WAS SIERPINSKI RIGHT III? CAN CONTINUUM-C.C. TIMES C.C.C. BE CONTINUUM-C.C.? SH481

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Abstract. We prove the consistency of: if B_1, B_2 are Boolean algebras satisfying the c.c.c. and the $2^{\aleph_0}$-c.c. respectively then $B_1 \times B_2$ satisfies the $2^{\aleph_0}$-c.c.

We start with a universe with a Ramsey cardinal (less suffice).

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§0 Introduction

We heard the problem from Velickovic who got it from Todorcevic, it says "are there P, a c.c.c. forcing notion, and Q is a $2^{\aleph_0}$-c.c. forcing such that $P \times Q$ is not $2^{\aleph_0}$-c.c.?" We can phrase it as a problem of cellularity of Boolean algebras or topological spaces.

We give a negative answer even for $2^{\aleph_0}$ regular, this by proving the consistency of the negation. The proof is close to [Sh 288],§3 which continues [Sh 276],§2 and is close to [Sh 289]. A recent use is [Sh 473].

We start with $V \models$ "λ is a Ramsey cardinal", use c.c.c. forcing blowing the continuum to λ. Originally the paper contained the consistency of e.g. $2^{\aleph_0} \to [\aleph_2]^2_3, 2^{\aleph_0}$ the first k^2_2-Mahlo, (weakly inaccessible)(remember $k^2_2 < \omega$) but the theorem presented arrive here to satisfactory state (for me) earlier. See more [Sh 546]. I thank Mariusz Rabus for corrections.

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What problems do [Sh 276], [Sh 288], [Sh 289], [Sh 473] and [Sh 481] raise? The most important are (we state the simplest uncovered case for each point):

A Question. 1) Can we get e.g. $CON(2^{\aleph_0} \to [\aleph_2]^2_3)$; more generally raise μ^+ to higher.
2) Can we get $CON(\aleph_\omega > 2^{\aleph_0} \to [\aleph_1]^2_3)$; generally lower 2^μ, the exact $\aleph_n$ seems to me less exciting.
3) Can get e.g. $CON(2^\mu > \lambda \to [\mu^+]^2_3)$?

Also concerning [Sh 473].

B Question. 1) Can we get the continuity on a non- meagre set for functions $f : {}^\kappa 2 \to {}^\kappa 2$?
2) what can we say on continuity of 2-place functions?
3) What about n-place functions (after [Sh 288]).

C Question. 1) Can we get e.g. for $\mu = \mu^{<\mu} > \aleph_0$,$CON$(if P is 2^μ-c.c.,Q is μ^+-c.c. then $P \times Q$ is 2^μ-c.c)?
2) Can we get e.g. CON (if P is $2^{\aleph_0}$-c.c.,Q is $\aleph_2$-c.c. then $P \times Q$ is $2^{\aleph_0}$-c.c.)
3) Can we get e.g. $CON(2^{\aleph_0} > \lambda > \aleph_0$, and if P is λ-c.c.,Q is $\aleph_2$-c.c. then $P \times Q$ is λ-c.c.)
On A1 see [Sh 546].

Discussion Maybe the solution to (A1) is by using squared demand and if in $\delta < \lambda, cf(\delta) > \mu$ we guess $\langle N_s : s \in [B]^{\leq 2}\rangle, \underset{\sim}{c}$, try to by Q_δ to add a large subset of B on which only two colours appear; but we want to do it also when otp$(B) > \mu^+$. Naturally we assume that if $\delta' < \delta, cf(\delta') > \mu^+, \delta' =$ Sup$(B \cap \delta')$ this was done $\langle N_s : s \in [B \cap \delta']^{\leq 2}\rangle$, but we need more: including dividing B to μ set on each only two colours (by Q_δ). To do this and have λ, k^2_3-Mahlo (rather than measurable or $(\lambda \to^+ (\omega_1)^{<\omega}_2)$ we have to use a very strong diamond.

For problem (A2) the natural thing is to use systems $\bar N = \langle N_s : s \in [B]^{\leq 2}\rangle$ which are not end extension systems. Then it is natural to use the forcing on this stronger; "specializing" not only the colouring but all $P_{N_s \cap \lambda}$-names of ordinals (as defined in §8). This required a suitable squared diamond; this has not yet been clarified (actually we need somewhat less than $\bar{N}, S$.

But for problem (A3) a weaker version of this suggest itself. As in the solution of 1, λ is k_2^2-Mahlo $\langle\langle N_s^\delta : s \in [B^\delta]^{\le 2}\rangle : \delta \in S\rangle$ is such that $N_s^\delta \subseteq H(\delta^+)$, (we think of N_s^δ as guessing the isomorphism type over $H(\delta)$. Now we have to define a preliminary forcing R, λ-complete or at least strategically λ-complete, satisfying the λ^+-c.c. So we have "copies" of $\langle N_s^\delta : s \in [B^\delta]^{\aleph 2}\rangle$ which behave like $\triangle$-systems. [Saharon].

But if want to get tree like systems (ease requirement on forcing) we need more (enough dependency). For simplicity $\lambda = cf(\chi) = \chi^\lambda$ and use the following instead forcing and do it with.
We can have in V (or force),
$\bar{C} = \langle C_\delta : \delta \in S\rangle$ a square, $S \subseteq \chi, \bar{S} =: \{\delta \in S : cf(\delta) = \lambda\} \subseteq \chi$ stationary, $[\delta \in S \Rightarrow \text{ otp}(C_\delta) \le \lambda)$, we have squared diamond $\bar{C} = \langle C_\delta : \delta \in S\rangle$, and we choose for $\delta \in S$,$B_\delta \prec \mathfrak{A}_\delta, \|B_\delta\| \le |\omega + \text{ otp}(C_\delta)|$,$\delta_1 \in C_{\delta_2} \Rightarrow B_{\delta_1} \prec B_{\delta_2}$ & $\langle B_\delta : \delta \in C_\delta \cup \{\delta_1\}\rangle \in B_{\delta_1}$, $\delta_2 = \text{ sup } C_{\delta_2} = B_{\delta_1} = \bigcup\limits_{\delta \in C_{\delta_1}} B_\delta^*$.
Now we can copy the squared diamond $\langle\langle N_{\delta,s} : s \in [B_\delta]^{\le 2}\rangle : \delta < \lambda\rangle$ getting $\left\langle\langle N_{\delta,s}^* : s \in [B_\delta^*]^{\le 2}\rangle : \delta \in S\right\rangle$. We then define $\langle P_i, Q_i, A_i : i < \chi\rangle, |a_i| \le \chi_2$ (or $|a_i| < \kappa$).

