The Universality Spectrum : Consistency for more Classes

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Abstract. We deal with consistency results for the existence of universal models in natural classes of models (more exactly–a somewhat weaker version). We apply a result on quite general family to $T_{\rm feq}$ and to the class of triangle-free graphs

§0 Introduction:

The existence of universal structures, for a class of structures in a given cardinality is quite natural as witnessed by having arisen in many contexts. We had wanted here to peruse it in the general context of model theory but almost all will interest a combinatorialist who is just interested in the existence of universal linear order or a triangle free graph. For a first order theory (complete for simplicity) we look at the universality spectrum $\mathrm{USP}_T = \{\lambda : T$ has a universal model in cardinal $\lambda\}$ (and variants). Classically we know

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that under GCH, every $\lambda > |T|$ is in USP_T, moreover $2^{<\lambda} = \lambda > |T| \Rightarrow \lambda \in \mathrm{USP}_T$ (i.e.–the existence of a saturated or special model, see e.g. [CK]). Otherwise in general it is "hard" for a theory T to have a universal model (at least when T is unstable). For consistency see [Sh100], [Sh175], [Sh 175a], Mekler [M] and parallel to this work Kojman-Shelah [KjSh 456] ; on ZFC nonexistence results see Kojman-Shelah [KjSh409], [KjSh447], [KjSh455]. We get ZFC non existence result (for T^*_{feq} under more restriction , essentially cases of failure of SCH) in §2, more on linear orders (in §3), consistency of (somewhat weaker versions of) existence results abstractly (in §4) derived consistency results and apply them to the class of models of T_{feq} (an indexed family of independent equivalence relations) and to the class of triangle free graphs (in §5). The general theorem in §4 was intended for treating all simple theories (in the sense of [Sh 93] , but this is not included as it is probably too much model theory for the expected reader here (and for technical reasons).

§**1**

1.1 Definition: For a class $\mathbf{K} = (\mathbf{K}, \leq_{\mathbf{K}})$ of models

1) $\mathbf{K}_\lambda = \{M \in \mathbf{K} : \|M\| = \lambda\}$

2) $\mathrm{univ}(\lambda, \mathbf{K}) = \mathrm{Min}\,\{|\mathcal{P}| : \mathcal{P}$ a set of models from $\mathbf{K}_\lambda$ such that for every $N \in \mathbf{K}_\lambda$ for some $N \in \mathcal{P}$, M can be $\leq_{\mathbf{K}}$-embedded into $N\}$.

3) $\mathrm{Univ}(\lambda, \mathbf{K}) = \mathrm{Min}\,\{\|N\| : N \in \mathbf{K}$, and every $M \in \mathbf{K}_\lambda$ can be $\leq_{\mathbf{K}}$-embedded into $N\}$.

4) If $\mathbf{K}$ is the class of models of T, T a complete theory, we write T instead $(\mathrm{mod}\, T, \prec)$ (i.e. the class of model of T with elementary embeddings). If $\mathbf{K}$ is the class of models of T, T a universal theory, we write T instead $(\mathrm{mod}\,(T), \subseteq)$.

1.2 Claim: 1) $\mathrm{univ}(\lambda, \mathbf{K}) = 1$ iff $\mathbf{K}$ has a universal member of cardinality λ.

2) Let T be first order complete, $|T| \leq \lambda$. *Then* we have $\text{univ}(\lambda, T) \leq \lambda$ implies $\text{univ}(\lambda, \mathbf{K}) = 1$ and $\text{Univ}(\lambda, T) \leq \text{univ}(\lambda, T) \leq \text{cf}\left(\mathcal{S}_{\leq\lambda}(\text{Univ}(\lambda, T), \subseteq)\right) = \text{cov}\,(\text{Univ}(\lambda, T), \lambda^+, \lambda^+, 2)$ (see [Sh-g] ; we can replace T with $\mathbf{K}$ with suitable properties).

§2 The universality Spectrum of T_{feq}

For T_{feq}, a prime example for a theory with the tree order property (but not the strict order property), we prove there are limitations on the universality spectrum; it is meaningful when SCH fails.

2.1 Definition: T^*_{feq} is the model completion of the following theory, T_{feq}. T_{feq} is defined as follows:

(a) it has predicates P, Q (unary) E (three place, written as $yE_x z\}$

(b) the universe (of any model of T) is the disjoint union of P and Q , each infinite

(c) $yE_x z \rightarrow P(x) \,\&\, Q(y) \,\&\, Q(z)$

(d) for any fixed $x \in P$, E_x is an equivalence relation on Q with infinitely many equivalence classes

(e) if $n < \omega$, $x_1, \ldots, x_n \in P$ with no repetition and $y_1, \ldots, y_n \in Q$ *then* for some $y \in Q$, $\bigwedge_{\ell=1}^{n} yE_{x_\ell} y_\ell$.

(Note: T_{feq} has elimination of quantifiers).

2.2 Claim: Assume:

(a) $\theta < \mu < \lambda$

(b) $\text{cf}\,\lambda = \lambda, \theta = \text{cf}\,\theta = \text{cf}\,\mu$, $\mu^+ < \lambda$

(c) $\chi =: pp_{\Gamma(\theta)}(\mu) > \lambda + |i^*|$

(d) there is $\{(a_i, b_i) : i < i^*\}$, $a_i \in [\lambda]^{<\mu}$, $b_i \in [\lambda]^\theta$ and $|\{b_i : i < i^*\}| \leq \lambda$ such that: for every $f : \lambda \to \lambda$ for some i, $f(b_i) \subseteq a_i$

then

(1) $T_{\rm feq}$ has no universal model in λ.

(2) Moreover, $\mathrm{univ}(\lambda, T_{\rm feq}) \geq \chi = pp_{\Gamma(\theta)}(\mu)$.

Proof: Let D be a θ-complete filter on θ, $\lambda_i = \mathrm{cf}\,\lambda_i < \mu = \sum_{i<\kappa} \lambda_i$, $\mathrm{tlim}_D \lambda_i = \mu$, $\chi =: \mathrm{tcf}(\prod_{i<\theta} \lambda_i / D) > i^*$ (and for (2), $\mathrm{tcf}\left(\prod_{i<\theta} \lambda_i / D\right) > \mathrm{univ}(\lambda, T_{\rm feq})$). Also let $\langle f_\alpha : \alpha < \chi \rangle$ be $<_D$-increasing cofinal in $\prod_{i<\theta} \lambda_i / D$. Let $S = \{\delta < \lambda : \mathrm{cf}\,\delta = \theta, \delta \text{ divisible by } \mu^{\omega+1}\}$. Let $\bar{C} = \langle C_\delta : \delta \in S \rangle$ be such that: C_δ a club of δ, $\mathrm{otp}(C_\delta) = \mu$ and $[\alpha \in C_\delta \Rightarrow \alpha > 0$ divisible by $\mu^\omega]$ and $\emptyset \notin \mathrm{id}^a(\bar{C})$ (i.e. for every club E of λ for stationary many $\delta \in S \cap E$, $C_\delta \subseteq E$) (exists-see [Sh 365,§2]).

For (1), let M^* be a candidate for being a universal model of $T_{\rm feq}$ of cardinality λ , for (2) let $\langle M^*_\zeta : \zeta < \kappa \rangle$ exemplify $\kappa =: \mathrm{univ}(\lambda, T_{\rm feq})$; for (1) let $\kappa = 1$, $M^*_0 = M_0$. Without loss of generality $|P^{M^*_\zeta}| = |Q^{M^*_\zeta}| = \lambda$, $P^{M^*_\zeta}$ is the set of even ordinals $< \lambda$, $Q^{M^*_\zeta}$ is the set of odd ordinals $< \lambda$.

For each $i < i^*$ and $\delta \in S$ and $z \in Q^{M^*_\zeta}$ let $a'_i = \{2\alpha : \alpha \in a_i\}$ and $d[z, \delta, i, \zeta] = \{\alpha : \alpha \in \mathrm{nacc}\, C_\delta$ and for some $x \in a'_i$ there is $y < \alpha$, such that $M^*_\zeta \models y E_x z$ but there is no $y < \sup(\alpha \cap C_\delta)$ such that $M^*_\zeta \models y E_x z\}$. Clearly $d[z, \delta, i, \zeta]$ is a subset of C_δ of cardinality $\leq |a_i| < \mu$.

Define $g_{z,\delta,i,\zeta} \in \prod_{j<\theta} \lambda_j$ by: if $|a_i| < \lambda_j, \beta \in C_\delta$, $\mathrm{otp}\,(\beta \cap C_\delta) = \lambda_j$ then $g_{z,\delta,i,\zeta}(j) = \mathrm{otp}\,(\varepsilon \cap C_\delta)$ where $\varepsilon \in C_\delta \cap \beta$ is $\mathrm{Min}\,\{\varepsilon : \varepsilon \in C_\delta \cap \beta,\ \varepsilon > \sup(d[z, \delta, i, \zeta] \cap \beta)\}$ and let $g_{z,\delta,i,\zeta}(j) = 0$ if $|a_i| \geq \lambda_j$. By the choice of $\langle f_\alpha : \alpha < \chi \rangle$ for some γ we have $g_{z,\delta,i,\zeta} <_D f_\gamma$, let $\gamma^* = \gamma^*[z,\delta,i,\zeta]$ be the first such γ. As $\mu = \text{tlim}_D \lambda_i$ clearly $\gamma^*[z,\delta,i,\zeta]$ is the first $\gamma < \chi$ such that for the D-majority of $i < \theta$, $\bigwedge_{\alpha \in d[z,\delta,i,\zeta]} \text{otp}\,(\alpha \cap C_\delta) \notin [f_\gamma(i), \lambda_i)$; clearly it is well defined. Wlog $\{b_i : i < i^*\} = \{b_i : i < i^*\} \cap \lambda$

As $\chi > \lambda + \kappa + |i^*|$, there is $\gamma(*) < \chi$ such that: $z \in Q^{M^*_\zeta}$, $\delta \in S$, $i < i^*$, $\zeta < \kappa \Rightarrow \gamma^*[z,\delta,i,\zeta] < \gamma(*)$. Now we can define by induction on $\alpha < \lambda$, N_α, γ_α such that:

(i) N_α is a model of T^*_{feq} with universe $\gamma_\alpha = \mu(1+\alpha)$,

(ii) all $x \in P^{N_\alpha}$ are even, all $y \in Q^{N_\alpha}$ are odd

(iii) N_α increasing continuous, $P^{N_\alpha} \neq P^{N_{\alpha+1}}$

(iv) for any $x \in P^{N_\alpha}$ there is a $y = y_{x,\alpha} \in Q^{N_{\alpha+1}} \setminus Q^{N_\alpha}$ such that $\neg(\exists z \in Q^{N_\alpha})[zE_x y]$,

(v) if $\alpha \in S, i < \alpha \cap i^* \cap \lambda$ and $b'_i \subseteq \text{Min}\,(C_\alpha)$ then there is a $z^i_\alpha \in Q^{N_{\alpha+1}} \setminus Q^{N_\alpha}$ such that Rang $f_{\gamma(*)} = \{\text{otp}\,(y \cap C_\alpha) : \text{for some } x \in b'_i\ ,\ y \text{ is minimal such that } yE_x z^i_\alpha\}$ where $b'_i \stackrel{\text{def}}{=} \{2\alpha : \alpha \in b_i\}$.

