More forcing notions imply diamond

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Abstract

We prove that the Sacks forcing collapses the continuum onto $\mathfrak{d}$, answering the question of Carlson and Laver. Next we prove that if a proper forcing of the size at most continuum collapses ω_2 then it forces $\diamondsuit_{\omega_1}$.

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0 Introduction

In 1979 Baumgartner and Laver proved that after adding ω_2 Sacks reals (by the countable support iteration) to a model of CH one gets a model in which the Sacks forcing forces CH (see theorem 5.2 of [2]). The question arose when the Sacks forcing may collapse cardinals and which of them. In 1989 Carlson and Laver posed a hypothesis that the Sacks forcing collapses the continuum at least onto the dominating number $\mathfrak{d}$ (see [3]). In the same paper they proved that, assuming CH, the Sacks forcing forces $\diamondsuit_{\omega_1}$. In the present paper we give an affirmative answer to the question of Carlson and Laver proving that the continuum is collapsed at least onto a cardinal number called here $\mathfrak{b}^{+\epsilon}$ when a Sacks real is added. The cardinal $\mathfrak{b}^{+\epsilon}$ is one of the cardinal invariants laying between the unbounded number $\mathfrak{b}$ and the dominating number $\mathfrak{d}$ which were introduced in [7]. After we got the answer we proved that if $\mathfrak{b}^{+\epsilon} = \omega_1$ then the Sacks forcing forces $\diamondsuit_{\omega_1}$. That naturally suggested the question if this is an accident and the answer we obtained says that it is a reflection of a more general theorem.

The main result of this paper says that if a proper forcing notion $\mathbb{P}$ of size not greater than the continuum collapses ω_2 then $\Vdash_{\mathbb{P}} \diamondsuit_{\omega_1}$.

Notation: Our notation is rather standard and is compatible with that of [5] or [4]. However, there are some exceptions. In a forcing notion $\mathbb{P}$ we write $p \leq q$ to say that "the condition q is stronger than p". The canonical $\mathbb{P}$-name for a generic filter is denoted by $\Gamma_{\mathbb{P}}$ or just Γ. For a formula φ of the forcing language and a condition $p \in \mathbb{P}$ we say that p *decides* φ ($p \parallel \varphi$) if either $p \Vdash \varphi$ or $p \Vdash \neg\varphi$.

A forcing notion $(\mathbb{P}, \leq)$ satisfies the Axiom A of Baumgartner (see [1]) if there are partial orders $\leq_n$ on $\mathbb{P}$ (for $n \in \omega$) such that

1. $p \leq_0 q$ if and only if $p \leq q$
2. if $p \leq_{n+1} q$ then $p \leq_n q$
3. if a sequence $\langle p_n : n \in \omega \rangle \subseteq \mathbb{P}$ satisfies $(\forall n \in \omega)(p_n \leq_n p_{n+1})$ then there exists a condition $p \in \mathbb{P}$ such that $(\forall n \in \omega)(p_n \leq p)$.
4. if $\mathcal{A} \subseteq \mathbb{P}$ is an antichain, $p \in \mathbb{P}$, $n \in \omega$ then there exists a condition $q \in \mathbb{P}$ such that $p \leq_n q$ and the set $\{r \in \mathcal{A} : q \text{ and } r \text{ are compatible}\}$ is countable.

It is well known that if $\mathbb{P}$ satisfies the Axiom A then $\mathbb{P}$ is proper.

The size of the continuum is denoted by $\mathfrak{c}$. We will use the quantifiers $(\forall^\infty n)$ and $(\exists^\infty n)$ as abbreviations for

$$(\exists m \in \omega)(\forall n > m) \ \text{ and } \ (\forall m \in \omega)(\exists n > m),$$

respectively. The Baire space ω^ω of all functions from ω to ω is endowed with the partial order $\leq^*$:

$$f \leq^* g \iff (\forall^\infty n)(f(n) \leq g(n)).$$

A family $F \subseteq \omega^\omega$ is unbounded in $(\omega^\omega, \leq^*)$ if

$$\neg(\exists g \in \omega^\omega)(\forall f \in F)(f \leq^* g)$$

and it is dominating in $(\omega^\omega, \leq^*)$ if

$$(\forall g \in \omega^\omega)(\exists f \in F)(g \leq^* f).$$

The unbounded number $\mathfrak{b}$ is the minimal size of an unbounded family in the partial order $(\omega^\omega, \leq^*)$, the dominating number $\mathfrak{d}$ is the minimal size of a dominating family in that order (for more information about these cardinals see [10] or [7]).

The set of all infinite subsets of ω is denoted by $[\omega]^\omega$. A tree on $\mathcal{X}$ is a set of finite sequences $T \subseteq \mathcal{X}^{<\omega}$ such that $s \subseteq t \in T$ implies $s \in T$. A tree T on $\mathcal{X}$ is perfect if for each $s \in T$ there are $t_0, t_1 \in T$ extending s, both in T and such that neither $t_0 \subseteq t_1$ nor $t_1 \subseteq t_0$. The body $[T]$ of a tree T is the set $\{x \in \mathcal{X}^\omega : (\forall n \in \omega)(x{\restriction}n \in T)\}$.

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1 Antichains of skew trees

The Sacks forcing $\mathbb{S}$ consists of all perfect trees $T \subseteq 2^{<\omega}$. These trees are ordered by inclusion (a stronger tree is the smaller one). For $T \in \mathbb{S}$ and $t \in T$ we say that t *ramifies in* T (or t is *a ramification point in* T) whenever both $t^\frown 0$ and $t^\frown 1$ are in T. For $s \in T \cap 2^n$, $n < k$ we say that s *ramifies in* T *below* k if there is $t \in T$ of length less than $k-1$ such that $s \subseteq t$ and t ramifies in T. A node $t \in T$ is a ramification point of rank n in T if t ramifies in T and exactly n initial segments of t ramify in T. Orders $\leq_n$ on $\mathbb{S}$ are defined by

$T \leq_n T'$ if and only if
$T \leq T'$ and if $t \in T$ is a ramification point of the rank $\leq n$ then $t \in T'$.

The Sacks forcing $\mathbb{S}$ together with orders $\leq_n$ (for $n \in \omega$) satisfies Axiom A of Baumgartner (see [1]).

For $T \in \mathbb{S}$ and $t \in T$ we put $(T)_t = \{s \in T : s \subseteq t \text{ or } t \subseteq s\}$.

Definition 1.1 *A tree $T \in \mathbb{S}$ is* skew *if for each $n \in \omega$ at most one node from $T \cap 2^n$ ramifies in T.*

Clearly the set of all skew perfect trees is dense in $\mathbb{S}$.

Carlson and Laver proved that CH implies $\Vdash_{\mathbb{S}} \diamondsuit_{\omega_1}$. A detailed analysis of their proof shows that the result can be formulated as follows.

Theorem 1.2 (T.Carlson, R.Laver, [3]) *Assume that $\mathfrak{b} = \omega_1$ and every maximal antichain $\mathcal{A} \subseteq \mathbb{S}$ consisting of skew trees is of the size $\mathfrak{c}$.*
Then $\Vdash_{\mathbb{S}} \diamondsuit_{\omega_1}$. ■

Since skew trees are very small (e.g. their bodies are both meager and null) the question appeared if the second assumption is always satisfied. The answer is negative:

Theorem 1.3 *It is consistent that there exists a maximal antichain $\{T_\alpha : \alpha < \omega_1\} \subseteq \mathbb{S}$ such that each tree T_α is skew while $\omega_1 < \mathfrak{c}$.*

Proof: Let $\bar{T} = \langle T_\alpha : \alpha < \alpha_0 \rangle$ be a sequence of skew trees, $\alpha_0 < \omega_1$ and let $\mathcal{S} = \{T \in \mathbb{S} : (\forall \alpha < \alpha_0)(T_\alpha, T \text{ are incompatible })\}$. We define a forcing notion $\mathbb{Q}(\bar{T})$:
Conditions are triples $(n, F, \bar{S})$ such that

$F \subseteq 2^{\leq n}$ is a finite skew tree of height $n \in \omega$,
$\bar{S} = \langle S_t : t \in F \cap 2^n \rangle$, $t \subseteq \text{root}(S_t)$ and $S_t \in \mathcal{S}$.

