ON MONK'S QUESTIONS

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Saharon Shelah

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

Rutgers University
Department of Mathematics
New Brunswick, NJ USA

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Anotated Content

§1 Introduction

§2 Existence of subalgebras with a preassigned algebraic density

[We first note (in 2.1) that if $\pi(B) \geq \theta = \mathrm{cf}(\theta)$ then for some $B' \subseteq B$ we have $\pi(B') = \theta$. Call this statement $(*)$. Then we give a criterion for $\pi(B) = \mu > \mathrm{cf}(\mu)$ (in 2.2) and conclude for singular μ that for a club of $\theta < \mu$ the $(*)$ above holds (2.2A), and investigate the criterion (in 2.3). Our main aim is, starting with $\mu = \mu^{<\mu}, \mathrm{cf}(\lambda) < \lambda$, to force the existence of a Boolean algebra B with $\pi(B) > \theta$ but for no $B' \subseteq B$ do we have $\pi(B') = \lambda$ (in fact $(\exists B' \subseteq B)[\pi(B') = \theta \Leftrightarrow \theta = \mathrm{cf}(\theta) \vee \mathrm{cf}(\theta) \leq \mu]$ for every $\theta \leq |B|$. Toward this, we define the forcing (Definition 2.5: how $\langle x_\alpha : \alpha \in W^P \rangle$ generate a Boolean algebra, $BA[p], W^p \in [\lambda]^{<\mu}$ with $x_\alpha > \theta$ has no non-zero member of $\langle x_\beta : \beta \in W^p \cap \alpha \rangle_{BA[p]}$ below it). We prove the expected properties of the generic (2.6), also the forcing has the expected properties (μ-complete, μ^+-c.c) (in 2.7). The main theorem (2.9) stated, the main point brings that if $\mu < \mathrm{cf}(\theta) < \theta$ for $B \subseteq BA[G], \pi(B) \neq \theta$; we use the criterion from above, a lemma related to $\triangle$-systems (see [Sh 430],6.6D, [Sh 513],6.1) quoted in 2.4; to reduce the problem to some special amalgamation of finitely many copies (the exact number is in relation to the arity of the term defining the relevant elements from the x_α's). The existence of such amalgamation was done separately earlier (2.8).

Lastly in 2.10 we show that the $\mathrm{cf}(\theta) > \mu$ above was necessary by proving the existence of a subalgebra with prescribed singular algebraic density $\lambda : \pi(B) > \lambda$ and $(\forall \mu < \lambda)[\mu^{<\,\mathrm{cf}(\lambda)} < \lambda]]$.

§3 On π and $\pi\chi$ of Products of Boolean Algebras

[If e.g. $\aleph_0 < \kappa = \mathrm{cf}(\chi) < \chi < \lambda = \mathrm{cf}(\lambda) < \chi^\kappa, (\forall \theta < \chi)(\theta^\kappa < \chi)$ we show that for some Boolean algebras B_i (for $i < \kappa$) : $\chi - \sum\limits_{i<\chi} \pi\chi(B_i) < \lambda$ but (for D a regular ultrafilter on κ) $\lambda = \pi\chi(\prod\limits_{i<\kappa} B_i/D)$ but $\prod\limits_{i<\kappa}(\pi\chi(B_i))/D = \chi^\kappa$. For this we use interval Boolean algebras on order of the form $\lambda_i \times \mathbb{Q}$.

We also prove for infinite Boolean algebras B_i (for $i < \kappa$) and D an ultrafilter on κ, if $n_i < \aleph_0, \mu = \prod\limits_{i<\kappa} n_i/D$ is regular (infinite) cardinal then $\pi\chi(\prod\limits_{i<\kappa} B_i/D) \geq \mu$.

§1 Introduction

Monk [M] asks: (problems 13, 15 in his list; π is the algebraic density, see 1.1 below)

For a (Boolean algebra) B, $\aleph_0 \le \theta \le \pi(B)$, does B have a subalgebra B' with $\pi(B') = \theta$?

If θ is regular the answer is easily positive (see 2.1), we show that in general it may be negative (see 2.9(3)), but for quite many singular cardinals - it is positive (2.10); the theorems are quite complementary. This is dealt with in §2.

In §3 we mainly deal with $\pi\chi$ (see Definition 3.2) show that the $\pi\chi$ of an ultraproduct of Boolean algebras is not necessarily the ultraproduct of the $\pi\chi$'s. Note: in Koppelberg Shelah [KpSh 415] ,Theorem 1.1 we prove that if SCH holds, $\pi(B_i) > 2^\kappa$ for $i < \kappa$ then $\pi\left(\prod\limits_{i<\kappa} B_i/D\right) = \prod\limits_{i<\kappa}(\pi(B_i))/D$.
We also prove that for infinite Boolean algebras $A_i (i < \kappa)$ and a non-principal ultrafilter D on κ: if $n_i < \aleph_0$ for $i < \kappa$ and $\mu =: \prod\limits_{i<\kappa} n_i/D$ is regular, then $\pi\chi(A) \ge \mu$. Here $A =: \prod\limits_{i<\kappa} A_i/D$. By a theorem of Peterson the regularity of μ is needed.

1.1 Notation: Boolean algebras are denoted by B and sometimes A.
For a Boolean algebra B

$$B^+ =: \{x \in B : x \ne 0\}$$

$$\pi(B) =: \text{ Min } \{|X| : X \subseteq B^+ \text{ is such that } \forall y \in B^+ \exists x \in X [x \le y]\}.$$

X like that is called dense in B. More generally if $X, Y \subseteq B$ we say X is dense in Y if $y \in Y \ \& \ y \ne 0 \Rightarrow (\exists x \in X)(0 < x \le y)$. For a $Y \subseteq B$, $\langle Y \rangle_B$ is the subalgebra of B which Y generates.
0_A is the constant function with domain A and value zero.
1_A is defined similarly.

§2 Existence of subalgebras with a preassigned algebraic density

Note

2.1 Observation. If $\pi(B) > \theta = \text{cf}(\theta) \geq |Y| + \aleph_0$ and $Y \subseteq B$ <u>then</u> for some subalgebra A of B, $Y \subseteq A$ and $\pi(A) = \theta$.

Proof. Without loss of generality $|Y| = \theta$. Let $Y = \{y_\alpha : \alpha < \theta\}$. Choose by induction on $\alpha \leq \theta$ subalgebras A_α of B, increasing continuous in α, $|A_\alpha| < \theta$, $y_\alpha \in A_{\alpha+1}$ such that: for each $\alpha < \theta$, some $x_\alpha \in A^+_{\alpha+1}$ is not above any $y \in A^+_\alpha$. This is possible because for no $\alpha < \theta$ can A^+_α be dense in B.
Now $A = A_\theta$ is as required. $\square_{2.1}$

2.2 Claim. *Assume B is a Boolean algebra, $\pi(B) = \mu >$ cf$(\mu) \geq \aleph_0$ (see Definition 1.1). <u>Then</u> for arbitrarily large regular $\theta < \mu$*

$(*)^B_\theta$ for some set Y we have:

$(*)^B_\theta[Y]$ $Y \subseteq B^+, |Y| = \theta,$ and there is <u>no</u> $Z \subseteq B^+$

$$\text{of cardinality } < \theta, \text{ dense in } Y \left(\text{i.e. } \bigwedge_{y \in Y} \bigvee_{z \in Z} z \leq y \right).$$

2.2A Conclusion. If B is a Boolean algebra, $\pi(B) = \mu >$ cf$(\mu) > \aleph_0$ and $\langle \mu_\zeta : \zeta <$ cf$(\mu)\rangle$ is increasing continuously with limit μ (so $\mu_\zeta < \mu$) <u>then</u> for some club C of cf(μ) for every $\zeta \in C$ for some $B' \subseteq B$ we have $\pi(B') = \mu_\zeta$.

Proof of 2.2. Let $Z^* \subseteq B^+$ be dense, $|Z^*| = \mu$.

If the conclusion fails, then for some $\theta^* < \mu$, for no regular $\theta \in (\theta^*, \mu)$ does $(*)^B_\theta$ hold. We now assume we chose such θ^*, and show by induction on $\lambda \leq \mu$ that:

$\otimes_\lambda$: if $Y \subseteq B^+$, $|Y| \leq \lambda$ then for some $Z \subseteq B^+$, $|Z| \leq \theta^*$ and Z is dense in Y.

First Case. $\lambda \leq \theta^*$.

Let $Z = Y$.

Second Case. $\theta^* < \lambda \leq \mu$ and cf$(\lambda) < \lambda$.

Let $Y = \bigcup\{Y_\zeta : \zeta <$ cf$(\lambda)\}, |Y_\zeta| < \lambda$. By the induction hypothesis for each $\zeta <$ cf(λ) there is $Z_\zeta \subseteq B^+$ of cardinality $\leq \theta^*$ which is dense in Y_ζ.

Now $Z' =: \bigcup_{\zeta < \text{cf}(\lambda)} Z_\zeta$ has cardinality $\leq \theta^* + \text{cf}(\lambda) < \lambda$, hence by the induction hypothesis there is $Z \subseteq B^+$ dense in Z' with $|Z| \leq \theta^*$. Easily Z is dense in $Y, |Z| \leq \theta^*$ and $Z \subseteq B^+$ so we finish the case.

Third Case. $\theta^* < \lambda \leq \mu,\ \lambda$ regular.

If for this Y, $(*)^B_\lambda[Y]$ holds, we get the conclusion of the claim. We are assuming not so; so there is $Z' \subseteq B^+$, $|Z'| < \lambda$, Z' dense in Y. Apply the induction hypothesis to Z' and get Z as required.
So we have proved $\otimes_\lambda$.

We apply $\otimes_\lambda$ to $\lambda = \mu,\ Y = Z^*$ and get a contradiction. $\square_{2.2}$

2.3 Claim. *1) If B, μ, θ, Y are as in 2.2 (so $(*)^B_\theta[Y]$ and θ is regular) <u>then</u> we can find $\bar y = \langle y_\alpha : \alpha < \theta \rangle$ contained in B^+ and a proper θ-complete filter D on θ containing all cobounded subsets of θ such that:*

$\otimes^B_{\bar y, D}$ *for every $z \in B^+$, $\{\alpha < \theta : z \leq y_\alpha\} = \emptyset \bmod D$.*

2) If in addition θ is a successor cardinal <u>then</u> we can demand that D is normal.

