# POSSIBLE PCF ALGEBRAS

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ABSTRACT. There exists a family $\{B_\alpha\}_{\alpha<\omega_1}$ of sets of countable ordinals such that

(1) $\max B_\alpha = \alpha$,
(2) if $\alpha \in B_\beta$ then $B_\alpha \subseteq B_\beta$,
(3) if $\lambda \le \alpha$ and $\lambda$ is a limit ordinal then $B_\alpha \cap \lambda$ is not in the ideal generated by the $B_\beta$, $\beta < \alpha$, and by the bounded subsets of $\lambda$,
(4) there is a partition $\{A_n\}_{n=0}^\infty$ of $\omega_1$ such that for every $\alpha$ and every $n$, $B_\alpha \cap A_n$ is finite.

## 1. Introduction.

In [3], [4], [5] and [6] the second author developed the theory of possible cofinalities (pcf), and proved, among others, that if $\aleph_\omega$ is a strong limit cardinal then $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$ as well as $2^{\aleph_\omega} < \aleph_{\omega_4}$. The latter inequality is established via an analysis of the structure of pcf; in particular, it is shown that if $\aleph_4 \le |\text{pcf}\{\aleph_n\}_{n=0}^\infty|$ then a certain structure exists on $\omega_4$, and then it is proved that such a structure is impossible. (Cf. [5], [1] and [2] for details.) One might hope that by investigating this structure one could possibly derive a contradiction for $\aleph_3$, $\aleph_2$ or even $\aleph_1$.

A major open problem in the theory of singular cardinals (or in the pcf theory) is whether it is consistent that $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} > \aleph_{\omega_1}$; or whether the set $\text{pcf}\{\aleph_n\}_{n=1}^\infty$ can be uncountable. If we make this assumption, we obtain a certain structure on $\omega_1$. The structure is described in Theorem 2.1. Unlike in the $\omega_4$-case, the structure so obtained is not impossible: in Theorem 3.1 we show that

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there exists a structure on $\omega_1$ which has the properties listed in the abstract, and consequently has the properties listed in Theorem 2.1.

In Section 2, all facts on Shelah's pcf theory not proved explicitly can be found in the expository articles [1] and [2]. In Section 3 we assume rudimentary knowledge of forcing.

## 2. A consequence of "pcf $\{\aleph_n\}_{n=0}^\infty$ is uncountable".

**Theorem 2.1.** *If pcf $\{\aleph_n\}_{n=0}^\infty$ is uncountable, then there exist sets $B_\alpha$, $\alpha < \omega_1$, of countable ordinals with the following properties:*

(a) *For every $\alpha < \omega_1$,* $\max\ B_\alpha = \alpha$.

(b) *For all $\alpha, \beta < \omega_1$, if $\alpha \in B_\beta$ then $B_\alpha \subseteq B_\beta$.*

(c) *For every limit ordinal $\lambda < \omega_1$, $B_\lambda \cap \lambda$ is unbounded in $\lambda$.*

(d) *There is a closed unbounded set $C$ of countable limit ordinals such that for all $\lambda \in C$ and for all $\alpha \geq \lambda$, the set $B_\alpha \cap \lambda$ is not in the ideal generated by the sets $B_\beta$, $\beta < \alpha$, and by bounded subsets of $\lambda$. (I.e. $B_\alpha \cap \lambda \not\subseteq \gamma \cup B_{\beta_1} \cup \cdots \cup B_{\beta_k}$, for any $\gamma < \lambda$, and any $\beta_1, \ldots, \beta_k < \alpha$.)*

(e) *Every unbounded set $X \subseteq \omega_1$ has an initial segment $X \cap \gamma$ that is not in the ideal generated by the sets $B_\alpha$, $\alpha < \omega_1$.*

(f) *Moreover, (e) remains true in every extension $M$ of the ground model that preserves cardinals and cofinalities, and has the property that every countable set of ordinals in $M$ is covered by a countable set in the ground model.*

