UNIVERSAL IN $(<\lambda)$-STABLE ABELIAN GROUP SH456

Saharon Shelah

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

Rutgers University
Department of Mathematics
New Brunswick, NJ USA

Abstract. A characteristic result is that if $2^{\aleph_0} < \mu < \mu^+ < \lambda = \mathrm{cf}(\lambda) < \mu^{\aleph_0}$, then among the separable reduced p-groups of cardinality λ which are $(<\lambda)$-stable there is no universal one.

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We deal with the existence of reduced separable (abelian) groups of cardinality λ under usual embeddings (or similar classes of abelian groups or modules). So this continues Kojman Shelah [KjSh 409], [KjSh 449], [KjSh 455], but we make the presentation self contained so we repeat some things from there. They deal with proving the non-existence of universal members in cardinality λ for various classes: [KjSh 409] deals with linear orders and f.o. theories with the strict order property; [KjSh 449] deals with unsuperstable f.o. theories both under elementary embeddings and [KjSh 455] deals with classes of abelian groups under pure embeddings.
If $\lambda = \lambda^{\aleph_0}$ there are universal groups of cardinality λ (among the reduced separable abelian p-groups): compact ones (see [Fu], [KjSh 455]).
If $2^{\aleph_0} < \lambda = \mathrm{cf}(\lambda)$ and $\mu^+ < \lambda < \mu^{\aleph_0}$ then by Kojman Shelah [KjSh 455] we have non-existence results for pure embeddings. Here we get the results for any embedding, restricting ourselves to $(< \lambda)$-stable groups (see Definition 3 below). Without this restriction we shall deal with it in [Sh 552].
If $\lambda < 2^{\aleph_0}$ on independence results for existence see [Sh 552]. More results in these directions may be found in [Sh 457], [Sh 500].

§1

1.1 Context. Fix $\bar{n} = \langle n_i : i < \omega \rangle$ a sequence of natural numbers > 1. We shall deal only with abelian groups, so we may omit "abelian".
For an (abelian) group G define a prenorm $\|x\| = \min\{2^{-i} : x \text{ divisible by } \prod_{j<i} n_j\}$ in G. Let d be the induced semi-metric; i.e. distance is $d(x,y) = \|x - y\|$. When in doubt use $d_{\bar{n}}, \|x\|_{\bar{n}}$.

1.2 Notation. Let "group" mean "abelian group".
We concentrate on the classes
(1) $\mathfrak{K}^{\rm rs}_p = \{G : G$ is a p-group which is reduced and separable$\}$
(where G is separable if every pure subgroup of rank 1 is a direct summand,
G is a p-group means $(\forall x \in G)(\exists n < \omega)[n \geq 1 \ \& \ p^n x = 0]$ and the norm here is $\|x\|_{<p:i<\omega>}$.
(2) $\mathfrak{K}^{\rm tfr}_{\bar{n}} = \{G : G$ is a torsion free group and ,
$d_{\bar{n}}$ is a metric (equivalently:$\| - \|_{\bar{n}}$ is a norm)$\}$.
We say that G is $\bar{n}$-reduced, if $G \in \mathfrak{K}^{\rm tfr}_{i\bar{n}}$; reduced means $\langle i : i < \omega \rangle$-reduced.
(3) Let $\mathfrak{K}_\lambda = \{G \in \mathfrak{K} : \|G\| = \lambda\}$.

1.3 Definition. We say that G is $(< \lambda)$-stable if:
$A \subseteq G, |A| < \lambda \Rightarrow \ \text{closure}_G(\langle A \rangle_G) = c\ell_G(\langle A \rangle_G) = \{x : d(x, \langle A \rangle_G) = 0\}$
has cardinality $< \lambda$ (where $\langle A \rangle_G =$ the subgroup of G generated by A and $d(x,A) = \inf\{d(x,y) : y \in A\}$).

1.4 Remark. $\| - \|$ is a norm on G if $\|x\| = 0 \Rightarrow x = 0$. For torsion free groups this means no non-zero homomorphic image of $\left\langle \left\{ \frac{1}{\prod_{j<i} n_j} : i < \omega \right\} \right\rangle$ is embeddable.