Remark. Part (2) is for curiousity only.

Proof. 1) Let $Y = \{y_\alpha : \alpha < \theta\}$. Define D:

$$\text{for } \mathcal{U} \subseteq \theta\,; \mathcal{U} \in D \ \underline{\text{iff}} \text{ for some } 0 < \zeta < \theta \text{ and } z_\epsilon \in B^+ \text{ for } \epsilon < \zeta, \text{ we have } \mathcal{U} \supseteq \{\alpha < \theta : \bigwedge_{\epsilon < \zeta} z_\epsilon \nleq y_\alpha\}.$$

Trivially D is closed under supersets and intersections of $< \theta$ members and every cobounded subset of θ belongs to it. Now $\emptyset \notin D$ because $(*)^B_\theta[Y]$.
2) Let $\theta = \sigma^+$. Assume there are no such $\bar y$, D. We try to choose by induction on $n < \omega$, Y^n_α $(\alpha < \theta)$ and club E_n of θ such that:

- **(a)** Y^n_α is a subset of B^+ of cardinality $< \theta$, increasing continuous in α
- **(b)** $Y^n_\alpha \subseteq Y^n_{\alpha+1}$
- **(c)** $Y^0_\alpha = \{y_\beta : \beta < \alpha\}$ (taken from part (1))
- **(d)** E_n is a club of θ, $E_{n+1} \subseteq E_n$, $E_0 = \{\delta < \theta : \delta \text{ divisible by } \sigma\}$
- **(e)** if $\delta \in E_{n+1}$ and $\delta \leq \alpha < \text{Min}(E_n \backslash (\delta + 1))$ <u>then</u> for every $y \in Y^n_\alpha$ there is $z \in Y^{n+1}_\delta$, $z \leq y$.

If we succeed, let $\beta^* = \bigcup_{n<\omega} \text{Min}(E_n)$ $(< \theta)$, and we shall prove that $\bigcup_{n<\omega} Y^n_{\beta^*}$ is dense in Y, getting a contradiction. For every $y \in \bigcup_{\substack{n<\omega \\ \alpha<\theta}} Y^n_\alpha$ let $\beta(y)$ be the minimal $\beta < \theta$ such that $\left(\exists z \in \bigcup_{n<\omega} Y^n_\beta\right) (z \leq y)$. Now β is well defined as $\langle \bigcup_{n<\omega} Y^n_\beta : \beta < \theta\rangle$ is increasing continuous and $y \in \bigcup_{\beta<\theta} \bigcup_{n<\omega} Y^n_\beta$. If $\beta(y) \leq \beta^*$ for every

$y \in Y \left(\subseteq \bigcup_{\alpha<\theta} Y^0_\alpha \right)$ we are done, assume not, so some $y^* \in Y = \bigcup_{\alpha<\theta} Y^0_\alpha$ exemplifies this. Now let $\beta = \beta(y^*)$ and $z \in \bigcup_{n<\omega} Y^n_\beta$ exemplifies this. Clearly $\langle \sup(\beta \cap E_n) : n < \omega \rangle$ is well defined; clearly it is a non-increasing sequence of ordinals hence eventually constant, say $n \geq n^* \Rightarrow \sup(\beta \cap E_n) = \gamma$. Now, without loss of generality $z \in Y^{n^*}_\beta$ (by clause (b)); note $\gamma \in E_n$ for $n \geq n^*$ (hence for every n). But by clause (e) there is $z' \in Y^{n^*+1}_\gamma$, $z' \leq z$, contradicting the choice of β.

So we cannot carry the construction, so we are stuck at some n. Fix such an n. Let $E_n \cup \{0\} = \{\delta_\epsilon : \epsilon < \theta\}$ (increasing with ϵ). Let $Y^n_{\delta_{\epsilon+1}} \backslash Y^n_{\delta_\epsilon} \subseteq \{y^\epsilon_\zeta : \zeta < \sigma\}$. For each $\zeta < \sigma$, let D_ζ be the normal filter generated by the family of subsets of θ of the form $\{\varepsilon < \theta : z \not\leq y^\varepsilon_\zeta\}$ for $z \in B^+$. If for every $\zeta < \sigma$, $\emptyset \in D_\zeta$ we can define Y^{n+1}_α, E_{n+1}, contradiction. So for some ζ, $\bar{y}^\zeta =: \langle Y^\varepsilon_\zeta : \zeta < \sigma \rangle, D_\zeta$ are as required in $\bigotimes^B_{\bar{y}^\zeta, D_\zeta}$. $\square_{2.3}$

2.4 Claim. *Suppose D is a filter on θ, σ-complete, $\theta = cf(\theta) \geq \sigma > 2^\kappa$, and for each $\alpha < \theta$, $\bar{\beta}^\alpha = \langle \beta^\alpha_\epsilon : \epsilon < \kappa \rangle$ is a sequence of ordinals. Then for every $\mathcal{U} \subseteq \theta$, $\mathcal{U} \neq \emptyset$ mod D there are $\langle \beta^*_\epsilon : \epsilon < \kappa \rangle$ (a sequence of ordinals) and $w \subseteq \kappa$ such that:*

- **(a)** $\epsilon \in \kappa \backslash w \Rightarrow cf(\beta^*_\epsilon) \leq \theta$
- **(b)** *if* $\bigwedge_{\alpha<\sigma} |\alpha|^\kappa < \sigma$ *then:*
  $\varepsilon \in \kappa \backslash w \Rightarrow \sigma \leq cf(\beta^*_\varepsilon)$
- **(c)** *if* $\beta'_\epsilon \leq \beta^*_\epsilon$ *for all* ε *and* $[\varepsilon \in w \equiv \beta'_\epsilon = \beta^*_\epsilon]$ *then*
  $\{\alpha \in \mathcal{U} : \beta'_\epsilon \leq \beta^\alpha_\epsilon \leq \beta^*_\epsilon$ *for all* ε *and* $[\varepsilon \in w \equiv \beta^\alpha_\epsilon = \beta^*_\epsilon]\} \neq \emptyset$ *mod* D.

Proof. [Sh 430],6.1D and better presented in [Sh 513],6.1.

2.5 Definition. 1) If $F \subseteq {}^w2$ let
$c\ell(F) = \{g \in {}^w2$: for every finite $u \subseteq w$ for some $f \in F$ we have $g \restriction u = f \restriction u\}$.
If $f \in {}^w2$, $w \subseteq$ Ord and $\alpha \in$ Ord let $f^{[\alpha]}$ be $(f \restriction (w \cap \alpha)) \cup 0_{w \backslash \alpha}$; let $f^{[\infty]} = f$.
2) Let $\mu = \mu^{<\mu} < \lambda$. We define a forcing notion $Q = Q_{\lambda,\mu}$:

- **(a)** the members are pairs $p = (w, F) = (w^p, F^p)$, $w \subseteq \lambda$, $|w| < \mu$, and F is a family of $< \mu$ functions from w to $\{0,1\}$ satisfying
  - (α) for every $\alpha \in w$, for some $f \in F$, $f(\alpha) = 1$
  - (β) if $f \in F$ and $\alpha \in w$ then $f^{[\alpha]} \in F$
- **(b)** the order: $p \leq q$ iff $w^p \subseteq w^q$ and
  - (α) $f \in F^q \Rightarrow f \restriction w^p \in c\ell(F^p)$
  - (β) $\forall f \in F^p \exists g \in F^q (f \subseteq g)$.

3) For $w \subseteq \lambda$, $F \subseteq {}^w2$ let $BA[w,F] = BA[(w,F)]$ be the Boolean algebra freely generated by $\{x_\alpha : \alpha \in w\}$ except that: if u, v are finite subsets of w and for no $f \in F$, $1_u \cup 0_v \subseteq f$ then $\bigcap_{\alpha \in u} x_\alpha - \bigcup_{\beta \in v} x_\beta = 0$.

4) If $G \subseteq Q_{\lambda,\mu}$ is generic over V then $BA[G]$ is $\bigcup_{p \in G} BA[p]$ (see 2.6(2),(3) below). Here $BA[p] =: BA[w^p, F^p]$.

2.6 Claim. *0) For $p \in Q_{\lambda,\mu}$, $BA[p]$ is a Boolean algebra; also for $f \in F^p$ and ordinal $\alpha \in w^p$ (or ∞) we have $f^{[\alpha]} \in F^p$.*
1) If $f \in F^p$, $p \in Q_{\lambda,\mu}$ then f induces a homomorphism we call f^{hom} from $BA[p]$ to the two members Boolean algebra $\{0,1\}$. In fact for a term τ in $\{x_\alpha : \alpha \in w^p\}$, $BA[p] \models$ "$\tau \neq 0$" iff for some $f \in F^p$, $f^{\text{hom}}(\tau) = 1$.
2) If $p \leq q$ then $BA[p]$ is a Boolean subalgebra of $BA[q]$.
3) Hence $BA[\underset{\sim}{G}]$ is well defined, $p \Vdash$ " $BA[p]$ is a Boolean subalgebra of $BA[\underset{\sim}{G}]$".
4) For $p \in Q_{\lambda,\mu}$, for $\alpha \in w^p, x_\alpha$ is a non-zero element which is not in the subalgebra generated by $\{x_\beta : \beta < \alpha\}$ nor is there below it a non-zero member of $\langle x_\beta : \beta < \alpha\rangle_{BA[p]}$.