*Proof.* Let $a = \text{pcf}\,\{\aleph_n\}_{n=0}^\infty$ and assume that $a$ is uncountable. Applying the pcf theory, one obtains (cf. [6], Main Theorem) sets $b_\lambda$, $\lambda \in a$, (*generators*) together with sequences of functions $f_i^\lambda$ $(i < \lambda)$ in $\prod a$. As $a$ contains all regular cardinals $\lambda < \aleph_{\omega_1}$, we let, for each $\alpha < \omega_1$

$$B_\alpha = \{\xi : \aleph_{\xi+1} \in b_{\aleph_{\alpha+1}}\}.$$

Property (a) is immediate. Property (b) is the *transitivity* of generators; such generators can be found (cf. [1], Lemma 6.9).

Property (c) is a consequence of the fact that for every countable limit ordinal $\lambda$, there exists an increasing sequence $\alpha_n$, $n<\omega$, with limit $\lambda$, and an ultrafilter $D$ on $\omega$ such that $\mathrm{cof}\left(\prod_{n=0}^{\infty}\aleph_{\alpha_n+1}/D\right)=\aleph_{\lambda+1}$ (cf. [1], Theorem 2.1).

Property (d): Let $\gamma_i$, $i<\omega_1$, be a continuous increasing sequence of countable ordinals constructed as follows: Given $\gamma_i$, we first note that $\aleph_{\omega_1+1}\in\mathrm{pcf}\,[\aleph_{\gamma_i+1},\ \aleph_{\omega_1})$ (by [1], Theorem 2.1), and by the Localization Theorem [6], there is a $\gamma_{i+1}<\omega_1$ such that $\aleph_{\omega_1+1}\in\mathrm{pcf}\,[\aleph_{\gamma_i+1},\ \aleph_{\gamma_{i+1}})$. Let $C$ be the set of all limit points of the sequence $\{\gamma_i\}_{i<\omega_1}$.

Now let $\lambda\in C$, $\alpha\geq\lambda$, $\gamma<\lambda$, and $\beta_1,\dots,\beta_k<\alpha$. We find $\gamma_i$ such that $\gamma<\gamma_i<\lambda$. By [1], Theorem 2.1, we have $\aleph_{\alpha+1}\in\mathrm{pcf}\,[\aleph_{\gamma_i+1},\aleph_{\gamma_{i+1}})$ and so there is an ultrafilter $D$ on $[\gamma_i+1,\gamma_i)$ such that $\mathrm{cof}\,(\prod\aleph_{\xi+1}/D)=\aleph_{\alpha+1}$. By the definition of generators, we have $B_\alpha\in D$ while $B_{\beta_i}\notin D$ $(i=1,\dots,k)$, and (d) follows.

Property (e): If $X\subseteq\omega_1$ is unbounded, then $\max\mathrm{pcf}\,\{\aleph_{\alpha+1}:\alpha\in X\}\geq\aleph_{\omega_1}$, and by the Localization Theorem, there is a countable $\gamma$ such that $\max\mathrm{pcf}\,\{\aleph_{\alpha+1}:\alpha\in X\cap\gamma\}\geq\aleph_{\omega_1}$. Now if $\alpha_1,\dots,\alpha_k$ are countable ordinals, we cannot have $X\cap\gamma\subseteq B_{\alpha_1}\cup\dots\cup B_{\alpha_k}$, because $\max\mathrm{pcf}\,(b_{\aleph_{\alpha_1}+1}\cup\dots\cup b_{\aleph_{\alpha_k}+1})=\max\{\aleph_{\alpha_i+1}:i=1,\dots,k\}<\aleph_{\omega_1}$.

Property (f): Let $M$ be an extension of the ground model $V$ that preserves cardinals and cofinalities, and assume further that every countable set of ordinals in $M$ is covered by a countable set in $V$.

To show that (e) is true in $M$, it suffices to show that the generators $b_\lambda$ are generators of the pcf structure in $M$. For that, it is enough to verify that the sequences $f_i^\lambda$ $(i<\lambda)$ are increasing cofinal sequences in $\prod a$ (modulo the appropriate ideals $J_{<\lambda}$). Since $M$ has the same cardinals and cofinalities, the claim follows upon the observation that for every regular $\lambda<\aleph_{\omega_1}$, every function $f\in\prod b_\lambda$ in $M$ is majorized by some function $g\in\prod b_\lambda$ in $V$.