The order is defined by

$(n^0, F^0, \bar{S}^0) \leq (n^1, F^1, \bar{S}^1)$ if and only if
$F^1 \restriction n^0 = F^0$ and $(\forall t \in F^0 \cap 2^{n_0})(\exists s \in F^1 \cap 2^{n_1})(S^1_s = (S^0_t)_s)$

Claim 1.3.1 *The forcing notion $\mathbb{Q}(\bar{T})$ satisfies the ccc.*

Why? Suppose that $\langle (n^i, F^i, \bar{S}^i) : i < \omega_1 \rangle \subseteq \mathbb{Q}(\bar{T})$. First we find $A \in [\omega_1]^{\omega_1}$, $n \in \omega$ and a finite skew tree $F \subseteq 2^{\leq n}$ of the height n such that for each $i \in A$ we have $n^i = n, F^i = F$. Next we find $A' \in [A]^{\omega_1}$, $n^* > n$ and a finite skew tree $F^* \subseteq 2^{n^*}$ such that $F^* \cap 2^n = F \cap 2^n$ and for each $i \in A'$

> each node $t \in F \cap 2^n$ ramifies in F^* (below n^*) and
> $S^i_t \cap 2^{\leq n^*} \supseteq (F^*)_t$.

For each $t \in F \cap 2^n$ choose two distinct $l(t), r(t) \in (F^*)_t \cap 2^{n^*}$. Let $i, j \in A'$. For $t \in F \cap 2^n$ put $S^*_{l(t)} = (S^i_t)_{l(t)}$ and $S^*_{r(t)} = (S^j_t)_{r(t)}$. Clearly $(n^*, F^*, \bar{S}^*) \in \mathbb{Q}(\bar{T})$ and this condition is stronger than both $(n^i, F^i, \bar{S}^i)$ and $(n^j, F^j, \bar{S}^j)$. The claim is proved.

Suppose that $G \subseteq \mathbb{Q}(\bar{T})$ is a generic filter over $\mathbf{V}$. Then a density argument shows that $T_G = \bigcup\{F : (\exists n, \bar{S})((n, F, \bar{S}) \in G)\}$ is a skew perfect tree. Let $\dot{T}_\Gamma$ be the canonical $\mathbb{Q}(\bar{T})$-name for the tree T_G.

Claim 1.3.2 *If $(n, F, \bar{S}) \in \mathbb{Q}(\bar{T})$, $t \in F \cap 2^n$*
then $(n, F, \bar{S}) \Vdash$ "$\dot{T}_\Gamma, S_t$ are compatible".

Why? Suppose $(n^0, F^0, \bar{S}^0) \in \mathbb{Q}(\bar{T})$, $t_0 \in F^0 \cap 2^{n^0}$. Take n^1 such that t_0 ramifies in $S^0_{t_0}$ below n^1. Take two distinct extensions t^0_0, t^1_0 of t_0, $t^0_0, t^1_0 \in S^0_{t_0} \cap 2^{n^1}$ and for $t \in (F^0 \cap 2^{n^0})$ fix an extension $t^1 \supseteq t$, $t^1 \in S^0_t$. Put

$$F^1 = \{t^1 \restriction m : m \leq n^1\} \cup \{t^i_0 \restriction m : m \leq n^1, i = 0, 1\},\ S^1_{t^1} = (S^0_t)_{t^1},\ S^1_{t^i_0} = (S^0_{t_0})_{t^i_0}.$$

Then $(n^1, F^1, \bar{S}^1) \in \mathbb{Q}(\bar{T})$ is a condition stronger than $(n^0, F^0, \bar{S}^0)$ and

$$(n^1, F^1, \bar{S}^1) \Vdash t^0_0, t^1_0 \in \dot{T}_\Gamma \cap S^0_{t_0}.$$

Since $S^1_{t^0_0}, S^1_{t^1_0} \subseteq S^0_{t_0}$ easy density argument proves the claim.

Claim 1.3.3 $\Vdash_{\mathbb{Q}(\bar{T})} (\forall \alpha < \alpha_0)(T_\alpha, \dot{T}_\Gamma$ *are incompatible*$)$.

Why? Let $\alpha < \alpha_0$, $(n, F, \bar{S}) \in \mathbb{Q}(\bar{T})$. Since each S_t (for $t \in F \cap 2^n$) is incompatible with T_α we find $n^* > n$ and $v(t) \in S_t \cap 2^{n^*}$ for $t \in F \cap 2^n$ such that $v(t) \notin T_\alpha$. Let

$$F^* = \{v(t) \restriction m : m \leq n^*, t \in F \cap 2^n\} \text{ and } S^*_{v(t)} = (S_t)_{v(t)} \text{ for } t \in F \cap 2^n.$$

Then $(n^*, F^*, \bar{S}^*) \geq (n, F, \bar{S})$ and $(n^*, F^*, \bar{S}^*) \Vdash \dot{T}_\Gamma \cap T_\alpha \subseteq F^*$. The claim is proved.

Now we start with $\mathbf{V} \models \neg$CH. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_1 \rangle$ be the finite support iteration such that

$$\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \mathbb{Q}(\langle \dot{T}_\beta : \beta < \alpha \rangle)$$

where $\dot{T}_\beta$ is the $\mathbb{P}_{\beta+1}$-name for the generic tree added by $\dot{\mathbb{Q}}_\beta$. Let $G \subseteq \mathbb{P}_{\omega_1}$ be a generic over $\mathbf{V}$. Since $\mathbb{P}_{\omega_1}$ satisfies ccc (by claim 1.3.1) we have $\mathbf{V}[G] \models \neg$CH. By claim 1.3.3, $\langle \dot{T}^G_\alpha : \alpha < \omega_1 \rangle$ is an antichain in $\mathbb{S}$. We claim that it is a maximal antichain (in $\mathbf{V}[G]$).
Suppose that $\dot{T}$ is a $\mathbb{P}_{\omega_1}$-name for an element of $\mathbb{S}$. Then $\dot{T}$ is a $\mathbb{P}_\alpha$-name for some $\alpha < \omega_1$. Assume that $p \in \mathbb{P}_{\omega_1}$ is such that

$$p \Vdash_{\omega_1} (\forall \alpha < \omega_1)(\dot{T}, \dot{T}_\alpha \text{ are incompatible}).$$

Take $\alpha_0 > \alpha$ such that $p \in \mathbb{P}_{\alpha_0}$. Since

$$p \Vdash_{\alpha_0} (\forall \alpha < \alpha_0)(\dot{T}, \dot{T}_\alpha \text{ are incompatible})$$

we can extend p to $q = p \cup \{(\alpha_0, (0, \{\emptyset\}, \langle \dot{T} \rangle))\} \in \mathbb{P}_{\omega_1}$. It follows from claim 1.3.2 that $q \Vdash_{\omega_1}$ "$\dot{T}_{\alpha_0}, \dot{T}$ are compatible" - a contradiction. The theorem is proved. ■

2 When Sacks forcing forces CH

In this section we show that if $\mathfrak{d} = \omega_1$ then $\Vdash_{\mathbb{S}}$CH, and hence applying the result of the next section we will be able to conclude $\Vdash_{\mathbb{S}} \diamondsuit_{\omega_1}$ provided $\mathfrak{d} = \omega_1$. [To be more precise, if CH holds then $\Vdash_{\mathbb{S}} \diamondsuit_{\omega_1}$ by theorem 1.2 of Carlson and Laver. If we are in the situation of $\mathfrak{d} = \omega_1 < \mathfrak{c}$ then, by corollary 2.6, $\mathfrak{c}$ is collapsed to ω_1 and hence ω_2 is collapsed (by forcing with $\mathbb{S}$). Now theorem 3.4 applies.] This answers the question of T.Carlson and R.Laver (see [3]).