[For carrying out this let $d_{\alpha,i} =^{df} \{\beta \in C_\alpha : \text{otp}\,(C_\alpha \cap \beta) = (f_{\gamma(*)}(j) + 1)$ for some $j < \theta\}$, so $d_{\alpha,i} \subseteq \text{nacc}\,(C_\alpha)$, now choose distinct $x_{\alpha,i,\beta} \in b'_i$ for $\beta \in d_{\alpha,i}$. Next choose $y_{\alpha,i,\beta} \in \beta \setminus \sup(C_\alpha \cap \beta)$ such that it is as in clause (iv) for $x_{\alpha,i,\beta}$ and $z^i_\alpha E_{x_{\alpha,i,\beta}} y_{\alpha,i,\beta}$.]

If $\zeta < \kappa$ and f is an embedding of $N = \bigcup_{\alpha<\lambda} N_\alpha$ into M^*_ζ, for some i we have $f(b'_i) \subseteq a'_i$ as we can define $f' : \lambda \to \lambda$ by $f(2\alpha) = 2f'(\alpha)$, well defined as f maps P^N into $P^{M^*_\zeta}$. Let $i_1 < \lambda$ be such that $b_{i_1} = b_i$. Let $E = \{\delta < \lambda : (M^*_\zeta{\restriction}\delta, N{\restriction}\delta, f) \prec (M^*_\zeta, N, f)$ and $\delta > i_1\}$, clearly it is a club of λ hence, by the choice of $\bar{C}$, for some $\delta \in S$ we have $C_\delta \subseteq E$. Let $z \stackrel{\text{def}}{=} f(z^{i_1}_\alpha)$, so $d[z,\delta,i_1,\zeta]$ is well defined. For each $j < \theta$ there are $\beta_0 < \beta_1$ in C_δ such that $\text{otp}\,(C_\delta \cap \beta_0) = f_{\gamma(*)}(j)$, $\text{otp}\,(C_\delta \cap \beta_1) = f_{\gamma(*)}(j) + 1$ and there is y in $(\beta_1 \setminus \beta_0) \cap Q^N$ and $x \in b'_{i_1} (= b'_i)$ such that $yE_x z^{i_1}_\alpha$, y minimal for those z^α_i, x. So $x^* = f(x) \in a'_i \subseteq M^*_\zeta$,

$f(z_i^\alpha) \in M_\zeta^*$, and letting $y^* = f(y)$ we have $y^* < \beta_1$, and $y^* E_{x^*} f(z_\alpha^i)$. Is there $y_1^* < \beta_0$ with those properties? if so $f(y) E_{f(x)} y_1^*$, $(M_\zeta^*, N, f) \models (\exists t)[t \in Q^{M_\zeta^*} \,\&\, f(t) E_{x^*}^{M_\zeta^*} y_1^*]$ so as $x < \beta_0, y_1^* < \beta_0 \in E_1$ there is such $t < \beta_0$, as E is an equivalence relation $f(t) E_{x^*} f(z_\alpha^i)$. Now as f is an embedding $t E_x^N z_\alpha^i$, contradicting the choice of y. So $y^* = f(y)$ witness $\beta_1 \in d[z, \delta, i, \zeta)$ hence $\mathrm{otp}\,(\beta_1 \cap C_\delta) \leq g_{f(z_\alpha^{i_1}), \delta, i_1, \zeta}(j)$

We easily get a contradiction.

$\square_{2.2}$

2.3 Claim: 1) In 2.2 we can replace clauses (c), (d) by (c)$^+$, (d)$^-$ below and the conclusions still hold.

(c)$^+$ $\chi = pp_D(\mu) > |i^*| + \lambda$, D a filter on θ, or at least for some $\langle \lambda_i : i < \theta \rangle$, $\lambda_i = \mathrm{cf}\,\lambda_i < \mu = \mathrm{tlim}_J \langle \mu_i : i < \theta \rangle$ and $\prod_{i<\theta} \lambda_i / D$ is χ-directed.

(d)$^-$ $\{(a_i, b_i) : i < i^*\}$, $a_i \in [\lambda]^{<\mu}$, $i^* \leq \lambda$ or at least $\{b_i : i < i^*\}$ has cardinality $\leq \lambda$, $b_i = \{\alpha_{i,\zeta} : \zeta < \theta\}$ and for every $f : \lambda \to \lambda$ for some i we have $\{\zeta < \theta : f(\alpha_{i,\zeta}) \in a_i\} \neq \emptyset \bmod D$.

2) Above we can weaken in (c)$^+$ the demand "$\prod_{i<\theta} \lambda_i / D$ is χ-directed" by "$\mathrm{cf}\,(\prod_{i<\theta} \lambda_i / D) \geq \chi$" if in clause (d)$^-$ we strengthen "$\models \emptyset \bmod D$" to "$\in D$".

Also similarly we can prove

2.4 Claim: Assume

(a) $\theta < \mu < \lambda \leq \lambda^*$

(b) $\theta = \mathrm{cf}\,(\theta) = \mathrm{cf}\,(\mu) = \mathrm{cf}\,\lambda$, $\mu^+ < \lambda$

(c) $\mathrm{pp}_D(\mu) > |i^*| + \mathrm{cov}\,(\lambda, \mu, \theta^+, \sigma)$, D is σ-complete.

(d)$^-$ $\{(a_i, b_i) : i < i^*\}$, $a_i \in [\lambda]^{<\mu}$, $i^* \le \lambda$ or at least $\{b_i : i < i^*\}$ has cardinality $\le \lambda$, $b_i = \{\alpha_{i,\zeta} : \zeta < \theta\}$ and for every $f : \lambda \to \lambda$ for some i we have $\{\zeta < \theta : f(\alpha_{i,\zeta}) \in a_i\} \ne \emptyset \bmod D$.

Then

(1) $T_{\rm feq}$ has no universal model in λ

(2) moreover $\mathrm{univ}(\lambda, T_{\rm feq}) \ge \mathrm{pp}_D(\mu)$

2.5 Remark: 1) When does (d) of 2.2 hold?; it is a condition on $\lambda > \mu > \theta$, assuming for simplicity $\theta > \aleph_0$, $i^* = \lambda$) e.g. it holds (even with $\bigwedge b_i = b_0$) if:

$(*)_1$ for some cardinal κ we have $\kappa^\theta \le \lambda$, $\kappa = \mathrm{cf}\,\kappa$, $\mathrm{cov}\,(\lambda, \kappa^+, \kappa^+, \kappa) \le \lambda$.

2) As for condition $(d)^-$ from claim 2.3, if D is the filter of co-bounded subsets of θ , it suffices to have

$(*)_2$ for some cardinal κ we have $\mathrm{cov}\,(\lambda, \mu, \kappa^+, \kappa) \le \lambda$, or equivalently, $\sigma \in [\mu, \lambda)$ and $\mathrm{cf}\,(\sigma) = \kappa$ imply $\mathrm{pp}_{\Gamma(\kappa)}(\sigma) \le \lambda$.

3) So if $\theta = \mathrm{cf}\,(\mu) < \beth_\omega(\theta) \le \mu < \mu^+ < \lambda = \mathrm{cf}\,(\lambda) < \mathrm{pp}_{\Gamma(\theta)}(\mu)$ then by [Sh 460] condition $(*)_1$ holds for some $\kappa < \beth_\omega(\theta)$

4) Why have we require $\theta > \aleph_0$? as then by [Sh-g, Ch. II , 5.4] we can describe the instances of cov by instances of pp ; now even without this restriction this usually holds (see there) and possibly it always hold ; alternatively , we can repeat the proof of 2.2 using cov

5) The parallel of 2.3(2) for 2.4 can be easily stated.

2.6 Conclusion: If $\theta = \mathrm{cf}\,\mu$; $\beth_\omega(\theta) \le \mu$, $\mu^+ < \lambda = \mathrm{cf}\,\lambda < \mathrm{pp}^+_{\Gamma(\theta)}(\mu)$ then $\mathrm{univ}(\lambda, T_{\mathrm{feq}}) \ge \mathrm{pp}_{\Gamma(\theta)}(\mu)$.

Proof: The next step is:

2.7 Question: Let T be f.o. with the tree property without the strict order property; (see [Sh-c]) does 2.2 hold for it?

§3 A consequence of the existence of a universal linear order.

This section continues, most directly, [KjSh 409].

3.1 Claim: Assume

(a)$_\lambda$ $\kappa < \lambda \le 2^\kappa$ and $2^{<\lambda} \le \lambda^+ < 2^\lambda$, λ is regular.

(b)$_\lambda$ in $\mu = \lambda^+$ there is a universal linear order

then

$\otimes_{\lambda,\mu}$ there are $f_\alpha : \lambda \to \lambda$ (for $\alpha < \mu$) such that:

$(*)_{\lambda,\mu}$ for no $f : \lambda \to \lambda$ do we have $\bigwedge_{\alpha<\mu} f_\alpha \ne_{J^{\mathrm{bd}}_\lambda} f$.

Proof: Assume $\otimes_{\lambda,\mu}$ fails. We use κ-tuples of elements to compute invariants. Note that $2^\kappa \le 2^{<\lambda} \le \lambda^+$ hence $2^\kappa \in \{\lambda, \lambda^+\}$ hence $(\lambda^+)^\kappa = \lambda^+$. Let $\langle \bar{x}^\varepsilon : \varepsilon < \lambda^+ \rangle$ list ${}^\kappa(\lambda^+)$. Let $\langle \eta_\alpha : \alpha < \lambda \rangle$ list λ distinct members of ${}^\kappa 2$ (not necessarily all of them). Note that as $2^{<\lambda} \le \lambda^+$ there is a stationary $S \in I[\lambda]$, $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}\,(\delta) = \lambda\}$ (see [Sh 365 , §2] for the definition of $I[S]$).

As $S \in I[\lambda]$ by [Sh365, §2] there is $\bar{C} = \langle C_\delta : \delta \in S \rangle$ an S-club system such that $\emptyset \notin \mathrm{id}_p(\bar{C})$, $\mathrm{otp}\, C_\delta = \lambda$ and

$\oplus$ for each $\alpha < \lambda$ we have $|\{C_\delta \cap \alpha : \alpha \in \operatorname{nacc} C_\delta\}| \le \lambda$.

Let M^* be a candidate for being a universal model of $T_{\rm ord}$ of cardinality λ^+, wlog with universe λ^+ .

For every linear order M with universe λ^+, for every $\bar{x} \in {}^\kappa M$ (a κ-tuple of members of M) and $\delta \in S$, we define a (possibly partial) function $g = g^{\bar{x}}_{M,\delta} : \operatorname{nacc} C_\delta \to \lambda$ as follows:

$(*)_0$ for $\alpha \in \operatorname{nacc} C_\delta$, $g(\alpha) = \beta$ *iff* for every $\zeta < \kappa$ we have :

$$\eta_\beta(\zeta) = 1 \iff (\forall \gamma < \alpha)(\exists \gamma' < \sup(\alpha \cap C_\delta))\ [\gamma <_M x_\zeta \Rightarrow \gamma <_M \gamma' <_M x_\zeta].$$

Clearly $g^{\bar{x}}_{M,\delta}(\alpha)$ can have at most one value . We call $(\delta, \bar{x})$ *good* in M if for every $\alpha \in \operatorname{nacc} C_\delta$ there is $\varepsilon < \delta$ such that : $\bar{x}^\varepsilon$, $\bar{x}$ realize the same $<_M$-Dedekind cut over $\{i : i < \sup(\alpha \cap C_\delta)\}$ (necessary if $2^{<\lambda} = \lambda^+$). (The meaning is that for every $\zeta < \kappa$, x^ε_ζ, x_ζ realize the same $<_M$-Dedekind cut over $\{i : i < \sup(\alpha \cap C_\delta)\}$).