Concerning (B1) the expected theorem holds. For 2-place function, note that the Sierpinski colouring can be viewed as a function from ${}^\mu 2$ to $\{0_\mu, 1_\mu\} \subseteq {}^\mu 2$. So the <u>best</u> we can hope for is

(B2)$'$ can we get the consistency of $(*)_\mu$ for any 2-place function f from ${}^\mu 2$ to ${}^\mu 2$ there are (everywhere) non-meagre $A \subseteq {}^\mu 2$ and continuous functions f_0, f_2 from A to ${}^\mu 2$ such that $(\forall \eta, \nu \in A)[f(\eta, \nu) \in \{f_0(\eta, \nu), f_1(\eta, nu)\}]$.

So we have to put together the proofs of [Sh 473] (continuity on non-meagre), [Sh 288] ($2^{\aleph_0} \to [\aleph_1]_3^2$), using the k_2^2- Mahlo only and replace $\aleph_0$ by μ.

For problem (B3), $\mu = \aleph_0$ we have to generalize [Sh 288],§3. But also for $\mu > \aleph_0$, we have to consider what can be said on the partition of trees (see [Sh 288],§4 for a positive answer for large cardinal (indestructible measurable $n^* < \omega$).

Concerning (C2), the problem with the approach to (A1) is "why should Q_δ from [Sh 481],1.7 satisfies Q_δ^2 is c.c.c."
Similarly (C3)(A3). A natural approach is to consider $\langle N_s : s \in [B]^{<\aleph_0}\rangle$ and use a subset $X \in [B]^{\text{otp}B}$ such that for different uses we use almost disjoint X's. This was not completed but we restrict ourselves to "not only P satisfies the $2^{\aleph_0}$- c.c. but even P^n (for each $n < \omega$).

Concerning (C1) we cannot replace ${}^\mu n$ elements of $\underset{\sim}{B}$ by 1, but we can use a directed system, so "P satisfies the 2^μ-c.c.", is replaced by "for $\sigma < \mu, P^\sigma$ satisfies the 2^μ-c.c." (or slightly less).

Another question is Velickovic's question answered for Borel c.c.c. forcing in [Sh 480]; i.e. (C4).

Preliminaries

0.A. *Let* $<^*_\chi$ be a well ordering of
$H(\chi) = \{x$:the transitive closure of x has cardinality $< \chi\}$ agreeing with the usual well ordering of the ordinals,
P (and Q, R) will denote forcing notions, i.e. partial order with a minimal element $\emptyset = \emptyset_P$.

A forcing notion P is λ-closed if every increasing sequence of members of P, of length less than λ, has an upper bound.

0.B. For sets of ordinals, A and B, define $H^{OP}_{B,A}$ as the maximal order preserving bijection between initial segments of A and B, i.e., it is the function with domain $\{\alpha \in A : otp(\alpha \cap A) < otp(B)\}$ and $H^{OP}_{A,B}(\alpha) = \beta$ if and only if $\alpha \in A$, $\beta \in B$ and $otp(\alpha \cap A) = otp(\beta \cap B)$.

Definition 0.1. $\lambda \to^+ (\alpha)^{<\omega}_\mu$ holds provided that: if whenever F is a function from $[\lambda]^{<\omega}$ to $\lambda, F(w) <$ min(w), $C \subseteq \lambda$ is a club then there is $A \subseteq C$ of order type α such that $\left[w_1, w_2 \in [A]^{<\omega}, |w_1| = |w_2| \Rightarrow F(w_1) = F(w_2)\right]$.
(See [Sh:f],XVII,4.x).

0.1A Remark. 1) If λ is Ramsey cardinal then $\lambda \to^+ (\lambda)^{<\omega}_\mu$.
2) If $\lambda =$ Min$\{\lambda : \lambda \to (\alpha)^{<\omega}_\mu\}$ then λ is regular and $\lambda \to^+ (\alpha)^{<\omega}_\mu$.

Definition 0.2. $\lambda \to [\alpha]^n_{\kappa,\theta}$ if for every function F from $[\lambda]^n$ to κ there is $A \subseteq \lambda$ of order type α such that $\{F(w) : w \in [A]^n\}$ has power $\leq \theta$.

Definition 0.3. A forcing notion P satisfies the Knaster condition (has property K) if for any $\{p_i : i < \omega_1\} \subseteq P$ there is an uncountable $A \subseteq \omega_1$ such that the conditions p_i and p_j are compatible whenever $i, j \in A$.

§1 Consistency of "c.c.c. $\times 2^{\aleph_0}$-c.c. $= 2^{\aleph_0}$ - c.c."

The a_i's are not really necessary but (hopefully) clarify.

1.1 Definition. 1) $\mathcal{K}_{\mu,\kappa}$ is the family of $\bar{Q} = \langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta : \gamma \le \alpha, \beta < \alpha\rangle$, where:

(a) $\langle P_\gamma, \underset{\sim}{Q}_\beta : \gamma \le \alpha, \beta < \alpha\rangle$ is a finite support iteration

(b) every P_γ, $\underset{\sim}{Q}_\gamma$ satisfies the c.c.c.

(c) $\underset{\sim}{Q}_\beta$ is a P_β-name which depends just on $G_{P_\beta} \cap P^*_{a_\beta}$ (see below; hence it is in $V[G_{P^*_\beta}]$), and $|\underset{\sim}{Q}_\beta| \le \kappa$ and its set of members $\subseteq V$ (for simplicity)

(d) $a_\beta \subseteq \beta$, $|a_\beta| \le \mu$ and $\gamma \in a_\beta \Rightarrow a_\gamma \subseteq a_\beta$.