Proof. Check. Part (0) should be clear, also part (1). Now part (2) follows by 2.5(2)(b) and the definition of $BA[p]$; so (3) should become clear. Lastly, concerning part (4), x_α is a non-zero member of $BA[p]$ by clause (α) of 2.5(2)(a). For $\alpha \in w^p$, by 2.5(2)(a)(α) there is $f^1 \in F^p$, $f^1(\alpha) = 1$, and by 2.5(2)(a)(β) there is $f^0 \in F^p$, $f^0(\alpha) = 0$, $f^0 \restriction (w \cap \alpha) = f^1 \restriction (w \cap \alpha)$; together with part (1) this proves the second phrase of part (4). As for the third phrase, let τ be a non-zero element of the subalgebra generated by $\{x_\beta : \beta < \alpha\}$, so for some $f \in F^p$, $f^{\text{hom}}(\tau) = 1$. By 2.5(2)(a)(β), letting $f_1 = f^{[\alpha]}$, we have $f_1(\alpha) = 0$ and $f_1 \in F^p$ and $f_1 \restriction (w \cap \alpha) \subseteq f$. Hence $f_1^{\text{hom}}(\tau) = f^{\text{hom}}(\tau) = 1$ and $f_1(\alpha) = 0$, hence $f_1^{\text{hom}}(x_\alpha) = 0$. This proves $BA[p] \models$ "$\tau \not\leq x_\alpha$". $\square_{2.6}$

2.7 Claim. *Assume $\mu = \mu^{<\mu} < \lambda$.*

(1) *$Q_{\lambda,\mu}$ is a μ-complete forcing notion of cardinality $\leq \lambda^{<\mu}$.*
(2) *$Q_{\lambda,\mu}$ satisfies the μ^+-c.c.*

Proof of 2.7. 1) The number of elements of $Q_{\lambda,\mu}$ is at most

$$\begin{aligned}
|\{(w,F) : & w \subseteq \lambda, |w| < \mu \text{ and } F \text{ is a family of } < \mu \text{ functions from} \\
& w \text{ to } \{0,1\}\}| \\
& \leq \sum_{\substack{w \subseteq \lambda \\ |w| < \mu}} |\{F : F \subseteq {}^w 2, \text{ and } |F| < \mu\}| \\
& \leq \sum_{\substack{w \subseteq \lambda \\ |w| < \mu}} (2^{|w|})^{<\mu} \leq |\{w : w \subseteq \lambda \text{ and } |w| < \mu\}| \times \mu \\
& = \lambda^{<\mu} + \mu = \lambda^{<\mu}.
\end{aligned}$$

As for the μ-completeness, let $\langle p_\zeta : \zeta < \delta\rangle$ be an increasing sequence of members of $Q_{\lambda,\mu}$ with $\delta < \mu$. Let $p_\zeta = (w_\zeta, F_\zeta)$, let $F'_\zeta = c\ell(F_\zeta)$, let $w = \bigcup_{\zeta < \delta} w_\zeta$ and let

$F' = \{f \in {}^w 2 : \text{for every } \zeta < \delta \text{ we have } f \restriction w_\zeta \in F'_\zeta\}$.
Clearly for every $\zeta < \delta$ and $f \in F_\zeta$ there is $g = g_f \in F'$ extending F. Lastly, let $F = \{g_f : f \in \bigcup_{\zeta<\delta} F_\zeta\}$. Then $p = (w, F) \in Q_{\lambda,\mu}$ is an upper bound of $\langle p_\zeta : \zeta < \delta\rangle$, as required.
2) By the Δ-system argument it suffices to prove that p^0, p^1 are compatible when:

- **(a)** $otp\left(w^{p^0}\right) = otp\left(w^{p^1}\right)$ and (letting $H = H^{OP}_{w^{p^1},w^{p^0}}$ be the unique order preserving function from w^{p^0} onto w^{p^1}),
- **(b)** H maps p^0 onto p^1 i.e.
$$f \in F^{p^0} \Leftrightarrow (f \circ H^{-1}) \in F^{p^1}$$
- **(c)** $\alpha \in w^{p^0} \Rightarrow \alpha \le H(\alpha)$
- **(d)** for $\alpha \in w^{p^0}$ we have $\alpha \in w^{p^1}$ iff $\alpha = H(\alpha)$.

We define now $q \in Q : w^q = w^{p^0} \cup w^{p^1}$,
$$F^q = \left\{(f \cup (f \circ H))^{[\beta]} : f \in F^{p^1}, \beta \in w^q \cup \{\infty\}\right\}. \qquad \square_{2.7}$$

2.8 Claim. *Suppose $Q = Q_{\lambda,\mu}$ and*

- **(a)** $p^\ell \in Q$ *for* $\ell < m$
- **(b)** $otp\left(w^{p^\ell}\right) = otp(w^{p^0})$, *and* $H_{\ell,k} = H^{OP}_{w^{p^\ell},w^{p^k}}$ *(see the proof of 2.7(2))*
- **(c)** $H_{\ell,k}$ *maps* p^k *onto* p^ℓ,
- **(d)** *for* $\alpha \in w^{p^0}$ *the sequence* $\langle H_{\ell,0}(\alpha) : \ell = 1, \dots, m-1\rangle$ *is either strictly increasing or constant; and* $\{\alpha, \beta\} \subseteq w^{p^0}$ & $\ell, k < m$ & $H_{\ell,0}(\alpha) = H_{k,0}(\beta)$ *implies* $\alpha = \beta$. *Lastly letting* $w^* = w^{p^0} \cap w^{p^1}$ *we have* $[\ell \ne k \Rightarrow w^{p^\ell} \cap w^{p^k} = w^*]$ *and* $H_{\ell,k} \restriction w^*$ *is the identity.*
- **(e)** $\tau(x_1, \dots, x_n)$ *is a Boolean term,* $\alpha_i^0 \in w^{p^0}$ *for* $i \in \{1, \dots, n\}, \alpha_1^0 < \dots < \alpha_n^0$, $\alpha_i^\ell = H_{\ell,0}(\alpha_i^0)$.
- **(f)** *In* $BA[p^0]$, $\tau\left(x_{\alpha_1^0}, \dots, x_{\alpha_n^0}\right)$ *is not zero and even not in the subalgebra generated by* $\{x_\alpha : \alpha \in w^*\}$.
- **(g)** $m - 1 > n + 1$.

Then there is $q \in Q$ such that:

- (α) $p^\ell \le q$ *for* $\ell < m$; *and* $w^q = \bigcup_{\ell<m} w^{p_\ell}$
- (β) $q \Vdash$ *"in* $BA[\underset{\sim}{G}]$, *there is a non-zero Boolean combination* τ^* *of* $\left\{\tau\left(x_{\alpha_1^\ell}, \dots, x_{\alpha_n^\ell}\right) : 1 \le \ell < m\right\}$ *which is* $\le \tau\left(x_{\alpha_1^0}, \dots, x_{\alpha_n^0}\right)$*".*

Proof. By assumption (f) (and 2.6(0),(4)) there are f_0^*, $f_1^* \in c\ell\left(F^{p^0}\right)$ such that:

- (A) $f_0^* \restriction w^* = f_1^* \restriction w^*$
- (B) in the two members Boolean algebra $\{0, 1\}$ we have

$$\tau\left(f_0^*(\alpha_1^0),\ldots,f_0^*(\alpha_n^0)\right)=0$$

$$\tau\left(f_1^*(\alpha_1^0),\ldots,f_1^*(\alpha_n^0)\right)=1.$$

Now there is $\gamma\in w^{p^0}\cup\{\infty\}$ such that $(f_0^*)^{[\gamma]}=f_0^*$ & $(f_1^*)^{[\gamma]}=f_1^*$ (e.g. $\gamma=\infty$). Choose such (γ,f_0^*,f_1^*) with γ minimal.

Let $w^q=\bigcup\limits_{\ell=0}^{m-1}w^{p^\ell}$. We define a function $g\in{}^{(w^q)}2$ as follows:

<u>*First Case*</u>. $g\restriction w^{p^0}=f_1^*$

<u>*Second Case*</u>. For odd $\ell\in[1,m)$, $g\restriction w^{p^\ell}=f_1^*\circ H_{0,\ell}$ and

<u>*Third Case*</u>. For even $\ell\in[1,m)$, (but not $\ell=0$!) $g^*\restriction w^{p^\ell}=f_0^*\circ H_{0,\ell}$.

Now g is well defined by clause (A) above. Let us define q:

$$F^q=\left\{\left(\bigcup_{\ell=0}^{m-1}f\circ H_{0,\ell}\right)^{[\alpha]}:\alpha\in w^q\cup\{\infty\}\text{ and }f\in F^{p_0}\right\}$$
$$\bigcup\left\{g^{[\alpha]}:\alpha\in w^q\cup\{\infty\}\right\}.$$

$$q=(w^q,F^q).$$

Let us check the requirements.

First Requirement: $q\in Q$.

Clearly $w^q\in[\lambda]^{<\mu}$. Also $F^q\subseteq{}^{(w^q)}2$, $|F^q|<\mu$ so we have to check the conditions (α) and (β) of Definition 2.5(2)(a):

Condition (α).

If $\alpha\in w^q$ then for some $\ell<m$, $\alpha\in w^{p^\ell}$, so as $p^\ell\in Q$ there is $f_\ell\in F^{p^\ell}$ such that $f_\ell(\alpha)=1$. Now for some $f_0\in F^{p^0}$, $f_0=f_\ell\circ H_{0,\ell}$ so
$f=:\bigcup\limits_{k<m}(f_0\circ H_{0,k})=\left(\bigcup\limits_{k<m}(f_0\circ H_{0,k})\right)^{[\infty]}$ belongs to F^q and
$f(\alpha)=(f_0\circ H_{0,\ell})(\alpha)=f_\ell(\alpha)=1$.

Condition (β).

As for $\alpha,\beta\in w\cup\{\infty\}$ and $f\in{}^{(w^q)}2$ we have $\left(f^{[\alpha]}\right)^{[\beta]}=f^{[Min\{\alpha,\beta\}]}$ and as $\left(\bigcup\limits_{\ell<m}f_\ell\right)^{[\alpha]}=\bigcup\limits_{\ell<m}(f_\ell)^{[\alpha]}$, this condition holds by the way we have defined F^q.

Second Requirement: For $\ell < m$, $p^\ell \le q$.