Let $h_\delta : \lambda \to \operatorname{nacc} C_\delta$ be: $h(i)$ is the $(i+1)$-th member of C_δ. We are assuming "$\otimes_{\lambda,\mu}$ fails", so $\{g^{\bar{x}}_{M^*,\delta} \circ h_\delta : \bar{x} \in {}^\kappa 2,\ \delta \in S\}$ cannot exemplify it. So we can find $h^*_{M^*} : \lambda \to \lambda$ such that:

$\otimes$ if $\bar{x} \in {}^\kappa(M^*)$, $\delta \in S$ is $(\delta, \bar{x})$ good in M^* then $(g^{\bar{x}}_{M^*,\delta} \circ h_\delta) \in {}^\lambda\lambda$ satisfies $h^* \neq_{J^{\rm bd}_\lambda} (g^{\bar{x}}_{M^*,\delta} \circ h_\delta)$.

Let $h^* = h^*_{M^*}$; let $g_\delta : \operatorname{nacc} C_\delta \to \lambda$ be $h^* \circ (h^{-1}_\delta) : \operatorname{nacc} C_\delta \to \lambda$. We now as in [KjSh 409, x.x?????] (using $S \in I[\lambda]$ i.e. $\oplus$) construct a linear order $N = M^{h^*}$ with universe λ^+, $N = \bigcup_{\alpha<\lambda} N_\alpha$, N_α increasing continuous in α with universe an ordinal $< \lambda^+$ and for each $\delta \in S$, there is a sequence $\bar{y}^\delta = \langle y^\delta_\zeta : \zeta < \kappa\rangle$ of members of $N_{\delta+1}$ such that

$(*)_1$ if $\alpha \in \operatorname{nacc} C_\delta$, $g_\delta(\alpha) = \beta$, $\zeta < \kappa$ *then*

$$\eta_\beta(\zeta) = 1 \Leftrightarrow (\forall \gamma \in N_\alpha)(\exists \gamma' \in N_{\sup(\alpha \cap C_\delta)})[\gamma <_N y^\delta_\zeta \Rightarrow \gamma <_N \gamma' <_N y^\delta_\zeta].$$

Suppose $f : \lambda^+ \to \lambda^+$ is an embedding of N into M^*, let $E = \{\delta < \lambda^+ : N_\delta$ universe is δ and δ is closed under f, $f^{-1}\}$. Clearly E is a club of λ^+, hence for some $\delta \in S$ the set $A = (\mathrm{acc}\, E) \cap (\mathrm{nacc}\, C_\delta)$ is unbounded in δ (so $\delta \in \mathrm{acc}\,\mathrm{acc}\, E$). Let $\bar{x} = \langle x_\zeta : \zeta < \kappa\rangle =: \langle f(y^\delta_\zeta) : \zeta < \kappa\rangle$, so we know (similarly to [KjSh 409 §3]????) that for $\alpha \in A$ and $\zeta < \kappa$ we have $g^{\bar{x}}_{M^*,\delta}(\alpha)(\zeta) = 1 \Leftrightarrow \eta_{g_\delta(\alpha)}(\zeta) = 1$. Hence $\alpha \in A \Rightarrow g^{\bar{x}}_{M^*,\delta}(\alpha) = g_\delta(\alpha) \Rightarrow (g^{\bar{x}}_{M^*,\delta} \circ h_\delta)(\mathrm{otp}\,(\alpha \cap C_\delta) - 1) = h^*(\mathrm{otp}\,(\alpha \cap C_\delta) - 1)$ contradicting the choice of h^*. $\square_{3.1}$

3.1A Claim: 1) In 3.1 if λ is a successor cardinal then we can get

$\oplus^0_\lambda$ there are $f_\alpha : \lambda \to \lambda$ for $\alpha < \lambda^+$ such that

$(*)^\lambda$ for every $f \in {}^\lambda\lambda$ for some $\alpha < \lambda^+$ we have $f_\alpha \neq_{D_\lambda} f$ (where D_λ is the club filter on λ).

2) If we allow $\mu > \lambda^+$, clause (a) of 3.1 holds and $(b)^*$ below then $\otimes_{\lambda,\mu}$ of 3.1 holds ; similarly in 3.1A(1), where

$(b)^*$ $\mathrm{univ}(\lambda^+, T_{\mathrm{ord}}) \le \mu$

Proof: 1) Use [Sh 413, 3.4].

2) The same proofs.

So from the existence of a universal linear order of cardinality λ^+, where λ is as in 3.1+3.1A(1), we get $\oplus^\lambda$, from this we get below a stronger guessing of clubs. $\square_{3.1A}$

3.2 Claim: Assume λ is regular uncountable , and

$\otimes^1_\lambda$ there are $f_\zeta : \lambda \to \lambda$ for $\zeta < \lambda^+$ such that: for every $f : \lambda \to \lambda$ for some ζ, $\{\alpha < \lambda : f_\zeta(\alpha) = f(\alpha)\}$ is stationary.

1) Let $S_1 \subseteq \{\delta < \lambda^+ \colon \operatorname{cf}(\delta) = \lambda\}$, $S_2 \subseteq \lambda$ be stationary, and $\delta \in S_1 \Rightarrow \delta = \sup(\delta \cap S_2)$. We can find $\bar{C} = \langle C^\zeta_\delta : \delta \in S_1, \zeta < \lambda^+ \rangle$, such that :

(a) C^ζ_δ is a club of δ of order type λ.

(b) $\operatorname{nacc} C^\zeta_\delta \subseteq S_2$.

(c) for every club E of λ^+, for stationarily many $\delta \in S_1$, for some $\zeta < \lambda^+$,

$$\delta = \sup\Big\{\alpha : \alpha \in \operatorname{nacc} C^\zeta_\delta \text{ and } \sup(\alpha \cap C^\zeta_\delta) \in \operatorname{nacc} C^\zeta_\delta,\ \operatorname{otp}(\alpha \cap C_\delta) \text{ is even and } \{\alpha, \sup(\alpha \cap C^\zeta_\delta)\} \subseteq E\Big\}.$$

2) Let $\lambda = \lambda^{<\lambda}$ and $S \subseteq \{\delta < \lambda^+ : \operatorname{cf} \delta = \lambda\}$ stationary. We can find $\bar{C} = \langle C^\zeta_\delta : \delta \in S, \zeta < \lambda^+ \rangle$ such that

(a) C^ζ_δ is a club of δ of order type λ.

(b) for every club E of λ^+ for stationary many $\delta \in S$, for some $\zeta < \lambda^+$, for every $\xi < \lambda$ we have E contains arbitrarily large (below λ) intervals of C_δ of length ξ

3) If λ is a successor cardinal then we can get (2) even if we omit "$\lambda = \lambda^{<\lambda}$" and weaken in $\otimes^1_\lambda$, "$f_\zeta(\alpha) = f(\alpha)$" to "$f_\zeta(\alpha) \geq f(\alpha)$".

4) In part(2), if $S_2 = \lambda^+$ we can omit "$\lambda = \lambda^{<\lambda}$" if we restrict ourselves in (b) to ξ a regular cardinal.

3.2A Remark 1) We can in 3.2(3) get the conclusion of 3.2(2) too if we fix ξ

2) We can replace in the assumptions and conclusions , λ^+ by μ is in 3.1A(2).

Proof: 1) Let $\langle C_\delta : \delta \in S_1 \rangle$ be such that: C_δ a club of δ, $\operatorname{otp} C_\delta = \lambda$ and $\operatorname{nacc}(C_\delta) \subseteq S_2$. If $\alpha < \beta < \lambda^+$, $S_2 \cap (\alpha, \beta)$ has at least two elements then let $\langle (\beta^\varepsilon_{\alpha,\beta}, \gamma^\varepsilon_{\alpha,\beta}) : \varepsilon < \lambda \rangle$ list all increasing pairs from $(S_2 \cap \beta \setminus \alpha)$ (maybe with repetitions). Let $\langle f_\zeta : \zeta < \lambda^+ \rangle$ exemplify $\otimes^1_\lambda$. Let $C_\delta = \{\alpha_{\delta,\varepsilon} : \varepsilon < \lambda\}$ (increasing). Let $e = e^\zeta_\delta \subseteq \lambda$ be a club of λ

such that: if $i < j$ are from e then $\gamma^{f_\zeta(i)}_{\alpha_{\delta,i},\delta} < \alpha_{\delta,j}$. Now for $\delta \in S_1$, $\zeta < \lambda^+$, we let: $C^\zeta_\delta = \{\alpha_{\delta,\varepsilon}, \beta^{f_\zeta(\varepsilon)}_{\alpha_{\delta,\varepsilon},\delta}, \gamma^{f_\zeta(\varepsilon)}_{\alpha_{\delta,\varepsilon},\delta} : \varepsilon \in e^\zeta_\delta\}$.

Clearly C^ζ_δ is a club of δ of order type λ. Now if E is a club of λ^+, then $E \cap S_2$ is a stationary subset of λ^+ so for some $\delta \in S_1$, $\delta = \sup(E \cap S_2)$ and define $g : \lambda \to \lambda$ by: $\beta^{g(\varepsilon)}_{\alpha_{\delta,\varepsilon},\delta}$, $\gamma^{g(\varepsilon)}_{\alpha_{\delta,\varepsilon}+1,\delta}$ are the first and second members of $(E \cap S_2) \setminus (\alpha_{\delta,\varepsilon}, \delta)\}$. By the choice of $\langle f_\zeta : \zeta < \lambda^+\rangle$ for some $\zeta < \lambda^+$, $(\exists^{\text{stat}}\,\varepsilon)(g(\varepsilon) = f_\zeta(\varepsilon))$. So C^ζ_δ is as required .

2) Similar proof (and we shall not use it).

3) In the proof of (1) for $\alpha < \lambda$ let $h(\alpha, -) : \lambda^- \xrightarrow{\text{onto}} \alpha$. We do the construction for each $\tau < \lambda^-$. The demand on $e = e^\zeta_\delta$ is changed to: if for $i < j$ are from e, then $\gamma^{h(f_\zeta(\alpha),\tau)}_{\alpha_{\delta,i},\delta} < \alpha_{\delta,j}$, and C^ζ_δ is changed accordingly. For some $\tau < \lambda$ we succeed (really this version of $\otimes^1_\lambda$ implies the original version .)

4) By the proof above we can get C^ζ_δ such that: for every regular $\xi < \lambda$ and club E of λ^+ for stationarily many $\delta \in S_1$, for unboundedly many $\alpha \in \text{nacc}\, C^\zeta_\delta$, we have: $\alpha \in E$, $\text{cf}\,(\alpha) = \xi$. Then we "correct" as usual (see[Sh365 §2]). $\square_{3.2}$.