1.5 Claim. *If G is a strongly λ-free group of cardinality λ <u>then</u> G is $(< \lambda)$-stable. If $(\forall \mu < \lambda)(\mu^{\aleph_0} < \lambda)$ <u>then</u> every*[1] *G is $(< \lambda)$-stable.*

Remark. The notion "G is strongly λ-free" is well known. It means:
$(\forall H)(\exists K)[H \subseteq G \ \& \ |H| < \lambda \rightarrow H \subseteq K \subseteq G \ \& \ K$ is free and G/K is λ free $]$.
See [Ek1].

1.6 Discussion. Now in the results of [KjSh 455], we can consider not necessarily pure embeddings. A difference from there is that here we add

$$A_\delta \subseteq \{\alpha \in C_\delta : cf(\alpha) > \aleph_0\}$$

(see [KjSh 455], Lemma 2, clause (iii).)

[1] i.e. $G \in \mathfrak{K}^{\rm rs}_p$, closure under $d_{\langle p:i<\omega \rangle}$ <u>or</u> $G \in \mathfrak{K}^{\rm tfr}_{\bar{n}}$, closure under $d_{\bar{n}}$, see 2.

1.6 Definition. Suppose that λ is a regular uncountable cardinal and that G is a group of cardinality λ.
1) A sequence $\bar{G} = \langle G_\alpha : \alpha < \lambda \rangle$ is called a λ-representation of G if and only if for all α:

(1) $G_\alpha \subseteq G_{\alpha+1}$
(2) G_α is of cardinality smaller than λ
(3) if α is limit then $G_\alpha = \bigcup\limits_{\beta<\alpha} G_\beta$
(4) $G = \bigcup\limits_{\alpha<\lambda} G_\alpha$.

2) Suppose $\bar{G} = \langle G_\alpha : \alpha < \lambda \rangle$ is a given representation of a group G. Suppose $c \subseteq \lambda$ is a set of ordinals, and the increasing enumeration of c is $\langle \alpha_i : i < i(*) \rangle$. Let $g \in G$ be an element. We define a way in which g chooses a subset of c:

$$\begin{aligned}\mathrm{Inv}_{\bar{G}}(g, c) = \{\alpha_i \in c : &\text{ for some } n \in [2, \omega) \text{ we have} \\ & g \in (G_{\alpha_i+1} + nG) \text{ but } g \notin (G_{\alpha_i} + nG))\}.\end{aligned}$$

We call $\mathrm{Inv}_{\bar{G}}(g, c)$ the invariant of the element g relative to the λ-representation $\bar{G}$ and the set of indices c.

Worded otherwise, $\mathrm{Inv}_{\bar{G}}(g, c)$, is the subset of those indices α_i such that by increasing the group G_{α_i} to the larger group $G_{\alpha_{i+1}}$, an n-congruent for g is introduced for some n.

3) Suppose that $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is a club guessing sequence; i.e. $C_\delta \subseteq \delta$ and for every club E of λ for stationarily many $\delta \in S$, we have $C_\delta \subseteq E$; we do not require that C_δ is a club of δ and that $\bar{G}$ is a λ-representation of a group G of cardinality λ. Let

(a) $P_\delta(\bar{G}, \bar{C}) = \{\mathrm{Inv}_{\bar{G}}(g, C_\delta) : g \in G\}$
(b) $P'_\delta(\bar{G}, \bar{C}) = \{\mathrm{Inv}_{\bar{G}}(g, C_\delta) : x \in G, \text{ moreover } x \in c\ell_G(G_\delta)\}$
(c) $\mathrm{INV}(G, \bar{C}) = [\langle P'_\delta(\bar{G}, \bar{C}) : \delta \in S \rangle]/id(\bar{C})$.