By the choice of q clearly $w^{p^\ell} \subseteq w^q$.
Also if $f \in F^{p^\ell}$ then $(f \circ H_{\ell,0}) \in F^{p^0}$ and

$$\bigcup_{k<m} ((f \circ H_{\ell,0}) \circ H_{0,k})^{[\infty]}$$

belongs to F^q and extends f.
Lastly, if $f \in F^q$ we shall prove that $f \restriction w^{p^\ell} \in c\ell(F^{p^\ell})$ (in fact, $\in F^{p^\ell}$); we have two cases: in the first case $f = \left(\bigcup_{\ell<m} (f_0 \circ H_{0,\ell})\right)^{[\alpha]}$ for some $f_0 \in F^{p^0}$, let $\beta =$ Min $[w^\ell \cup \{\infty\} \backslash \alpha]$, so $f^{[\alpha]} \restriction w^\ell = (f_0 \circ H_{0,\ell})^{[\beta]}$, clearly $f_0 \circ H_{0,\ell} \in F^{p^\ell}$ hence $(f_0 \circ H_{0,\ell})^{[\beta]} \in F^{p^\ell}$ is as required. The second case is $f = g^{[\alpha]}$, let $\beta =$ Min $\left[w^{p^\ell} \cup \{\infty\} \backslash \alpha\right]$, now $f \restriction w^{p^\ell}$ is $f_0^* \circ H_{0,\ell}$ or $f_1^* \circ H_{0,\ell}$ so $f \restriction w^{p^\ell}$ is $(f_0^* \circ H_{0,\ell})^{[\beta]}$ or $(f_1^* \circ H_{0,\ell})^{[\beta]}$ hence belongs to F^{p^ℓ}.

Third Requirement: There is a non-zero Boolean combination of $\left\{\tau\left(x_{\alpha_1^\ell}, \dots, x_{\alpha_n^\ell}\right) : \ell = 1, m-1\right\}$ which is $\le \tau\left(x_{\alpha_1^0}, \dots, x_{\alpha_n^0}\right)$ in $BA[q]$.

The required Boolean combination will be

$$\tau^* = \bigcap_{\ell=0}^{\left[\frac{m-2}{2}\right]} \tau\left(x_{\alpha_1^{2\ell+1}}, \dots, x_{\alpha_n^{2\ell+1}}\right) - \bigcup_{\ell=1}^{\left[\frac{m-1}{2}\right]} \tau\left(x_{\alpha_1^{2\ell}}, \dots, x_{\alpha_n^{2\ell}}\right).$$

So we have to prove the following two assertions.

<u>First assertion</u>: $BA[q] \models$ "$\tau^* \neq 0$".

Now $g = g^{[\infty]} \in F^q$ satisfies, for each $\ell \in \left[0, \left[\frac{m-2}{2}\right]\right]$:

$$g^{\text{hom}}\left(\tau\left(x_{\alpha_1^{2\ell+1}}, \dots, x_{\alpha_n^{2\ell+1}}\right)\right) = (f_1^* \circ H_{0,2\ell+1})^{\text{hom}}\left(\tau\left(x_{\alpha_1^{2\ell+1}}, \dots, x_{\alpha_n^{2\ell+1}}\right)\right) = 1;$$

also for each $\ell \in \left[1, \left[\frac{m-1}{2}\right]\right]$,

$$g^{\text{hom}}\left(\tau\left(x_{\alpha_1^{2\ell}}, \dots, x_{\alpha_n^{2\ell}}\right)\right) = (f_0^* \circ H_{0,2\ell})^{\text{hom}}\left(\tau\left(x_{\alpha_1^{2\ell}}, \dots, x_{\alpha_n^{2\ell}}\right)\right) = 0.$$

Putting the two together, we get the assertion.

<u>Second assertion</u>: $BA[q] \models$ "$\tau\left(x_{\alpha_1^0}, \dots, x_{\alpha_n^0}\right) \ge \tau^*$".

So we have to prove just that:

$$f \in F^q \Rightarrow f^{\text{hom}}\left(\tau^* - \tau\left(x_{\alpha_1^0}, \dots, x_{\alpha_n^0}\right)\right) = 0.$$

<u>*First Case*</u>. For some $\alpha \in w^q \cup \{\infty\}$ and $f_0 \in F^{p^0}$ we have

$$f = \left(\bigcup_{\ell < m} f_0 \circ H_{0,\ell} \right)^{[\alpha]}.$$

Let $\beta_\ell = Min\left(w^{p^\ell} \cup \{\infty\} \backslash \alpha\right)$, and let $\gamma_\ell \in w^{p^0}$ be such that $\gamma_\ell = H_{0,\ell}(\beta_\ell)$ or $\gamma_\ell = \beta_\ell = \infty$.
Now by the assumption on $\langle w^{p^\ell} : \ell < m \rangle$ we have $\langle \gamma_\ell : \ell < m \rangle$ is non-increasing. Let for $\ell < m,\ j_\ell = Min\left\{ j : j = n+1 \text{ or } j \in \{1, \ldots, n\} \text{ and } \alpha^0_j \geq \gamma_\ell \right\}$. So $\langle j_\ell : \ell < m \rangle$ is non-increasing and there are $\leq n+1$ possible values for each j_ℓ. But by assumption (g), $m - 1 > n + 1$, so for some $k, 0 < k < k+1 < m$ and $j_k = j_{k+1}$. So (as $\alpha^i_1 < \cdots < \alpha^i_n$)

$$\bigwedge_{j=1}^n f\left(x_{\alpha^k_j}\right) = f\left(x_{\alpha^{k+1}_j}\right),$$

hence

$$\bigwedge_{j=1}^n f^{\text{hom}}\left(x_{\alpha^k_j}\right) = f^{\text{hom}}\left(x_{\alpha^{k+1}_j}\right)$$

hence

$$f^{\text{hom}}\left(\tau\left(x_{\alpha^k_1}, \ldots, x_{\alpha^k_n}\right)\right) = f^{\text{hom}}\left(\tau\left(x_{\alpha^{k+1}_1}, \ldots, x_{\alpha^{k+1}_n}\right)\right),$$

hence (see Definition of τ^*) $f^{\text{hom}}(\tau^*) = 0$ hence

$$f^{\text{hom}}\left(\tau^* - \tau\left(x_{\alpha^0_1}, \ldots, x_{\alpha^0_n}\right)\right) = 0,$$

as required.

<u>*Second Case*</u>. For some $\alpha \in w^q \cup \{\infty\}$, $f = g^{[\alpha]}$.

Let again $\beta_\ell = \text{ Min}\left(w^{p^\ell} \cup \{\infty\} \backslash \alpha\right)$, $\gamma_\ell = H_{0,\ell}(\beta_\ell)$ (or $\gamma_\ell = \beta_\ell = \infty$), and

$$\gamma'_\ell = \begin{cases} \gamma_\ell & \text{if} \quad \gamma_\ell < \gamma \\ \infty & \text{if} \quad \gamma_\ell \geq \gamma \end{cases}$$

$j_\ell = \text{ Min}\left\{ j : j = n+1 \text{ or } j \in \{1, \ldots, n\} \text{ and } \alpha^0_j \geq \gamma'_\ell \right\}$. So $\langle \gamma_\ell : \ell < m \rangle, \langle \gamma'_\ell : \ell < m \rangle$ are non-increasing and so is $\langle j_\ell : \ell < m \rangle$. Here γ is the ordinal we chose before defining q_1 just after (B) in the proof.

If for some $k, 0 < k < k+1 < m$, $j_k = j_{k+1} \leq n$, (hence $\gamma'_{k+1} \leq \gamma'_k < \gamma$) then $f^{\text{hom}}\left(\tau^* - \tau\left(x_{\alpha^0_1}, \ldots, x_{\alpha^0_n}\right)\right) = 0$ as $f^{\text{hom}}(\tau^*) = 0$ which holds because $f^{\text{hom}}\left(\tau\left(x_{\alpha^k_1}, \ldots, x_{\alpha^k_n}\right)\right) = f^{\text{hom}}\left(\tau\left(x_{\alpha^{k+1}_1}, \ldots, x_{\alpha^{k+1}_n}\right)\right)$ (the last equality holds by the choice of γ; i.e. if inequality holds then the triple $(\gamma_k, (f^*_0)^{[\gamma_k]}, (f^*_1)^{[\gamma_k]})$ contradicts the choice of γ as minimal). But $j_\ell\,(\ell = 1, m-1)$ is non-increasing hence we can show inductively on $\ell = 1, ..., n+1$ that $j_{m-\ell} \geq \ell$. So necessarily $j_1 = n+1$ but as j_ℓ is non-increasing clearly $j_0 = n+1$ hence

$$f^{\text{hom}}\left(\tau\left(x_{\alpha_1^0}, \dots, x_{\alpha_n^0}\right)\right) = g^{[\alpha]}\left(\tau\left(x_{\alpha_1^0}, \dots, x_{\alpha_n^0}\right)\right) = g\left(\tau\left(x_{\alpha_1^0}, \dots, x_{\alpha_n^0}\right)\right)$$
$$= f_1^*\left(\tau\left(x_{\alpha_1^0}, \dots, x_{\alpha_n^0}\right)\right) = 1$$

hence

$$f^{\text{hom}}\left(\tau^* - \tau\left(x_{\alpha_1^0}, \dots, x_{\alpha_n^0}\right)\right) = 0,$$

as required. $\square_{2.8}$

2.9 Theorem. *Suppose $\mu = \mu^{<\mu} < \lambda$, $Q = Q_{\lambda,\mu}$ and $V \models$ G.C.H. (for simplicity). Then:*

(1) *Q is μ-complete, μ^+ - c.c. (hence forcing with Q preserves cardinals and cofinalities).*
(2) *$\Vdash_Q$ "$2^\mu = (\lambda^\mu)^V$", $|Q| = \lambda^{<\mu}$, so cardinal arithmetic in V^Q is easily determined.*
(3) *Let $G \subseteq Q$ be generic over V. Then $BA[G]$ (see Definition 2.5(4)) is a Boolean algebra of cardinality λ such that:*
   (a) *if $\theta \leq \lambda$ is regular then for some subalgebra B of $BA[G]$, $\pi(B) = \theta$*
   (b) *if $\theta \leq \lambda$ and $\theta >$ cf$(\theta) > \mu$ then for no $B \subseteq BA[G]$ is $\pi(B) = \theta$*
   (c) *$BA[G]$ has μ non-zero pairwise disjoint elements but no μ^+ such elements (so its cellularity is μ)*
   (d) *if $a \in B^+$ then $BA[G] \restriction a$ satisfies (a), (b), (c) above (also (e))*
   (e) *if $\theta \leq \lambda$ and cf$(\theta) \leq \mu$ then for some $B' \subseteq BA[G]$ we have $\pi(B') = \theta$*
   (f) *in $BA[G]$ for every $\alpha < \lambda$, $\{x_\beta : \alpha \leq \beta < \alpha + \mu^+\} \subseteq B^+$ is dense in $\langle\{x_\beta : \beta < \alpha\}\rangle_{BA[G]}$.*

2.9A Remark. 1) This shows the consistency of a negative answer to problems 13 + 15 of Monk [M].
2) We could of course make 2^μ bigger by adding the right number of Cohen subsets of μ.