3.3 Claim: Assume:

(a) λ regular, $S \subseteq \lambda$ stationary, $\lambda^\kappa = \lambda$.

(b) $\bar{C} = \langle C_\delta : \delta \in S\rangle$, C_δ a club of δ.

(c) $\bar{\mathcal{P}} = \langle \mathcal{P}_\delta : \delta \in S\rangle, \mathcal{P}_\delta \subseteq \mathcal{P}(\text{nacc}\,(C_\delta))$ is closed upward.

(d) for every club E of λ for some δ, $E \cap \text{nacc}\, C_\delta \in \mathcal{P}_\delta$

(e) $\kappa < \lambda$, $T_\delta = \bigcup\{T_{\delta,\beta,\gamma} : \beta \leq \gamma, \{\beta, \gamma\} \subseteq \text{nacc}\, C_\delta\}$, for $\beta < \gamma \in \text{nacc}\, C_\delta$, $T_{\delta,\beta,\gamma} \subseteq {}^{\gamma \cap \text{nacc}\, C_\delta}({}^\kappa\beta)$, $|T_{\delta,\beta,\gamma}| \leq \lambda$, and even for each γ the set $\bigcup\{T_{\delta,\beta,\gamma} : \gamma \in \text{nacc}\, C_\delta,\ \beta \in \gamma \cap \text{nacc}\, C_\delta\}$ has cardinality $\leq \lambda$.

(f) If $A \in \mathcal{P}_\delta$, for $\zeta < \lambda^+$ we have $f_\zeta \in {}^{\mathrm{nacc}\, C_\delta}({}^\kappa\delta)$ and $\Big[\beta < \gamma$ are from $A \Rightarrow f_\zeta{\restriction}\beta \in T_{\delta,\beta,\gamma}\Big]$. *Then* for some $f^* \in {}^{\mathrm{nacc}\, C_\delta}({}^\kappa\delta)$ we have $[\beta < \gamma$ from $A \Rightarrow f^*{\restriction}\beta \in T_{\delta,\beta,\gamma}]$ and for every $\zeta < \lambda^+$, $\{\beta \in A : f_\zeta(\beta) = f^*(\beta)\} \notin \mathcal{P}_\delta$.

Then there is no universal linear order of cardinality λ^+.

Proof: Similar to the previous one.

3.5 Conclusion: If $2^\lambda > \lambda^+, \lambda = \mathrm{cf}\,\lambda > \aleph_0$, $\bar{C} = \langle C_\delta : \delta \in S\rangle$, $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}\,\delta = \lambda\}$ stationary, $\lambda^+ \notin \mathrm{id}^a(\bar{C})$ and for each α we have $|\{C_\delta \cap \alpha : \alpha \in \mathrm{nacc}\, C_\delta\}| \leq \lambda$ *then*

(a) there is no universal linear order in λ^+

(b) moreover , $\mathrm{univ}(\lambda^+, T_{\mathrm{ord}}) \geq 2^\lambda$).

3.6 Discussion: (1) The condition $\otimes_\lambda$ from 3.1 holds in the models (of ZFC) constructed in [Sh 100, §4] where $\lambda = \aleph_0$, $2^{\aleph_0} = \aleph_2$ and there is a non meager subset of ${}^\omega 2$ of cardinality $\aleph_1$.

(2) It is clear from 3.5 that the existence of a universal graph in μ does not imply the existence of a universal linear order in μ every for $\mu = \lambda^+$, $\lambda = \lambda^{<\lambda}$: as by [Sh 175], [Sh 175a], if $V \models \mathrm{GCH}$, $\lambda = \lambda^{<\lambda}$, $\bar{C} = \langle C_\delta : \delta < \lambda^+, \mathrm{cf}\,\delta = \lambda\rangle$ guesses clubs, for some λ^+-c.c. forcing notion P we have $V^P \models_P$ "there is a universal graph in λ^+". But in V^P the property of $\bar{C}$, guessing clubs, is preserved and it shows that there is no universal linear order.

(3) We can look at this from another point of view:

(a) Considering the following three proofs of consistency results on the existence of universal structures: [Sh 100, §4] (universal linear order in $\aleph_1$), [Sh 175, §1] (universal graphs in λ^+, $\lambda = \lambda^{<\lambda}$ and [Sh 175a] (universal graphs in other cardinals), the first result cannot be gotten by the other two proofs.

(b) For theories with the strict order property it is "harder" to have universal models than for simple theories (see [Sh93]) as the results of [Sh500, §1] on simple theories fail for the theory of linear order (by 3.5) and even all (f.o.) theories with the strict order property (as in [KjSh 409, x.x])

(4) Concerning 3.5(b) , note that (for any complete first order T) we have $\text{Univ}(\mu, T) \le 2^{<\mu}$ hence $\text{cf}\left(\mathcal{S}_{\le\mu}(2^{<\mu}), \subseteq\ \right) \ge \text{univ}(\mu, T)$ so under reasonable hypotheses we get in 3.5(b) equality (i.e., $\mu = \lambda^+$).

§4 Toward the consistency for simple theories

The aim of this proof was originally to deal with the universality spectrum of simple countable theories and as a first approximation to characterize $\{\lambda : \text{univ}(\lambda^+, T) \le \lambda^{++} < 2^\lambda\}$, but we shall do it more generally and have more consequences. On simple theories see [Sh 93]. The reader may well read the "smooth" version, i.e. add in Definition 4.1, the $(<\lambda)$-smoothness from 4.2(4), (5), and so we can omit clauses $(e)(\beta), (\gamma), (\delta) + (1)$ from Definition 4.1. He can also assume in 4.1 that $\tau_i = \tau_0$.

4.0 Notation: (1) For a set $u \in \mathcal{S}_{<\lambda}(\lambda^+) =: \{u \subseteq \lambda^+ : |u| < \lambda\}$ let $\sup_\lambda(u) = \{\alpha + \lambda : \alpha \in u\}$ also let $S^{\lambda^+}_\lambda = \{\delta < \lambda^+ : \text{cf}\,\delta = \lambda\}$

(2) If $u_1, u_2 \in \mathcal{S}_{<\lambda}(\lambda^+)$, $h : u_1 \to u_2$ is *legal* if it is one to one, onto, and there is a unique h^+ such that: h^V is one to one order preserving from $\sup_\lambda(u_1)$ onto $\sup_\lambda(u_2)$ and for $\alpha \in u_1$, $h^+(\alpha + \lambda) = h(\alpha) + \lambda$.

(3) We say that h is *lawful* if in addition h^+ is the identity . We sometimes use “legal” and “lawful” for functions $h : u_1 \to u_2$ when $u_i \subseteq \lambda^+, |u_i| \geq \lambda$.

(4) Wide λ^+-trees $\mathcal{T} = (\mathcal{T}, <)$ are here-just subsets of ${}^{\lambda^+>}(\lambda^+)$ of cardinality $\leq \lambda^+$ closed under initial segments with the order being initial segment. A *branch* is a maximal linearly ordered subset, a λ^+-*branch* is one of order type λ^+. (So the trees are automatically normal).

4.1 Definition: $K_{\rm ap} = (K_{\rm ap}, \leq_{K_{\rm ap}})$ is a λ-approximation family, *if* for some sequence $\bar\tau$ $(= \langle \tau_i : i < \lambda^+ \rangle$ of vocabularies , $|\tau_i| \leq \lambda$, τ_i increasing with i, $M \restriction i$ means $(M \restriction \tau_i) \restriction i$; τ_i can have relations and functions with infinite arity but $< \lambda$ (you may concentrate on the case $\tau_i = \tau$ for all $i < \lambda$) the following hold :

(a) $K_{\rm ap}$ is a set of τ-model with a partial order $\leq = \leq_{K_{\rm ap}}$ (or μ is a $\tau_{\sup(M)}$-model).

(b) if $M \in K_{\rm ap}$ then $|M|$ is a subset of λ^+ of cardinality $< \lambda$ and $M \leq_{K_{ap}} N \Rightarrow M \subseteq N$.

(c) if $M \in K_{\rm ap}, \delta \in S^{\lambda^+}_\lambda$ then $M \restriction \delta \in K_{\rm ap}$ and $M \restriction \delta \leq_{K_{\rm ap}} M$; also $M \restriction 0 \in K_{\rm ap}$ (this is just to say we have the joint embedding property).

(d) any $\leq_{K_{\rm ap}}$-increasing chain in $K_{\rm ap}$ of length $< \lambda$ has an upper bound.

(e) (α) if $\delta \in S^{\lambda^+}_\lambda$, $M_0 = M_2 \restriction \delta$, $M_0 \leq_{K_{\rm ap}} M_1$, $|M_1| \subseteq \delta$ *then* M_1, M_2 has a common $\leq_{K_{\rm ap}}$-upper bound M_3 , such that $M_3 \restriction \delta = M_1$

(β) *if* we have $M_{1,i} (i < i^* < \lambda)$, $M_{1,i} \in K_{\rm ap}$ increasing with i , $|M_{1,i}| \subseteq \delta_i \in S^{\lambda^+}_\lambda$ and $M_2 \restriction \delta_i \leq_{K_{\rm ap}} M_{1,i}$, *then* there is a common upper bound M_3 to $\{M_2\} \cup \{M_{1,i} : i < i^*\}$

(γ) *if* we have $M_1 \in K_{\rm ap}$, $M_{2,i} \in K_{\rm ap}$ for $i < i^* < \lambda$ increasing with i , $\delta \in S^{\lambda^+}_\lambda$, $M_{2,i} \restriction \delta \leq M_1$ *then* there is a common $\leq_{K_{\rm ap}}$-upper bound to $\{M_1\} \cup \{M_{2,i} : i < i^*\}$ such that $M_3 \restriction \delta = M_1$.

(δ) *if* (i) $\langle \delta_i : i \leq i^* \rangle$ is a strictly increasing sequence of members of $S^{\lambda^+}_\lambda$,

(ii) we have $M_{1,i}(i < i^* < \lambda)$, $M_{1,i} \in K_{\rm ap}$ increasing with i ,

(iii) $[i(1) < i(2) \Rightarrow M_{i(1)} = M_{i(2)} \restriction \delta_{i(1)}]$

(iv) $|M_{1,i}| \subseteq \delta_i$

(v) $M_{2,j} \in K_{\rm ap}$ for $j < j^*$ has universe $\subseteq \delta_{i^*}$, and is $<_{K_{\rm ap}}$-increasing in j

(vi) $M_{2,j} \restriction \delta_i \leq_{K_{\rm ap}} M_{1,i}$,

then there is a common upper bound M_3 to $\{M_{2,j} : j < j^*\} \cup \{M_{1,i} : i < i^*\}$ such that for every $i < i^*$ we have $M_3 \restriction \delta_i = M_{1,i}$

(f) For $\alpha < \lambda^+$, $\{M \in K_{\rm ap} : |M| \subseteq \alpha\}$ has cardinality $\leq \lambda$.