The second item should read "the equivalence class of the sequence of P_δ's modulo the ideal $\mathrm{id}(\bar{C})$", where two sequences are equivalent modulo an ideal if the set of coordinates in which the sequences differ is in the ideal and

$$\mathrm{id}(\bar{C}) = \mathrm{id}^a(\bar{C}) = \{A \subseteq \lambda : \text{ for every club } E \text{ of } \lambda \text{ for no } \delta \in A \cap E, C_\delta \subseteq E\}.$$

1.7 Fact. For $\lambda, G, \bar{G}, \bar{C}, C$ as above:

(1) $\mathrm{Inv}_{\bar{G}}(y, C)$ is a countable subset of C
(2) $P_\delta(\bar{G}, \bar{C})$ is a family of countable subsets of C_δ of cardinality $\leq |C|^{\aleph_0}$ and $\leq \lambda$
(3) $P'_\delta(\bar{G}, \bar{C})$ is a subset of $P_\delta(\bar{G}, \bar{C})$, if $G \in \mathfrak{K}_p^{\mathrm{rs}}$ then $P'_\delta(\bar{G}, \bar{C}) = P_\delta(\bar{G}, \bar{C})$.
(4) If in addition G is $(< \lambda)$-stable then $P'_\delta(\bar{G}, \bar{C})$ has cardinality $< \lambda$.
(5) $\mathrm{INV}(G, \bar{C})$ really does not depend on the choice of the representation $\bar{G}$.

Proof. Straight.

1.8 Lemma. *Let $\lambda > 2^{\aleph_0}$ be regular, $\bar{C} = \langle C_\delta : \delta \in S\rangle$ be a club guessing sequence, $\bar{A} = \langle A_\delta : \delta \in S\rangle$, $A_\delta \subseteq C_\delta$, $otp(A_\delta) = \omega$ and $\alpha \in A_\delta \Rightarrow cf(\alpha) > \aleph_0$.*
Then there is a $(<\lambda)$-stable separable p-group G, and $|G| = A$ such that

- $(*)$ *if H is $(<\lambda)$-stable separable p-group H, $|H| = \lambda, G$ embeddable into $H, \bar{H}$ a representation of H then for $id(\bar{C})$-almost every $\delta \in S$ we have*
- $(**)$ *$A_\delta \subseteq B$ for some $B \in P_\delta(\bar{H}, \bar{C})$.*

Remark. The proof is closely related to [Sh:e], III, 7.15.

Proof. Let for $\delta \in S, \eta_\delta$ be an increasing ω-sequence of ordinals enumerating A_δ.
Let
$\Gamma_n = \left\{\rho : \rho \in {}^n\omega, \rho \text{ strictly increasing sequence of length } n, \rho(\ell) \in \omega\backslash\{0\}\right\}$
and let $\Gamma = \bigcup\limits_{n<\omega} \Gamma_n$ and $\Gamma_\omega = \{\rho : \rho \in {}^\omega\omega \text{ and } n < \omega \Rightarrow \rho \restriction n \in \Gamma_n\}$.
Let G^* be generated as a group by $x_{\eta,\rho}$ where for some $n < \omega$ we have $\eta \in {}^n\lambda$ and $\rho \in \Gamma_n$ freely except:

$$p^{g(\rho)+1} x_{\eta,\rho} = 0 \text{ where } g(\rho) =: \sum\{\rho(m) : m < \ell g(\rho)\}.$$

Let G be generated by $x_{\eta,\rho}$ (where $\bigvee\limits_n (\eta \in {}^n\lambda \ \ \& \ \ \rho \in \Gamma_n)$), $y_{\delta,\rho,n}$ ($\delta \in S$, $\rho \in \Gamma$ and $n < \omega$) freely except:

$$\oplus_1 \qquad p^{g(\rho)+1} x_{\eta,\rho} \equiv 0 \text{ (for } \eta \in {}^n\lambda \text{ and } \rho \in \Gamma_n)$$

$$\oplus_2 \qquad y_{\delta,\rho,n} - p^{\rho(n)} y_{\delta,\rho,n+1} = x_{\eta_\delta \restriction n, \rho\restriction n} \text{ (for } \delta \in S, \rho \in \Gamma_\omega).$$

Let G_α be the subgroup of G generated by

$$Y_\alpha = \{x_{\eta,\rho} : (\exists n)(\eta \in {}^n\alpha \ \ \& \ \ \rho \in \Gamma_n\} \cup \{y_{\delta,\rho,n} : \delta \in S \cap \alpha, \rho \in \Gamma_\omega, n < \omega\}.$$

Clearly G is from $\mathfrak{K}_p^{\text{tr}}$, has cardinality λ and $\bar{G} = \langle G_\alpha : \alpha < \lambda\rangle$ is a representation of G. In fact G_α is the group generated by Y_α freely except the equations of the form $\oplus_1, \oplus_2$ required above which involve only its generators.

