$Aut(M)$ has a large dense free subgroup for saturated M

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Abstract

We prove that for a stable theory T, if M is a saturated model of T of cardinality λ where $\lambda > |T|$, then $Aut(M)$ has a dense free subgroup on 2^λ generators. This affirms a conjecture of Hodges.

1 Introduction

A subgroup G of the automorphism group of a model M is said to be dense if every finite restriction of an automorphism of M can be extended to an automorphism in G. In this paper we present Shelah's proof of a conjecture of Hodges that for a cardinal λ with $\lambda > |T|$, if M is a saturated model of T of size λ then the automorphism group of M, $Aut(M)$, has a dense free subgroup of cardinality 2^λ. Wilfrid Hodges had noted that the theorem was true for λ such that $\lambda \geq |T|$ and $\lambda^{<\lambda} = \lambda$. Peter Neumann then pointed out to him that de Bruijn had shown that independently of any set theoretic assumptions on λ, $Sym(\lambda)$, the group of permutations of λ, has a free subgroup on 2^λ generators. On checking the proof, Hodges found one could also make the subgroup dense, so the natural conjecture was that

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for any cardinal $\lambda > |T|$ if there is a saturated model M of cardinality λ, then $Aut(M)$ has a dense free subgroup on 2^λ generators. As Shelah likes questions, Hodges asked him about the conjecture when Shelah went to work with Hodges in the Summer of 1989. The proof presented in this paper is simpler than the one from 1989, thought of by Shelah while he was helping Melles with his earlier proof. While the proof is not complicated, Melles thinks it is a nice application of the non forking relation for stable theories. By the following theorem of Shelah, the only open case was for T stable, although for completeness, we also include here a proof for the case that $|M| = \lambda = \lambda^{<\lambda}$.

Theorem 1 *T has a saturated model in λ iff one of the following hold*

1. $\lambda = \lambda^{<\lambda} + |D(T)|$
2. *T is stable in λ*

PROOF [Sh c] VIII 4.7.

Definition 2 *A subgroup G of $Aut(M)$ is $< \lambda$ dense if every elementary permutation of a subset A of M such that $|A| < \lambda$ has an extension in G.*

Melles asked Shelah about the natural stregthening of Hodges question; Can one find a subroup G of $Aut(M)$ for M a saturated model of cardinality λ, such that G is $< \lambda$ dense and of cardinality 2^λ? Shelah quickly found a proof affirming this stronger conjecture.
Throughout this paper we work in $\mathfrak{C}^{eq}$ and follow the notation from [Sh c]. See there for the definitions of any notions left undefined here. We denote the identity map by id.

2 $|M| = \lambda = \lambda^{<\lambda}$

Lemma 3 *Let I be an infinite order and let $\langle a_i \mid i \in I\rangle$ be a sequence indiscernible over A, f an elementary map with domain $A \cup \bigcup \left\{ a_i \mid i \in I \right\}$, and B a set. Then there is a sequence $\langle c_i \mid i \in I\rangle$ which realizes $tp(\langle f(a_i) \mid i \in I\rangle / f(A))$ such that $\langle c_i \mid i \in I\rangle$ is indiscernible over $B \cup f(A)$.*

PROOF By compactness and Ramsey's theorem.

Lemma 4 *Let $\tau = f_n^{\epsilon_n} \dots f_0^{\epsilon_0}$ be a term (intended to represent a composition of functions with f^1 meaning f and f^{-1} meaning the inverse of f) such that $\epsilon_i \in \{-1,1\}$ and $f_i = f_{i+1} \Rightarrow \epsilon_i = \epsilon_{i+1}$. Let M, N be models such that M is saturated of cardinality λ, and $M \prec N$ with N being λ^+ saturated and λ^+ homogenous. Let $\left\{ f_{\nu_0} \dots f_{\nu_n} \right\}$ be a finite set of automorphisms of M with $f_{\nu_i} = f_{\nu_{i+1}}$ iff $f_i = f_{i+1}$ in τ. Then there are automorphisms of N, $\left\{ F_{\nu_0} \dots F_{\nu_n} \right\}$ such that each F_{ν_i} extends f_{ν_i} and $F_{\nu_n}^{\epsilon_n} \dots F_{\nu_0}^{\epsilon_0} \neq id_N$.*