(g) We call $h : M_1 \to M_2$ a *lawful* (*legal*) $K_{\rm ap}$-isomorphism if h is an isomorphism from M_1 onto M_2 and h is lawful (legal). We demand:

(α) if $M_1 \in K_{\rm ap}$, $u_1 = |M_1|$, $u_2 \subseteq \lambda^+$ and h a lawful mapping from u_1 onto u_2 *then* for some $M' \in K_{\rm ap}$, $|M'| = u_2$ and h is a lawful $K_{\rm ap}$-isomorphism from M onto M'.

(β) lawful $K_{\rm ap}$-isomorphisms preserve $\leq_{K_{\rm ap}}$.

(h) If $M \in K_{\rm ap}$ and $\beta < \lambda^+$ *then* for some $M' \in K_{\rm ap}$ we have $M \leq_{K_{\rm ap}} M'$ and $\beta \in |M'|$

(i) [Amalgamation] Assume $M_\ell \in K_{\rm ap}$ for $\ell < 3$ and $M_0 \leq_{K_{\rm ap}} M_\ell$ for $\ell = 1,2$. *Then* for some $M \in K_{\rm ap}$ and lawful function f we have: $M_1 \leq_{K_{\rm ap}} M$, the domain of f is M_2 , $f \restriction |M_0|$ is the identity and f is a $\leq_{K_{\rm ap}}$-embedding of M_2 into M, i.e. $f^{d,d}????(M_2) \leq_{K_{\rm ap}} M$

(j) If $M_i \in AP$ for $i < i^* < \lambda$ is $<_{K_{\rm ap}}$-increasing, $\bigwedge_{i<i^*} \bigwedge_{\ell<2} M_i \leq_{K_{\rm ap}} N^\ell \in K_{\rm ap}$ *then* there is N^+, $N^2 \leq_{K_{\rm ap}} N^+ \in K_{\rm ap}$ and a $\leq_{K_{\rm ap}}$-embedding f of N into N^+ over

$\bigcup_{i<i^*} M_i$.

4.1A Remark: 1) This is similar to λ^+-uniform λ forcing, see [Sh107], [ShHL 162] see also [Sh326, AP], [Sh405, AP].

2) From (g)$(\alpha), (\beta)$ we can deduce

(γ) if h is a lawful $K_{\rm ap}$- isomorphism from $M_1 \in K_{\rm ap}$ onto $M_2 \in K_{\rm ap}$, and $M_1 \le_{K_{\rm ap}} M'_1$ and h can be extended to some lawful h^+ with domain $|M'_1|$ *then* for some h', M'_2 we have $M_2 \le_{K_{\rm ap}} M'_2$, $h \subseteq h'$ and h' a lawful $K_{\rm ap}$-isomorphism from M'_1 onto M'_2.

3) We can use a linear order $<^*$ of λ^+ is $<^* \restriction [\lambda\alpha, \lambda\alpha + \lambda)$ is a saturated model of $\mathrm{Th}(\mathbb{Q}, <)$ and demand legal (and lawful) maps to preserve it. No real change.

4.1B Definition We call $K_{\rm ap}$ *homogeneous* if in clause (g) of definition 4.1 we can replace "lawful" by "legal".

4.2 Definition: 1) For $K_{\rm ap}$ is a λ-approximation family, we let:

$K_{\rm md} = \{\Gamma :$ (i) Γ is a $\le_{K_{\rm ap}}$-directed subset of $K_{\rm ap}$

(ii) Γ is maximal in the sense that : for every $\beta < \lambda^+$ for some $M \in \Gamma$ we have $\beta \in |M|$

(iii) if $M \in \Gamma$, $M \le_{K_{\rm ap}} M'$, *then* for some $M'' \in \Gamma$, there is a lawful $K_{\rm ap}$-isomorphism h from M' onto M'' over $M\}$.

2) $K_{\rm ap}$ is a simple λ-*approximation* if: (it is a λ-approximation family and) *for every* $\Gamma \in K_{\rm md}$ and $\{(M_i, N_i) : i < \lambda^+\}$ satisfying $M_i \in \Gamma$, $M_i \le_{K_{\rm ap}} N_i \in K_{\rm ap}$ *there is* a club C of λ^+ and pressing down $h : C \to \lambda^+$ such that:

(∗) if $\delta_1 < \delta_2$ are in $C \cap S^{\lambda^+}_\lambda$, $h(\delta_1) = h(\delta_2)$ and $M_{\delta_1} \le_{K_{\rm ap}} M \in \Gamma$, $M_{\delta_2} \le_{K_{\rm ap}} M \in \Gamma$ *then* we can find $N \in K_{\rm ap}$, $M \le_{K_{\rm ap}} N$, and a lawful $\le_{K_{\rm ap}}$-embeddings $f_{\delta_1}, f_{\delta_2}$ of $N_{\delta_1}, N_{\delta_2}$ into N over $M_{\delta_1}, M_{\delta_2}$ respectively such that $f_{\delta_1} \restriction (N_{\delta_1} \restriction \delta_1) = f_{\delta_2} \restriction (N_{\delta_2} \restriction \delta_2)$.

Of course by refining h we can demand on δ_1, δ_2 also that

(∗∗) $M_{\delta_1} \restriction \delta_1 = M_{\delta_2} \restriction \delta_2$, $N_{\delta_1} \restriction \delta_1 = N_{\delta_2} \restriction \delta_2$, $|M_{\delta_1}| \subseteq \delta_2$, $(|N_{\delta_1}| \subseteq \delta_2$ and some f is a lawful isomorphism from N_1 onto N_2 mapping M_1 onto M_2.

3) We define $K^\alpha_{\rm md}$ as before but $M \in \Gamma \Rightarrow |M| \subseteq \lambda\alpha$.

4) $K_{\rm ap}$ is θ-closed if $\theta = {\rm cf}\,\theta < \lambda$ and: if $\langle M_i : i < \theta \rangle$ is $\le_{K_{\rm ap}}$-increasing in $K_{\rm ap}$ *then* $\bigcup_{i<\theta} M_i \in K_{\rm ap}$ is an $\le_{K_{\rm ap}}$- upper bound; moreover $(\forall i < \theta)[M_i \le_{K_{\rm ap}} N]$ implies $\cup_{i<\theta} M_i \le_{K_{\rm ap}} N$.

5) $K_{\rm ap}$ is $(< \lambda)$-closed if it is θ-closed for every $\theta < \lambda$

6) $K_{\rm ap}$ is *smooth* if

(α) it is $(< \lambda)$-closed;

(β) all vocabularies τ_i are finitary;

(γ) in clauses (c),(e)(α), and (e)(γ) we can replace "$\delta \in S^{\lambda^+}_\lambda$" to " $\delta > 0$ is divisible by λ".

7) $K_{\rm ap}$ is a λ-approximation^x family if from Definition 4.1 it satisfies clauses (a), (b), (c), is smooth??, (g), (h), (i), and

(i)′ if $M_2 \restriction \delta \le M_1 \le \delta_1$, then M_1, M_2 have an upper bound

(j)′ if $M_i \le_{K_{\rm ap}}$ is $\le_{K_{\rm ap}}$-increasing then $\bigcup_{i<\delta} M_i \le K_{\rm ap}$.

8) $K_{\rm ap}$ is nice if whenever $M_0 \le_{K_{\rm ap}} M_1$, $\delta \in S^{\lambda^+}_\lambda$, $|M_1| \subseteq \delta$, $M_0 = M_2 \restriction \delta$ and $M_\ell \le M' \in K_{\rm ap}$ for $\ell < 3$, *then* we can find $M_3 \le_{K_{\rm ap}} M''$ such that $M' \le_{K_{\rm ap}} M''$, $M_\ell \le_{K_{\rm ap}} M_3$ for

$\ell < 3$ and $M_3 \restriction \delta = M_1$.

9) $K_{\rm ap}$ is weakly nice if whenever for $\ell = 1, 2$, $M_0 = M_\ell \restriction \delta_\ell$, $\delta_\ell \in S^{\lambda^+}_\lambda$, $|M_1| \subseteq \delta_2$, $\delta = \delta_1$ and M' as above, we can find M'' as above.

4.2A Observation: 1) If $M, N \in K^\alpha_{\rm md}$, $\alpha < \lambda^+$, then some lawful f is an isomorphism from M onto N.

2)???

4.3 Lemma: Suppose that

(A) $\lambda = \lambda^{<\lambda}$;

(B) $K_{\rm ap}$ is a λ-approximation family;

(C) $\Gamma^*_\alpha \in K_{\rm md}$ for $\alpha < \alpha^*$;

(D) $\mathcal{T}$ is a wide λ^+-tree, A_α a λ^+-branch of T for $\alpha < \alpha^*$ and for $\alpha \neq \beta (< \alpha^*)$ we have $A_\alpha \neq A_\beta$, and we let $\varepsilon(\alpha, \beta) =$ the level of the $<_{\mathcal{T}}$-last member of $A_\alpha \cap A_\beta$, $\zeta(\alpha, \beta) = (\varepsilon(\alpha, \beta) + 1)\lambda$.

Then there is a forcing notion Q such that:

(a) Q is λ-complete of cardinality $|\alpha^*|^{<\lambda}$

(b) Q satisfies the version of λ^+-c.c. from [Sh 288 §1] (for simplicity - here always for $\varepsilon = \omega$ but by smoothness we actually have lub).

(c) For some Q-names $\underset{\sim}{h}_\alpha$ and $\underset{\sim}{\Gamma}'_\alpha$ (for $\alpha < \alpha^*$) we have: $\Vdash_Q$ "for $\alpha < \alpha^*$ we have $\underset{\sim}{\Gamma}'_\alpha \in K_{\rm md}$, $\underset{\sim}{h}_\alpha$ is lawful , maps λ^+ onto λ^+ , and maps Γ_α onto Γ'_α such that for $\alpha < \beta < \alpha^*$, $\underset{\sim}{\Gamma}'_\alpha \restriction \zeta(\alpha, \beta) = \underset{\sim}{\Gamma}'_\beta \restriction \zeta(\alpha, \beta)$, so for every $M \in \Gamma_\alpha$ we have $h_\alpha \restriction (|M|)$ is lawful and is an isomorphism from M onto some $M' \in \underset{\sim}{\Gamma}'_\alpha$".

4.3A Remark: 1) Our freedom is in permuting $(\lambda\alpha, \lambda\alpha+\lambda)$; up to such permutation $\Gamma_\alpha\restriction(\lambda i) = \{M \in \Gamma'_\alpha : |M| \subseteq \lambda i\}$ is unique.

2) If we demand that $K_{\rm ap}$ be smooth the proof is somewhat simplified.

3) We can replace assumption (B) by

(B)′ $K_{\rm ap}$ is a λ-approximation^x family.