Proof. By Claim 2.7 clearly parts (1), (2) hold. We are left with part (3), by 2.6(3) $BA[G]$ is a Boolean algebra, by 2.6(4) it has cardinality λ. As for clause (a), it is exemplified by $\langle x_\alpha : \alpha < \theta\rangle_{BA[G]}$ (by 2.6(4)). Clause (c), the first statement is easy by the genericity of G (i.e. as for $p \in Q$, $\alpha \in \lambda \backslash w^p$ we can find q, $p \leq q \in Q$, $w^q = w^p \cup \{\alpha\}$ and in $BA[q]$, x_α is disjoint to all $y \in J$, for any ideal J of $BA[p]$). As for clause (c), second statement, it follows from the $\triangle$-system argument and the proof of 2.7(2).

Clause (d): concerning the generalization of clause (a), let $a \in (BA[G])^+$, so we can find finite disjoint $u, v \subseteq \lambda$ such that $0 < \bigcap_{\alpha \in u} x_\alpha - \bigcap_{\alpha \in v} x_\alpha \leq a$, choose $\beta = \sup(u \cup v)$, let

$$U = \Big\{ x_\gamma : \beta < \gamma < \beta + \mu, \text{ and } BA[G] \models \text{“} x_\gamma \leq \Big(\bigcap_{\alpha \in u} x_\alpha - \bigcap_{\alpha \in v} x_\alpha \Big) \text{”} \Big\}.$$

This set is forced to be of cardinality μ and the subalgebra of $(BA[G]) \restriction a$ generated by $\{x_\gamma \cap a : \gamma \in [\beta, \beta + \mu)\}$ is as required.
Clause (d), the generalization of clause (b) will follow from clause (b). Clause (d) the generalization of clause (c); the cellularity $\leq \mu$ follows from clause (c), the existence of μ pairwise disjoint elements follows from: for every $p \in Q_{\lambda,\mu}, \alpha < \lambda$ and $a \in BA[p]$ such that $a \in \langle x_\beta : \beta \in w^p \cap \alpha \rangle_{BA[p]}$ and $\beta \in [\alpha, \lambda) \backslash w^p$ there is q such that $p \leq q \in Q_{\lambda,\mu}$ and $BA[q] \models$ “ $\bigwedge_{\gamma \in w^p \backslash \alpha} x_\beta \cap x_\gamma = 0 \ \& \ x_\beta \leq a$”.

As for clause (e), (and the generalization in clause (d)), let $a \in (BA[G])^+$, let $u \subseteq \lambda$ be finite such that $a \in \langle x_\alpha : \alpha \in u \rangle_{BA[G]}$. Then we can find $\langle a_i : i < \mu \rangle$ pairwise disjoint non-zero member of $\langle x_\alpha : \alpha \in [\sup(u), \sup(u) + \mu) \rangle_{BA[G]}$ which are below a. Let $\theta = \sum_{\zeta < \operatorname{cf}(\theta)} \theta_\zeta$, each θ_ζ regular, let $B_\zeta \subseteq (BA[G]) \restriction a_\zeta$ be a subalgebra with $\pi(B_\zeta) = \theta_\zeta$, and lastly let B be the subalgebra of $BA[G] \restriction a$ generated by $\{a_i : i < \operatorname{cf}(\theta)\} \cup \bigcup_{\zeta < \operatorname{cf}(\theta)} B_\zeta$; check that $\pi(B) = \theta$.

Clause (f) follows by a density argument. The real point (and the only one left) is to prove Clause (b) of part (3). So suppose toward contradiciton that $\mu < \operatorname{cf}(\chi) < \chi \leq \lambda$ and $p \in Q$ but $p \Vdash_Q$ “$\underset{\sim}{B} \subseteq BA[\underset{\sim}{G}]$ is a subalgebra, $\pi(\underset{\sim}{B}) = \chi$”.

So by Claim 2.2(1)+2.3(1)
$p \Vdash$ “for arbitrarily large regular $\theta < \chi$, there is $\bar{y} = \langle \bar{y}_\alpha : \alpha < \theta \rangle$
(a sequence of non-zero elements of $\underset{\sim}{B}$) and θ-complete proper filter D on θ
(containing the cobounded subsets of θ) such that $\otimes^B_{\bar{y},D}$ holds (see 2.3(1))”.
Let $\kappa = \operatorname{cf}(\chi)$, so we can find regular $\theta_\zeta \in (\operatorname{cf}(\chi), \chi)$, (so $\theta_\zeta > \mu$) increasing with ζ, $\chi = \sum_{\zeta < \kappa} \theta_\zeta$, for $i < \kappa$, $\Big(\sum_{j < i} \theta_j \Big)^\kappa < \theta_i$ (remember $V \models$ G.C.H.) and for each $\zeta < \kappa$, condition $p_\zeta, p \leq p_\zeta \in Q$, and $\underset{\sim}{\bar{y}}^\zeta = \langle \underset{\sim}{\bar{y}}^\zeta_\alpha : \alpha < \theta_\zeta \rangle$, and ($Q$-name of a) proper θ_ζ-complete filter $\underset{\sim}{D}_\zeta$ on θ containing the co-bounded subsets of θ such that $p_\zeta \Vdash$ “ $\otimes^B_{\underset{\sim}{\bar{y}}^\zeta, \underset{\sim}{D}_\zeta}$ ” (and without loss of generality $\Vdash$ “$\underset{\sim}{y}^\zeta_\alpha \in (BA[\underset{\sim}{G}])^+$”). For each $\zeta < \kappa$ and $\alpha < \theta_\zeta$ there is a maximal antichain $\bar{p}^{\zeta,\alpha} = \langle p_{\zeta,\alpha,\epsilon} : \epsilon < \mu \rangle$ of members of Q above p_ζ and terms $\tau_{\zeta,\alpha,\epsilon} = \tau'_{\zeta,\alpha,\epsilon} \big(x_{\beta(\zeta,\alpha,\epsilon,0)}, \ldots, x_{\beta(\zeta,\alpha,\epsilon,n_\alpha(\zeta,\epsilon))} \big)$ (i.e. Boolean terms in $\{x_\alpha : \alpha < \lambda\}$) such that: $p_\zeta \leq p_{\zeta,\alpha,\epsilon}$ and $p_{\zeta,\alpha,\epsilon} \Vdash_Q$ “$\underset{\sim}{y}^\zeta_\alpha = \tau_{\zeta,\alpha,\epsilon}$”.
Without loss of generality $\{\beta(\zeta, \alpha, \epsilon, \ell) : \ell \leq n_\alpha(\zeta, \epsilon)\} \subseteq w[p_{\zeta,\alpha,\epsilon}]$.
Clearly for each $\zeta < \kappa$, $p_\zeta \Vdash$ “θ_ζ is the disjoint union of $\{\alpha < \theta_\zeta : p_{\zeta,\alpha,\epsilon} \in \underset{\sim}{G}\}$ for $\epsilon < \mu$” so for some Q-name $\underset{\sim}{\epsilon}_\zeta < \mu$, we have $p_\zeta \Vdash_Q$ “ $\Big\{ \alpha < \theta : p_{\zeta,\alpha,\underset{\sim}{\epsilon}_\zeta} \in \underset{\sim}{G} \Big\} \neq \emptyset \bmod \underset{\sim}{D}_\zeta$”. So there are $\epsilon_\zeta < \mu$ and q_ζ satisfying $p_\zeta \leq q_\zeta \in Q$, such that $q_\zeta \Vdash$ “ϵ_ζ is as

above" and let $p_{\zeta,\alpha} =: p_{\zeta,\alpha,\epsilon_\zeta}$. So we have a Q-name $\underset{\sim}{A}_\zeta$ such that $q_\zeta \Vdash_Q$ "$\underset{\sim}{A}_\zeta \subseteq \theta_\zeta$, $\underset{\sim}{A}_\zeta \neq \emptyset \mod \underset{\sim}{D}_\zeta$ and $\alpha \in \underset{\sim}{A}_\zeta \Leftrightarrow p_{\zeta,\alpha} \in \underset{\sim}{G}_Q$".

By possibly replacing $\theta_\zeta, \underset{\sim}{A}_\zeta$ by $A^*_\zeta \in [\theta_\zeta]^{\theta_\zeta}$, $\underset{\sim}{A}'_\zeta = \underset{\sim}{A}_\zeta \cap A^*_\zeta$ respectively, and increasing q_ζ, we can assume:
$\mathrm{otp}(w^{p_{\alpha,\zeta}}) = i^*_\zeta (< \mu)$, and letting $w^{p_{\alpha,\zeta}} = \left\{\beta_{\alpha,\zeta,i} : i < i^*_\zeta\right\}$, (increasing with i) and (by Claim 2.4) for some $w^*_\zeta \subseteq i^*_\zeta$, $\langle \beta^*_{\zeta,i} : i < i^*_\zeta\rangle$ and τ^*_ζ we have: $\tau'_{\zeta,\alpha,\epsilon_\zeta} = \tau^*_\zeta$, and for some strictly increasing $\langle j(\zeta,\ell) : \ell \leq n_\zeta\rangle$ we have:

$$q_\zeta \Vdash_Q \text{“}(a)\ \alpha \in \underset{\sim}{A}_\zeta \ \ \&\ \ i \in w^*_\zeta \Rightarrow \beta_{\alpha,\zeta,i} = \beta^*_{\zeta,i}$$

$$(b)\ \beta(\zeta,\alpha,\epsilon_\zeta,\ell) = \beta_{\zeta,\alpha,j(\zeta,\ell)} \text{ and } n_\alpha(\zeta,\epsilon_\zeta) = n_\zeta$$

$$(c)\ \text{ for every } \beta'_{\zeta,i} < \beta^*_{\zeta,i} (\text{ for } i \in i^*_\zeta \backslash w^*_\zeta) \text{ we have}$$

$$\Big\{\alpha < \theta_\zeta : \alpha \in \underset{\sim}{A}_\zeta, [i \in w^*_\zeta \Rightarrow \beta_{\alpha,\zeta,i} = \beta^*_{\zeta,i}] \text{ and for } i \in i^*_\zeta \backslash w^*_\zeta$$

$$\text{we have } \beta'_{\zeta,i} < \beta_{\alpha,\zeta,i} < \beta^*_{\zeta,i}\Big\} \neq \emptyset \mod \underset{\sim}{D}_\zeta\text{”}.$$

Also $[i \in i^*_\zeta \backslash w^*_\zeta \Rightarrow \left(2 + \sum_{j<i} \theta_j\right)^\kappa < cf(\beta^*_{\zeta,i}) \leq \theta_\zeta]$ (remember $\underset{\sim}{D}_\zeta$ is a θ_ζ-complete filter) on θ_ζ.