PROOF If $\epsilon_0 = 1$, let $A_0 = \{a_i^0 \mid i < \omega\} \subseteq N$ be an infinite indiscernible sequence over M. Let F be an extension of f_{ν_0} to an automorphism of N and let $A_1 \subseteq N$ realize $tp(F[A_0]/F[M])$ such that A_1 is indiscernible over $A_0 \cup M$. Let $F_{\nu_0}^0$ be the elementary map extending f_{ν_0} such that A_0 is sent to A_1. If $\epsilon_0 = -1$, then let $A_0 = \{a_i^0 \mid i < \omega\} \subseteq N$ be an infinite indiscernible sequence over M. Let F be an extension of $(f_{\nu_0})^{-1}$ to an automorphism of N and let $A_1 \subseteq N$ realize $tp(F[A_0]/F[M])$ such that A_1 is indiscernible over $A_0 \cup M$. Let $(F_{\nu_0}^0)^{-1}$ be the elementary map extending $(f_{\nu_0})^{-1}$ such that $(F_{\nu_0}^0)^{-1}$ sends A_0 to A_1. Now by induction on $0 < i \leq n$ we define infinite sequences A_{i+1} indiscernible over $M \cup \bigcup_{j \leq i} A_j$ and elementary maps $F_{\nu_i}^i$ such that

1. $F_{\nu_i}^i$ extends f_{ν_i}
2. If $j < i$ and $\nu_j = \nu_i$, then $F_{\nu_j}^j \subseteq F_{\nu_i}^i$
3. $F_{\nu_i}^{\epsilon_i} \dots F_{\nu_0}^{\epsilon_0}(A_0) = A_{i+1}$

Now suppose $0 < i \leq n$ and $F_{\nu_j}^j$ have been defined for all $j < i$. Suppose $\epsilon_i = 1$ and there is a $j < i$ such that $\nu_j = \nu_i$. Let j^* be the largest such j. Let F be an extension of $F_{\nu_{j^*}}^{j^*}$ to an automorphism of N. By the construction A_i is indiscernible over the domain of $F_{\nu_{j^*}}^{j^*}$. So we can find A_{i+1} realizing $tp(F[A_i]/dom\, F_{\nu_{j^*}}^{j^*})$ such that A_{i+1} is indiscernible over $M \cup \bigcup_{j \leq i} A_j$. Let $F_{\nu_i}^i$ be the elementary map extending $F_{\nu_{j^*}}^{j^*}$ taking A_i to A_{i+1}. If $\epsilon_i = -1$ or if j^* does not exist, the induction step is similar. Now let F_{ν_j} be an automorphism of N extending $F_{\nu_i}^i$ where i is the largest index such that $\nu_i = \nu_j$. $F_{\nu_n}^{\epsilon_n} \dots F_{\nu_0}^{\epsilon_0} \neq id_N$ since $F_{\nu_n}^{\epsilon_n} \dots F_{\nu_0}^{\epsilon_0}(A_0) = A_{n+1}$ and $A_0 \cap A_{n+1} = \emptyset$ since A_{n+1} is indiscernible over $A_0 \cup M$.

Theorem 5 *Let T be a complete theory, M a saturated model of T of cardinality λ with $|T| \leq \lambda = \lambda^{<\lambda}$. Then $Aut(M)$ has a dense free subgroup on 2^λ generators.*

PROOF Let $TR = {}^{<\lambda}\lambda$. For $\alpha < \lambda$, let $TR_\alpha = {}^{<\alpha}\lambda$ and let $\bar{0}_\alpha$ be the function with domain α and range $\{0\}$. We define by induction on $\alpha < \lambda$ a model M_λ of T and $f_\eta \in Aut(M_\alpha)$ for $\eta \in TR_\alpha$ such that