## 3. Existence of the family $\{B_\alpha\}_{\alpha<\omega_1}$.

**Theorem 3.1.** *There exist a partition $\{A_n\}_{n=0}^{\infty}$ of $\omega_1$, and a family $\{B_\alpha\}_{\alpha<\omega_1}$ of countable sets of countable ordinals such that*

(a) *For every* $\alpha < \omega_1$, $\max B_\alpha = \alpha$.

(b) *For all* $\alpha, \beta < \omega_1$, *if* $\alpha \in B_\beta$ *then* $B_\alpha \subseteq B_\beta$.

(c) *For every limit ordinal* $\lambda < \omega_1$ *and for all* $\alpha \geq \lambda$, $B_\alpha \cap \lambda \not\subseteq \gamma \cup B_{\beta_1} \cup \cdots \cup B_{\beta_k}$ *for any* $\gamma < \lambda$ *and any* $\beta_1, \ldots, \beta_k < \alpha$.

(d) *For all* $\alpha < \omega_1$ *and all* $n$, $B_\alpha \cap A_n$ *is finite.*

**Corollary 3.2.** *If* $M$ *is any* $\aleph_1$*-preserving extension of* $V$, *then every unbounded set* $X \subseteq \omega_1$ *in* $M$ *has an initial segment* $X \cap \gamma$ *that is not in the ideal generated by the sets* $B_\alpha$, $\alpha < \omega_1$.

*Proof.* By (d), any set in the ideal has a finite intersection with each $A_n$. If $X \subseteq \omega_1$ is unbounded then some $X \cap A_n$ is uncountable, and so some $(X \cap \gamma) \cap A_n$ is infinite. Hence $X \cap \gamma$ is not in the ideal.

To construct the structure described in Theorem 3.1 we shall first define a forcing notion and prove that it forces such a structure to exist in the generic extension. The forcing notion that we use satisfies the countable chain condition and consists of finite conditions consisting of countable ordinals and relations between countable ordinals. Using a general method due to the second author [7] we then conclude that such a structure exists in $V$.

**Definition 3.3.**

A forcing condition is a quadruple $p = (S_p,\ \pi_p,\ b_p,\ u_p)$ such that

(i) $S_p$ is a finite subset of $\omega_1$,

(ii) $b_p$ is a function from $S_p \times S_p$ into $\{0,1\}$ such that

$$b_p(\alpha,\alpha) = 1 \qquad (\alpha \in S_p)$$

$$b_p(\alpha,\beta) = 0 \qquad (\alpha, \beta \in S_p, \quad \alpha < \beta)$$

$$\text{if } b_p(\alpha,\beta) = 1 \text{ and } b_p(\beta,\gamma) = 1 \text{ then } b_p(\alpha,\gamma) = 1 \qquad (\alpha,\beta,\gamma \in S_p)$$

(iii) $u_p$ is a natural number,

(iv) $\pi_p$ is a function from $S_p$ into $\{0, ..., u_p - 1\}$ such that for all $\alpha$ and $\beta$ in $S_p$, if $b_p(\alpha,\beta) = 1$ and $\beta < \alpha$ then $\pi_p(\beta) \neq \pi_p(\alpha)$, [Motivation: $S$ is the support of the condition, $\pi(\alpha) = n$ forces $\alpha \in A_n$, $b(\alpha, \beta) = 1$ forces $\beta \in B_\alpha$ and $b(\alpha, \beta) = 0$ forces $\beta \notin B_\alpha$.]

A condition $r = (S_r,\ \pi_r,\ b_r,\ u_r)$ is stronger than $p = (S_p,\ \pi_p,\ b_p,\ u_p)$ if

(i) $S_r \supseteq S_p$,

(ii) $b_r$ extends $b_p$,

(iii) $\pi_r$ extends $\pi_p$

(iv) $u_r \geq u_p$,

(v) for all $\alpha \in S_p$ and all $\beta \in S_r - S_p$, if $b_r(\alpha, \beta) = 1$ then $\pi_r(\beta) \geq u_p$.