Proof: We define Q as follows:

$p \in Q$ *iff* $p = \langle (M^p_\alpha, h^p_\alpha) : \alpha \in w^p\rangle$ where

(a) $w_p \in [\alpha^*]^{<\lambda}$;

(b) $M^p_\alpha \in \Gamma_\alpha$;

(c) h^p_α a lawful mapping, Dom $h^p_\alpha = |M^p_\alpha|$;

(d) if $\alpha \neq \beta$ are in w^p, then : $h_\alpha(M^p_\alpha\restriction\zeta(\alpha,\beta))$ and $h_\beta(M^p_\beta\restriction\zeta(\alpha,\beta))$ are $\leq_{K_{\rm ap}}$-comparable;

(e) for every $\alpha \in w^p$, for some $n < \omega$, $0 = i_0 < i_1 < \ldots < i_n = \lambda^+$, we have: for $\ell \in [1,n)$, $i_\ell \in S^{\lambda^+}_\lambda$ and for every $\ell < n$

$(*)_\ell$ for every $\beta \in w$ for which $\zeta(\alpha,\beta) \in [i_\ell, i_{\ell+1})$ and $j \in [i_\ell, i_{\ell+1}) \cap S^{\lambda^+}_\lambda$ there is $\gamma \in w$ such that: $j \leq \zeta(\alpha,\gamma) \in [i_\ell, i_{\ell+1})$ and $M^p_\beta\restriction\zeta(\alpha,\beta) \leq_{K_{\rm ap}} M^p_\gamma\restriction\zeta(\alpha,\gamma)$

The order is $p \leq q$ iff: $w^p \subseteq w^q$ and for $\alpha \in w^p$: $M^p_\alpha \leq_{K_{\rm ap}} M^q_\alpha$, $h^p_\alpha \subseteq h^q_\alpha$ and $M^p_\alpha \neq M^q_\alpha \Rightarrow \bigwedge_{\beta\in w^p} h_\beta(M^p_\beta\restriction\zeta(\alpha,\beta)) \leq_{K_{\rm ap}} h_\alpha(M^q_\alpha\restriction\zeta(\alpha,\beta))$.

The lemma will follow from the facts 4.4-4.7 below.

4.4 Fact: Any increasing chain in Q of length $< \lambda$ has an upper bound.

Proof: Let $\langle p_i : i < \delta\rangle$ be an increasing sequence in Q, $\delta < \lambda$ a limit ordinal. Let

$w = \bigcup\{w^{p_i} : i < \delta\}$, and list w as $\{\alpha_j : j < j^*\}$. We now choose by induction on $j < j^*$, a member M_j of $K_{\rm ap}$ and a lawful mapping h_j with domain $|M_j|$ such that :

$\otimes$ (a) if $\langle (M^{p_i}_{\alpha_i}, h^{p_i}_{\alpha_i}) : i < \delta$ but $\alpha_j \in w^{p_i}\rangle$ is eventually constant, then this value is (M_j, h_j).

(b) Otherwise let $h_j(M_j) \in \Gamma_{\alpha_j}$ be a $\leq_{K_{\rm ap}}$-upper bound of $\{h^{p_i}_{\alpha_i}(M^{p_i}_{\alpha_i}) :\ i < \delta$ but $\alpha_j \in w^{p_i}\} \cup \{h_{j_1}(M_{j_1})\restriction\zeta(\alpha_j, \alpha_{j_1}) : j_1 < j\}$.

If we succeed $q =^{df} \langle (M_j, h_j) : j \in w\rangle$ is a member of Q as required. Why? First we check that $q \in Q$. Clauses (a),(b),(c) are obvious; for clause (d) let $\alpha \neq \beta$ be in w , so let $\{\alpha, \beta\} = \{\alpha_{j_1}, \alpha_{j_2}\}$, $j_1 < j_2$; now if $(*)$(b) holds for j_2 just note that $h_{j_1}(M_{j_1})\restriction\zeta(\alpha_{j_1}, \alpha_{j_2}) \leq h_{j_2}(M_{j_2})$ by the choice of the later; and if $(*)$(a) holds for j_2, then for some $i < \delta, (M_{j_2}, h_{j_2}) = (M^{p_i}_{j_2}, h^{p_i}_{j_2})$ and now check the choice of (M_{j_1}, h_{j_1}). If for it too clause (b) holds for some $i(1) < \delta$, $(M_{j_1}, h_{j_1}) = (M^{P_{i(1)}}_{j_1},\ h^{P_i(1)}_{j_2})$ and use $p_{\max\{i(1),i\}} \in Q$. If for j_1 clause (b) holds then by its choice $h^{p_i}_{j_2}(M^{P_i}_{j_2})\restriction\zeta(j_1, j_2) \leq h_{j_2}(M_{j_1})$ hence $h_{j_1}(M_{j_2})\restriction\zeta(j_1, j_2) = h^{p_i}_{j_1}(M^{p_i}_{j_1})\restriction\zeta(j_1, j_2) \leq h_{j_1}(M_{j_2})\restriction\zeta(j_1, j_2)$ as required. So we are left with the case clause (b) of $(*)$ apply to j_2, which is even easier. For clause (e) , clearly it is enough to prove :

$(*)$ for every $i_1 \in (S^{\lambda^+}_\lambda \cup \{\lambda^+\})$ there is $i_0 \in i_1 \cap (S^{\lambda^+}_\lambda \cup \{0\})$ such that $(*)_\ell$ of clause (e) of the definition of Q holds with i_0, i_1 taking the role of $i_\ell, i_{\ell+1}$.

Let $i_1 \in S^{\lambda^+}_\lambda \cup \{\lambda^+\}$ be given ; for each $i < i_1$ let $f(i) =^{df} \sup\{\zeta(\beta, \alpha) + 1 : \beta \in w, \zeta(\beta, \alpha) \in [i, i_1)\}$ (if the supremum is on an empty set - we are in a trivial case). Clearly $[j_1 < j_2 <$

$i_1 \Rightarrow f(j_1) \geq f(j_2)]$, so for some $i_0 \in i_1 \cap (S^{\lambda^+}_\lambda \cup \{0\})$ for all $i \in [i_0, i_1) \cap (S^{\lambda^+}_\lambda \cup \{0\})$ we have $f(i) = f(i_0)$. Now for each $i < i_1$ let $g(i) =^{df} \sup\{j+1 : j < j^*, \zeta(\alpha_j, \alpha) \in [i, i_1)$ and in $\otimes$ case (b) occurs for $j\}$, note: if the suremum is on the empty set then the value is zero; again it is clear that g decrease with i hence wlog for all $i \in [i_0, i_1)$ we have $g(i) = g(i_0)$

case 1 $g(i_0) > 0$; this means that for every $i \in [i_0, i_1)$ there is $\beta \in w$ such that : $\zeta(\beta, \alpha) \in [i_0, i_1)$ and letting $\beta = \alpha_j$ and in (*) above case (b) occurs.

Check

case 2 not case 1

For every $\gamma \in w$ let ξ_γ be the first ordinal δ such that $\langle (M^{p_i}_\gamma, h^{p_i}_\gamma) : i < \delta, i \geq \xi_\gamma \rangle$ is constant, and again wlog for some ε^* for every $i \in [i_0, i_1), \varepsilon' < \varepsilon^*, \zeta < f(i_0)$ and $j < g(i_0)$ there is $\beta \in w$ such that $\zeta \leq \zeta(\beta, \alpha) \in [i, i_1), \beta \in \{\alpha_{j'} : j \leq j' < g_a(i_0)\}$ and $j_\beta \geq \varepsilon'$, the rest should be clear.

So we have proved that $q \in Q$; now $p_i \leq_{K_{\rm ap}} q$ is straightforward. So now we have only to prove that we can carry the inductive definition from (*).

In the choice of M_j, h_j we first have chosen $h_j(M_j)$. We do it by choosing $h(M_j \restriction \zeta)$ for $\zeta \in \{\zeta(\alpha_j, \beta) : \beta \in w\}$; there we use clause (e)(δ) of Definition 4.1. Having chosen $h_j(M_j)$ we can find M_j, h_j by clauses (g)(α) + (β) of Definition 4.1. $\square_{4.4}$

4.5 Fact: 1) if $p \in Q$, $\alpha \in w^p$ and $N \in \Gamma_\alpha$ then for some q: $p \leq q$, $w^q = w^p$ and $\bigwedge_{\beta \in w^p \setminus \{\alpha\}} (M^p_\beta, h^p_\beta) = (M^q_\beta, h^q_\beta)$ and $N \leq M^q_\alpha$.

2) If $p \in Q$, $\alpha < \alpha^*$ then for some q, $p \leq q \in Q$ and $\alpha \in w^q$.

Proof: 1) Easier than the previous one (or let $\delta = 1$, $p_0 = p$ and $\{\alpha_j : j < j^*\}$ list w^p with

$\alpha = \alpha_0$, repeat the proof of 4.4 , just use q to choose (M_0, h_0). $\square_{4.5}$

2) Easier.

Note the following

4.6 Fact: If $K_{\rm ap}$ is θ-closed, then the following set is Q' dense in Q: $\{p \in Q$: if $\alpha, \beta \in w^p$, then $h^p_\alpha(M^p_\alpha)\restriction\zeta(\alpha,\beta) = h^\ell_\alpha(M^p_\beta)\restriction\zeta(\alpha,\beta)\}$.

Proof: Follows easily from the previous Facts.

4.7 Fact: The chain condition $(*)_{\lambda^+,\omega}$ from [Sh 288 §1] holds.

Proof: For simplicity assume $K_{\rm ap}$ is $\aleph_0$-closed so we can use 4.6. Suppose $p(\delta) \in Q$ for $\delta \in S^{\lambda^+}_\lambda$. For some pressing down function $h : S^{\lambda^+}_\lambda \to \lambda^+$ and $\langle \omega_\gamma : \gamma < \lambda^+\rangle$ we have:

$(*)$ if $h(\delta^1) = h(\delta^2)$, $\delta^1 < \delta^2$ *then:*

(a) $\text{otp}\,(w^{p(\delta^1)}) = \text{otp}\,(w^{p(\delta^2)})$ and $w^{p(\delta^1)} \cap w^{p(\delta^2)} = w_{h(\delta^1)}$

(b$_1$) $OP_{w^{p(\delta^1)},w^{p(\delta^2)}}$ is[†] the identity on $w^{p(\delta^1)} \cap w^{p(\delta^2)}$

(b$_2$) for $\alpha, \beta \in w^{p(\delta^1)}$ the following are equivalent:

(i) $\zeta(\alpha,\beta) < \delta^1$;

(ii) $\zeta(\alpha',\beta') < \delta^2$ where $\alpha' =^{df} \text{OP}_{w^{p(\delta^1)},w^{p(\delta^2)}}(\alpha)$, $\beta' =^{df} \text{OP}_{w^{p(\delta^1)},w^{p(\delta^2)}}(\beta)$;

(iii) $\zeta(\alpha',\beta') = \zeta(\alpha,\beta)$ where α', β' are as in (ii).

(c) $|M^{p(\delta^1)}_\alpha|$ is bounded in δ_2 and also $\sup\{\zeta(\alpha,\beta) : \alpha \neq \beta$ are in $w^{p(\delta^1)}\} < \delta_2$

(d) if $\alpha^2 = \text{OP}_{w^{p(\delta^1)},w^{p(\delta^2)}}(\alpha^1)$ then

---

[†] OP_{u_1,u_2} is the unique order preserving function f such that (Dom f) an initial segment of u_1 Rang (f) an initial segment of u_2 and Dom $(f) = u_1 \vee$ Rang $(f) = u_2$.

(α) OP$_{|M^{p(\delta^1)}_{\alpha^1}|,|M^{p(\delta^2)}_{\alpha^2}|}$ is an isomorphism from $M^{p(\delta^1)}_{\alpha^1}$ onto $M^{p(\delta^2)}_{\alpha^2}$ which is lawful.