1. $M_\alpha \models T$
2. $|M_\alpha| = \lambda$
3. $\langle M_\alpha \mid \alpha < \lambda \rangle$ is increasing continuous
4. If α is a successor, then M_α is saturated
5. $\nu \lhd \eta \rightarrow f_\nu \subseteq f_\eta$
6. If $\alpha = \beta + 1$, then $\langle f_\eta \mid \eta \in TR_\alpha \backslash \left\{ \bar{0}_\beta \frown i \mid i < \lambda \right\} \rangle$ is free

For $\alpha = \beta + 1$ we let $\langle f_{\bar{0}_\beta \frown i} \mid i < \lambda \rangle$ be a sequence of automorphisms of M_α such that each finite partial automorphism of M_β has an extension in

$$\left\{ f_{\bar{0}_\beta \frown i} \mid i < \lambda \right\}.$$

If we succeed in doing the induction then for $\eta \in {}^\lambda\lambda$ if

$$f_\eta = \bigcup \left\{ f_\nu \mid \nu = \eta \restriction \alpha, \ \alpha < \lambda \right\}$$

then the f_η and the $\bigcup\limits_{\alpha < \lambda} M_\alpha$ are as required by the theorem. The only problem in the induction is at successor steps, so let $\alpha = \beta + 1$. Let

$$\Gamma = \left\{ \tau \mid \tau \text{ is a term of the form } f_{\nu_n}^{\epsilon_n} \dots f_{\nu_0}^{\epsilon_0} \right\}$$

such that

1. $\forall i < n+1, \ \epsilon_i \in \{-1, 1\}$
2. $\forall i < n+1, \ \nu_i \in TR_\alpha \backslash \left\{ \bar{0}_\beta \frown i \mid i < \lambda \right\}$
3. $\forall i < n+1, \ \nu_i = \nu_{i+1} \ \Rightarrow \ \epsilon_i = \epsilon_{i+1}$

Let $\langle \tau_i \mid i < \lambda \rangle$ be a well ordering of Γ. Let N be a λ^+ saturated, λ^+ homogenous model of T containing M_β. By induction on $\gamma < \lambda$ we define elementary submodels $M_{\beta,\gamma}$ of N and for every $\nu \in TR_\alpha \backslash \left\{ \bar{0}_\beta \frown i \mid i < \lambda \right\}$, $f_{\nu,\gamma}$ such that

1. $M_{\beta,0} = M_\beta$
2. $f_{\nu,0} = f_{\nu \restriction \alpha}$
3. If $\gamma = \zeta + 1$, $M_{\beta,\gamma}$ is saturated of cardinality λ
4. $\zeta < \gamma \ \Rightarrow \ f_{\nu,\zeta} \subseteq f_{\nu,\gamma}$
5. If $\gamma = \zeta + 1$ and $\tau_\zeta = f_{\nu_n}^{\epsilon_n} \ldots f_{\nu_0}^{\epsilon_0}$ then $f_{\nu_n,\gamma}^{\epsilon_n} \ldots f_{\nu_0,\gamma}^{\epsilon_0} \neq id_{M_{\beta,\gamma}}$

If we succeed in the induction then we can let $M_\alpha = \bigcup\limits_{\alpha<\lambda} M_{\beta,\gamma}$ and $f_\nu = \bigcup\limits_{\alpha<\lambda} f_{\nu,\gamma}$. The only non-trivial step in the induction is for successor steps, so let $\gamma = \zeta + 1$. By lemma 4 we can find automorphisms $F_{\nu_0,\gamma}, \ldots, F_{\nu_n,\gamma}$ of N extending $f_{\nu_0,\zeta}, \ldots, f_{\nu_n,\zeta}$ such that if $\tau_\zeta = f_{\nu_n}^{\epsilon_n} \ldots f_{\nu_0}^{\epsilon_0}$ then

$$F_{\nu_n,\gamma}^{\epsilon_n} \ldots F_{\nu_0,\gamma}^{\epsilon_0} \neq id_N$$

For $\nu \in TR_\alpha \backslash \left\{ \bar{0}_\beta \frown i \mid i < \lambda \right\}$, but not in $\{\nu_0, \ldots, \nu_n\}$, let $F_{\nu,\gamma}$ be an arbitrary extension of $f_{\nu,\zeta}$ to N. Let $M_{\beta,\gamma} \prec N$ be a saturated model of size λ such that