As we can replace $\langle \theta_\zeta : \zeta < \kappa\rangle$ by any subsequence of length κ, and $\kappa = cf(\kappa) > \mu$ without loss of generality: $i^*_\zeta = i^*$, $w^*_\zeta = w^*$, $\mathrm{otp}(w^{q_\zeta}) = j^*$. Let
$w^{q_\zeta} = \{\beta^*_{\zeta,i} : i^* \leq i < j^*\}$. Now we apply 2.4 to κ, D_κ (filter of closed unbounded sets) and $\langle\langle \beta^*_{\zeta,i} : i < j^*\rangle : \zeta < \kappa\rangle$ and get $\langle \beta^\otimes_i : i < j^*\rangle$ and $w^\otimes \subseteq j^*$. Without loss of generality the q_ζ are pairwise isomorphic. Note
$[i \in i^* \backslash w^* \ \ \&\ \ \zeta < \xi < \kappa \Rightarrow \beta^*_{\zeta,i} \neq \beta^*_{\xi,i}]$ (as $cf(\beta^*_{\zeta,i}) \leq \theta_\zeta$, $cf(\beta^*_{\xi,i}) > \theta_\zeta$).
Hence $w^\otimes \cap i^* \subseteq w^*$. For every $\zeta < \kappa$ and $i \in i^* \backslash w^*$, let $\beta^-_{\zeta,i} < \beta^*_{\zeta,i}$ be such that the interval $[\beta^-_{\zeta,i}, \beta^*_{\zeta,i})$ is disjoint to $\{\beta^*_{\xi,j} : \xi < \kappa, j < i^*\} \cup \{\beta^\otimes_i : i < i^*\}$, and as we can omit an initial segment of $\langle \theta_i : i < \kappa\rangle$ without loss of generality $[\beta^*_{\zeta,i}, \beta^\otimes_i)$ is disjoint to $\{\beta^\otimes_j : j < i^*\}$. Choose for each $\zeta < \kappa$, $\alpha_\zeta \in A^*_\zeta$, such that $\left[i \in i^* \backslash w^* \Rightarrow \beta_{\zeta,\alpha_\zeta,i} \in [\beta^-_{\zeta,i} + \mu, \beta^*_{\zeta,i})\right]$. Let
$\underset{\sim}{Y}$ = the Boolean subalgebra generated by $\left\{\tau_{\zeta,\alpha_\zeta} : q_\zeta \in \underset{\sim}{G} \text{ and } p_{\zeta,\alpha_\zeta} \in \underset{\sim}{G}\right\}$. This set has cardinality $\leq \kappa$, and we shall prove

$$(*) \qquad q_0 \Vdash_Q \text{“}\underset{\sim}{Y} \backslash \{0\} \text{ is dense in } \left\{\tau_{0,\beta} : \beta \in \underset{\sim}{A}_0\right\}\text{”}.$$

This contradicts the choice $q_0 \Vdash$ "$(*)^{\underset{\sim}{B}}_{\bar{y}^0, D_0}$, $\underset{\sim}{A} \neq \emptyset \mod \underset{\sim}{D}_0$".
To prove $(*)$ assume $q_0 \leq r_0 \in Q$, we can choose $\zeta^* < \kappa$ and r^+_ζ, for $\zeta \in [\zeta^*, \kappa)$ such that $r_0 \leq r^+_0$, $q_\zeta \leq r^+_\zeta$, $p_{\zeta,\alpha_\zeta} \leq r^+_\zeta$ and $\langle r_\zeta : \zeta = 0 \text{ or } \zeta \in [\zeta^*, \kappa)\rangle$ is as in 2.8; apply 2.8 and get a contradiction. $\square_{2.9}$

A theorem complementary to 2.9 is:

2.10 Theorem. *If* $\pi(B) > \lambda$ *and*

(A) $cf(\lambda) = \aleph_0$ *or*

(B) $\bigwedge_{\mu<\lambda} \mu^{< \mathrm{cf}(\lambda)} < \lambda$ *or*

(C) $\bigwedge_{\mu<\lambda} 2^\mu < \pi(B)$ *(or just)* $\bigwedge_{\mu<\lambda} 2^\mu < |B|$ & $\lambda \le \pi(B)$.

Then B has a subalgebra B' such that $\lambda = \pi(B') = |B'|$.

Remark. The conclusion of 2.10 implies that $\lambda \in \pi_{Ss}(B)$.

Proof. Case (C) is easier so we ignore it. By 2.1 without loss of generality $\pi(B) = \lambda^+ = |B|$. We try to choose by induction on $\alpha < \lambda$, a_α such that:

(a) $a_\alpha \in B^+$

(b) for $\beta < \alpha$ we have $B \models$ "$a_\alpha \cap a_\beta = 0$"

(c) $\pi(B \restriction a_\alpha) < \lambda^+$.

Let a_α be defined iff $\alpha < \alpha^*$.

Case a. $\alpha^* \ge \lambda$.
Let B' be the subalgebra generated by $\{a_\alpha : \alpha < \lambda\}$ clearly $|B'| = \pi(B') = \lambda$.

Case b. Not Case a but $\sum_{\alpha<\alpha^*} \pi(B \restriction a_\alpha) \ge \lambda$.

So we can find distinct $\alpha_\zeta < \alpha^*$ for $\zeta < \mathrm{cf}(\lambda)$ such that $\sum_\zeta \pi\left(B \restriction a_{\alpha_\zeta}\right) \ge \lambda$.
We can find regular $\theta_\zeta \le \pi\left(B \restriction a_{\alpha_\zeta}\right)$, such that $\sup_{\zeta < \mathrm{cf}(\lambda)} \theta_\zeta = \lambda$ then find $B_\zeta \subseteq B \restriction a_{\alpha_\zeta}$, such that $|B_\zeta| = \theta_\zeta$ and $\pi(B_\zeta) = \theta_\zeta$ (by 2.1). Let B' be the subalgebra of B generated by $\bigcup_{\zeta< \mathrm{cf}(\lambda)} B_\zeta \cup \{a_{\alpha_\zeta} : \zeta < \mathrm{cf}(\lambda)\}$.
Clearly $|B'| = \pi(B') = \lambda$.

Case c. Cases a, b fail.

Let $I = \{a \in B : \bigwedge_{\alpha<\alpha^*} a \cap a_\alpha = 0\}$, so I is an ideal of B and $a \in I \Rightarrow \pi(B \restriction a) \ge \lambda$.
Also $I \ne \{0\}$ (as if $I = \{0\}$ then $\pi(B) \le \sum_{\alpha<\alpha^*} \pi(B \restriction a_\alpha) < \lambda$). So easily without loss of generality:

(∗) $$\text{if } a \in I \cap B^+ \text{ then } \pi(B \restriction a) > \lambda.$$

(∗∗) $$\text{if } a \in I \cap B^+ \text{ then } B \restriction a \text{ is an atomless Boolean algebra}$$

Now without loss of generality B satisfies $(\mathrm{cf}(\lambda))$-c.c. (otherwise act as in Case b), so we have finished if Case (A) of the hypothesis holds.
So case (B) of the hypothesis holds, hence we can use Lemma 4.9, p.88 of [Sh:92] and find a free subalgebra B' of B of cardinality $(\lambda^{< \mathrm{cf}(\lambda)})^+$ hence of cardinality λ, this B' is as required. $\square_{2.10}$

§3 On π and $\pi\chi$ of Products of Boolean Algebras

3.1 Theorem. *Suppose*

$$\otimes \ \aleph_0 < \kappa = cf(\chi) < \chi < \lambda = cf(\lambda) < (\chi^\kappa) \text{ and } \bigwedge_{\theta<\chi} \theta^\kappa < \chi.$$

Then there are Boolean Algebras B_i (for $i < \kappa$) such that (on $\pi(F, B)$, $\pi\chi(B)$ see below)

(∗)(a) $\pi\chi(B_i) < \chi = \sum_{j<\kappa} \pi\chi(B_j)$

(b) *for any uniform ultrafilter D on κ*

$$\lambda = \pi\chi\left(\prod_{i<\kappa} B_i/D\right)$$

(c) *if D is a regular ultrafilter on κ then $\prod_{i<\kappa}(\pi\chi(B_i))/D = \chi^\kappa$.*

3.2 Definition. 1) For a Boolean algebra B and ultrafilter F of B, let

$$\pi(F, B) = \text{ Min } \left\{|X| : X \subseteq B^+ \text{ and } (\forall y \in F)(\exists x \in X)[x \le y]\right\}.$$

We say X is dense in F (though possibly $x \notin F$).
2) For a Boolean algebra B,

$$\pi\chi(B) = \text{ sup } \{\pi(F, B) : F \text{ an ultrafilter of } B\}$$

$$\pi\chi^+(B) = \cup\{\pi(F, B)^+ : F \text{ an ultrafilter of } B\}.$$

3.3 Remark. 1) If $\kappa = \aleph_0$ the theorem holds almost always and probably always, but we omit it to simplify the statement. (The theorem holds for $\kappa = \aleph_0$, e.g. if $\chi < \lambda = \text{cf}(\lambda) <$ (first fix point $> \chi$), more generally if

$$\otimes' \ \kappa = \text{ cf}(\chi) < \chi < \lambda = \text{ cf}(\lambda) < pp^+_{J^{bd}_\kappa}(\chi) \text{ and } 2^\kappa < \chi$$

see [Sh:g]. The point is that [Sh 355],5.4 deals with uncountable cofinalities).