It is easy to verify that "stronger than" is a transitive relation.

**Definition 3.4.**

If $p = (S_p,\ \pi_p,\ b_p,\ u_p)$ is a condition and $\eta$ is a countable ordinal, we let

$$p \restriction \eta = (S_p \cap \eta,\ \pi_p \restriction \eta,\ b_p \restriction (\eta \times \eta),\ u_p).$$

Clearly, $p \restriction \eta$ is a condition and $p$ is stronger than $p \restriction \eta$.

**Lemma 3.5 (Amalgamation).** *If $p$ and $q$ are conditions and $\eta$ a countable ordinal such that $q$ is stronger than $p \restriction \eta$ and $S_q \subseteq \eta$ then there exists a condition $r$ such that $r$ is stronger than both $p$ and $q$ (and such that $S_r = S_p \cup S_q$).*

*Proof.* Note that $u_q \geq u_p$. We let $S_r = S_p \cup S_q$, $\pi_r = \pi_p \cup \pi_q$ and $u_r = u_q$. We define $b_r$ as follows: if $\alpha$ and $\beta$ are both in $S_p$ (both in $S_q$) the we let $b_r(\alpha, \beta) = b_p(\alpha, \beta)$ (we let $b_r(\alpha, \beta) = b_q(\alpha, \beta)$.) If $\alpha \geq \eta$ is in $S_p$ and if $\beta < \eta$ is in $S_q - S_p$ then we let $b_r(\alpha, \beta) = 1$ if and only if there exists a $\gamma < \eta$ in $S_p$ such that $b_p(\alpha, \gamma) = 1$ and $b_q(\gamma, \beta) = 1$. Otherwise we let $b_r(\alpha, \beta) = 0$.

Next we verify that $r$ is a condition. It is easy to see that requirement (ii) from the definition is satisfied. To verify (iv), the only case we need to worry about is when $b_r(\alpha, \beta) = 1$ where $\alpha \geq \eta$ is in $S_p$ and $\beta < \eta$ is in $S_q - S_p$. In this case, $\pi_q(\beta) \geq u_p$ (because $q$ is stronger than $p \restriction \eta$ and $b_q(\gamma, \beta) = 1$ for some $\gamma \in S_p \cap \eta$) while $\pi_p(\alpha) < u_p$, and so $\pi_r(\beta) \neq \pi_r(\alpha)$.

Since $r \restriction \eta = q$, $r$ is stronger than $q$. In order to show that $r$ is stronger than $p$ we only need to verify condition (v), and only for the case when $\alpha \geq \eta$ is in $S_p$

and $\beta < \eta$ is in $S_q - S_p$. This is however exactly the argument in the preceding paragraph.

**Lemma 3.6.** *The forcing satisfies the countable chain condition.*

*Proof.* Given $\aleph_1$ conditions, we first find $\aleph_1$ of them whose supports form a $\Delta$-system, with a root $A$, i.e. $S_{p_\xi} \cap S_{p_\eta} = A$ whenever $\xi < \eta$, and such that $\beta < \alpha$ whenever $\beta \in S_{p_\xi}$ and $\alpha \in S_{p_\eta} - A$. Then $\aleph_1$ of them have the same restrictions of $\pi$ and $b$ to the root $A$, and the same $u$.

Now it follows from Lemma 3.5 that any two such conditions are compatible.

Let $G$ be a generic set of conditions. In $V[G]$, we let, for each $\alpha < \omega_1$ and each $n < \omega$,

$$B_\alpha = \{\beta : b(\alpha, \beta) = 1 \text{ for some condition } (S,\ \pi,\ b,\ u) \in G\}, \tag{3.7}$$

$$A_n = \{\alpha : \pi(\alpha) = n \text{ for some condition } (S,\ \pi,\ b,\ u) \in G\}. \tag{3.8}$$

Clearly, $\max B_\alpha = \alpha$, and if $\alpha \in B_\beta$ then $B_\alpha \subseteq B_\beta$. The sets $A_n$ are mutually disjoint subsets of $\omega_1$.