Suppose $f : G \to H$ is an embedding and H (not necessarily pure) $\bar{H}$ are in $(*)$ of the Lemma.
Let

$$\begin{aligned} E =: \Big\{\alpha < \lambda : & f(G_\alpha) \subseteq H_\alpha, f^{-1}(H_\alpha \cap \text{ Rang}(f)) \subseteq G_\alpha, \\ & \alpha \text{ a limit ordinal and for every } \beta < \alpha \\ & \text{for some } \gamma, \beta < \gamma < \alpha \text{ and} \\ & c\ell_G(G_\beta) \subseteq G_\gamma, c\ell_H(H_\beta) \subseteq H_\gamma, \text{ for every } x_{\eta,\rho} \in G \\ & \text{we have } [x_{\eta,\rho} \in G_\alpha \Leftrightarrow \eta \in {}^{\omega>}\alpha]\Big\}. \end{aligned}$$

Hence

$\oplus_3$ if $\alpha \in E, y \in H\backslash H_\alpha$ and $\mathrm{cf}(\alpha) > \aleph_0$ then $d(g, H_\alpha) > 0$

$\oplus_4$ if $\alpha \in E, \mathrm{cf}(\alpha) > \aleph_0, (\exists n)[\eta \in {}^n\lambda \ \& \ \rho \in \Gamma_n]$ but $\eta \notin {}^{\omega>}\alpha, x_{\eta,\rho} \in G$ then: $p^{g(\rho)} x_{\eta,\rho} \in G\backslash G_\alpha$ hence $p^{g(\rho)} f(x_{\eta,\rho}) = f(p^{g(\rho)} x_{\eta,\rho}) \in H\backslash H_\alpha$ hence $d(p^{g(\rho)} x_{\eta,\rho}, H_\alpha) > 0$.

Let $\delta \in E$ be such that $C_\delta \subseteq E$. Define a function $h : \Gamma \to \omega$ such that:

$$(*)_0 \qquad \text{for } \rho \in {}^{n+1}\omega \text{ we have } d\left(f(p^{g(\rho)} x_{\eta_\delta \restriction (n+1),\rho}), H_{\eta_\delta(n)}\right) \geq 2^{-h(\rho)}.$$

Note: h is defined by $\oplus_4$.
Next choose a strictly increasing function $\rho^* \in {}^\omega\omega$ such that:

$$\text{for every } n < \omega \text{ we have } \rho^*(n) > h(\rho^* \restriction n)$$

(this is an overkill). Remembering $g(\eta) = \sum\limits_{m<\ell g(\eta)} \rho^*(m)$ we clearly can prove by induction on n that (use $\oplus_2$ and the definition of g):

$$(*)_1 \ \ y_{\delta,\rho^*,0} = \sum_{m<n} p^{g(\rho^* \restriction m)} x_{\eta_\delta \restriction m, \rho^* \restriction m} + p^{g(\rho^* \restriction n)} y_{\delta,\rho^*,n}.$$

Note: $x_{\eta_\delta \restriction (n+1), \rho^* \restriction (n+1)} \in G_{\eta_\delta(n+1)} \backslash G_{\eta_\delta(n)}$ and $\eta_\delta(\ell) < \eta_\delta(n)$ for $\ell < n$ hence $\sum\limits_{m<n} p^{g(\rho^* \restriction m)} x_{\eta_\delta \restriction m, \rho^* \restriction m} \in G_{\eta_\delta}(n)$. By this and $(*)_1$ for each $n < \omega$ (use $n+2$ above)

$$(*)_2 \ \ y_{\delta,\rho^*,0} - p^{g(\rho^* \restriction (n+1))} x_{\eta_\delta \restriction (n+1), \rho^* \restriction (n+1)} \in G_{\eta_\delta(n)} + p^{g(\rho^* \restriction (n+2))} G$$

