Cardinalities of countably based topologies

Saharon Shelah

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Department of Mathematics, The Hebrew University, Jerusalem, Israel
[1] Department of Mathematics, Rutgers University, New Brunswick NJ, USA

ABSTRACT

Let T be the family of open subsets of a topological space (not necessarily Hausdorff or even T_0). We prove that if T has a countable base and is not countable, then T has cardinality at least continuum.

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Topological spaces are not assumed to be Hausdorff, or even T_0.

Theorem 1 *Let T be the set of open subsets of a topological space, and suppose that T has a countable base B (more precisely, B is a countable subset of T which is closed under finite intersections, and the sets in T are the unions of subsets of B). Then the cardinality of T is either $2^{\aleph_0}$ or $\leq \aleph_0$.*

This answers a question of Kishor Kale. We thank Wilfrid Hodges for telling us the question and for writing up the proof from notes. In a subsequent

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work we shall deal with the case $\lambda \leq |B| < 2^\lambda$, $|T| > |B|$, λ strong limit of cofinality $\aleph_0$ and prove that $|T| \geq 2^\lambda$

Our proof begins with some notation. A set Ω is given, together with a countable family B of subsets of Ω; $\Omega = \bigcup B$ and B is closed under finite intersections. We write T for the set of all unions of subsets of B. Thus T is a topology on Ω and B is a base for this topology.

We write X, Y etc. for subsets of Ω. We write $T(X)$ for the set $\{X \cap Y : Y \in T\}$, and likewise $B(X)$ with B in place of T. We say X is *small* if $|T(X)| \leq \omega$, and *large* otherwise.

Lemma 2 *If $|\Omega| = \aleph_0$ and $|T| > \aleph_0$ then $|T| = 2^{\aleph_0}$.*

PROOF: Identify Ω with the ordinal ω, and list the set B by a function ρ with domain ω, so that $B = \{\rho(m) : m < \omega\}$. Then a set X is in T if and only if

$$(\exists Y \subseteq \omega)(\forall n \in \omega)\ (n \in X \leftrightarrow \exists m(m \in Y \wedge n \in \rho(m))).$$

Thus B is an analytic set, and so its cardinality must be either $2^{\aleph_0}$ or $\leq \aleph_0$ (cf. Mansfield & Weitkamp [1] Theorem 6.3). $\square_2$

Lemma 3 *Suppose Ω is linearly ordered by some ordering $\preceq$ in such a way that the sets in T are initial segments of Ω and any initial segment of the form $(-\infty, x)$ is open. If $|T| > \aleph_0$ then $|T| = 2^{\aleph_0}$.*

PROOF: As B is countable, the linear order has a countable dense subset D, but as T is countable, the rationals are not embeddable in D, i.e. D is scattered. By Hausdorff's structure theorem for scattered linear orderings, D has at most countably many initial segments (cf. Mansfield & Weitkamp [1] Theorem 9.21). $\square_3$

Henceforth we assume that Ω is uncountable and large, and that $|T| < 2^{\aleph_0}$, and we aim for a contradiction. Replacing Ω by a suitable subset if necessary, we can also assume:

Hypothesis The cardinality of Ω is $\aleph_1$.

Finally we can assume without loss that if x, y are any two distinct elements of Ω then there is a set in B which contains one but not the other. (Define x

and y to be *equivalent* if they lie in exactly the same sets in B. Choose one representative of each equivalence class.)