We start with the following general observation.

Lemma 2.1 *Let $\mathbb{P}$ be a forcing notion, κ a cardinal. Suppose that there exist antichains $\mathcal{A}_\zeta \subseteq \mathbb{P}$ for $\zeta < \kappa$ such that*

(*) $(\forall p \in \mathbb{P})(\exists \zeta < \kappa)(|\{q \in \mathcal{A}_\zeta : p \leq q\}| = |\mathbb{P}|)$.

Then $\Vdash_{\mathbb{P}} |\mathbb{P}^{\mathbf{V}}| \leq \kappa$.

PROOF: For each $\zeta < \kappa$, by an easy induction, one can construct a function $\phi_\zeta : \mathcal{A}_\zeta \longrightarrow \mathbb{P}$ such that for every $p, p' \in \mathbb{P}$

> if $|\{q \in \mathcal{A}_\zeta : p \leq q\}| = |\mathbb{P}|$
> then $\phi_\zeta(q) = p'$ for some $q \in \mathcal{A}_\zeta$, $q \geq p$.

Now let $\dot{\phi}$ be a $\mathbb{P}$-name for a function from κ into $\mathbb{P}^{\mathbf{V}}$ such that

$$q \Vdash \dot{\phi}(\zeta) = \phi_\zeta(q) \quad \text{for } \zeta < \kappa,\ q \in \mathcal{A}_\zeta.$$

Clearly for each $p, p' \in \mathbb{P}$, if $\zeta < \kappa$ witnesses **(*)** for p then there is $q \geq p$ such that $q \Vdash \dot{\phi}(\zeta) = p'$. Consequently $\Vdash_{\mathbb{P}} \mathrm{rng}(\dot{\phi}) = \mathbb{P}^{\mathbf{V}}$ and we are done. ■

Thus to prove that the Sacks forcing collapses continuum we will construct the respective sequence of antichains in $\mathbb{S}$. The sequence will be produced from a special family of subsets of $[\omega]^\omega$

For a set $X \in [\omega]^\omega$ let $\mu_X : \omega \xrightarrow{\text{onto}} X$ be the increasing enumeration of the set X.

Definition 2.2 (1) *A family $\mathcal{F} \subseteq [\omega]^\omega$ is* dominating in $[\omega]^\omega$ *if*

$$(\forall Y \in [\omega]^\omega)(\exists X \in \mathcal{F})(\forall^\infty n)(|[\mu_X(n), \mu_X(n+1)) \cap Y| \geq 2).$$

(2) *A family $\mathcal{F} \subseteq [\omega]^\omega$ is* weakly dominating in $[\omega]^\omega$ *if for every set $Y \in [\omega]^\omega$*

$$(\exists X \in \mathcal{F})(\exists^\infty i)(\forall j < 2^i)(|[\mu_X(2^i + j), \mu_X(2^i + j + 1)) \cap Y| \geq 2).$$

(3) $\mathfrak{b}^{+\epsilon} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega$ *is weakly dominating*$\}$.

REMARKS: 1) Note that if $\mathcal{F}$ is a dominating family in $[\omega]^\omega$ then $\{\mu_X : X \in \mathcal{F}\}$ is a dominating family in the order $\langle \omega^\omega, \leq^* \rangle$. And conversely, if $F \subseteq \omega^\omega$ is a dominating family of increasing functions, $X_{f,n} = \{f(n), f(f(n)), f(f(f(n)))\ldots\}$ (for $f \in F, n \in \omega$) then $\{X_{f,n} : f \in F, n \in \omega\}$ is a dominating family in $[\omega]^\omega$. In particular the minimal size of a dominating family in $[\omega]^\omega$ is the dominating number $\mathfrak{d}$. Clearly each dominating family is weakly dominating. Consequently $\mathfrak{b}^{+\epsilon} \leq \mathfrak{d}$.
2) We have the following inequalities:

$$\mathfrak{b} \leq \mathfrak{b}^{+\epsilon} \leq \min\{|X| : X \subseteq 2^\omega \text{ is not meager}\}.$$

Moreover, the inequality $\mathfrak{b} < \mathfrak{b}^{+\epsilon}$ is consistent with ZFC (see [7]; $\mathfrak{b}^{+\epsilon}$ is the cardinal $\mathfrak{d}(S_{+\epsilon})$ of that paper).
3) One can replace "≥ 2" in the definition of a weakly dominating family (and $\mathfrak{b}^{+\epsilon}$) by "≥ 1" (and replace the function $i \mapsto 2^i$ by any other increasing function) and still the results of this section could be carried on (with this new $\mathfrak{b}^{+\epsilon}$). The reason why we use this definition of $\mathfrak{b}^{+\epsilon}$ is that it fits to a more general schema of cardinal invariants studied in [7]. For example note that the unbounded number $\mathfrak{b}$ equals to

$$\min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \ \&\ (\forall Y \in [\omega]^\omega)(\exists X \in \mathcal{F})(\exists^\infty i)(|[\mu_X(i), \mu_X(i+1)) \cap Y| \geq 2)\}$$

and "≥ 2" in the above cannot be replaced by "≥ 1".

Definition 2.3 *Let $T \in \mathbb{S}$, $X \in [\omega]^\omega$. We say that the condition T* weakly obeys *the set X if*

$$(\exists^\infty i)(\forall j < 2^i)(\forall t \in T \cap 2^{\mu_X(2^i+j)})(t \text{ ramifies in } T \text{ below } \mu_X(2^i + j + 1)).$$

Lemma 2.4 *Suppose $X \in [\omega]^\omega$. Then there exists an antichain $\mathcal{A} \subseteq \mathbb{S}$ such that*

$(*_X)$ *if $T \in \mathbb{S}$ weakly obeys X then $|\{S \in \mathcal{A} : T \leq S\}| = \mathfrak{c}$.*

PROOF: Let $\{T_\alpha : \alpha < \mathfrak{c}\} = \{T \in \mathbb{S} : T \text{ weakly obeys } X\}$ be an enumeration with $\mathfrak{c}$ repetitions. Let $\{h_\alpha : \alpha < \mathfrak{c}\} \subseteq \omega^\omega$ be a family of functions such that

$$(\forall \alpha < \mathfrak{c})(\forall i < \omega)(h_\alpha(i) < 2^i) \text{ and } (\forall \alpha < \beta < \mathfrak{c})(\forall^\infty i)(h_\alpha(i) \neq h_\beta(i)).$$

Since T_α weakly obeys X we have that for infinitely many i, for each $j < 2^i$ each node $t \in T_\alpha \cap 2^{\mu_X(2^i+j)}$ ramifies in T_α below $\mu_X(2^i+j+1)$. Consequently, for each $\alpha < \mathfrak{c}$ we can construct a condition $S_\alpha \geq T_\alpha$ such that for every $i \in \omega$:

if some $t \in S_\alpha \cap 2^{\mu_X(2^i+j)}$ ramifies below $\mu_X(2^i+j+1)$, $j < 2^i$
then $j = h_\alpha(i)$.

Note that $(\forall^\infty n)(h_\alpha(n) \neq h_\beta(n))$ implies that conditions S_α, S_β are incompatible. Thus $\mathcal{A} = \{S_\alpha : \alpha < \mathfrak{c}\}$ is an antichain. Clearly this $\mathcal{A}$ works. ■

Theorem 2.5 $\Vdash_{\mathbb{S}} \mathfrak{c} = |(\mathfrak{b}^{+\epsilon})^{\mathbf{V}}|$.