(β) $M^{p(\delta^1)}_{\alpha^1}\restriction(\delta^1\lambda) = M^{p(\delta^2)}_{\alpha^2}\restriction(\delta^2\lambda)$.

Now we have to prove : $h(\delta^1) = h(\delta^2) \Rightarrow p(\delta^1), p(\delta^2)$ are compatible. In the list $\{\alpha_j : j < j^*\}$ put $w^{p(\delta^1)} \cap w^{p(\delta^2)}$ an initial segment. Say $\{\alpha_j : j < i^*\}$. First we restrict ourselves further by assuming $K_{\rm ap}$ is nice (see Definition 4.2(7)). We define a common upper bound p; we let $w^p = w^{p(\delta^1)} \cup w^{p(\delta^2)}$. For $\alpha \in w^{p(\delta^\ell)} \setminus w^{p(\delta^{1-\ell})}$ let $(M^p_\alpha, h^p_\alpha) = (M^{p(\delta^\ell)}_\alpha, h^{p(\delta^\ell)}_\alpha)$. For $\alpha \in w^{p(\delta^\ell)} \cap \omega^{p(\delta^\ell)}$ first choose $M^p_\alpha \in \Gamma_\alpha$ such that $M^{p(\delta^\ell)}_\alpha \leq_{K{\rm ap}} M^p_\alpha$ for $\ell = 1, 2$, and $M^p_\alpha\restriction\delta^1 = M^{p(\delta^\ell)}_\alpha\restriction\delta^2$ [Why? -by Definition 4.1 clause (e)(γ), now we can find such $M^p_\alpha \in K_{\rm ap}$, now we can find one in Γ_α by "$K_{\rm ap}$ is nice" (see Definition 4.2(7))].

Second, we deal with the case $K_{\rm ap}$ is not nice. Without loss of generality there is $\delta_0 \in S^{\lambda+}_\lambda$, $M^{p(\delta^\ell}_{\alpha_j}\restriction\delta_1 \subseteq \delta_0$ and $A_{\alpha_j}\restriction\delta_0 \neq A_{\alpha_i}\restriction\delta_0$ for $j < i < i^*$. We choose by induction on $j \leq i^*$ a condition q^j, increasing with j, $w^{q^j} = \{\alpha_i : i < j\}$, q^j, $\bigwedge_{\ell=1,2}\bigwedge_{i<j} M^{p(\delta^\ell)}_{\alpha_i} \leq_{K_{\rm ap}} M^{q_j}_{\alpha_i}$. The bookmkeeping is as in the proof of 4.4, the successor case as in the proof above (for nice $K_{\rm ap}$) but using amalgamation (=clause (i) of Definition 4.1 in the end). $\square_{4.7}$

This finishes the proof of 4.3. $\square_{4.3}$

The simplicity of $K_{\rm ap}$ is referred to only in 4.8 below, but it is needed to get the universality results later.

4.8 Claim: Assume $K_{\rm ap}$ is a simple λ^+-approximation system. If $\Gamma_0 \subseteq K_{\rm ap}$ is directed and $\alpha < \lambda^+ \Rightarrow \lambda = |[\lambda\alpha, \lambda\alpha+\lambda) \setminus \bigcup_{M\in P}|M|$, *then* for some forcing notion Q satisfying the λ^+-c.c. of [Sh 288 §1], $|Q| = \lambda^+$, $\vdash_Q$ "there is a Γ and a lawful f such that $f(\Gamma_0) \subseteq \Gamma \in K_{\rm md}$".

Proof: Natural. By renaming, without loss of generality $A \stackrel{\rm def}{=} \cup\{|M| : M \in \Gamma_0\} = \{2\alpha : \alpha < \lambda^+\}$. $Q = \{M : M \in K_{\rm ap}$ and $M \restriction A \in K_{\rm ap}$ and $M \restriction A \le_{K_{\rm ap}} M\}$ order by $\le_{K_{\rm ap}}$. $\square_{4.8}$

4.9 Conclusion: Assume $\lambda = \lambda^{<\lambda} < 2^{\lambda^+} = \chi$, and a λ^+-tree $\mathcal{T}$ with $\ge \chi$ branches is given[††] For simplicity we assume that λ^+ is the set of members of $\mathcal{T}$, 0 is the root and $\alpha <_{\mathcal{T}} \beta \Rightarrow \alpha < \beta$ for $t \in \mathcal{T}$ and let $u_t = \{[\alpha\lambda, \alpha\lambda + \lambda) : \alpha \le_T t\}$. *Then* there is a forcing notion P such that:

(a) P is λ-complete, satisfies the λ^+-c.c. and has cardinality χ (so the cardinals in V^P are the same and cardinal arithmetic should be clear).

(b) for any λ-approximation system $K_{\rm ap}$ there are $\langle \Gamma^\zeta_t, M_t : t \in \mathcal{T}_\gamma\rangle$ for $\zeta < \lambda^{++}$ such that:

   (α) $\Gamma^\zeta_t \in K^{\lambda(\ell g(t)+1)}_{\rm md}$

   (β) $t <_T s \Rightarrow \Gamma^\zeta_t \subseteq \Gamma^\zeta_s$

   (γ) for every $\Gamma \in K_{\rm md}$ for some $\zeta < \lambda^{++}$ and λ^+-branch $B = \{t_\alpha : \alpha < \lambda^+\}$ of T and lawful function from λ^+ onto λ^+ mapping Γ onto $\bigcup_{\alpha<\lambda^+} \Gamma_{t_\alpha}$.

(c) Is $R \in V^P$ is $(< \lambda)$-complete, satisfies the version of the λ^+-c.c.$(*)_{\lambda^+,\omega}$ from [Sh288§1] and $D_i \subseteq (i < \lambda^+)$ is a dense subset of R and $|R| \le \lambda^+$, *then* for some directed $G \subseteq R$, $\bigwedge_i D_i \cap G \ne \emptyset$.

Proof: We use iterated forcing of length $\chi \times \lambda^{++}$, $(< \lambda)$-support , each iterand satisfying

[††] if $\lambda = \lambda^{<\lambda}, 2^\lambda = \lambda^+ < \chi = \chi^{\lambda^+}$, and we add χ Cohen subsets to λ^+ (i.e. force by $\{f : f$ a partial function from χ to $\{0,1\}$ of cardinality $< \lambda^+\}$, then in V^P those assumptions hold.

the λ^+-c.c.$(*)_{\lambda^+,\omega}$ from [Sh 288 §1], $\langle P_i, \underset{\sim}{Q}_j : i \le \chi+\lambda^{++}, j < \chi\times\lambda^{++}\rangle$ such that: for every $K_{\rm ap}$ (from V or from some intermediate universe) for unboundedly many $i < \chi \times \lambda^{++}$, we use the forcings from 4.3 or 4.8. $\square_{4.9}$

§5 Applications

5.1 Lemma: Suppose

(A) T is first order , complete, for simplicity with elimination of quantifiers (or just inductive theory with the amalgamation and disjoint embedding property).

(B) $K_{\rm ap}$ is a simple λ-approximation system such that every $M \in K_{\rm ap}$ is a model of T hence every M_Γ , where for $\Gamma \in K_{\rm md}$ we let $M_\Gamma = \bigcup\{M : M \in \Gamma\}$.

(C) every model M of T of cardinality λ^+ can be embedded into M_Γ for some $\Gamma \in K_{\rm md}$ with $\bigcup_{M\in P} |M| = \{2\alpha : \alpha < \lambda^+\}$.

Then:

(a) in 4.9 in V^P, there is a model of T of cardinality λ^{++} universal for models of T of cardinality λ^+.

(b) So in V^P , univ$(\lambda^+, T) \le \lambda^{++}$ but there is a club guessing sequence $\langle C_\delta : \delta \in S^{\lambda^+}_x\rangle$.

Proof: Straightforward. $\square_{5.1}$

Though for theories with the strict order property, the conclusion of §4 (and 5.1) fails, for some non simple theories we can succeed. Note that in 5.1 we have some freedom in choosing $K_{\rm ap}$ even after T is fixed .

5.2 Lemma: Let $T = T^*_{\rm feq}$; it satisfies the assumption of 5.1 (hence its conclusions). In fact we can find a smooth nice simple λ-approximation system $K_{\rm ap}$ such that every model M of T of cardinality λ^+ is embeddable into some $M \in K^{\rm md}_{\rm ap}$.

5.2A Remark 1) Note that there $\text{univ}(\lambda, T^*_{\rm feq}) = \text{univ}(\lambda, T_{\rm feq})$. Actually the λ-approximation family we get is also homogeneous.

2) The situation is similar for T_3 in 5.3.

Proof: By 5.2A(1) we deal with models of $T_{\rm feq}$. Condition (A) of 5.1 clearly holds. The main point is to define $K_{\rm ap}$.

(α) $M \in K_{\rm ap}$ iff:

(i) M is a model of T

(ii) $|M| \in [\lambda^+]^{<\lambda}$

(β) $M_1 \le_{K_{\rm ap}} M_2$ iff

(i) $M_1 \subseteq M_2$

(ii) if $\delta \in S^{\lambda^+}_\lambda$, $a \in P^{M_1} \cap \delta$, $b \in Q^{M_1} \setminus \delta$ and $(\forall c \in M_1)[M_1 \models$ "$bE_a c$"$\Rightarrow c \notin \delta]$ *then* $(\forall c \in M_2)$ $[M_2 \models$ "$bE_a c$"$\Rightarrow c \notin \delta]$.

Also condition (C) of 5.1 is easy and we turn to condition (B). The checking of "$(K_{\rm ap}, \le_{K_{\rm ap}})$ is a λ-approximation family" (see Definition 4.10) as well as smoothness is staightforward. E.g. let us check the amalgamation (Definition 4.1 clause(i)). So assume $M_\ell \in K_{\rm ap}$ for $\ell < 3$, $M_0 \le_{K_{\rm ap}} M_1$, $M_0 \le_{K_{\rm ap}} M_2$; by Definition 4.2 clause (g)(α) without loss of generality

$|M_1| \cap |M_2| = |M_0|$. Now we shall define a model M with universe $|M_1| \cup |M_2|$, as follows: $P^M =^{df} P^{M_1} \cup P^{M_2}$, $Q^M =^{df} Q^{M_1} \cup Q^{M_2}$, and for each $x \in P^M$, we let E_x be the closure to an equivalence relation of the set of cases occurring in M_1 and/or M_2, now check. The checking is straightforward.