2) For such $\bar{Q}$ we call $a \subseteq \ell g(\bar{Q})$,$\bar{Q}$-closed if $[\beta \in a \Rightarrow a_\beta \subseteq a]$ and let

$$P^*_a = P^{\bar{Q}}_a =: \Big\{p \in P_\alpha : \mathrm{Dom}(p) \subseteq a \text{ and for all } \beta \in \mathrm{Dom}(p) : p(\beta) \in V$$
$$\text{(not a name) and } p \restriction a_\beta \Vdash \text{“}p(\beta) \in \underset{\sim}{Q}_\beta\text{”}\Big\}$$

(so we are defining P^*_a by induction on $\sup(a)$) ordered by the order of $P_{\sup(a)}$.

3) $\mathcal{K}^k_{\mu,\kappa}$ is the class of $\bar{Q} \in \mathcal{K}_{\mu,\kappa}$ such that if $\beta < \gamma \le \ell g(\bar{Q})$,$cf(\beta) \ne \aleph_1$ then P_γ/P_β satisfies the Knaster condition (actually we can use somewhat less). Let $\mathcal{K}^n_{\mu,\kappa} = \mathcal{K}_{\mu,\kappa}$.

4) If defining $\bar{Q}$ we omit P_α we mean $\bigcup\limits_{\beta<\alpha} P_\beta$ if α is limit, $P_\beta * \underset{\sim}{Q}_\beta$ if $\alpha = \beta + 1$.

5) We do not lose, if we assume $\underset{\sim}{Q}_\beta \subseteq [\kappa]^{<\aleph_0}$ and the order $\subseteq$; (then 1.2(1)(g) becomes trivial as for closed $p, q \in P^*_j, p \restriction a \le q \restriction a$).

1.2 Claim. *1) Assume $x \in \{n, k\}$ and $\bar{Q} = \langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha, \gamma \le \alpha\rangle \in \mathcal{K}^x_{\mu,\kappa}$. <u>Then</u>*

(a) *for $\alpha^* < \alpha$, $\bar{Q} \restriction \alpha^* =: \langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha^*, \gamma \le \alpha^*\rangle$ belongs to $\mathcal{K}^x_{\mu,\kappa}$*

(b) *P^*_α is a dense subset of P_α*

(c) *for any $\bar{Q}$-closed $a \subseteq \alpha$, $P^*_a \lessdot P_\alpha$ (in particular P^*_α is a dense subset of P_α); moreover, if $p \in P^*_\alpha$ then $p \restriction a \in P^*_a$ and $[p \restriction a \le q \in P^*_a \Rightarrow r =: q \cup p \restriction (\alpha \backslash a) \in P_\alpha \ \& \ p \le r \ \& \ q \le r]$*

(d) *for a $\bar{Q}$-closed $a \subseteq \alpha$, $\langle P^*_{a\cap\gamma}, \underset{\sim}{Q}_\beta, a_\beta : \beta \in a, \gamma \in a\rangle$ belongs to $\mathcal{K}^x_{\mu,\lambda}$ (except renaming; not used)*

(e) *if $\underset{\sim}{Q}_\alpha$ is a P^*_a-name of a c.c.c. forcing notion of cardinality $\le \kappa$, each member of $\underset{\sim}{Q}_\alpha$ is from V, $a \subseteq \alpha$ is $\bar{Q}$-closed, $|a| \le \mu$ and $P_{\alpha+1} = P_\alpha * \underset{\sim}{Q}_\alpha$ and $\underset{\sim}{Q}_\alpha$ satisfies the Knaster condition or at least $\beta < \alpha \Rightarrow P_\alpha * \underset{\sim}{Q}_\alpha/P_{\beta+1}$ satisfies the Knaster condition <u>then</u> $\langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha + 1, \gamma \le \alpha + 1\rangle \in \mathcal{K}^x_{\mu,\lambda}$*

(f) *if* $n < \omega, p_1, \dots, p_n \in P_{\alpha^*}$ *and*

(∗) *for every* $\beta \in \bigcup_{\ell=1}^{n} Dom(p_\ell)$ *for some* $m = m_{\beta,\ell} \in \{1, \dots, n\}$ *we have* $p_m \restriction \beta \Vdash$ "$p_\ell(\beta) \leq_{\underset{\sim}{Q}_\beta} p_m(\beta)$ *for* $\ell = \{1, \dots, n\}$"

then $p_1, \dots, p_n$ *has a least common upper bound* p *which is defined by:* $Dom(p) = \bigcup_{\ell=1}^{n} Dom(f), p_\ell(\beta) = p_{m_{\beta,\ell}}(\beta)$, *so in particular* $p \in P_{\alpha^*}$ *and* $\bigwedge_{\ell=1}^{n} p_\ell \in P^*_{\alpha^*} \Rightarrow p \in P^*_{\alpha^*}$

(g) *if* $p_\ell \leq p$ *and* $p_\ell \in P^*_\gamma$ *for* $\ell < n$, *and* a_k *is* $\bar{Q}$*-closed for* $k < m$ *then there is* $p' \in P^*_\gamma$, *such that* $p \leq p'$ *and* $P^*_{a_k} \models p_\ell \restriction a_k \leq p' \restriction a_k$ *for* $\ell < n, k < m$.

2) If $x \in \{n, k\}$ *and* $\delta < \lambda$ *is a limit ordinal, for* $\alpha < \delta$ *we have* $\langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha, \gamma \leq \alpha\rangle \in \mathcal{K}^x_{\mu,\lambda}$ *and* $P_\delta = \bigcup_{\gamma<\delta} P_\gamma$ *then* $\langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta : \beta < \delta, \gamma \leq \delta\rangle$ *belongs to* $\mathcal{K}^x_{\mu,\lambda}$.

Proof. Straightforward.

Essentially by [Sh 289],2.4(2),p.176 (which is slightly weaker and its proof left to the reader, so we give details here).