hence (as f is an embedding of G into H mapping $G_{\eta_\delta(n)}$ into $H_{\eta_\delta(n)}$ as $\eta_\delta(n) \in A_\delta \subseteq C_\delta \subseteq E$):

$$(*)_3 \ \ f(y_{\delta,\rho^*,0}) - p^{g(\rho^* \restriction (n+1))} f(x_{\eta_\delta \restriction (n+1), \rho^* \restriction (n+1)}) \in H_{\eta_\delta(n)} + p^{g(\rho^* \restriction (n+2))} H.$$

By the choice of ρ^* we have

$$\rho^*(n+1) > h(\rho^* \restriction (n+1))$$

and hence by the choice of h (i.e. by $(*)_0$) we conclude

$$(*)_4 \ \ p^{g(\rho^* \restriction (n+1)} f(x_{\eta_\delta \restriction (n+1), \rho^* \restriction (n+1)}) \notin H_{\eta_\delta(n)} + p^{g(\rho^* \restriction (n+2)} H.$$

By $(*)_3$ and $(*)_4$ we get

$$(*)_5 \ \ f(y_{\delta,\rho^*,0}) \notin H_{\eta_\delta(n)} + p^{g(\rho^* \restriction (n+2))} H.$$

Now use $(*)_1$ for $\delta, \rho^*, n+2$: note that $\sum\limits_{m<n+2} p^{g(\rho^* \restriction m)} x_{\eta_\delta \restriction m, \rho^* \restriction m}$ belongs to G_γ if $\bigcup\limits_{m<n+2} \mathrm{Rang}(n_\delta \restriction m) \subseteq \gamma \in E$, but the maximal ordinal in $\bigcup\limits_{m<n+2} \mathrm{Rang}(\eta_\delta \restriction m)$ is $\eta_\delta(n)$; hence

$$(*)_6 \ \ y_{\delta,\rho^*,0} \in H_{\min(C_\delta \backslash (\eta_\delta(n)+1))} + p^{g(\rho^* \restriction (n+2)} H.$$

Apply f on $(*)_6$ as $C_\delta \subseteq E$ we have

$(*)_7$ $f(y_{\delta,\rho^*,0}) \in H_{\min(C_\delta \setminus (\eta_\delta(n)+1)} + p^{g(\rho^* \restriction (n+2))} H$.

So by $(*)_5 + (*)_7$ we deduce $\eta_\delta(n) \in \mathrm{Inv}_{\bar H}\,(f(y_{\delta,\rho^*,0}), C_\delta)$ for each n, so $A_\delta \subseteq \mathrm{Inv}_{\bar H}\,(f(y_{\delta,\rho^*,0}), C_\delta)$, but the later belongs to $P_\delta(\bar H, C_\delta)$ as required. $\square_{1.8}$

1.9 Conclusion. Assume $\lambda = cf(\lambda) > 2^{\aleph_0}, \mu^+ < \lambda < \mu^{\aleph_0}$. Then in the class

$$\mathfrak{K}^{\mathrm{rs,st}}_{p,\lambda} =: \{G : G \text{ is an abelian reduced, separable } p\text{-group, and } G \text{ is } (<\lambda)\text{-stable of cardinality } \lambda\}$$

there is no universal member under (usual) embedding.

Proof. By [Sh 420], 1.8 we can find stationary $S \subseteq \lambda$ and $\bar C = \langle C_\delta : \delta \in S\rangle, C_\delta$ a subset of $\delta, \mathrm{otp}(C_\delta) = \mu \times \omega$ (in fact, C_δ closed in $\sup(C_\delta)$, which is not necessarily δ) such that $\mathrm{id}^a(\bar C)$ is a proper ideal on λ (i.e. $\lambda \notin \mathrm{id}^a(\bar C)$): remember

$$\mathrm{id}^a(C) = \{A \subseteq \lambda : \text{for some club } E \text{ of } \lambda \text{ for some } \delta \text{ is } C_\delta \subseteq E\}.$$