1. $M_{\beta,\zeta} \prec M_{\beta,\gamma}$
2. $M_{\beta,\gamma}$ contains witnesses to $F_{\nu_n,\gamma}^{\epsilon_n} \ldots F_{\nu_0,\gamma}^{\epsilon_0} \neq id_N$
3. $M_{\beta,\gamma}$ is closed under the $F_{\nu,\gamma}$

For each $\nu \in TR_\alpha \backslash \left\{ \bar{0}_\beta \frown i \mid i < \lambda \right\}$, let $f_{\nu,\gamma}$ be $F_{\nu,\gamma} \restriction M_{\beta,\gamma}$.

3 $|M| = \lambda < \lambda^{<\lambda}$

Throughout this section, by theorem 1 mentioned in the introduction, we can assume that T is stable. Although the proofs in this section are simple, there is an hidden element of complexity covered over by theorem 1.

Theorem 6 *Let $\langle M_i \mid i < \delta \rangle$ is an increasing elementary chain of models of T that are λ saturated with $cf\,\delta \geq \kappa_r(T)$. Then $\bigcup_{i<\delta} M_i$ is a λ saturated model of T.*

PROOF [Sh c] III 3.11

Lemma 7 *Let $\left\{C_i \mid i \in I\right\}$ be independent over A and let $\left\{D_i \mid i \in I\right\}$ be independent over B. Suppose that for each $i \in I$, $tp(C_i/A)$ is stationary. Let f be an elementary map from A onto B, and let for each $i \in I$, f_i be an elementary map extending f which sends C_i onto D_i. Then*

$$\bigcup_{i\in I} f_i$$

is an elementary map from $\bigcup_{i\in I} C_i$ onto $\bigcup_{i\in I} D_i$.

PROOF Left to the reader.

Theorem 8 *Let T be a complete stable theory and let M be a saturated model of T of cardinality $\lambda > |T|$. Then*

1. *$Aut(M)$ has a dense free subgroup G of cardinality 2^λ*
2. *In fact, if $\sigma \leq \lambda$ is regular, then there is a free subgroup G of $Aut(M)$ such that any partial automorphism f of M with $|dom\, f| < \sigma$ can be extended to an element of G.*

PROOF Let $\sigma + \kappa_r(T) \leq \kappa = cf(\kappa) \leq \lambda$. We define by induction on $i \leq \kappa$ an increasing continuous elementary chain of models M_i of T, ordinals α_i of cardinality 2^λ and families $\left\{g^i_\alpha \mid \alpha < \alpha_i\right\}$ of automorphisms of M_i and such that

1. $\alpha_0 = 2^\lambda$
2. $|M_i| = \lambda$
3. $\langle \alpha_j \mid j \leq i \rangle$ is increasing continuous
4. $\forall g \in Aut(M_i) \quad \bigvee_{\alpha<\alpha_i} g \subseteq g^{i+1}_\alpha$
5. For a fixed α, the g^i_α form an elementary chain

6. $\langle g_\alpha^{i+1} \mid \alpha < \alpha_{i+1} \rangle$ is free

7. If $i = j+1$, or $i = 0$ then M_i is saturated.

If we succeed in doing the induction then by theorem 6 $M_\kappa = \bigcup_{i<\kappa} M_i$ is a saturated model of cardinality λ. If we let for $\alpha < \bigcup_{i<\kappa} \alpha_i$,

$$g_\alpha = \bigcup \left\{ g_\alpha^j \mid \alpha_j > \alpha, j < \kappa \right\}$$

then $\left\{ g_\alpha \mid \alpha < \bigcup_{i<\kappa} \alpha_i \right\}$ is free by item 6. in the construction and is dense (in the strong sense of 2. of the theorem) by item 5.