PROOF: Since $\Vdash_{\mathbb{S}} \mathfrak{c} = |\mathfrak{c}^{\mathbf{V}}|$ it is enough to show that

$\Vdash_{\mathbb{S}}$ "there exists a function ϕ from $(\mathfrak{b}^{+\epsilon})^{\mathbf{V}}$ onto $\mathfrak{c}^{\mathbf{V}}$".

By the definition of the cardinal $\mathfrak{b}^{+\epsilon}$ there exists a sequence $\langle X_\zeta : \zeta < \mathfrak{b}^{+\epsilon}\rangle \subseteq [\omega]^\omega$ which is weakly dominating. Apply lemma 2.4 to construct antichains $\mathcal{A}_\zeta \subseteq \mathbb{S}$ such that

if $T \in \mathbb{S}$ weakly obeys X_ζ then $|\{S \in \mathcal{A}_\zeta : T \leq S\}| = \mathfrak{c}$.

Since each tree $T \in \mathbb{S}$ weakly obeys some X_ζ we can conclude the assertion from lemma 2.1. ■

Corollary 2.6 *Assume that* $\mathfrak{d} = \omega_1$. *Then* $\Vdash_{\mathbb{S}} CH$. ■

The Marczewski ideal $\mathcal{S}_0$ is a σ-ideal of subsets of the Cantor space 2^ω. This ideal is connected with the Sacks forcing. It consist of all sets $A \subseteq 2^\omega$ such that

$$(\forall T \in \mathbb{S})(\exists T' \geq T)([T'] \cap A = \emptyset),$$

where $[T'] = \{x \in 2^\omega : (\forall n \in \omega)(x{\restriction}n \in T')\}$.

Some connections between the Marczewski ideal $\mathcal{S}_0$ and the Sacks forcing $\mathbb{S}$ were established in [6].

Corollary 2.7 $\mathbf{add}(\mathcal{S}_0) \leq \mathfrak{b}^{+\epsilon}$

PROOF: The crucial fact for this inequality is the existence of a sequence $\langle \mathcal{A}^*_\zeta : \zeta < \mathfrak{b}^{+\epsilon}\rangle \subseteq \mathbb{S}$ of maximal antichains in $\mathbb{S}$ such that

$$(\forall T \in \mathbb{S})(\exists \zeta < \mathfrak{b}^{+\epsilon})([T] \setminus \bigcup\{[S] : S \in \mathcal{A}^*_\zeta\} \neq \emptyset).$$

For this first, as in the proof of theorem 2.5, find antichains $\mathcal{A}_\zeta \subseteq \mathbb{S}$ for $\zeta < \mathfrak{b}^{+\epsilon}$ such that

$$(\forall T \in \mathbb{S})(\exists \zeta < \mathfrak{b}^{+\epsilon})(|\{S \in \mathcal{A}_\zeta : T \leq S\}| = \mathfrak{c}).$$

Now fix $\zeta < \mathfrak{b}^{+\epsilon}$. To construct $\mathcal{A}^*_\zeta$ take an enumeration $\{T_\alpha : \alpha < \mathfrak{c}\}$ of $\mathbb{S}$ and an enumeration $\{T^*_\alpha : \alpha < \mathfrak{c}\}$ of $\{T \in \mathbb{S} : |\{S \in \mathcal{A}_\zeta : T \leq S\}| = \mathfrak{c}\}$. Next by induction on $\alpha < \mathfrak{c}$ choose trees $S_\alpha \in \mathbb{S}$ and branches $x_\alpha \in 2^\omega$ such that (for $\alpha < \mathfrak{c}$):

$x_\alpha \in [T^*_\alpha] \setminus \bigcup_{\beta<\alpha}[S_\beta]$,

either $(\exists S \in \mathcal{A}_\zeta)(S_\alpha \geq S)$ or $\mathcal{A}_\zeta \cup \{S_\alpha\}$ is an antichain,

if T_α is incompatible with all S_β (for $\beta < \alpha$) then $S_\alpha \geq T_\alpha$,

S_α is incompatible with each S_β for $\beta < \alpha$ and

$[S_\alpha] \cap \{x_\beta : \beta \leq \alpha\} = \emptyset$.

At stage $\alpha < \mathfrak{c}$ we easily find a suitable $x_\alpha \in [T^*_\alpha]$ since continuum many members of $\mathcal{A}_\zeta$ is stronger than T^*_α and each S_β (for $\beta < \alpha$) is either stronger than some member of $\mathcal{A}_\zeta$ or incompatible with all elements of $\mathcal{A}_\zeta$. (Remember that two conditions $S, T \in \mathbb{S}$ are incompatible in $\mathbb{S}$ if and only if $[S] \cap [T]$ is countable.) If the condition T_α is compatible with some S_β for $\beta < \alpha$ then we put $S_\alpha = S_\beta$. Otherwise we choose $S \in \mathbb{S}$ such that T_α and S are compatible and either $S \in \mathcal{A}_\zeta$ or S is incompatible with all members of $\mathcal{A}_\zeta$. As each perfect set contains continuum many disjoint perfect sets we can find a tree $S_\alpha \geq T_\alpha, S$ such that $[S_\alpha] \cap \{x_\beta : \beta \leq \alpha\} = \emptyset$.
Then $\{S_\alpha : \alpha < \mathfrak{c}\} = \mathcal{A}^*_\zeta$ is a maximal antichain (note that there could be repetitions in $\{S_\alpha : \alpha < \mathfrak{c}\}$). The points x_α (for $\alpha < \mathfrak{c}$) witness that no $[T^*_\alpha]$ is covered by $\bigcup\{[S] : S \in \mathcal{A}^*_\zeta\}$.

Now, having antichains $\mathcal{A}^*_\zeta$ as above, we put $A_\zeta = \bigcup\{[T] : T \in \mathcal{A}^*_\zeta\}$. Since $\mathcal{A}^*_\zeta$ is a maximal antichain the complement of A_ζ is in the ideal $\mathcal{S}_0$. Moreover, for each $T \in \mathbb{S}$ there is $\zeta < \mathfrak{b}^{+\epsilon}$ with $[T] \setminus A_\zeta \neq \emptyset$. Hence $\bigcup\limits_{\zeta<\mathfrak{b}^{+\epsilon}} (2^\omega \setminus A_\zeta) \notin \mathcal{S}_0$.

■

Remark: Recently P. Simon has proved that in the results of these section one can replace $\mathfrak{b}^{+\epsilon}$ by the unbounded number $\mathfrak{b}$.

3 Collapse ω_2 – the continuum will fall down

In this section we will prove that if the Sacks forcing (or any proper forcing of size $\leq \mathfrak{c}$) collapses ω_2 then it forces $\diamondsuit_{\omega_1}$. First we will give combinatorial tools needed for the proof. Let us start with fixing some notation.

For an ordinal κ by $\mathrm{IS}(\kappa)$ we will denote the set of finite incresing sequences with values in κ. χ stands for a "sufficiently large" cardinal, $\mathcal{H}(\chi)$ is the family of all sets hereditarily of the cardinality less than χ.

For $\zeta < \omega_1$ let $\zeta = \{e^\zeta_n : n \in \omega\}$ be an enumeration.

Let $S^2_i = \{\delta < \omega_2 : \mathrm{cf}(\delta) = \omega_i\}$ for $i = 0, 1$.

Lemma 3.1 (S.Shelah, see 2.3 of [9]) *There exists a ("club–guessing") sequence $\bar{C} = \langle C_\delta : \delta \in S^2_0\rangle$ such that*

1. *$C_\delta \subseteq \delta$, $\sup C_\delta = \delta$,*
2. *the order type of C_δ is ω,*
3. *for every closed unbounded subset E of ω_2 there exist $\delta \in S^2_0$ such that $C_\delta \subseteq E$.* ■

We fix a club–guessing sequence $\bar{C} = \langle C_\delta : \delta \in S^2_0\rangle$ as in 3.1. For $\delta \in S^2_0$ let $C_\delta = \{\alpha^\delta_n : n \in \omega\}$ be the increasing enumeration.