1.3 Claim. *Assume* $\lambda \rightarrow^+ (\omega\alpha^*)^{<\omega}_\mu$ *(e.g.* λ *a Ramsey cardinal,* $\alpha^* = \lambda$*)* $\chi > \lambda$, $x \in H(\chi)$.
1) There is an end extension strong $(\chi, \lambda, \alpha^*, \mu, \aleph_0, \omega)$*-system for* x *(see Definition 1.3A).*
2) There is an end extension $(\chi, \lambda, \alpha, \mu, \aleph_0, \omega)$*- system for* x *if* x *is Ramsey or* $\lambda =$ *Min*$\{\lambda : \lambda \rightarrow (\omega\alpha^*)^{<\omega}_\mu\}$ *(also then the condition holds for every* $\mu' < \mu$*).*

1.3A Definition. 1) We say $\bar{N} = \langle N_s : s \in [B]^{<1+n}\rangle$ is a $(\chi, \lambda, \alpha, \theta, \sigma, n)$-system if:

(a) $N_s \prec (H(\chi), \in)$ (or of some expansion) $\theta + 1 \subseteq N_s, \|N_s\| = \theta, {}^{\sigma>}(N_s) \subseteq (N_s)$
(b) $B \subseteq \lambda, \mathrm{otp}(B) = \alpha$
(c) $n \leq \omega$ (equally is allowed but $1 + \omega = \omega$ so s is always finite)
(d) $N_s \cap N_t \subseteq N_{s \cap t}$
(e) $N_s \cap B = s$
(f) if $|s| = |t|$ then $N_s \cong N_t$ say $H_{s,t}$ is an isomorphism from N_s onto N_t (necessarily $H_{s,t}$ is unique)
(g) if $s' \subseteq s$, $t' = \{\alpha \in t : (\exists \beta \in s')[|\beta \cap s| = |\alpha \cap t|]\}$ then $H_{s',t'}, H_{s,t}$ are compatible functions; $H_{s,s} = id$, $H_{s,t} \supseteq H^{OP}_{s,t}$, $H_{s_0,s_1} \circ H_{s_1,s_2} = H_{s_0,s_2}, H_{t,s} = (H_{s,t})^{-1}$
(h) $\sup(N_s \cap \lambda) <$ Min$\{\alpha \in B : \bigwedge_{\gamma \in s} \gamma < \alpha\}$.

2) We add the adjective "strong" if in strengthen clause (d) by

(d)$^+$ $N_s \cap N_t = N_{s \cap t}$ (so in clause (g), $H_{s',t'} \subseteq H_{s,t}$).

3) We add the adjective "end extension" if

(i) $s \lhd t \Rightarrow N_s \cap \lambda \lhd N_t \cap \lambda$ (where $A \lhd B$) means $A = B \cap \min(B \backslash A)$

4) We add "for x" if $x \in N_s$ for every $s \in [B]^{<1+n}$, and $H_{s,t}(x) = x$.

1.3B Remark. If λ is a Ramsey cardinal (or much less see [Sh:f],XVII,4.x,[Sh 289],§4) then we have if $\gamma \in s \cap t, s \cap \gamma = t \cap \gamma$ and $y \in N_s$ then in $(H(\chi), \in, <^*_\chi)$ the elements y and $H_{t,s}(y)$ realizes the same type over $\{i : i < \gamma\}$. [prove?]

Proof. 1) Let $C = \{\delta < \lambda$: for every $\alpha < \delta$ there is $N \prec (H(\chi), \in, <^*_\chi)$ such that $\mu + 1 + \alpha \subseteq N$ and $\sup(N \cap \lambda) < \delta\}$. Clearly C is a club of λ.

Let $B_0 = \{\alpha_i : i < \omega\alpha^*\} \subseteq C$, ($\alpha_i$ strictly increasing) be indiscernible in $(H(\chi), \in, <^*_\chi, x)$ (see Definition 0.1). Let $B = \{\alpha_i : i < \omega\alpha^*$ limit$\}$. For $s \in [B_0]^{<\aleph_0}$ let N^0_s = the Skolem Hull of $s \cup \{i : i \leq \mu\} \cup \{x, \lambda\}$ under the definable functions of $(H(\chi), \in, <^*_\chi)$ and

$N_s = \cup \left\{ N^0_{t_1} \cap N^0_{t_2} : t_1, t_2 \in [\{\alpha_i : i < \omega\alpha^*\}]^{<\aleph_0} \text{ and } s = t_1 \cap t_2 \right\}$.

Clearly

$(*)$ $\|N_s\| \leq \mu$ and $\{x, \lambda\} \subseteq N_s$.

Now we shall show

$(*)_1$ if $s \in [B]^{<\aleph_0}, y \in N_s$ then for every finite $t \subseteq B_0$ there is $u \in [B_0]^{<\aleph_0}$ such that $s \subseteq u, u^* \cap t \subseteq s$ and $y \in N^0_u$.

As $y \in N_s$ there are $s_1, s_2 \in [B_0]^{<\aleph_0}$ such that $y \in N^0_{s_1} \cap N^0_{s_2}$ and $s = s_1 \cap s_2$. Let $s_1 \cup s_2 = \{\alpha_{i_0}, \dots, \alpha_{i_{m-1}}\}$ (increasing), and let $n^* = \sup\{n$: for some $\beta, \beta + n \in t\} + 1$, and define for $\ell \leq m$ a function f_ℓ with domain $s_1 \cup s_2$, such that

$$f_\ell(\alpha_{i_k}) = \begin{cases} \alpha_{i_k+1} & \underline{\text{if}} \quad k \geq m - \ell \text{ and } i_k \notin s \\ \alpha_{i_k} & \underline{\text{otherwise}} \end{cases}$$

Note that

$\bigotimes_1$ for $\ell < m, f_\ell \restriction s_1 = f_{\ell+1} \restriction s_1$ or $f_\ell \restriction s_2 = f_{\ell+1} \restriction s_2$ (or both) [why? as $i_\ell \in s_2 \backslash s_1 \backslash s$ or $i_\ell \in s_2 \backslash s_1 \backslash s$ or $i_\ell \in s = s_1 \cap s_2$].
$\bigotimes_2$ f_ℓ is order preserving with domain $s_0 \cup s_1, f_\ell \restriction s =$ the identity.