3.4 Proof of Theorem 3.1. Let for a linear order $\mathcal{I}$, $BA[\mathcal{I}]$ be the Boolean algebra of subsets of $\mathcal{I}$ generated by the close- open intervals $[a, b) = \{x \in \mathcal{I} : a \le x < b\}$ where we allow $a \in \{-\infty\} \cup \mathcal{I}$, $b \in \mathcal{I} \cup \{\infty\}$, (and $a \le b$). Now clearly

$(*)_1$ if F is an ultrafilter on $\mathcal{I}$, <u>then</u> there is a Dedekind cut $(\mathcal{I}^d, \mathcal{I}^u)$ of $\mathcal{I}$ (i.e. $\mathcal{I}^d \cap \mathcal{I}^u = \emptyset$, $\mathcal{I}^d \cup \mathcal{I}^u = \mathcal{I}$ and $(\forall x_0 \in \mathcal{I}^d)(\forall x_1 \in \mathcal{I}^u)[x_0 < x_1]$) such that for $x \in BA[\mathcal{I}]$, $x \in F$ <u>iff</u> for some $a_0 \in \mathcal{I}^d$, $a_1 \in \mathcal{I}^u$ we have $[a_0, a_1) \le x$.

$(*)_2$ if $\mathcal{I}, F, (\mathcal{I}^d, \mathcal{I}^u)$ are as above then

$$\pi(F, BA(\mathcal{I})) \text{ is } : \begin{cases} \text{Max } \{\text{cf}(\mathcal{I}^d), \text{cf}((\mathcal{I}^u)^*)\} & \underline{\text{if}} \quad \text{cf}(\mathcal{I}^d), \text{cf}((\mathcal{I}^u)^*) \ge \aleph_0 \\ \text{cf}(\mathcal{I}^d) & \underline{\text{if}} \quad \text{cf}((\mathcal{I}^u)^*) = 1 \\ \text{cf}((\mathcal{I}^u)^*) & \underline{\text{if}} \quad \text{cf}(\mathcal{I}^d) = 1 \\ 1 & \underline{\text{if}} \quad \text{cf}(\mathcal{I}^d) = \text{ cf}((\mathcal{I}^u)^*) = 1 \end{cases}$$

note also

$$\pi\chi(BA(\mathcal{I})) = \text{ Max}\{\text{cf}(\mathcal{I}^d), \text{cf}((\mathcal{I}^u)^*) : (\mathcal{I}^d, \mathcal{I}^u) \text{ a Dedekind cut of } \mathcal{I}\}.$$

Now by the assumption $\otimes$ (and [Sh:g],II,5.4 + VIII,§1]), we can find a (strictly) increasing sequence $\langle \lambda_i : i < \kappa \rangle$ of regular cardinals, $\kappa < \lambda_i < \chi$, $\chi = \sum_{i<\kappa} \lambda_i$ such that $\prod_{i<\kappa} \lambda_i / J^{bd}_\kappa$ has true cofinality λ (where J^{bd}_κ is the ideal of bounded subsets of κ).

Let $\mathbb{Q}$ be the rational order and $\mathcal{I}_i$ be $\lambda_i \times \mathbb{Q}$ (i.e. the set of elements is $\{(\alpha, q) : \alpha < \lambda_i, q \in \mathbb{Q}\}$, the order is lexicographical). Let $B_i = BA[\mathcal{I}_i]$, so by $(*)_1, (*)_2$ we know that $\pi\chi(B_i) = \lambda_i$. Moreover, if F is an ultrafilter of B_i, then $\pi(F, B) = \aleph_0$ except when F is the ultrafilter F_i generated by $\{x^i_\alpha = [(\alpha, 0), \infty) : \alpha < \lambda_i\}$. Let $x^i_{\alpha,q} =: [(\alpha, q), \infty)$. Let D be a uniform ultrafilter on κ, so $\prod_{i<\kappa} \lambda_i/D$ has cofinality λ. Also if D is regular, then (see [CK]) we know $\chi^\kappa = \prod_{i<\kappa} \lambda_i/D = \prod_{i<\kappa} (\pi\chi B_i)/D$. So parts (a) and (c) of $(*)$ of Theorem 3.1 are satisfied. To prove part (b) of $(*)$, let D be a uniform ultrafilter on κ and let $B =: \prod_{i<\kappa} B_i/D$. Let F be such that $(B, F) = \prod_{i<\kappa} (B_i, F_i)/D$. Clearly F is an ultrafilter of B, it is generated by $X = \prod_{i<\kappa} X_i/D$ where $X_i = \{x^i_{\alpha,q} : \alpha < \lambda, q \in \mathbb{Q}\} \subseteq B_i$, which is linearly ordered in B, and this linear order has the same cofinality as $\prod_{i<\kappa} \lambda_i/D$, which has cofinality λ. So $\pi\chi(F, B) = \lambda$ hence $\pi\chi(B) \geq \lambda$. Let F' be an ultrafilter of B, $F' \neq F$.
Let $X_d := \{x \in X : x \in F'\}$, and
$X_u := \{x \in X : x \notin F'$ (i.e. $1_B - x \in F')\}$. Clearly (X_d, X_u) is a Dedekind cut of X (which is linearly ordered: as a subset of B, or as $\prod_{i<\kappa} X_i/D$ where $X_i \subseteq B_i$ inherit the order from B_i, so $x^i_{\alpha,a} < x^i_{\beta,b} \Leftrightarrow (\alpha, a) < (\beta, b)$.) If $X_d = X$ easily $F' = F$ contradiction, so $X_u \neq \emptyset$.

We shall prove now that $\pi\chi(F, B) \leq 2^\kappa$. If not, we shall choose by induction on $\zeta < (2^\kappa)^+$ a set Y_ζ, subsets Z^i_ζ of λ_i for $i < \kappa$, increasing continuous in ζ and y_ζ such that:

$$|Z^i_\zeta| \leq 2^\kappa$$

$$\xi < \zeta \Rightarrow Z^i_\xi \subseteq Z^i_\zeta$$

$$Y_\zeta = \prod_{i<\kappa} (Z^i_\zeta \times \mathbb{Q})/D \backslash \{0\}, \text{ so } |Y_\zeta| \leq 2^\kappa$$

$$y_\zeta \in F'$$

$$y_\zeta \in Y_{\zeta+1}$$

$$(\forall y \in Y_\zeta)(y > 0 \Rightarrow \neg y \leq y_\zeta).$$

There is no problem in doing this for $i = 0$, let $Z^i_\zeta = \{0\}$, for i limit let $Z^i_\zeta = \bigcup\limits_{\varepsilon<\zeta} Z^i_\varepsilon$.
Now having defined $\langle Z^i_\zeta : i < \kappa\rangle$ (hence Y_ζ) let

$$y_\zeta = \langle y^i_\zeta : i < \kappa\rangle/D,$$

$$y^i_\zeta = \bigcup_{\ell<n_{i,\zeta}} \left[x^i_{\alpha_{i,\zeta,2\ell},q_{i,\zeta,2\ell}}, x^i_{\alpha_{i,\zeta,2\ell+1},q_{i,\zeta,2\ell+1}}\right)$$

where $\langle(\alpha_{i,\zeta,\ell}, q_{i,\zeta,\ell}) : \ell < 2n_{i,\zeta}\rangle$ is a strictly increasing sequence of members of $Z^i_\zeta \cup \{-\infty, +\infty\}$ (we write $-\infty = (-\infty, 0), +\infty = (\infty, 0)$.) Let
$Z^i_{\zeta+1} = Z^i_\zeta \cup \{\alpha_{i,\zeta,\ell} : \ell < 2n_{i,\zeta}\}$.

For some unbounded $\mathcal{U} \subseteq (2^\kappa)^+$;

(a) $q_{i,\zeta,\ell} = q_{i,\ell}$ and $n_{i,\zeta} = n_i$,

apply 2.4, we get an easy contradiction. $\square_{3.4}$

3.4A Remark. We can similarly analyze (when $B_i = BA[\mathcal{I}_i]$)

$$\left\{\pi(F, \prod_{i<\kappa} B_i/D) : F \text{ an ultrafilter of } \prod_{i<\kappa} B_i/D\right\}\backslash(2^\kappa)^+ =$$

$$\Big\{\lambda : \lambda'_i = \text{cf}(\lambda'_i) > 2^\kappa \text{ and in } \mathcal{I}_i \text{ there is a Dedekind cut}$$
$$(X_d, X_u) \text{ such that } (\text{cf}(X_d), \text{cf}(X^*_u)) = (\lambda^d_i, \lambda^u_i) \text{ such that}$$
$$\lambda = \text{ Min } [\{\text{cf}(\Pi\lambda^u_i/D), \text{cf}(\Pi\lambda^d_i/D)\}\backslash\{1\}]\Big\}.$$

Note in comparison that by Koppelberg Shelah [KpSh 415],Th.1.1

3.5 Theorem. *Assume D is an ultrafilter on κ, for $i < \kappa$, A_i is a Boolean Algebra, $\lambda_i = \pi(A_i)$. Assume the Strong Hypothesis - [Sh 420],6.2, i.e. $pp(\mu) = \mu^+$ for all singulars or just SCH. If $2^\kappa < \lambda_i$ (or just $\{i : 2^\kappa < \lambda_i\} \in D$) then*
$\pi(\prod\limits_{i<\kappa} A_i/D) = \prod\limits_{i<\kappa} \lambda_i/D$.

3.6 Claim. *Assume that for $i < \kappa$, A_i is an infinite Boolean algebra, D is a non-principal ultrafilter on κ and $A =: \prod_{i<\kappa} A_i/D$. If $n_i < \omega$ for $i < \kappa$ and $\mu =: \prod_{i<\kappa} n_i/D$ and μ is a regular cardinal then $\pi\chi(A) \geq \mu$.*

Proof. Let χ be a large enough regular cardinal (i.e. such that κ, D, A_i, A belong to $H(\chi)$). Let $\mathfrak{C}_i = (H(\chi), \in, <^*)$ and $\mathfrak{C} = \prod_i \mathfrak{C}_i/D$, so A is a member of $\mathfrak{C}$.

