\xi^*} < \kappa_{\zeta_l(p)}$, then

$$\min\left(\frac{f_{\zeta_l(p)}(k_l)}{g_{\xi^*}(k_l)}, \frac{f_{\xi^*}(k_l)}{g_{\xi^*}(k_l)}\Big/ h_{\zeta_l(p)}(k_l)\right) < \frac{1}{|\text{Level}_{k_l}(p)|}$$

is dense almost open.

Proof: Write F_ζ for the function $\min\left(\frac{f_\zeta}{g_{\xi^*}}, \frac{f_{\xi^*}}{g_{\xi^*}} \Big/ h_\zeta\right)$. Recall that if $\kappa_\zeta < \kappa_{\xi^*}$, then F_ζ tends to 0.
Fix a condition p, We will use the game $G(P,p)$. *spendthrift* will use the following strategy: Whenever $\alpha_n \in A_\zeta$ and $\kappa_\zeta < \kappa_{\xi^*}$, then *spendthrift* first find m_0 such that for all $m \geq m_0$ we have $F_\zeta(m) < 1/|\mathrm{Level}_{h_{n-1}}(p_{n-1})|$. Now find $\nu^n \supseteq \eta^n$ of length $> m_0$ with a large enough norm, and play any condition p_n obeying the rules of the game. In particular, we must have $\left|\mathrm{Level}_{|\nu^n|}(p^n)\right| = \left|\mathrm{Level}_{|\eta^n|}(p^n)\right|$.
Clearly the condition resulting from the game satisfies the requirements. ☺ 3.32

3.33 Proof that $\Vdash_P \mathbf{c}(f_\xi, g_\xi) \leq \kappa_\xi$: Fix ξ. We will write f for f_ξ, etc.
Let

$$A := \bigcup\{A_\zeta : \kappa_\zeta \leq \kappa_\xi\}.$$

We will show that the g-slaloms from $V^{P\restriction A}$ already cover $\prod f$. This is sufficient, because $\Vdash_P (2^{\aleph_0})^{V^{P\restriction A}} \leq |A| = \kappa_\xi$.
Let p_0 be an arbitrary condition. Let $\underset{\sim}{\tau}$ be a name of a function $< f$. Find a condition $p \geq p_0$ satisfying (I)–(VI).
For each l we now define sets $F_{\eta_l} \subseteq \mathrm{succ}_{p(\alpha_l)}(\eta_l)$ as follows:

(1) If $\alpha_l \in A$, then $F_{\eta_l} = \mathrm{succ}_{p(\alpha_l)}(\eta_l)$.
(2) If $f_{\zeta_l}(k_l) \leq g_\xi(k_l)/|\mathrm{Level}_{k_l}(p)|$, then again $F_{\eta_l} = \mathrm{succ}_{p(\alpha_l)}(\eta_l)$.
(3) Otherwise, we thin out the set $\mathrm{succ}_{p(\alpha_l)}(\eta_l)$ such that each $\bar\eta$ in $\mathrm{Level}_{k_l}(p)$ decides $\underset{\sim}{\tau}(k_l)$ up to at most $g(k_l)/|\mathrm{Level}_{k_l}(p)|$ many values.

Here is a more detailed description of case (3): Let $k = k_l$, $\zeta = \zeta_l$.
Note that if neither (1) nor (2) holds, then letting $c := f_\xi(k)$, $d := g_\xi(k)/|\mathrm{Level}_k(p)|$, we have $c/d \leq h_\zeta(k)$.
Using (c,d)-completeness of the norm $\|\cdot\|_{\zeta,k}$we define a sequence

$$\mathrm{succ}_{p(\alpha_l)}(\eta_l) = L(0) \supseteq L(1) \supseteq \cdots \supseteq L(|\mathrm{Level}_k(p)|)$$

as follows. Let $\bar\eta_0, \ldots, \bar\eta_{|\mathrm{Level}_k(p)|-1}$ be an enumeration of $\mathrm{Level}_k(p)$.
Given $L(i)$, we know that for each $\nu \in L(i)$ the sequence $\bar\eta_i^{+\nu}$, (i.e., the condition $p^{[\bar\eta_i^{+\nu}]}$) decides $\underset{\sim}{\tau}(k)$. (See 3.27.) since there only $\leq c$ many possible values for $\underset{\sim}{\tau}(k)$, we can use (c,d)-completeness to find a set $L(i+1) \subseteq L(i)$ and a set $C(i)$ such that

(a) $\|L(i+1)\| \geq \|L(i)\| - 1$
(b) $|C(i)| \leq d$.
(c) For every $\nu \in L(i+1)$, $p^{[\bar\eta_i^{+\nu}]} \Vdash \underset{\sim}{\tau}(k) \in C(i)$.