The only difficulty in the induction is for $i = j+1$. Let $\left\{ p_\zeta \mid \zeta < \zeta^* \right\}$ list $S^1(acl\,\emptyset)$. (So $\zeta^* \le \lambda$) Let $\left\{ a_\gamma^\zeta \mid \zeta < \zeta^*, \gamma < \lambda \right\}$ be a set of elements independent over M_j such that $tp(a_\gamma^\zeta / M_j)$ is a nonforking extension of p_ζ. For every $g \in Aut(M_j)$ let $f^{[g]}$ be the permutation of ζ^* such that

$$f^{[g]}(\zeta) = \xi \;\Leftrightarrow\; g(p_\zeta) = p_\xi$$

List $Aut(M_j) \backslash \left\{ g_\alpha^j \mid \alpha < \alpha_j \right\}$ as

$$\langle g_\alpha^j \mid \alpha_j \le \alpha < \alpha_i \rangle$$

Let

$$A_i = M_j \cup \left\{ a_\gamma^\zeta \mid \zeta < \zeta^*, \gamma < \lambda \right\}$$

and let

$$\left\{ h_\alpha^i \mid \alpha < \alpha_i \right\}$$

be a set of free permutations of $Sym(\lambda)$. Define for $\alpha < \alpha_i$ a permutation $g_\alpha^{j,*}$ of A_i by letting $g_\alpha^{j,*} \restriction M_j = g_\alpha^j$ and

$$g_\alpha^{j,*}(a_\gamma^\zeta) = a_{h_\alpha^i(\gamma)}^{f^{[g_\alpha^j]}(\zeta)}$$

By lemma 7 each $g_\alpha^{j,*}$ is an elementary map. The $\langle g_\alpha^{j,*} \mid \alpha < \alpha_i \rangle$ are free. For suppose not. Then for some $\{\alpha_1, \ldots, \alpha_n, \alpha_{n+1}\} \subseteq \alpha_i$ we would have

$$g_{\alpha_1}^{j,*} \cdots g_{\alpha_n}^{j,*} = g_{\alpha_{n+1}}^{j,*}$$

so for every $\zeta < \zeta^*$

$$f[g^{j,*}_{\alpha_1} \dots g^{j,*}_{\alpha_n}](\zeta) = f[g^{j,*}_{\alpha_{n+1}}](\zeta)$$

and for every $\gamma < \lambda$,

$$h_{\alpha_1} \dots h_{\alpha_n}(\gamma) = h_{\alpha_{n+1}}(\gamma)$$

a contradiction to the freeness of the h_{α_i}. Let M_i be a model such that

1. $A_i \subseteq M_i \prec \mathfrak{C}^{eq}$
2. $(M_i, a)_{a \in A_i}$ is saturated
3. $|M_i| = \lambda$

This is possible as the theory of $(\mathfrak{C}^{eq}, a)_{a \in A_i}$ is stable in λ. As $(M_i, a)_{a \in A_i}$ is saturated we can for each $\alpha < \alpha_i$, let g^i_α be an extension of $g^{j,*}_\alpha$ to an automorphism of M_i.

4 $<\lambda$ Denseness

Theorem 9 *Let M be a saturated model of cardinality $\lambda > |T|$. Then $Aut(M)$ has a free $<\lambda$ dense free subgroup on 2^λ generators.*