Now we are left with the main point: the simplicity of K_{ap} (see Definition 4.2(2)). Choose h as implicit in $(**)$ of Definition 4.2(2); so let $\delta_1 < \delta_2$, $M_{\delta_1}, M_{\delta_2}, M, N_{\delta_1}, N_{\delta_2}, f$ be as there. Let $f_{\delta_1}, f_{\delta_2}$ be lawful mappings such that $f_{\delta_1}\restriction(N_{\delta_1}\restriction\delta_1) = f_{\delta_2}\restriction(N_{\delta_2}\restriction\delta_2)$ and $N'_0 \stackrel{\text{def}}{=} M$ and for $\ell = 1, 2$ the $f_{\delta_\ell}(N_{\delta_\ell} \setminus M_{\delta_\ell})$ is disjoint to M for $\ell = 1, 2$; let $N'_\ell = f_{\delta_\ell}(N_{\delta_\ell})$. Now we define $N \in K_{\text{ap}}$; it is a model with universe $|N'_0| \cup |N'_1| \cup |N'_2|$, $P^N = p^{N'_0} \cup P^{N'_1} \cup P^{N'_2}$, $Q^N = Q^{N'_0} \cup Q^{N'_1} \cup Q^{N'_2}$. Lastly for $x \in P^N$, we let E_x be the finest equivalence relation on Q^N which extend each $E_x^{N'_\ell}$, (if $x \in N'_\ell$). Why is N a model of T_{feq}? Clearly P^N, Q^N is a partition of N (as this holds for $N'_\ell(\ell < 3)$ and as any two of those models agree on their intersection) and each $E^N_x (x \in P^N)$ is an equivalence relation on Q^N (by its choice). Why N'_ℓ is a submodel of N? concerning P and Q there are no problems. So assume $x \in P^{N'_\ell}$, and we shall prove $E^N_x \restriction |N'_\ell| = E^{N'_\ell}_x$, the inclusion $\supseteq$ is by the choice of E^N_x. For the other inclusion, if x belongs (and the proof of amalgamation) to only one N'_m it is totally trivial. If it belongs to exactly two of them, say N'_{m_1}, N'_{m_2} just note $N'_{m_1} \cap N'_{m_2} \le N'_{m_1}, N'_{m_2}$. So assume $x \in \bigcap_{\ell=0}^{2} P^{N'_\ell}$ and here we shall use clause (β)(ii) of the definition of $\le_{K_{\text{ap}}}$. So suppose $y_0, \dots, y_{m(*)}$ are such that $N'_{\ell(*)} \models y_m E_x y_{m+1}$, but $N'_{\ell(m(*))} \models \neg y_0 E_x y_{m(*)}$; wlog $m(*)$ is minimal. Of course without loss of generality $\ell(m) \neq \ell(m+1)$ (as then we can omit y_{m+1}) and $\ell(m) \neq \ell(m+2)$ (otherwise $y_{m+1} \in N'_{\ell(m)} \cap N'_{\ell(m+1)} = N'_{\ell(m+1)} \cap N'_{\ell(m+2)} \to y_{m+2}$ and we can omit y_{m+1}. So necessarily

$\ell(m) = \ell(m+3)$ and $\{\ell(m), \ell(m+1), \ell(m+2)\} = \{0, 1, 2\}$, hence enough to deal with the case $m(*) = 3$. As $x \bigcap_{m<\zeta} N'_{\ell(m)}$ =??? $\in \bigcap_{\ell=0}^{2} N'_\ell$, clearly $x \in N'_1 \cap N'_2 \subseteq \delta_1$, and for some m, $y_m \in N'_1 \cap N'_2 (\subseteq \delta_1)$, and for some $m_1 \in \{m-1, m+1\}$ and $k \in \{1, 2\}$, $y_{m_1} \in N'_0 \cap N'_k = M \cap N'_k = M \cap N_k$, so by the choice of $\leq_{K_{\rm ap}}$ there is $y'_{m_1} \in N'_0 \cap N'_1 \cap \delta_k \cap Q^N$, $y'_{m_1} E_x^{N'_0} y_m$, but so $y'_{m_1} \in \bigcap_{\ell<3} N'_\ell$ (as $N'_1 \cap \delta_1 = N'_2 \cap \delta_2$), and we are done. $\square_{5.2}$

5.3 Lemma: $T_{\rm trf}$, the theory of triangle free graphs satisfies the assumption of 5.1 (hence its conclusions).

Proof: Let xRy mean $\{x, y\}$ is an edge. The main point is to define $K_{\rm ap}$

(α) $M \in K_{\rm ap}$ iff

- (i) M is a model of T
- (ii) $|M| \in [\lambda^+]^{<\lambda}$

(β) $M_1 \leq_{K_{\rm ap}} M_2$ iff

- (i) $M_1 \subseteq M_2$
- (ii) if $\delta \in S_\lambda^{\lambda^+}$, $a, b \in M_1 \cap \delta$ and there is no $c \in M_1 \cap \delta$, $M_1 \models cRa \,\&\, cRb$ *then* for no $c \in M_2 \cap \delta$, $M_2 \models cRa \,\&\, cRb$.

Let us check Definition 4.1, i.e. that $(K_{\rm ap}, \leq_{K_{\rm ap}})$ is a λ-approximation system

Clause (a), (b), (c) are immediate.

Clause (d) holds in a strong form: the natural union is a lub; and even $K_{\rm ap}$ is smooth.

Clause (e) follows from (d)$^+$ and (i) (amalgamation)

Clauses (f) and (g) are immediate (as in g) the demand is on lawful h only).

Clause (h) is trivial.

Clause (i): Using a lawful f without loss of generality $|M_1| \cap |M_2| = |M_0|$. Define M_3 : $|M_3| = |M_1| \cup |M_2|$, $R^{M_3} = R^{M_1} \cup R^{M_2}$. Clearly $M_3 \in K_{\rm ap}$ as for $M_\ell \leq_{K_{\rm ap}} M_3$, by transitivity and symmetry it is enough to prove $M_1 \leq_{K_{\rm ap}} M_3$, clearly $M_1 \subseteq M_3$, (i.e. clause (i) of (β) above). For proving clause (ii) let $\delta \in S^{\lambda^+}_\lambda$, $a, b \in M_1$ and $c \in M_3 \cap \delta$ be such that $aR^{M_3}c\&bR^{M_3}c$. If $c \in M_2 \setminus M_1$, necessarily $a, b \in M_2$ hence $a, b \in M_0$ and use $M_0 \leq_{K_{\rm ap}} M_2$, but if $c \in M_1$ there is nothing to prove.

Clause y follows from smoothness.

Next let us show that $K_{\rm ap}$ is simple. Let $\delta_1 < \delta_2$ (from $S^{\lambda^+}_\lambda$), $M_{\delta_1}, M_{\delta_2}, N_{\delta_1}, N_{\delta_2}, M, f$ be as in Definition 4.2(2) (**). Without loss of generality $M \cap N_{\delta_1} = M_{\delta_1}$, $M \cap N_{\delta_2} = M_{\delta_2}$. Define a model N.

$$|N| = |N_{\delta_1}| \cup |N_{\delta_2}| \cup |M|$$

$$R^N = R^{N_{\delta_1}} \cup R^{N_{\delta_2}} \cup R^{N_{\delta_3}}$$

Clearly N extends each of the models $N_{\delta_1}, N_{\delta_2}, M$ (hence $M_{\delta_1}, M_{\delta_2}$ too).

Clearly for proving $N \in K_{\rm ap}$ it suffices to show

$(*)_1$ if N there is no triangle.

Why? Clearly the only case we should consider is $a \in N_{\delta_1} \cap N_{\delta_2} \setminus M$, $b \in N_{\delta_1} \cap M \setminus N_{\delta_2}$, $c \in N_{\delta_2} \cap M \setminus N_{\delta_1}$. (hence $b \in M_{\delta_1} \setminus \delta_1$, $c \in M_{\delta_2} \setminus \delta_2$). So for some $c' \in M_{\delta_1} \setminus \delta_1$, $f(c') = c$ but also $f(a) = a$ hence $aR^{N_{\delta_1}}c^1$, also $aR^{M_{\delta_1}}b$, so as $M_{\delta_1} \leq_{K_{\rm ap}} N_{\delta_1}$ clearly for some $a' \in M_{\delta_1} \cap \delta_1$ we have $a'R^{M_{\delta_1}}c'Ra'R^{M_{\delta_1}}b$. Applying again f we get $a'R^{M_{\delta_2}}c$. So (by the last two sentences) in M we have $a'R^Mc\&a'R^Mb$. But by the choice of abc (and as $M \subseteq N$) we get a', b, c is a triangle in M which belongs to $K_{\rm ap}$, contradiction.

So $N \in K_{\rm ap}$; also $M \le_{K_{\rm ap}} N$. [Why? being submodels sould be clear. So suppose δ, a, b, c contradicts clause (ii) of the definition of $\le_{K_{\rm ap}}$, so $c \in N \setminus M$, so $c \in N_{\delta_\ell} \setminus M_{\delta_\ell}$ for some $\ell \in \{1,2\}$.

If $c \in N_{\delta_\ell} \setminus M_{\delta_2} \setminus N_{\delta_{1-\ell}}$ then necessarily (as $aR^N c, bR^N c$) we have $a, b \in N_{\delta_\ell}$ hence $a, b \in N_{\delta_\ell} \cap M = M_{\delta_\ell}$ using $M_{\delta_\ell} \le_{K_{\rm ap}} N_{\delta_\ell}$ we have $c' \in M_{\delta_\ell} \cap \delta$ such that $M_{\delta_\ell} \models aRc' \& bRc'$, c' is as required).

So necessarily $c \in N_{\delta_\ell} \cap N_{\delta_{1-\ell}} \setminus M$ hence $a, b \in M_{\delta_1} \cup M_{\delta_2}$; if $a, b \in M_{\delta_1}$ do as above, also if $a, b, \in M_{\delta_2}$ do as above, so by symmetry without loss of generality $a \in M_{\delta_1}$, $b \in M_{\delta_2}$. Now use f as in the proof of "N is triangle free".)

The case $N_{\delta_\ell} \le N$ is similar.

Having proved $K_{\rm ap}$ is a simple λ-approximation family, it is easy to check the assumptions (A) and (C) of 5.1 hold. $\square_{5.3}$

5.4 Discussion: The similarity between the proofs of 5.2, 5.3 is not incidental. For a complete first order T, let e.g. $M^* \in \mathfrak{B}(\chi)$ be a λ^+-saturated model of T, choose by induction on $\zeta < \lambda$, an elementary submodel $\mathfrak{B}_\zeta$ of $(\mathfrak{B}(\chi), \in, <^*_\chi)$ of cardinality λ^+ such that $\{M^*, \mathfrak{B}_\varepsilon : \varepsilon < \zeta\} \cup (\lambda^+ + 1) \subseteq \mathfrak{B}_\zeta$, $\mathfrak{B}_\zeta^{<\lambda} \subseteq \mathfrak{B}_\zeta$, $f_\zeta \in \mathfrak{B}_{\zeta+1}$ a mapping from $\mathfrak{B}_\zeta \cap \lambda^+$ onto $\{\delta + \varepsilon : \varepsilon \le \zeta\delta < \lambda^+ \text{ divisible by } \lambda\}$, extending $\bigcup_{\varepsilon<\zeta} f_\varepsilon$. In the end let N^* be the model with universe λ^+ such that $\bigcup_{\zeta<\lambda^+} f_\zeta$ as an isomorphism for $\bigcup_{\zeta<\lambda^+} \mathfrak{B}_\zeta \cap M^*$ onto N^*. Let E be a thin enough club of λ^+. Let $K'_{\rm ap} = \{N : N \prec N^*$ and $(N, \delta)_{\delta \in N \cap {\rm nacc}\, E} \prec (N^*, \delta)_{\delta \in N \cap {\rm nacc}\, E}\}$. (only nacc replaces $S^{\lambda^+}_\lambda$).

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