Definition 3.2 *Let $\delta \in S^2_0$ and let $\zeta < \omega_1$ be limit.*

1. *A sequence $\langle N_\eta : \eta \in \mathrm{IS}(\omega)\rangle$ is* a semi-(ζ, δ)-creature *(for the sequence $\bar{C}$) if*
   - α) *N_η is a countable elementary submodel of $\mathcal{H}(\chi)$, $N_\eta \cap \omega_1 \subseteq \zeta$ and $\bigcup_{n\in\omega} N_{h\restriction n} \cap \omega_1 = \zeta$ for every increasing function $h \in \omega^\omega$,*
   - β) *if $\eta \subseteq \nu$ then $N_\eta \prec N_\nu$,*
   - γ) *$N_\eta \cap \omega_2 \subseteq \alpha^\delta_0 \cup \bigcup_{n<\mathrm{lh}(\eta)} [\alpha^\delta_{\eta(n)}, \alpha^\delta_{\eta(n)+1})$,*
   - δ) *for each $n < \mathrm{lh}(\eta)$ the intersection $N_\eta \cap [\alpha^\delta_{\eta(n)}, \alpha^\delta_{\eta(n)+1})$ is non empty.*
2. *Let $\mathbb{P}$ be a forcing notion, $X \in \mathcal{H}(\chi)$.* A (ζ, δ)-creature for $\mathbb{P}, X$ *is a system $\{(N_\eta, \tau_\eta, k_\eta) : \eta \in \mathrm{IS}(\omega)\}$ such that*

$\alpha)$ *the sequence* $\langle N_\eta : \eta \in \mathrm{IS}(\omega)\rangle$ *is a semi-*(ζ,δ)*-creature and* $X, \mathbb{P}, \leq_{\mathbb{P}}, \omega_2, \omega_1, \ldots \in N_\emptyset$,

$\beta)$ $k_\eta \in \omega$, $\{e^\zeta_k : k < k_\eta\} \subseteq N_\eta$, *and for every increasing function* $h \in \omega^\omega$ *the sequence* $\langle k_{h\restriction n} : n \in \omega\rangle$ *is unbounded,*

$\gamma)$ τ_η *is a function such that* $\mathrm{dom}(\tau_\eta) \in [\mathbb{P} \times \omega]^{\leq\omega}$, *for each* $k \in \omega$ *the set* $\{p \in \mathbb{P} : (p,k) \in \mathrm{dom}(\tau_\eta)\}$ *is an antichain in* $\mathbb{P}$ *and* $\mathrm{rng}(\tau_\eta) \subseteq 2$,

$\delta)$ *if* $\eta \subseteq \nu$ *then* $k_\eta \leq k_\nu$ *and* $\tau_\eta \subseteq \tau_\nu$.

3. *Let* $\mathrm{CR}^\zeta_\delta(\mathbb{P}, X)$ *be the family of all* (ζ,δ)*-creatures for* $\mathbb{P}, X$.

REMARKS: *1.* A $\mathbb{P}$-name for a subset of $\zeta < \omega_1$ can be thought of as a function τ such that $\mathrm{rng}\tau \subseteq 2$ and $\mathrm{dom}\tau \subseteq \mathbb{P} \times \zeta$ has the following property:

> for each $\xi \in \zeta$ the set $\{p \in \mathbb{P} : (p,\xi) \in \mathrm{dom}\tau\}$ is an antichain in $\mathbb{P}$

(and then for $(p,\xi) \in \tau$: $p \Vdash \xi \in \tau$ if $\tau(p,\xi) = 1$ and $p \Vdash \xi \notin \tau$ otherwise). If the forcing notion $\mathbb{P}$ is proper every such a name can be (above each condition) forced to be equal to a countable name.

2. Thus in a (ζ,δ)-creature $\{(N_\eta, \tau_\eta, k_\eta) : \eta \in \mathrm{IS}(\omega)\}$ for $\mathbb{P}$ the functions τ_η can be thought of as approximations of a name for a subset of ζ. Note that we demand no relations between functions τ_η and models N_η. The last are only "side parameters". The parameter will decide above which conditions the name is described by the functions τ determined by a branch through the creature.

Lemma 3.3 *For every* $X \in \mathcal{H}(\chi)$ *and a closed unbounded set* $D \subseteq \omega_1$ *for some* $\zeta \in D$ *and* $\delta \in S^2_0$ *there exists a semi-*(ζ,δ)*-creature* $\langle N^*_\eta : \eta \in \mathrm{IS}(\omega)\rangle$ *such that* $X \in N^*_\emptyset$.

PROOF: The following special case of theorem 2.2 of [8] is a main tool for constructing semi-creatures:

Claim 3.3.1 (M.Rubin and S.Shelah, [8]) *Suppose that $\mathcal{T} \subseteq \omega_2^{<\omega}$ is a tree such that for each node $t \in \mathcal{T}$ the set $\mathrm{succ}_{\mathcal{T}}(t)$ of successors of t is of the size ω_2. Assume that $\phi : \mathcal{T} \longrightarrow \omega_1$. Then there exists a subtree $\mathcal{T}_0$ of $\mathcal{T}$ such that*

$$(\forall t \in \mathcal{T}_0)(|\mathrm{succ}_{\mathcal{T}_0}(t)| = \omega_2) \quad \textit{and} \quad \sup \phi[\mathcal{T}_0] < \omega_1.$$

If additionally ϕ is increasing (i.e. $t \subseteq s \in \mathcal{T}$ implies $\phi(t) \leq \phi(s)$) then we can demand that $\lim_n \phi(x{\restriction}n)$ is constant for all infinite branches $x \in [\mathcal{T}_0]$.

For $v \in \mathrm{IS}(S_1^2)$ choose N_v such that

(0) $X \in N_\emptyset$;

(1) N_v is an elementary countable submodel of $\mathcal{H}(\chi)$;

(2) $N_v \cap \omega_1 \in D$, $[\max(v), \omega_2) \cap N_v \neq \emptyset$;

(3) if $v \subseteq w$ then $N_v \prec N_w$.

Now we will inductively define a tree $\mathcal{T} \subseteq \mathrm{IS}(S_1^2)$ and ordinals $\delta_v < \omega_2$ for $v \in \mathcal{T}$ such that:

(4) if $v \in \mathcal{T}$ then $|\mathrm{succ}_{\mathcal{T}}(v)| = \omega_2$ and

(5) if $v \in \mathcal{T}$, $\phi_v : S_1^2 \xrightarrow{\text{onto}} \mathrm{succ}_{\mathcal{T}}(v)$ is the increasing enumeration of $\mathrm{succ}_{\mathcal{T}}(v)$ then for every $\alpha \in S_1^2$ and $w \in \mathcal{T}$, $w \supseteq v^\frown\phi_v(\alpha)$

$$N_w \cap \alpha \subseteq \delta_v.$$

To start with we put $\emptyset \in \mathcal{T}$. For each $v \in \mathrm{IS}(S_1^2)$ let $\rho_v = \sup(N_v \cap v(0))$. Applying claim 3.3.1 for each $\alpha \in S_1^2$ we find a tree $\mathcal{T}^{\langle\alpha\rangle} \subseteq \mathrm{IS}(S_1^2)$ and $\rho^\alpha < \alpha$ such that

(6) $\mathrm{root}(\mathcal{T}^{\langle\alpha\rangle}) = \langle\alpha\rangle$;

(7) each node extending $\langle\alpha\rangle$ has ω_2 successors in $\mathcal{T}^{\langle\alpha\rangle}$;

(8) for each $v \in \mathcal{T}^{\langle\alpha\rangle}$, $\rho_v < \rho^\alpha$.