As $y \in N^0_{s_1} \cap N^0_{s_2}$ there are terms τ_1, τ_2 such that

$$y = \tau_1(\dots, \alpha_{i_k}, \dots)_{\alpha_{i_k} \in s_1} = \tau_2(\dots, \alpha_{i_k}, \dots)_{\alpha_{i_k} \in s_2}.$$

Using the indiscernibility of B_0 we can prove by induction on $\ell \leq m$ that

$\bigotimes_{3,\ell}$ $y = \tau_1(\dots, f_\ell(i_{\alpha_{i_k}}, \dots)_{\alpha_{i_k} \in s_1} = \tau_2(\dots, f_\ell(\alpha_{i_k}), \dots)_{\alpha_{i_k} \in s_2}.$

[Why? For $\ell = 0$ this is given by the choice of τ_1, τ_2. For $\ell + 1$ note that by $\otimes_2$, $f_{\ell+1} \circ f_\ell^{-1}$ is an order preserving function from $\mathrm{Rang}(f_\ell)$ onto $\mathrm{Rang}(f_{\ell+1})$.
By $\otimes_{3,\ell}$ and "B_0 is indiscernible" we know
$\tau_1(\ldots, f_\ell(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_1} = \tau_2(\ldots, f_\ell(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_2}$. By the last two sentences and the indiscernibility of B_0

$$\tau_1(\ldots, (f_{\ell+1} \circ f_\ell^{-1})(f_\ell(\alpha_{i_k})), \ldots)_{\alpha_{i_k} \in s_1} = \tau_2(\ldots, (f_{\ell+1} \circ f_\ell^{-1})(f_\ell(\alpha_{i_k})), \ldots)_{\alpha_{i_k} \in s_2}.$$

But $(f_{\ell+1} \circ f_\ell^{-1})(f_\ell(\alpha_{i_k})) = f_{\ell+1}(\alpha_{i_k})$ so

$$\tau_1(\ldots, f_{\ell+1}(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_1} = \tau_2(\ldots, f_{\ell+1}(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_2}.$$

But by $\otimes_1$ for some $e \in \{1, 2\}$ we have $f_\ell \restriction s_e = f_{\ell+1} \restriction s_e$, so
$\tau_e(\ldots, f_{\ell+1}(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_e} = \tau_e(\ldots, f_\ell(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_e}$ but the latter is equal to y (by the induction hypothesis), hence the former so by the last sentence

$$y = \tau_1(\ldots, f_{\ell+1}(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_1} = \tau_2(\ldots, f_{\ell+1}(\alpha_{i_k}), \ldots)_{\alpha_{i_k} \in s_2}.$$

So we have caried the induction on $\ell \leq m$, and for $\ell = m$ we get $y \in N^0_{f_m(s_1)}$, but by the choice of n^* and f_m clearly $f_m(s_1) \cap t \subseteq s$, and we have proved $(*)_1$.
Now we can note

$(*)_2$ if $s \in [B]^{<\aleph_0}$ and $y_1, \ldots, y_n \in N_s$ then for some $s_1, s_2 \in [B_0]^{<\aleph_0}$ we have: $s = s_1 \cap s_2$ and $y_1, \ldots, y_n \in N^0_{s_1} \cap N^0_{s_2}$.

[Why? We can find $u_1, \ldots, u_n \in [B_0]^{<\aleph_0}$ such that $s \subseteq u_\ell, y_\ell \in N^0_{u_\ell}$ (as $y_\ell \in N_s$). Now by $(*)_1$ for each $\ell = 1, 2, \ldots, n$ we can find $v_\ell \in [B_0]^{<\aleph_0}$ such that $s \subseteq v_\ell, s = v_\ell \cap (\bigcup\limits_{m=1}^n u_m)$ and $y_\ell \in N^0_{v_\ell}$. Let $u = \bigcup\limits_{i=1}^n u_\ell, v = \bigcup\limits_{\ell=1}^n u_\ell$, clearly $y_1, \ldots, y_n \in N^0_u \cap N^0_v$ and $u \cap v = s$, as required].
Now as we have Skolem functions $(*)_2$ implies

$(*)_3$ $N_s \prec (H(\chi), \in, <^*_\chi)$

Also trivially

$(*)_4$ $N^0_s \prec N_s$ hence $\mu + 1 \subseteq N_s$

$(*)_5$ $s \subseteq t \Rightarrow N_s \prec N_t$.

Also

$(*)_6$ $N_{s_1} \cap N_{s_2} = N_{s_1 \cap s_2}$ for $s_1, s_2 \in [B]^{<\aleph_0}$.

[Why? The inclusions $N_{s_1 \cap s_2} \subseteq N_{s_1} \cap N_{s_2}$ follows from $(*)_5$; for the other direction let $y \in N_{s_1} \cap N_{s_2}$. By $(*)_1$ as $y \in N_{s_1}$ there is t_1 such that $s_1 \subseteq t_1 \in [B_0]^{<\aleph_0}, t_1 \cap (s_1 \cup s_2) = s_2$ and $y \in N^0_{t_1}$. By $(*)_1$, as $y \in N_{s_2}$ there is t_2 such that $s_2 \subseteq t_2 \in [B_0]^{<\aleph_0}, t_2 \cap (s_1 \cup s_2 \cup t_1) = s_1$ and $y \in N^0_{t_2}$. So $y \in N^0_{t_1} \cap N^0_{t_2}$, but easily $t_1 \cap t_2 = s_1 \cap s_2$].

$(*)_7$ $\sup(N_s \cap \lambda) <$ $\mathrm{Min}\{\alpha \in B : \bigwedge\limits_{\gamma \in s} \gamma < \alpha\}$.

[why? as $B_0 \subseteq C$ and see the Definition of C].