Lemma 4 *If for each $n < \omega$, X_n is a small subset of Ω, then $\bigcup_{n<\omega} X_n$ is small.*

PROOF: Each X_n has a countable subset Y_n such that if V, W are elements of T with $V \cap X_n \neq W \cap X_n$ then there is some element $y \in Y_n$ which is in exactly one of V, W. Now if V, W are elements of T which differ on $\bigcup_{n<\omega} X_n$, then they already differ on some X_n and hence they differ on $Y = \bigcup_{n<\omega} Y_n$. But Y is countable; so Lemma 2 implies that either Y is small or $|T(Y)| = 2^{\aleph_0}$. The latter is impossible since $|T| < 2^{\aleph_0}$, and so Y is small, hence $\bigcup_{n<\omega} X_n$ is small. $\square_4$

Our main argument lies in the next lemma, which needs some further notation. Let Z be a subset of Ω. The *Z-closure* of a subset X of Z is the set $\mathrm{cl}_Z(X)$ of all elements y of Z such that every set in B which contains y meets X. Given an element x of Z and a subset X of Z, we write $\mathrm{back}_Z(x, X)$ for the set $\{y \in Z : y \notin X \cup \mathrm{cl}_Z\{x\}\}$.

Lemma 5 *Suppose Z is a large subset of Ω. Then there are an element x of Z and a set $X \in B$ such that $x \in X$ and $\mathrm{back}_Z(x, X)$ is large.*

PROOF: Assume Z is a counterexample; we shall reach a contradiction. By a *Z-rich set* we mean a subset N of $Z \cup \mathcal{P}Z$ such that

- N is countable.
- If $x \in N$ and $X \in B$ then $\mathrm{back}_Z(x, X) \in N$.
- If U is a subset of Z which is a member of N and is small, and V, W are elements of T such that $V \cap U \neq W \cap U$, then there is some element of $N \cap U$ which lies in exactly one of V and W.

Since Z has cardinality at most ω_1, we can construct a strictly increasing continuous chain $\langle N_i : i < \omega_1 \rangle$ of Z-rich sets, such that $Z \subseteq \bigcup_{i<\omega_1} N_i$.

Let us say that an element x of Z is *pertinent* if there is some $i < \omega_1$ such that $x \in N_{i+1} \setminus N_i$, and x lies in some small subset of Z which is in N_i. If z is not pertinent, we say it is *impertinent*.

We claim that if V, W are any two distinct members of $T(Z)$ then some impertinent element is in exactly one of V and W. For this, consider the least $i < \omega_1$ such that some element z of $N_{i+1} \setminus N_i$ is in the symmetric difference of V and W. If z is pertinent, then by the last clause in the definition of Z-rich sets, some element of N_i already distinguishes V and W, contradicting the choice of i. This proves the claim.

Now let I be the set of all impertinent elements of Z. Since Z is large and $N_0 \cap Z$ is small, the claim implies that I is large. Thinning the chain if necessary, we can arrange that for each $i < \omega_1$, $N_{i+1} \setminus N_i$ contains infinitely many elements of I.

We can partition I into countably many sets, so that for every $i < \omega_1$, each set meets $I \cap (N_{i+1} \setminus N_i)$ in exactly one element. By Lemma 4 above, since I is large, at least one of these partition sets must be large. Let J be a large partition set. We define a binary relation $\preceq$ on J by:

$$x \preceq y \ \Leftrightarrow \text{ for all } U \in B \text{, if } y \in U \text{ then } x \in U.$$

We shall reach a contradiction with Lemma 3 by showing that $\preceq$ is a linear ordering and $T(J)$ is a set of initial segments of J under $\preceq$ which contains all the initial segments of the form $\{x : x \prec y\}$.

The relation $\preceq$ is clearly reflexive and transitive. We made it antisymmetric by assuming that no two distinct elements of Ω lie in exactly the same sets in B. We must show that if x and y are distinct elements of Z then either $x \preceq y$ or $y \preceq x$.

Let x, y be a counterexample, so that there are sets $X, Y \in B$ with $x \in X \setminus Y$ and $y \in Y \setminus X$. By symmetry and the choice of J we can assume that for some $i < \omega_1$, $x \in N_i$ and $y \in N_{i+1} \setminus N_i$. Since y is impertinent, no small set containing y is in N_i. In particular $\text{back}_Z(x, X)$ contains y and hence is not both small and in N_i. But since N_i is Z-rich, it contains $\text{back}_Z(x, X)$. Also we assumed that Z is a counterexample to the lemma; this implies that $\text{back}_Z(x, X)$ is small. We have a contradiction.