Applying Fodor's lemma we find $\delta_\emptyset$ and $A_\emptyset$ such that

(9) $A_\emptyset \in [S^2_1]^{\omega_2}$;

(10) $\delta_\emptyset = \rho^\alpha$ for $\alpha \in A_\emptyset$.

We put $A_\emptyset = \mathrm{succ}_{\mathcal{T}}(\emptyset)$ and we decide that $(\mathcal{T})_{\langle\alpha\rangle} \subseteq \mathcal{T}^{\langle\alpha\rangle}$ for each $\alpha \in A_\emptyset$. Note that at this moment we are sure that if $\langle\phi_\emptyset(\alpha)\rangle \subseteq w$ then $N_w \cap \alpha \subseteq N_w \cap \phi_\emptyset(\alpha) \subseteq \delta_\emptyset$ for each $\alpha \in S^2_1$.

Suppose we have decided that $v \in \mathcal{T}$ and $(\mathcal{T})_v \subseteq \mathcal{T}^v$.

Let $\phi'_v : S^2_1 \xrightarrow{\text{onto}} \mathrm{succ}_{\mathcal{T}^v}(v)$ be the increasing enumeration. For each $\alpha \in S^2_1$ we apply claim 3.3.1 to find $\rho^\alpha < \alpha$ and a tree $\mathcal{T}^{v^\frown\phi'_v(\alpha)} \subseteq \mathcal{T}^v$ such that

(11) $\mathrm{root}(\mathcal{T}^{v^\frown\phi'_v(\alpha)}) = v^\frown\phi'_v(\alpha)$;

(12) each node in $\mathcal{T}^{v^\frown\phi'_v(\alpha)}$ extending $v^\frown\phi'_v(\alpha)$ has ω_2 successors in $\mathcal{T}^{v^\frown\phi'_v(\alpha)}$;

(13) for each $w \in \mathcal{T}^{v^\frown\phi'_v(\alpha)}$, $w \supseteq v^\frown\phi'_v(\alpha)$ we have $\sup(N_w \cap \alpha) < \rho^\alpha$.

Next we choose δ_v and A_v such that

(14) $A_v \in [S^2_1]^{\omega_2}$;

(15) $\delta_v = \rho^\alpha$ for all $\alpha \in A_v$.

We put $\mathrm{succ}_{\mathcal{T}}(v) = \phi'_v[A_v]$ and we decide that for $\alpha \in A_v$

$$(\mathcal{T})_{v^\frown\phi'_v(\alpha)} \subseteq \mathcal{T}^{v^\frown\phi'_v(\alpha)}.$$

Note that at this moment we are sure that if $w \in \mathcal{T}$, $v^\frown\phi_v(\alpha) \subseteq w$ then $N_w \cap \alpha \subseteq N_w \cap \beta \subseteq \delta_v$, where $\phi'_v(\beta) = \phi_v(\alpha)$ (clearly $\alpha \le \beta$). This finishes the construction of the tree $\mathcal{T}$ (satisfying **(4), (5)**).

For $v \in \mathcal{T}$ let $\zeta_v = N_v \cap \omega_1 \in D$. We apply claim 3.3.1 once again to find $\zeta < \omega_1$ and a tree $\mathcal{T}^* \subseteq \mathcal{T}$ such that each node in $\mathcal{T}^*$ has ω_2 successors in $\mathcal{T}^*$ and for each ω-branch z through $\mathcal{T}^*$ we have $\sup\{\zeta_{z\restriction n} : n \in \omega\} = \zeta$. Then $\zeta \in D$. For $v \in \mathcal{T}^*$ let $\psi_v : S^2_1 \xrightarrow{\text{onto}} \mathrm{succ}_{\mathcal{T}^*}(v)$ be the increasing enumeration and let $\delta^*_v = \sup(N_v \cap \omega_2)$. Let

$$\begin{aligned} E = \{\delta < \omega_2 :\ & \delta \text{ is limit } \&\ (\forall v \in \mathrm{IS}(\delta) \cap \mathcal{T}^*)(\delta_v < \delta\ \&\ \delta^*_v < \delta)\ \& \\ & \&\quad (\forall v \in \mathcal{T}^* \cap \mathrm{IS}(\delta))(\forall \beta < \delta)(\exists \gamma \in S^2_1)(\beta < \gamma \le \psi_v(\gamma) < \delta)\}. \end{aligned}$$

Since E is a closed unbounded subset of ω_2 we find $\delta \in E$ such that $C_\delta \subseteq E$.

Now we may define the semi-(ζ,δ)-creature we are looking for by constructing an embedding $\pi : \mathrm{IS}(\omega) \longrightarrow \mathcal{T}^*$ such that $\mathrm{lh}(\pi(\eta)) = \mathrm{lh}(\eta)$ and choosing corresponding models $N_{\pi(\eta)}$. This is done by induction on the length of a sequence $\eta \in \mathrm{IS}(\omega)$:
Put $\pi(\emptyset) = \emptyset$. Note that $\delta_\emptyset < \alpha^\delta_0$ (as $\alpha^\delta_0 \in E$).
Suppose we have defined $\pi(\eta) \in \mathcal{T}^*$ such that $\delta_{\pi(\eta)} < \alpha^\delta_{n_{k-1}+1}$, where $\eta = \langle n_0, n_1, \ldots, n_{k-1}\rangle$. Given $n_k > n_{k-1}$.
Take any $\gamma \in (\alpha^\delta_{n_k}, \alpha^\delta_{n_k+1}) \cap S^2_1$ and put $\pi(\eta^\frown n_k) = \pi(\eta)^\frown \psi_{\pi(\eta)}(\gamma) \in \mathcal{T}^*$. By the choice of γ we have $\delta_{\pi(\eta^\frown n_k)} < \alpha^\delta_{n_k+1}$.

Finally let $N^*_\eta = N_{\pi(\eta)}$ for $\eta \in \mathrm{IS}(\omega)$.

Since $\delta_\emptyset < \alpha^\delta_0$, $\delta^*_{\pi(\langle n_0\rangle)} < \alpha^\delta_{n_0+1}$ we have that for every $n_0 \in \omega$

$$N^*_{\langle n_0\rangle} \cap \omega_2 \subseteq \alpha^\delta_0 \cup [\alpha^\delta_{n_0}, \alpha^\delta_{n_0+1}) \quad \text{and} \quad N^*_{\langle n_0\rangle} \cap [\alpha^\delta_{n_0}, \alpha^\delta_{n_0+1}) \neq \emptyset$$

(we use here (2) and (5)). Similarly, if $\eta = \langle n_0, \ldots, n_{k-1}, n_k\rangle \in \mathrm{IS}(\omega)$ then

$$N^*_\eta \cap \alpha^\delta_{n_{i+1}} \subseteq \alpha^\delta_{n_i+1} \text{ for } i < k \text{ and}$$

$$N^*_\eta \cap [\alpha^\delta_{n_i}, \alpha^\delta_{n_i+1}) \neq \emptyset \text{ for } i \leq k.$$

Consequently the sequence $\langle N^*_\eta : \eta \in \mathrm{IS}(\omega)\rangle$ is a semi-(ζ,δ)-creature (and we are done as $X \in N^*_\emptyset$, $\zeta \in D$). ■

Theorem 3.4 *Assume $\mathbb{P}$ is a proper forcing notion, $|\mathbb{P}| \leq \mathfrak{c}$. Suppose $\Vdash_{\mathbb{P}} |\omega_2^{\mathbf{V}}| = \omega_1$. Then $\Vdash_{\mathbb{P}} \diamondsuit_{\omega_1}$.*

Proof: Let $\mathbb{P}$ be a proper forcing notion collapsing ω_2 and of size $|\mathbb{P}| \leq \mathfrak{c}$. Since $\mathbb{P}$ collapses ω_2 and $|\mathbb{P}| \leq \mathfrak{c}$ we have $\mathfrak{c} \geq \omega_2$. Let Θ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \text{“}\Theta : \omega_1 \longrightarrow \omega_2^{\mathbf{V}} \text{ is an increasing unbounded function”}.$$