Now let F_{η_l} be $L(|\mathrm{Level}_k(p)|)$, and let

$$(\oplus) \qquad B_k := \bigcup_i C(i).$$

So $|B_k| \leq |\mathrm{Level}_k(p)| \cdot d \leq g(k)$.

Clearly $\|F_{\eta_l}\|_{\zeta_l,k_l} \geq \|p\|_{k_l} - |\mathrm{Level}_{k_l}(p)| > \frac{1}{2}\|p\|_{k_l}$.
This completes the definition of the sets F_{η_l}.

Let $q \geq p$ be the condition defined from p using the F_{η_l} (see 3.21). We will find a $P{\restriction}A$-name for a g-slalom $\underset{\sim}{\bar{B}} = \langle \underset{\sim}{B}_k : k \in \omega\rangle$ such that

$$q \Vdash \underset{\sim}{\bar{B}} \text{ covers } \underset{\sim}{\tau}.$$

If k is not a splitting level, then every $\bar{\eta}$ in level k decides $\underset{\sim}{\tau}(k)$ by (Va). So in this case we can let

$$B_k := \{i : \exists\bar{\eta} \in \mathrm{Level}_k(p), p^{[\bar{\eta}]} \Vdash \underset{\sim}{\tau}(k) = i\}$$

This set is of size $\leq |\mathrm{Level}_k(p)| < g(k)$, and clearly $q \Vdash \underset{\sim}{\tau}(k) \in B_k$.
If k is a splitting level, $k = k_l$, then there are three cases.
Case 1: $\alpha_l \in A$: We define $\underset{\sim}{B}_k$ to be a $P{\restriction}A$-name satisfying the following:

$$\Vdash_{P\restriction A} \underset{\sim}{B}_k = \{i : \exists\bar{\eta} \in \mathrm{Level}_{k+1}(p), V \models p^{[\bar{\eta}]} \Vdash \underset{\sim}{\tau}(k) = i, \bar{\eta}(\alpha_l) \subseteq r_{\alpha_l}\}$$

Thus, we only admit those $\bar{\eta}$ which agree with the generic real added by the forcing Q_{α_l}. Clearly $\Vdash_{P\restriction A}$ $|B_k| \leq \mathrm{Level}_k(p) < g(k)$, and $p \Vdash_P \underset{\sim}{\tau}(k) \in B_k$.
Case 2: $f_{\zeta_l}(k) \leq g_\xi(k)/|\mathrm{Level}_k(p)|$.
So we have $|\mathrm{Level}_{k+1}(p)| \leq f_{\zeta_l}(k) \cdot |\mathrm{Level}_k(p)| \leq g(k)$, so we can let

$$B_k := \{i : \exists\bar{\eta} \in \mathrm{Level}_{k+1}(p), p^{[\bar{\eta}]} \Vdash \underset{\sim}{\tau}(k) = i\}$$

This set is of size $\leq |\mathrm{Level}_{k+1}(p)| \leq g(k)$, and again $p \Vdash \underset{\sim}{\tau}(k) \in B_k$.
Case 3: Otherwise. We have already defined B_{k_l} in $(\oplus)$. By condition (c) above, $q \Vdash \underset{\sim}{\tau}(k) \in B_k$.

So indeed $q \Vdash$"$\underset{\sim}{\bar{B}} = \langle \underset{\sim}{B}_k : k \in \omega\rangle$ is a g-slalom covering $\underset{\sim}{\tau}$" ☺ 3.33 ☺ 3.1 ☺ [GSh 448]

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