Now check that (a)-(h) of Definition 1.3A holds.
Now $\langle N_s : s \in [B]^{<\aleph_0}\rangle$ is as required.
2) If λ is Ramsey, without loss of generality otp$(B_0) = \lambda$ and it is easy to check 1.3A(i). The other case is like [Sh 289],§4. $\square_{1.3}$

1.4 Theorem. *Assume $\aleph_0 < \mu \le \kappa < \lambda = cf(\lambda)$,$\lambda$ strongly inaccessible, λ a Ramsey cardinal, and $\diamondsuit_{\{\delta<\lambda: cf(\delta)=\aleph_1\}}$ (can be added by a preliminary forcing). Then we have P such that:*

(a) *P is a c.c.c. forcing of cardinality λ adding λ reals (so the cardinals and cardinal arithmetic in V^P should be clear), in particular in V^P we have $2^{\aleph_0} = \lambda$*
(b) $\Vdash_P$ *"MA holds for c.c.c. forcing notions of cardinality $\le \mu$ and $< \lambda$ dense sets (and even for c.c.c. forcing notions of cardinality $\le \kappa$ which are from $V[A]$ for some $A \subseteq \mu$)"*
(c) $\Vdash_P$ *"if B is a λ-c.c. Boolean algebra, $x_i \in B\backslash\{0\}$ for $i < \lambda$ then for some $Z \subseteq \lambda$, $|Z| = \aleph_1$ and $\{x_i : i \in Z\}$ generates a proper filter of B (i.e. no finite intersection is 0_B)"*
(d) $\Vdash_P$ *"if B_1 is a c.c.c. Boolean algebra, B_2 is a λ-c.c. Boolean algebra then $B_1 \times B_2$ is a λ-c.c. Boolean algebra."*

Proof. Let $\langle A_\delta : \delta < \lambda, cf(\delta) = \aleph_1\rangle$ exemplifies the diamond. We choose by induction on $\alpha < \lambda$, $\bar{Q}^\alpha = \langle P_\gamma, \underset{\sim}{Q}_\beta, a_\beta : \gamma \le \alpha, \beta < \alpha\rangle \in \mathcal{K}^n_{\mu,\kappa}$ such that $\alpha^1 < \alpha \Rightarrow \bar{Q}^{\alpha^1} = \bar{Q}^\alpha \restriction \alpha^1$. In limits α use 1.2(2), for $\alpha = \beta + 1$, $cf(\beta) \ne \aleph_1$ take care of (b) by suitable bookkeepping using 1.2(1)(e). If $\alpha = \beta + 1$, $cf(\beta) = \aleph_1$ and A_β codes $p \in P_\beta$ and P_β-names of a Boolean algebra $\underset{\sim}{B}_\beta$ and sequence $\langle \underset{\sim}{x}^\beta_i : i < \beta\rangle$ of non-zero members of $\underset{\sim}{B}_\beta$, and p forces ($\Vdash_{P_\beta}$) that there is in $V[\underset{\sim}{G}_{P_\beta}]$ some c.c.c. forcing notion Q of cardinality $\le \mu$ adding some $Z \subseteq \beta$, $|Z| = \aleph_1$ with $\{x^\beta_i : i \in Z\}$ generating a proper filter of $\underset{\sim}{B}_\beta$ then we choose $\underset{\sim}{Q}_\beta$, if $p \in \underset{\sim}{G}_{p_\beta}$, as such Q. If $p \notin \underset{\sim}{G}_{P_\beta}$ or there is no such Q in $V[\underset{\sim}{G}_{p_\beta}]$, then $\underset{\sim}{Q}_\beta$ is e.g. Cohen forcing.

So every $\bar{Q}^\alpha$ is defined, let $P = \bigcup_{\gamma<\lambda} P_\gamma$. Clearly ($\alpha$) + (b) holds and (d) follows by (c). So the rest of the proof is dedicated to proving (c).
So let $p \in P$, $p \Vdash$ "$\underset{\sim}{B}$ a λ-c.c. Boolean algebra, $\underset{\sim}{x}_i \in B\backslash\{0_B\}$ for $i < \lambda$" without loss of generality the set of members of $\underset{\sim}{B}$ is λ.

Let $x = \left\langle P, p, \underset{\sim}{B}, \langle \underset{\sim}{x}_i : i < \lambda\rangle\right\rangle$, $\chi = \lambda^+$, by Claim 1.3 there are $A \in [\lambda]^\lambda$ and $\langle N_s : s \in [A]^{<\aleph_0}\rangle$ as there (for $\kappa = \mu + \kappa$ here standing for μ there). Let

$$C = \Big\{\delta < \lambda : \delta \text{ a strong limit cardinal } > \kappa + \mu, [\alpha < \delta \Rightarrow \bar{Q} \restriction \alpha \in H(\delta)],$$
$$\delta = \sup(A \cap \delta), s \in [A \cap \delta]^{<\aleph_0} \Rightarrow \sup(\lambda \cap N_s) < \delta,$$
$$\underset{\sim}{B} \restriction \delta \text{ a } P_\delta\text{-name, and for } i < \delta \text{ we have } \underset{\sim}{x}_i \text{ a } P_\delta\text{-name}\Big\}.$$

For some accumulation point δ of C, $cf(\delta) = \aleph_1$ and A_δ codes $\left\langle p, \underset{\sim}{B} \restriction \delta, \langle \underset{\sim}{x}_i : i < \delta \rangle \right\rangle$. We shall show that for some q, $p \leq q \in P_\delta$ and $q \Vdash_{P_\delta}$ "there is Q as required above". By the inductive choice of $\underset{\sim}{Q}_\delta$ this suffices.

Let $A^* \subseteq A \cap \delta$, $\mathrm{otp}(A^*) = \omega_1$, $\delta = \sup(A^*)$ and $\langle \delta_i : i < \omega_1 \rangle$ increasing continuous, $\delta = \bigcup\limits_{i<\omega_1} \delta_i$, $\delta_i \in C$, $A^* \cap \delta_0 = \emptyset$, $|A^* \cap [\delta_i, \delta_{i+1})| = 1$.