Our aim is to construct a sequence $\langle \dot{A}_\zeta : \zeta < \omega_1\rangle$ of $\mathbb{P}$-names which witnesses $\diamondsuit_{\omega_1}$ in $\mathbf{V}^{\mathbb{P}}$. In the construction we will use (ζ,δ)-creatures which can be thought of as countable “trees” of possible fragments of names for

subsets of ζ (together with some parameters for controlling their behaviour). Each infinite branch through the creature will define a (countable) name for a subset of ζ. Next we will choose continuum many branches together with conditions in $\mathbb{P}$. Our choice will ensure that the conditions form an antichain in $\mathbb{P}$ and all antichains involved in the name determined by a single branch (in important cases) are predense above the corresponding condition. This will define the name $\dot{A}_\zeta$ for a subset of ζ. The main difficulty will be in proving that the sequence $\langle \dot{A}_\zeta : \zeta < \omega_1 \rangle$ is (a name for) a $\diamondsuit_{\omega_1}$–sequence. But this we will obtain right from the existence of creatures which was proved in lemma 3.3.

Before we define the names $\dot{A}_\zeta$ we have to identify some creatures (as the set $\mathrm{CR}^\zeta_\delta(\mathbb{P}, \Theta)$ can be very large):

For a (ζ, δ)-creature $S = \{(N_\eta, \tau_\eta, k_\eta) : \eta \in \mathrm{IS}(\omega)\} \in \mathrm{CR}^\zeta_\delta(\mathbb{P}, \Theta)$ let

$$U(S) = \bigcup_{\eta \in \mathrm{IS}(\omega)} N_\eta \cap \mathbb{P}.$$

Clearly $U(S)$ is a countable subset of $\mathbb{P}$ and hence there is at most $\mathfrak{c}$ possibilities for $U(S)$. Let $S^i = \{(N^i_\eta, \tau^i_\eta, k^i_\eta) : \eta \in \mathrm{IS}(\omega)\} \in \mathrm{CR}^\zeta_\delta(\mathbb{P}, \Theta)$, $i = 0, 1$. We say that the creatures S^0, S^1 are *equivalent* ($S^0 \equiv S^1$) whenever

(i) $U(S^0) = U(S^1)$ and

(ii) for each $\eta \in \mathrm{IS}(\omega)$: $N^0_\eta \cap \mathbb{P} = N^1_\eta \cap \mathbb{P}$, $k^0_\eta = k^1_\eta$, $\tau^0_\eta = \tau^1_\eta$ and
$\{A^0 \cap U(S^0) : A^0 \in N^0_\eta \text{ is a maximal antichain in } \mathbb{P}\} =$
$= \{A^1 \cap U(S^1) : A^1 \in N^1_\eta \text{ is a maximal antichain in } \mathbb{P}\}$.

(Note that actually condition (ii) implies (i).) Since for each $\eta \in \mathrm{IS}(\omega)$ there is at most $\mathfrak{c}$ possibilities for k_η, τ_η, $N_\eta \cap \mathbb{P}$ and $\{A \cap U(S) : A \in N_\eta$ is a maximal antichain in $\mathbb{P}\}$ the relation $\equiv$ has at most $\mathfrak{c}$ equivalence classes.

The following claim should be clear:

Claim 3.4.1 *Let $S^i = \{(N^i_\eta, \tau^i_\eta, k^i_\eta) : \eta \in \mathrm{IS}(\omega)\} \in \mathrm{CR}^\zeta_\delta(\mathbb{P}, \Theta)$ (for $i = 0, 1$) be equivalent creatures. Let $h \in \omega^\omega$ be an increasing function. Then*

1. *$\bigcup_{n \in \omega} N^i_{h\restriction n}$ is an elementary (countable) submodel of $\mathcal{H}(\chi)$,*
2. *$\bigcup_{n \in \omega} N^0_{h\restriction n} \cap \mathbb{P} = \bigcup_{n \in \omega} N^1_{h\restriction n} \cap \mathbb{P}$,*

3. *if* $A^0 \in N^0_{h\restriction n}$ *is a maximal antichain in* $\mathbb{P}$ *then for some maximal antichain* $A^1 \in N^1_{h\restriction n}$ *we have* $A^0 \cap \bigcup_{n\in\omega} N^0_{h\restriction n} = A^1 \cap \bigcup_{n\in\omega} N^1_{h\restriction n}$,

4. $\{A^0 \cap \bigcup_{n\in\omega} N^0_{h\restriction n} : A^0 \in \bigcup_{n\in\omega} N^0_{h\restriction n}$ *is a maximal antichain in* $\mathbb{P}\} =$
$= \{A^1 \cap \bigcup_{n\in\omega} N^1_{h\restriction n} : A^1 \in \bigcup_{n\in\omega} N^1_{h\restriction n}$ *is a maximal antichain in* $\mathbb{P}\}$,

5. *if* $p \in \mathbb{P}$ *is* $(\bigcup_{n\in\omega} N^0_{h\restriction n}, \mathbb{P})$*-generic then it is* $(\bigcup_{n\in\omega} N^1_{h\restriction n}, \mathbb{P})$*-generic.*

Fix a limit ordinal $\zeta < \omega_1$.

We are going to define a name $\dot{A}_\zeta$ for a subset of ζ.

Suppose that $\bigcup_{\delta\in S^2_0} \mathrm{CR}^\zeta_\delta(\mathbb{P}, \Theta) \neq \emptyset$.

Let $p^i \in \mathbb{P}$, $S^i \in \bigcup_{\delta\in S^2_0} \mathrm{CR}^\zeta_\delta(\mathbb{P}, \Theta)$ (for $i < \mathfrak{c}$) be such that $\{(p^i, [S^i]_\equiv) : i < \mathfrak{c}\}$ lists all members of $\mathbb{P} \times (\bigcup_{\delta\in S^2_0} \mathrm{CR}^\zeta_\delta(\mathbb{P}, \Theta)/\equiv)$ with $\mathfrak{c}$ repetitions. Take any family $\{h_i : i < \mathfrak{c}\} \subseteq \omega^\omega$ of increasing functions such that for distinct $i, j < \mathfrak{c}$ the intersection $\mathrm{rng}(h_i) \cap \mathrm{rng}(h_j)$ is finite.

Now for each $i < \mathfrak{c}$ we put $M_i = \bigcup_{n<\omega} N^i_{h_i\restriction n}$, $\tau_i = \bigcup_{n\in\omega} \tau^i_{h_i\restriction n}$.

Each M_i is a countable elementary submodel of $\mathcal{H}(\chi)$ and τ_i is a function. Since $\mathbb{P}$ is proper we find $p_i \in \mathbb{P}$ such that p_i is $(M_i, \mathbb{P})$-generic. If we can find such a condition p_i above the condition p^i then we also demand $p_i \geq p^i$. Note that $M_i \cap \omega_1 = \zeta$ and $M_i \cap \omega_2 \subseteq \delta^i$ is cofinal in δ^i (what is a consequence of (γ), (δ) of definition 3.2(1)), where $\delta^i \in S^2_0$ is such that $S^i \in \mathrm{CR}^\zeta_{\delta_i}(\mathbb{P}, \Theta)$. Hence

$$p_i \Vdash \text{“}\mathrm{rng}(\Theta\restriction\zeta) \subseteq M_i \text{ is unbounded in } \delta^i\text{”}.$$

If $\delta^i \neq \delta^j$ then the conditions p_i, p_j force inconsistent sentences (unboundness of $\mathrm{rng}(\Theta\restriction\zeta)$ in δ^i, δ^j, respectively). If $\delta^i = \delta^j$ but $i \neq j$ then the choice of the functions h_i, h_j guaranties (by (γ), (δ) of 3.2(1)) that sets $M_i \cap [\alpha, \delta^i)$ and $M_j \cap [\alpha, \delta^i)$ are disjoint for some $\alpha < \delta^i$. Consequently if $i \neq j$ then p_i, p_j are incompatible.