PROOF If $\lambda^{<\lambda} = \lambda$, the proof given gives a $<\lambda$ dense free subgroup. So we can assume that T is stable in λ. We work in $\mathfrak{C}^{eq}$. Let $\langle p_i \mid i < i^* \leq \lambda \rangle$ list all types over $acl\,\emptyset$. Let $\left\{ a_{i,\zeta,\xi} \mid i < i^*,\ \zeta < \lambda,\ \xi < \lambda \right\}$ be independent over $\emptyset$, with $a_{i,\zeta,\xi}$ realizing p_i and

$$(M, a_{i,\zeta,\xi})_{(i,\zeta,\xi) \in i^* \times \lambda \times \lambda}$$

is saturated. Let $\left\{ f_\alpha \mid \alpha < 2^\lambda \right\}$ be a free subgroup of $Sym(\lambda)$. Let $\left\{ g_\alpha \mid \alpha < 2^\lambda \right\}$ be a list of permutations of subsets of M of cardinality $<\lambda$ such that for every $\alpha < 2^\lambda$, $acl\,\emptyset \subseteq dom\, g_\alpha$. Let $C_\alpha = dom\, g_\alpha\ (= ran\, g_\alpha)$. For some subset u_α of $i^* \times \lambda \times \lambda$ such that $|u_\alpha| \leq |C_\alpha| + \kappa_r(T)$,

$$C_\alpha \qquad \bigcup_{\{a_{i,\zeta,\xi} \mid (i,\zeta,\xi) \in u_\alpha\}} \qquad \{a_{i,\zeta,\xi} \mid (i,\zeta,\xi) \in i^* \times \lambda \times \lambda\}$$

We can find a $D_\alpha \supseteq C_\alpha$ and $v_\alpha \supseteq u_\alpha$ such that $|D_\alpha| = |C_\alpha|$, $|v_\alpha| \leq |C_\alpha| + \kappa_r(T)$, and for some extension g'_α of g_α, g'_α is an automorphism of D_α with $D_\alpha \supseteq \{a_{i,\zeta,\xi} \mid (i,\zeta,\xi) \in v_\alpha\}$ and

$$D_\alpha \underset{\{a_{i,\zeta,\xi} \mid (i,\zeta,\xi) \in v_\alpha\}}{\bigcup\!\!\!\!\!\!\bigcup} \{a_{i,\zeta,\xi} \mid (i,\zeta,\xi) \in i^* \times \lambda \times \lambda\}$$

Since

$$\{a_{i,\zeta,\xi} \mid (i,\zeta,\xi) \in i^* \times \lambda \times \lambda - v_\alpha\} \underset{\emptyset}{\bigcup\!\!\!\!\!\!\bigcup} \{a_{i,\zeta,\xi} \mid (i,\zeta,\xi) \in v_\alpha\}$$

we have

$$D_\alpha \underset{\emptyset}{\bigcup\!\!\!\!\!\!\bigcup} \{a_{i,\zeta,\xi} \mid (i,\zeta,\xi) \in i^* \times \lambda \times \lambda - v_\alpha\}$$

For each $\alpha < 2^\lambda$ let

$$G^\alpha_\zeta = \{\zeta < \lambda \mid \forall \xi < \lambda \, \forall i < i^* \; a_{i,\zeta,\xi} \notin D_\alpha\}$$

Let h_α be the map taking $a_{i,\zeta,\xi}$ to $a_{i',\zeta,f_\alpha(\xi)}$ for $\zeta \in G^\alpha_\zeta$ if $g_\alpha(p_i) = p_{i'}$. Since

$$D_\alpha \underset{\emptyset}{\bigcup\!\!\!\!\!\!\bigcup} \{a_{i,\zeta,\xi} \mid (i,\zeta,\xi) \in i^* \times \lambda \times \lambda - v_\alpha\}$$

and g'_α and h_α agree on $acl\,\emptyset$, $g'_\alpha \cup h_\alpha$ is an elementary map. Let g''_α be an extension of $g'_\alpha \cup h_\alpha$ to an automorphism of M. (This is possible as

$$(M,c)_{c \in D_\alpha \cup dom\, h_\alpha \cup ran\, h_\alpha}$$

is saturated.) If $\{\alpha_0, \ldots, \alpha_n\} \subseteq 2^\lambda$ then

$$G^{\alpha_0}_\zeta \cap \ldots \cap G^{\alpha_n}_\zeta \neq \emptyset$$

so the g''_α are free, and by construction g''_α extends g_α so the g''_α are $< \lambda$ dense.

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