Clearly $\mu > \aleph_1$ (as we can replace C_δ by any $C'_\delta \subseteq C_\delta$ of order type μ) so without loss of generality if $\alpha \in \mathrm{nacc}(C_\delta) =: \{\beta \in C_\beta : \beta > \sup(\beta \cap C_\beta)\}$ then $\mathrm{cf}(\beta) > \aleph_0$. Suppose $H \in \mathfrak{K}^{\mathrm{rs,st}}_{p,\lambda}$ is universal in it, let $\bar H = \langle H_i : i < \lambda\rangle$ be a representation of H. Let

$$P_\delta =: \{\mathrm{Inv}_{\bar H}(y, C_\delta) : y \in H\}$$

$$P'_\delta =: \{A \subseteq C_\delta : \text{for some } B \text{ we have } A \subseteq B \in P_\delta\}.$$

So P'_δ is a family of $\leq \lambda$ subsets of C_δ, hence there is $A_\delta \subseteq \mathrm{nacc}(C_\delta)$, unbounded in C_δ or order type ω such that $A_\delta \notin P'_\delta$. Now apply Lemma 1.8. $\square_{1.9}$

1.10 Claim. *In Lemma 1.8, instead of using Γ_ω use any $\Gamma' \subseteq \Gamma_\omega$ such that:*

$\otimes_1$ $(\forall h)(h$ *a function from* ${}^{\omega>}\omega$ *to* $\omega \Rightarrow$
$(\exists \rho^* \in \Gamma')(\forall^\infty n)(\rho^*(n) > h(\rho^* \restriction n))$

is enough.
So $|\Gamma| = \mathfrak{d}$ *is O.K.*

Proof. Reflect.

1.11 Claim. *1) We can phrase Lemma 1.6 by an invariant, letting $C_\delta = \{\alpha_i : i < otp(C_\delta)\}$ (increasing) by defining:*

$$P(x, C_\delta, \bar{G}) =: \Big\{ A \subseteq C_\delta : otp(A) = \omega, A = \{\alpha_{i(n)} : n < \omega\}, i(n) < i(n+1),$$

every $\alpha \in A$ has cofinality $> \aleph_0$, and there are $T \subseteq {}^{\omega >}\omega$, downward closed (by $\triangleleft$), $<> \in T$ such that $\eta \in T \Rightarrow (\exists^\infty n)[\eta\hat{\ } < n > \in T]$ and $x_\rho \in G_{\alpha_{i(n)+1}}$ has order $g(\rho)+1$, and for some $h : T \to \omega$, for every $\rho \in Lim(T) (\subseteq {}^\omega\omega)$ satisfying $(\forall n)[\rho(n) > h(\rho \restriction n)]$ there is y_ρ such that

$$y_\rho = \sum_{\ell < n} p^{g(\rho \restriction \ell)} x_{\rho \restriction \ell} \text{ mod } p^{g(\rho \restriction (n+1))} G \text{ for each } n \Big\}.$$

2) We can restrict ourselves to $\Gamma' \subseteq \Gamma_\omega$ as in Claim 1.10.

1.12 Claim. *1) We can deal similarly with the class $\mathfrak{K}^{\rm tf}_{\bar{n}}$, i.e. this class has no universal member in λ (under usual embedding if $2^{\aleph_0} < \mu, \mu^+ < \lambda = cf(\lambda) < \mu^{\aleph_0}$. In particular the class of reduced torsion free groups (using $\bar{n} = \langle i! : i < \omega \rangle$,etc).*
2) Similarly, for the class of metric spaces with embedding meaning one to one functions preserving $\lim x_n = x$.
3) If in Lemma 1.8 we weaken $(**)$ to

$(**)^-$ *$A_\delta \cap B$ is infinite for some $B \in P_\delta(\bar{H}, \bar{C})$, then this:*
- (a) *suffices in the proof of conclusion (a)*
- (b) *in order that it holds, the requirement in 1.10(1) can be weakened to:*

$\otimes_2$ *for every function h from ${}^{\omega >}\omega$ to ω we have $(\exists \rho^* \in \Gamma')(\exists^\infty n)[\rho^*(n) > h(\rho^* \restriction n)]$.*