Let $\mathcal{A}_\zeta$ be a maximal antichain in $\mathbb{P}$ extending $\{p_i : i < \mathfrak{c}\}$ and let $\dot{A}_\zeta$ be a name for a subset of ζ such that for each $(p, k) \in \mathrm{dom}(\tau_i)$

$$p_i \Vdash \text{“if } p \in \Gamma_{\mathbb{P}} \text{ then } \dot{A}_\zeta(e^\zeta_k) = \tau_i(p, k)\text{”}$$

(we identify a subset of ζ with its characteristic function).

If $\bigcup_{\delta\in S^2_0} \mathrm{CR}^\zeta_\delta(\mathbb{P}, \Theta) = \emptyset$ then take any maximal antichain and a name for a subset of ζ.

We want to show that the sequence $\langle \dot{A}_\zeta : \zeta < \omega_1 \rangle$ is a (name for a) $\diamondsuit_{\omega_1}$-sequence. For this suppose that $\dot{A}$ is a $\mathbb{P}$-name for a subset of ω_1, $\dot{D}$ is a $\mathbb{P}$-name for a closed unbounded subset of ω_1, $p \in \mathbb{P}$. We have to prove:

Claim 3.4.2 *There exist a limit ordinal $\zeta < \omega_1$ and a condition $p^* \in \mathcal{A}_\zeta$ such that $p^* \geq p$ and $p^* \Vdash$ "$\dot{A}_\zeta = \dot{A} \cap \zeta$ & $\zeta \in \dot{D}$".*

To prove the claim we use lemma 3.3 to find a semi-(ζ, δ)-creature $\langle N^*_\eta : \eta \in \mathrm{IS}(\omega) \rangle$ such that $\Theta, \dot{A}, \dot{D}, \mathbb{P}, p, \ldots \in N^*_\emptyset$. Next:

let $k_\eta = \min\{l : e^\zeta_l \notin N^*_\eta\}$.

For each $\eta \in \mathrm{IS}(\omega)$ and $k < k_\eta$ we fix a maximal antichain $\mathcal{B}^k_\eta$ in $\mathbb{P}$ such that $\mathcal{B}^k_\eta \in N^*_\eta$ and $(\forall p \in \mathcal{B}^k_\eta)(p \parallel e^\zeta_k \in \dot{A})$. Moreover we demand that $\eta \subseteq \nu$ implies $\mathcal{B}^k_\eta = \mathcal{B}^k_\nu$ (for $k < k_\eta$). Now we define functions τ_η for $\eta \in \mathrm{IS}(\omega)$ by

$$\mathrm{dom}(\tau_\eta) = \bigcup_{k<k_\eta}((\mathcal{B}^k_\eta \cap N^*_\eta) \times \{k\}),$$
$$\tau_\eta(p,k) = 1 \text{ if and only if } p \Vdash e^\zeta_k \in \dot{A}.$$

It should be clear that $S = \{(N^*_\eta, \tau_\eta, k_\eta) : \eta \in \mathrm{IS}(\omega)\}$ is a (ζ, δ)-creature for $\mathbb{P}, \Theta$. Thus we find $i < \mathfrak{c}$ such that $S \equiv S^i$ and $p = p^i$ (where $S^i \in \mathrm{CR}^\zeta_\delta(\mathbb{P}, \Theta)$, $p^i \in \mathbb{P}$ are as in the definition of the antichain $\mathcal{A}_\zeta$ and the name $\dot{A}_\zeta$). Then the condition $p^* = p_i \in \mathcal{A}_\zeta$ is $(\bigcup_{n\in\omega} N^*_{h_i\restriction n}, \mathbb{P})$-generic, $p^* \geq p^i = p \in N^*_\emptyset$. The name $\dot{A}_\zeta$ agrees with decissions of $\tau_{h_i\restriction n}$ (or $\mathcal{B}^k_{h_i\restriction n}$). By the genericity of p^* we conclude that $p^* \Vdash$ "$\dot{A} \cap \zeta = \dot{A}_\zeta$ & $\zeta \in \dot{D}$". Thus the claim is proved.

The theorem follows from the claim. ■

4 Laver forcing, Miller forcing, Silver forcing...

Results of the second section can be formulated for other forcing notions. Without any problems we can prove the respective facts for the Silver forcing (and generally for forcing notions consisting of compact trees).

Recall that the Silver forcing notion consists of partial functions p such that $\mathrm{dom}(p) \subseteq \omega$, $\omega \backslash \mathrm{dom}(p)$ is infinite and $\mathrm{rng}(p) \subseteq 2$. These functions are ordered by the inclusion.

Theorem 4.1 *The Silver forcing notion forces "*$\mathfrak{c} = |(\mathfrak{b}^{+\epsilon})^{\mathbf{V}}|$*".* ■

We have to be more carefull when we work with trees on ω. Nevertheless even in this case we get the similar result

The Laver forcing $\mathcal{L}$ consists of infinite trees $T \subseteq \omega^{<\omega}$ such that for each $t \in T$, $\mathrm{root}(T) \subseteq t$ we have $|\mathrm{succ}_T(t)| = \omega$.

Definition 4.2 *We say that a condition $T \in \mathcal{L}$* weakly obeys *a set $X \in [\omega]^\omega$ whenever for each ramification point $t \in T$*

$$(\exists^\infty i)(\forall j < 2^i)(\mathrm{succ}_T(t) \cap [\mu_X(2^i + j), \mu_X(2^i + j + 1)) \neq \emptyset).$$

Fix $T \in \mathcal{L}$. Take $X_0 \in [\omega]^\omega$ such that for each ramification point $t \in T$

$$(\forall^\infty i)(\mathrm{succ}_T(t) \cap [\mu_{X_0}(i), \mu_{X_0}(i+1)) \neq \emptyset).$$

Suppose that $X \in [\omega]^\omega$ is such that

$$(\exists^\infty i)(\forall j < 2^i)(|[\mu_X(2^i + j), \mu_X(2^i + j + 1)) \cap X_0| \geq 2).$$

Then clearly T weakly obeys X. Consequently if $\mathcal{F} \subseteq [\omega]^\omega$ is a weakly dominating family then T weakly obeys some $X \in \mathcal{F}$.

Suppose now that T weakly obeys $X \in [\omega]^\omega$ and $h : \omega \longrightarrow \omega$ is such that $(\forall i)(h(i) < 2^i)$. Then we can easily construct a condition $T^h \geq T$ such that

> if $t \in T^h$ is a ramification point in T^h, $t^\frown n \in T^h$ and $j < 2^i$, $2^i + j \leq n < 2^i + j + 1$
> then $h(i) = j$.

Moreover, if h_0, h_1 are such that $(\forall^\infty i)(h_0(i) \neq h_1(i))$ then the respective conditions T^{h_0}, T^{h_1} are incompatible – their intersection has no node with infinitely many immediate successors. Consequently we can repeat the proof of 2.4 and we get

Theorem 4.3 $\Vdash_{\mathcal{L}} \mathfrak{c} = |(\mathfrak{b}^{+\epsilon})^{\mathbf{V}}|$. ■

The argument above applies for the Miller forcing too. Recall that this order consists of perfect trees $T \subseteq [\omega]^{<\omega}$ such that

$$(\forall t \in T)(\exists s \in T)(t \subseteq s \ \&\ |\mathrm{succ}_T(s)| = \omega).$$

Thus we can conclude

Theorem 4.4 *The Miller forcing collapses the continuum onto* $(\mathfrak{b}^{+\epsilon})^{\mathbf{V}}$.
■

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