**Lemma 3.9.** *For every $\alpha < \omega_1$ the set of all conditions $p$ with $\alpha \in S_p$ is dense. For every $n$ the set of all conditions $p$ with $u_p \geq n$ is dense.*

*Proof.* If $q$ is a condition and $\alpha \notin S_q$ then let $S_p = S_q \cup \{\alpha\}$, let $b_p(\alpha, \alpha) = 1$, $u_p = u_q + 1$ and $\pi_p(\alpha) = u_q$. Then $p$ is a condition stronger than $q$. The proof of the second statement is similar.

**Corollary 3.10.** $\{A_n\}_{n=0}^{\infty}$ *is a partition of* $\omega_1$.

**Lemma 3.11.** *For all $\alpha < \omega_1$ and all $n$, $B_\alpha \cap A_n$ is finite.*

*Proof.* Let $\alpha$ and $n$ be given, and let $p = (S_p,\ \pi_p,\ b_p,\ u_p)$ be a condition. We shall find a stronger condition $q$ that forces that $B_\alpha \cap A_n$ is finite.

There is a condition $q = (S_q,\ \pi_q,\ b_q,\ u_q)$ stronger than $p$ such that $\alpha \in S_q$ and that $u_q > n$. We claim that $q$ forces that $B_\alpha \cap A_n \subseteq S_q$.

If $\beta$ is an ordinal not in $S_q$ and if $r = (S_r,\ \pi_r,\ b_r,\ u_r)$ is a stronger condition that forces $\beta \in B_\alpha$ then because $b_r(\alpha,\beta) = 1$, we have $\pi_r(\beta) \geq u_q > n$, and so $r$ forces $\beta \notin A_n$. Thus $q$ forces $B_\alpha \cap A_n \subseteq S_q$.

**Lemma 3.12.** *Let $\lambda < \omega_1$ be a limit ordinal, let $\alpha \geq \lambda$, and let $\gamma < \lambda$ and $\alpha_1, \dots, \alpha_k < \alpha$. There exists a $\beta \geq \gamma$, $\beta < \lambda$, such that $\beta \in B_\alpha$ and $\beta \notin B_{\alpha_1}, \dots, \beta \notin B_{\alpha_k}$.*

*Proof.* Let $p = (S_p,\ \pi_p,\ b_p,\ u_p)$ be a condition. We may assume that $\alpha, \alpha_1, \dots, \alpha_k \in S_p$. Let $\beta < \lambda$ be such that $\beta \geq \gamma$ and $\beta \notin S_p$.

Let $\eta = \alpha + 1$ and $S = S_p \cap \eta$. We let $S_q = S \cup \{\beta\}$, $u_q = u_p + 1$, $\pi_q \restriction S = \pi_p \restriction S$, $\pi_q(\beta) = u_p$, $b_q \restriction (S \times S) = b_p \restriction (S \times S)$, $b_q(\alpha,\beta) = b_q(\beta,\beta) = 1$, and $b_q(\beta,\xi) = b_q(\xi,\beta) = 0$ otherwise. The condition $q = (S_q,\ \pi_q,\ b_q,\ u_q)$ is stronger than $p \restriction \eta$, has $S_q \subseteq \eta$ and forces $\beta \in B_\alpha$, $\beta \notin B_{\alpha_1}, \dots, \beta \notin B_{\alpha_k}$. By Lemma 3.5 there is a condition $r$ that is stronger than both $p$ and $q$.

This concludes the proof that the forcing from Definition 3.3 adjoins a structure described in Theorem 3.1. That such a structure exists in $V$ is a consequence of the general theorem (Theorem 1.9) in [7]. Our forcing is $\omega_1$-uniform in the sense of Definition 1.1 in [7] and the dense sets needed to produce the $B_\alpha$ and the $A_n$ in Theorem 3.1 conform to Definition 1.4 in [7] and hence the method of [7] applies.

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