In V^{P_δ} we define:

$$\underset{\sim}{Q} = \left\{ u : u \in [A^*]^{<\aleph_0}, \text{ and } \underset{\sim}{B} \models \text{"} \bigcap_{i \in u} \underset{\sim}{x}_i \neq 0_{\underset{\sim}{B}} \text{"} \right\}$$

ordered by inclusion. It suffices to prove that some q, $p \leq q \in P_\delta$, q forces that: $\underset{\sim}{Q}$ is c.c.c. with $\cup \underset{\sim}{G}_{\underset{\sim}{Q}}$ an uncountable set; now clearly q forces that $\{\underset{\sim}{x}_i : i \in \cup \underset{\sim}{G}_{\underset{\sim}{Q}}\}$ generates a proper filter of $\underset{\sim}{B}$.

If not, we can find q_i, u_i such that:

$$p \leq q_i \in P^*_\delta \text{ and } q_i \Vdash_{P_\delta} \text{"} u_i \in \underset{\sim}{Q} \text{"} \text{ (where } u_i \in [A^*]^{<\aleph_0})$$

and $\langle (q_i, u_i) : i < \omega_1 \rangle$ are pairwise incompatible in $P_\delta * \underset{\sim}{Q}$.

Let v_i be a finite subset of A^* such that: $u_i \subseteq v_i$, and

$(*)$ $[v \subseteq A^* \ \& \ v \text{ finite } \ \& \ \gamma \in (\mathrm{Dom}\ q_i) \cap N_v \Rightarrow \gamma \in (\mathrm{Dom}\ q_i) \cap N_{v \cap v_i}]$.

By Fodor's Lemma for some stationary, $S \subseteq \omega_1, u^*, v^*, n^*$ and $i(*)$ we have: for $i < j$ in S,

$$v_i \cap \delta_i = v^* \subseteq \delta_{i(*)}, v_i \subseteq \delta_j, |v_i| = n^*, u_i \cap \delta_i = u^*$$

$$i(*) = \mathrm{Min}(S)$$

$$\{|\gamma \cap v_i)| : \gamma \in u_i\} \text{ does not depend on } i$$

$$q_i \restriction \delta_i \in P^*_{\delta_{i(*)}}$$

$$q_i \in P^*_{\delta_j}.$$

Let $b_i =: N_{v_i} \cap \lambda$, so b_i is necessarily $\bar{Q}^\delta$-closed and $|b_i| = \kappa$. Let $q^1_i = q_i \restriction b_i$, so necessarily $q^1_i \in P^*_{b_i}$ (see 2.2(1)(c)). Easily $P^*_{b_i} \subseteq N_{v_i}$ (though do not belong to it) so $q^1_i \in N_{v_i}$.

Let $q^2_i =: H_{v_{i(*)}, v_i}(q^1_i)$, so $q^2_i \in P^*_{b_{i(*)}}$; let $q^3_i =: (q_i \restriction \delta_{i(*)}) \cup \left[q^2_i \restriction (N_{v_{i(*)}} \cap \lambda \backslash \delta_{i(*)}) \right]$ by 1.2(1)(c) we know $q^3_i \in P^*_{\sup(b_{i(*)})+1}$ and $q^2_i \leq q^3_i$, even without loss of generality $q^2_i \leq q^3_i \restriction b_{i(*)}$. As $P^*_{\sup b_{i(*)}+1} \lessdot P_\delta$ and P_δ satisfies the c.c.c. clearly for some $i < j$ from S, q_i^3, q_j^3, are compatible in $P^*_{\sup(b_{i(*)})+1}$, so let $r \in P^*_{\sup(b_{i(*)})+1}$ be a common upper bound. So $q_i^3 \restriction (\delta_{i(*)} \cap b_{i(*)}) \le r \restriction (\delta_{i(*)} \cap b_{i(*)})$ and $q_j^3 \restriction (\delta_{i(*)} \cap b_{i(*)}) \le r \restriction (\delta_{i(*)} \cap b_{i(*)})$ and $q_i^3 \restriction b_{i(*)} \le r \restriction b_{i(*)}, q_j^3 \restriction b_{i(*)}$.

Without loss of generality $\mathrm{Dom}(r) \subseteq b_{i(*)} \cup \delta_{i(*)}$ (allowed as $b_{i(*)}$ and $\delta_{i(*)}$ are closed, see 1.2(1)(c)); let $r_i = H_{v_i, v_{i(*)}}(r \restriction b_{i(*)})$ and similarly $r_j = H_{v_j, v_{i(*)}}(r \restriction b_{i(*)})$.

Note that $r_i \in P^*_{\delta_j}, r_j \in P^*_\delta, r_j \restriction \delta_j = r_i \restriction \delta_i = r \restriction \delta_{i(*)}$. Hence $r_i \cup r_j \in P^*_\delta$.

Case 1. $r_i \cup r_j$ do not force (i.e. $\Vdash_{P_\delta}$) that

$$\underset{\sim}{B} \models \text{“} \bigcap_{\alpha \in u_i \cup u_j} \underset{\sim}{x}_\alpha = 0_{\underset{\sim}{B}} \text{”}.$$

Then there is $r' \in P_\delta$, $r_i \le r'$, $r_j \le r'$ forcing the negation. So without loss of generality $r' \in P^*_\delta$, and (as all parameters appearing in the requirements on r' are in $N_{v_i \cup v_j}$ also) $r' \in P^*_{\lambda \cap (N_{v_i \cup v_j})}$. Now
r', r, q_i, q_j has an upper bound $r'' \in P_\delta$.
[Why? By 1.2(1)(f), we have to check the condition $(*)$ there, so let
$\beta \in \mathrm{Dom}(r') \cup \mathrm{Dom}(r) \cup \mathrm{Dom}(q_i) \cup \mathrm{Dom}(q_j)$].