4) If
- (a) *$\lambda = cf(\lambda) > \min\{\Gamma' \subseteq \Gamma_\omega : \Gamma'$ satisfies $\otimes_1\}$ or just*
- (a)$^-$ *$\lambda = cf(\lambda) > \min\{|\Gamma'| : \Gamma' \subseteq \Gamma_\omega, \Gamma'$ satisfies $\otimes_2\}$ and*
- (b) *for some $\mu < \lambda$ we have $cf([\mu]^{\aleph_0}, \subseteq) > \lambda$*

<u>then</u> $\mathfrak{K}^{\rm rs,st}_{p,\lambda}$ has no universal member.

(in the above definitions (a) or (a)$^-$ we can replace Γ_ω by ${}^\omega\omega$, does not matter).

Proof. E.g.
(3)(a) Just note that there is $\langle B^\delta_i : i < \mu^{\aleph_0}\rangle, B^\delta_i \subseteq C_\delta$ is unbounded in it and $i \neq j \Rightarrow B^\delta_i \cap B^\delta_j$ is finite. So every $B \in P_\delta$ (see the proof of Conclusion 1.9) we have $\{i < \mu^{\aleph_0} : B \cap B^\delta_i$ is infinite$\}$ has cardinality $\leq 2^{\aleph_0}$.
(3) The point is that we can find an order $<^*$ on $\mu \times \omega$ such that $(\mu \times \omega, <^*)$ is a

tree with ω levels the $(n+1)$-th level is $(\mu \times n, \mu \times (n+1))$, level 0 is $\{0\}$, level 1 is $[1, \mu)$, and each node has μ immediate successor, we can have $g : \mu \times \omega \to [\mu \times \omega]^{\aleph_0}$ such that $w \in [\mu \times \omega]^{<\aleph_0} \Rightarrow \mu \times \omega = \text{ otp}\{\alpha : g(\alpha) = w\}$.
If $\mathcal{P} \subseteq [\mu \times \omega]^{\aleph_0}$, $|\mathcal{P}| < \text{cf}([\mu]^{\aleph_0}, \subseteq)$ without loss of generality

$$B \in \mathcal{P} \ \& \ \alpha \in B \Rightarrow g(\alpha) \subseteq B.$$

Choose $A \in [\mu \times \omega]^{\aleph_0}$ not included in any $B \in A$, let $A = \{\alpha_n : n < \omega\}$, choose $\gamma_n, \gamma_{n+1} > \gamma_n + \mu, g(\gamma_n) = \{\alpha_\ell : \ell < n\}$.

§2 Large cofinality and singulars

We want to prove that $\mathfrak{K}^{\rm rs}_{p,\lambda}$ has large cofinality (parallel to [KjSh 409] §4). We meet a problem here as we have to deal with $y_{\delta,\rho,0}(\rho \in \Gamma_\omega)$ not just with $y_{\delta,0}$. Remember: existence of universal is equivalent to cofinality being 1, i.e. cofinality of $((\mathfrak{K}^{\rm rs}_p)_\lambda$, embedability) and, what is closely related, prove non-existence of universal in singular cardinals.

2.1 Definition. For an ideal J on λ and $\chi \geq \lambda$ let

$$\begin{aligned}\chi^{\langle J\rangle} = \ &\text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\chi]^\lambda, \text{ and for every } f \in {}^\lambda\chi \\ &\text{there is } B \in \mathcal{P}, \text{ such that} \\ &\{i < \lambda : f(i) \in B\} \neq \emptyset \bmod J\}\end{aligned}$$

$$\begin{aligned}\chi^{[\lambda]} = \ &\text{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\chi]^\lambda, \text{ and every } A \in [\chi]^\lambda \\ &\text{is included in the union of } < \lambda \text{ members of } \mathcal{P}\}.\end{aligned}$$