Thus it follows that $\preceq$ is a linear ordering of J, and the definition of $\preceq$ then implies that $T(J)$ is a set of initial segments of $\preceq$. As B separates points, every set $\{x : x \prec y\}$ is open. This contradicts Lemma 3 and so proves the present lemma. $\square_5$

PROOF OF THEOREM 1: Now we can finish the proof of the theorem. We shall find elements x_n of Ω and sets $X_n \in T$ $(n < \omega)$ such that $x_m \in X_n$

if and only if $m = n$. By taking arbitrary unions of the sets X_n it clearly follows that $|T| = 2^\omega$.

We define x_n and X_n by induction on n. Writing Z_{-1} for Ω and Z_n for $\mathrm{back}_{Z_{n-1}}(x_n, X_n)$, we require that $x_{n+1} \in Z_n$ and each set Z_n is large. Since Ω is large, Lemma 5 tells us that we can begin by choosing x_0 and X_0 so that $\mathrm{back}_\Omega(x_0, X_0)$ is large.

After x_n and X_n have been chosen, we use Lemma 5 again to choose x_{n+1} in Z_n and Y_{n+1} in B so that $x_{n+1} \in Y_{n+1}$ and $\mathrm{back}_{Z_n}(x_{n+1}, Y_{n+1})$ is large. For each $m \leq n$, x_{n+1} is in Z_m and hence it is not in $\mathrm{cl}_{Z_{m-1}}\{x_m\}$, so that there is some set $U_m \in B$ which contains x_{n+1} but not x_m. Put $X_{n+1} = \bigcap_{m \leq n} U_m \cap Y_{n+1}$. (Note that this is the one place where we use the fact that B, and hence also T, is closed under finite intersections.) Since $X_{n+1} \subseteq Y_{n+1}$, $\mathrm{back}_{Z_n}(x_{n+1}, X_{n+1})$ is large.

We must show that this works. First, $x_n \in X_n$ for each n by construction. Next, if $m \leq n$ then x_{n+1} is in Z_m and hence it is not in X_m. Finally if $m \leq n$ then $x_m \notin X_{n+1}$ by the definition of X_{n+1}. $\square_5$

The following theorem has a similar proof. We omit details, except to say that (i) "countable" is replaced by "of cardinality at most $|B|$", and ω_1 by $|B|^+$, and (ii) a more complicated analogue of Lemma 2 is needed.

Theorem 6 *Let T be the set of open subsets of a topological space Ω (not neccessarily Hausdorff, nor even T_0), and suppose that T has a base B which is closed under finite intersections, and $|T| > |B| + \aleph_0$. Then*

(1) there are $x_n \in \Omega$ and $X_n \in B$ for $n < \omega$ such that for all $m, n < \omega$, $x_n \in X_m$ iff $m = n$, and

(2) $|T| \geq 2^{\aleph_0}$.

One naturally asks whether we can let B in Theorem 1 be any set such that T is the set of unions of sets in B, without the requirement that B is closed under finite intersections. The answer is no, for the following reason.

Lemma 7 *Suppose there is a tree S with δ levels, μ nodes and at least λ branches of length δ, where $\lambda \geq \mu$; suppose also that S is normal (i.e. at each limit level there are never two or more nodes with the same predecessors). Then there are a set Ω of cardinality λ and a family of μ subsets of Ω which has exactly λ unions.*

CONSTRUCTION: Let Ω be a set of λ branches of length δ; for each $s \in S$ let U_s be $\{x \in \Omega : s \notin x\}$. Lastly let B be the family of sets $\{U_s : s \in S\}$, so that $|B| = \mu$. Now the sets in T are: members of B, Ω itself and complements of singletons; so $|T| = \lambda$. $\square_7$

Thus by starting with the full binary tree of height ω, we can build examples where B is countable and T is any cardinal between ω and 2^ω.

References

[1] Richard Mansfield and Galen Weitkamp, Recursive aspects of descriptive set theory, Oxford Univ. Press 1985.