Clearly $\omega^* =: \langle \omega : i < \kappa \rangle/D$ is considered by $\mathfrak{C}$ a limit ordinal and, from the outside, has a cofinality, which we call λ. Without loss of generality $i < \kappa \Rightarrow n_i > 2$.

The proof is divided to two cases.

Case 1. There are no $\mu_0 < \mu$ and $n^0_i < n_i$ such that $\aleph_0 \leq \mu_0 = \left|\prod_{i<\kappa} n^0_i/D\right|$.

We can find n^*_i such that:

(1) $\mu = \prod_{i<\kappa} n^*_i/D$

(2) $\mu = \prod_{i<\kappa} 2^{(n^*_i)^{(n^*_i)}}/D$

Let for $i < \kappa$, $\langle a^i_k : k < 2^{(n^*_i)^{(n^*_i)}} \rangle$ be pairwise disjoint non-zero members of A_i with union 1_{A_i}. Let P^i be the Boolean subalgebra generated by $\{a^i_k : k < 2^{(n^*_i)^{(n^*_i)}}\}$. Let $R^i =: \{a^i_k : k < 2^{(n^*_i)^{(n^*_i)}}\}$. Let for $k < n^*_i$, $Q^i_k \subseteq P^i$ be such that: Q^i_k is a set of n^*_i pairwise disjoint non-zero elements of P^i such that if $\langle b_k : k < n^*_i \rangle \in \prod_{k<n^*_i} Q^i_k$ then $\cap\{b_k : k < n^*_i\}$ is not zero.
Let $F^i(x) =: \cup\{a^i_k : x \cap a^i_k \neq 0_{A_i}$ and $\ell < k \Rightarrow x \cap a^i_\ell = 0_{A_i}\}$ so the union is on at most one element and $F^i(x) = 0_{A_i} \Leftrightarrow x = 0_{A_i}$.
Let $(A, P, Q, R, F, n^*) =: \prod_{i<\kappa} (A_i, P^i, Q^i, R^i, F^i, n^*_i)$. (We consider Q as a two place relation). Note that

$(*)_1$ P is a Boolean subalgebra of A

$(*)_2$ If D is a subset of P^+ then its density in A is equal to its density in P.

[Why? If $Y \subseteq A^+$ is dense in D, then $\{F(c) : c \in Y\}$ is a subset of P^+ dense in D of cardinality $\leq |Y|$, for the other direction use the same set].
Now let us enumerate the members of n^* as $\{k_\alpha : \alpha < \mu\}$ (no repetitions) we also list the members of P^+ as $\{c_\alpha : \alpha < \mu\}$. Now we choose by induction on $\alpha < \mu$ a member b_α of Q_{k_α} such that it contains (in A) no one among $\{c_\beta : \beta < \alpha\}$. As each c_β can "object" to at most one $b \in Q_{k_\alpha}$ (as the candidates are pairwise disjoint) and Q_{k_α} has cardinality $\mu > |\alpha|$ we can do this. Also by the choice of the Q^i's there is a filter of P to which b_α belongs for every $\alpha < \mu$, so we are done.

Case 2[1]. There is $\mu_0 < \mu$ and $n^0_i < n_i$ such that $\aleph_0 \leq \mu_0 = \prod_{i<\kappa} n^0_i/D$.

We can define X_i, Y_i such that: X_i is the family of those subsets of Y_i with exactly

[1] in this case the regularity of μ is not used

n^0_i elements and $|Y_i| = n_i \times n^0_i + 1$ and e.g. Y_i is a set of natural numbers; note that $|X_i| > n_i$. Let $X =: \langle X_i : i < \kappa\rangle/D$, $Y =: \langle Y_i : i < \kappa\rangle/D$, for $y \in Y$ (in $\mathfrak{C}$'s sense) let $S_y =: \{x \in X : y \in x\}$. Let $n^1_i =: |X_i|$, note $\left|\prod\limits_{i<\kappa} n^1_i/D\right| \geq \mu$; let $\langle a^i_k : k < n^1_i\rangle$ be a partition of 1_{A_i} to non- zero members of A_i and h_i be a one to one function from X_i onto $R_i =: \{a^i_k : k < n^1_i\}$ and for $y \in Y_i$ let $b^i_y =: \bigcup\{h_i(x) : x \in S_y\} \in A_i$; we define $h, n^1, \langle b_y : y \in Y\rangle \in \mathfrak{C}$ naturally, let P^*_i be the subalgebra of A_i generated by $\{a^i_k : k < n^1_i\}$ and $P^* = \prod\limits_{i<\kappa} P_i/D$ as in the other case. By a cardinality argument if $k < \omega$, $[n^0_i > k \;\; \& \;\; y_0, \ldots, y_{k-1} \in Y \Rightarrow A_i \models$ "$b^i_{y_0} \cap \cdots \cap b^i_{y_{k-1}} \neq 0_{A_i}$"] hence $\{b_y : y \in Y\} \subseteq P^*$ generates a filter of P^*. Let D be an ultrafilter of P^* containing b_y for $y \in Y$. If $Z \subseteq A \setminus \{0\}$ exemplifies the density of D in A^* and is of cardinality $\mu_2 < \mu$, as in case 1 without loss of generality $Z = \{a_f : f \in F\} \subseteq P^*$ where $F \subseteq \prod\limits_{i<\kappa} n^1_i$, $|F| < \mu$, $a_f =: \langle a^i_{f(i)} : i < \kappa\rangle/D$. Let $n = \langle n_i : i < \kappa\rangle/D$, $n^0 = \langle n^0_i : i < \kappa\rangle/D$.
Each $f \in F$, $h^{-1}(f)$ is from X, so is a subset of Y of cardinality n_0 from inside ("considered" by $\mathfrak{C}$ to be so) so μ_0 from the outside; from the inside has cardinality n and from the outside n has cardinality μ so there is a member of X disjoint to all of the $h^{-1}(f)$, contradiction to the density. So D has in A density μ, hence for every ultrafilter F of A extending D, $\pi(F, A) \geq \mu$.
Hence $\pi\chi(A) \geq \mu$ as required. $\square_{3.6}$

Remark. 1) If we ignore regularity, case (1) suffices as for every $\bar{n}/D \in \omega^\kappa/D$, $\mu = \Pi\bar{n}/D \geq \aleph_0$ we can find $\bar{n}^\ell/D \in \omega^\kappa/D$ such that $\omega^\kappa/D \models$ "$2^{\bar{n}^{\ell+1}/D} < \bar{n}^\ell/D$", so $\langle|\bar{n}^\ell/D| : \ell < \omega\rangle$ is eventually constant.
2) If each A_i is of cardinality $\aleph_0$, $\mu = \aleph^\kappa_0/D$ is regular the proof above gives $\pi\chi\left(\prod\limits_{i<\kappa} A_i/D\right) = \mu$ (if 3.6 does not apply, ω^κ/D is μ-like, so we can apply case 2 with $|X_i| = |Y_i| = \aleph_0$).

ADDED IN PROOF

We can add to claim 2.3:

CLAIM 2.3(3): Assume that B is a Boolean algebra and $\sigma < \theta = cf(\theta)$ and $(*)^B_\sigma[Y]$ and for no $\tau \in (\sigma, \theta)$ and Y' do we have $(*)^B_\tau[Y']$.

Then in the conclusion of 2.3(1) we can add: D is normal (hence in 2.2 we get that for arbitrarily large $\theta < \mu$, there are a normal filter D on θ and $\langle y_i : i < \theta \rangle$ as in 2.3(1)).

Remark: If $\theta = \tau^+$ then 2.3(2) gives the conclusion.

Proof: Let $Y = \{y_i : i < \theta\}$, we choose by induction of $n < \omega$ a club E_n of θ and sequence $\overline{y}^n = \langle y^n_i : i < \theta \rangle$ of non zero members of B such that:

(a) letting $Y_n = \{y^n_i : i < \theta\}$, we have $Y = Y_0, Y_n \subseteq Y_{n+1}$ and $E_{n+1} \subseteq E_n$

(b) if $\delta \in E_{n+1}$ and if $\delta < \alpha < \min(E_n \backslash (\delta + 1))$ then for some $\beta < \delta, y^{n+1}_\beta \leq_B y^n_\alpha$.

Let for $n = 0$, $\overline{y}^0$ list Y and $E_0 = \{\delta < \theta : \delta$ a limit ordinal divisible by $\sigma\}$.

For $n = m+1$, for each $\delta \in E_n$, let $\gamma_\delta = \min(E_n \backslash (\delta + 1))$ and let Z^n_δ be a subset of B^+ of cardinality $\leq \sigma$ dense in $\{y^m_i : \delta \leq i < \gamma_\delta\}$ which exist by the proof of 2.2. Let $Z^n_\delta = \{z^n_{\delta+i} : i < \sigma\}$. (no double use of the same index).

Also for each $\zeta < \sigma$ let D^n_ζ be the normal filter on θ generated by the subsets of θ of the form $\{i < \theta : z \not\leq y^n_{i+\zeta}\}$ for $z \in B^+$; by our assumption toward contradiction there are $z^n_{\zeta_\epsilon} \in B^+$ for $\epsilon < \theta$ and club E^n_ζ of θ such that if $\delta \in E^n_\zeta$ and $\zeta < \sigma$ then for some $\epsilon < \delta$ we have $z_{\zeta,\epsilon} \leq y^n_{\delta+\zeta}$

Let E_{n+1} be a club of θ included in E_n and in each E^n_ζ and choose $\overline{y}^{n+1}$ such that: its range include the range of $\overline{y}^n$ and for $\delta_1 < \delta_2 \in E_{n+1}$ we have $\{y^{n+1}_{\delta+\xi}\} : \xi < \sigma\}$ include $\{y^n_{\delta+\xi} : \xi < \sigma\} \cup Z^n_\delta$ and $\{y^{n+1}_i : i < \delta\}$ include each $z^n_{\zeta,\epsilon}$ for $\zeta < \sigma$ and $\epsilon < \delta$. The rest as as in the proof of 2.3(2).

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