2.2 Definition. For $\lambda = \text{cf}(\lambda) > \aleph_0$, and a club guessing sequence $\bar C = \langle C_\delta : \delta \in S\rangle$ we define $J = J^\ell_*[\bar C]$ (for $\ell = 1,2$). It is the following family of subsets of $\lambda \times ({}^\omega\omega)$:

$$\Big\{A \subseteq \lambda \times ({}^\omega\omega) : \text{for some } B \in \text{id}^a[\bar C] \text{ for every} \\ \delta \in S \backslash B \text{ there is } h : {}^{\omega>}\omega \to \omega \text{ such that for no } \rho \\ \text{do we have: } (\delta,\rho) \in A \text{ and:} \\ \ell = 0 \Rightarrow \text{ for infinitely many } n < \omega, \rho(n) > h(\rho \restriction n) \\ \ell = 1 \Rightarrow \text{ for every large enough } n, \\ \rho(n) > h(\rho \restriction n)\Big\}.$$

2.3 Claim. *Assume that $\lambda = cf(\lambda) > 2^{\aleph_0}$, $\bar C = \langle C_\delta : \delta \in S\rangle$ is a club guessing sequence, $S \subseteq \lambda$, $\mu^+ < \lambda \leq \chi_0 \leq \chi_1 < \mu^{\aleph_0}$, $H_j \in \mathfrak{K}^{\rm rs,st}_{p,\leq \chi_0}$ for $j < \chi_1$, and $\chi^{[J^0_*[\bar C]]} < \mu^{\aleph_0}$. Then there is $G \in \mathfrak{K}^{\rm rs,st}_{p,\lambda}$ which is not not embeddable in H_γ for every $\gamma < \chi_1$.*

Proof. Without loss of generality $|H_\gamma| = \chi_0$ and the set of elements of H_γ is χ_0, let $\chi^*_0 = \chi_0^{[J^1_*[\bar C]]}$ and let $\mathcal{P} = \{A_\varepsilon : \varepsilon < \chi^*_0\}$ exemplifies $\chi^{[J^1_*[\bar C]]} = \chi^*_0$. Let $H_{\gamma,\varepsilon}$ be a pure subgroup of H_γ of cardinality λ including A_ε and $\bar H_{\gamma,\varepsilon} = \langle H_{\gamma,\varepsilon,i} : i < \lambda\rangle$ be a representation of $H_{\gamma,\varepsilon}$. Let $\bar C = \langle C_\delta : \delta \in S\rangle$ be as in the proof of Lemma

1.8, and choose $A_\delta \subseteq$ nacc(C_δ) unbounded of order type ω such that for no $\gamma < \chi$, and $\varepsilon < \chi_0^*$ is there a set B such that $B \in P_\delta(\bar{H}_{\gamma,\varepsilon}, \bar{C})$ and $A \cap B$ infinite (possible by cardinality considerations). Let G be as in Lemma 1.8 for $\lambda, \bar{C}, \langle A_\delta : \delta \in S\rangle$. If G is embeddable into say H_γ, let f be such embedding, so for some $\zeta < \chi_0^*$ for $\zeta < \zeta(*)$ we have

$$\{(\delta, \rho) : \delta \in S, \rho \in {}^\omega\omega\} \notin J_*^0[\bar{C}]\}.$$

Now repeat the proof of Lemma 1.8. $\square_{2.3}$

2.4 Conclusion. Assume $\lambda = \text{cf}(\lambda) > 2^{\aleph_0}, \mu^+ < \lambda < \chi_0 \leq \chi_1 < \mu^{\aleph_0}$ and $\chi_0^{[J_*^0[\bar{C}]]} < \mu^{\aleph_0}$ for some guessing club system $\bar{C} = \langle C_\delta : \delta \in S\rangle$, $S \subseteq \lambda$ stationary.

Then for every cardinal λ' (possibly singular) $\lambda \leq \lambda' \leq \chi_0$, the class $\mathfrak{K}_{p,\lambda}^{\text{rs,st}}$ has no universal member (under usual embeddings).

2.5 Question. Can we combine $1.10(4) + 1.2.4$?

* * *

Remark. The above can also be interpreted as giving negative results on the existence of models of cardinality $\chi_0 > \lambda$ universal for $\mathfrak{K}_{p,\lambda}^{\text{rs,st}}$, etc. For more results on non existence of universals see [Sh 552].

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