Subcase a. $\beta \in \delta_{i(*)} \setminus N_{v_i \cup v_j}$. Note that $N_{v_i \cup v_j} \cap \delta_{i(*)} = N_{v^*} \cap \lambda = b_{i(*)}$ (see choice of the N_u's and definition of the b_ε's) but $\mathrm{Dom}(r') \subseteq N_{v_i \cup v_j} \cap \lambda$, so $\beta \notin \mathrm{Dom}(r')$. Now

$$q_i \restriction \delta_i = q_i \restriction \delta_{i(*)} = q_i^3 \restriction \delta_{i(*)} \le r$$

$$q_j \restriction \delta_j = q_j \restriction \delta_{i(*)} = q_j^3 \restriction \delta_{i(*)} \le r.$$

So $r \restriction \beta \Vdash_{P_\beta}$ “$q_i(\beta) \le r(\beta), q_j(\beta) \le r(\beta)$” and $\beta \notin \mathrm{Dom}(r')$. So we have confirmed $(*)$ from 1.2(1)(f) for this subcase.

Subcase b. $\beta \in \delta_{i(*)} \cap N_{v_i \cup v_j}$.

Exactly as above:
$N_{v_i \cup v_j} \cap \delta_{i(*)} = N_{v^*} \cap \lambda = b_{i(*)}$, so $\beta \in N_{v^*}, \beta \in \delta_{i(*)} \cap b_{i(*)}$. Also
$q_i \restriction b_{i(*)} = q_i^1 \restriction \delta_{i(*)} = q_i^2 \restriction \delta_{i(*)} = q_i^3 \restriction (\delta_{i(*)} \cap b_{i(*)}) \le r \restriction (\delta_{i(*)} \cap b_{i(*)})$ and
$q_j \restriction b_{i(*)} = q_j^1 \restriction \delta_{i(*)} = q_j^2 \restriction \delta_{i(*)} = q_j^3 \restriction (\delta_{i(*)} \cap b_{i(*)}) \le r \restriction (\delta_{i(*)} \cap b_{i(*)})$ and
$r \restriction (\delta_{i(*)} \cap b_{i(*)}) \le r'$ (as $H_{v_i, v_{i(*)}}$ is the identity on $\delta_{i(*)} \cap b_{i(*)}$).

The last three inequalities confirm the requirement in 1.2(1)(f) (as $\beta \in \delta_{i(*)} \cap b_{i(*)}$, see above).

Subcase c. $\beta \in (\delta \setminus \delta_{i(*)}) \setminus N_{v_i \cup v_j}$.

In this case $\beta \notin \mathrm{Dom}(r')$ (as $r' \in N_{v_i \cup v_j}$). Also $\delta_{i(*)} < \delta_i < \delta_j < \delta$ and:

$$\mathrm{Dom}(r) \setminus \delta_{i(*)} \subseteq (b_{i(*)} \cup \delta_{i(*)}) \setminus \delta_{i(*)} \subseteq [\delta_{i(*)}, \delta_i)$$

$$\mathrm{Dom}(q_i) \setminus \delta_{i(*)} \subseteq [\delta_i, \delta_j)$$

$$\mathrm{Dom}(q_j)\backslash\delta_{i(*)} \subseteq [\delta_j, \delta).$$

So β belongs to at most one of $\mathrm{Dom}(r'), \mathrm{Dom}(r), \mathrm{Dom}(q_i), \mathrm{Dom}(q_j)$ so the requirement $(*)$ from 1.2(1)(f) holds trivially.

Subcase d. $\beta \in (\delta\backslash\delta_{i(*)}) \cap N_{v_i \cup v_j}$.

Clearly $\beta \notin \mathrm{Dom}(r)$.

We know $q_i \restriction b_i = q_i^1, r_i \leq r', H_{v_{i(*)},v_i}(q_i^1) = q_i^2 \leq q_i^3 \restriction b_{i(*)} \leq r \restriction b_{i(*)}$ hence $q_i^1 \leq H^{-1}_{v_{i(*)},v_i}(r \restriction b_{i(*)}) = H_{v_i,v_{i(*)}}(r \restriction b_{i(*)}) = r_i$ but $r_i \leq r'$, so together $q_i^1 \leq r'$, and similarly $q_j^1 \leq r'$. As we have noted $\beta \notin \mathrm{Dom}(r)$ we have finished confirming condition $(*)$ from 1.2(1)(f).

So really r, r', q_i, q_j has a least common upper bound, hence $(r'', u_i \cup u_j) \in P_\delta * \underset{\sim}{Q}$ exemplified (q_i, u_i), (q_j, u_j) are compatible, as required.

Case 2. Not 1.

Let $\langle s_\beta : \beta < \lambda\rangle$ be such that:

$$s_\beta \in [A]^{<\aleph_0}, v^* \subseteq s_\beta, |s_\beta\backslash v^*| = |v_i\backslash v^*|, \sup(v^*) < \delta_{i(*)} < \min(s_\beta\backslash v^*)$$

$$\delta < \min(s_\beta\backslash v^*) \text{ (for simplicity)}$$

$$\beta < \gamma \Rightarrow \max(s_\beta) < \min(s_\gamma\backslash v^*).$$

As the truth value of $\bigcap_{\alpha\in u_i} x_\alpha$ is a P^*_a-name for some closed $a \in N_{v_i}$ of cardinality $\leq \mu$, and $q_i \Vdash [\underset{\sim}{B} \models \text{“}\bigcap_{\alpha\in u_i} x_\alpha \neq 0_{\underset{\sim}{B}}\text{”}]$ clearly $q_i^1 \Vdash [\underset{\sim}{B} \models \text{“}\bigcap_{\alpha\in u_i} x_\alpha \neq 0_{\underset{\sim}{B}}\text{”}]$.

For $\beta < \lambda$ let $\gamma^\beta = H_{s_\beta,v_{i(*)}}(r \restriction b_{i(*)}$, and $u'_\beta = H_{s_\beta,v_{i(*)}}(u_0)$. Let

$$\underset{\sim}{Y} = \{\beta < \lambda : r^\beta \in \underset{\sim}{G}_P\}.$$

Clearly:

$$r^\beta \Vdash_{P_\lambda} [\underset{\sim}{B} \Vdash \text{“}\bigcap_{i\in u'_\beta} \underset{\sim}{x}_i \neq 0_{\underset{\sim}{B}}\text{”}].$$

Clearly $p \leq r^\beta$ and for some β we have $r^\beta \Vdash$ “$\underset{\sim}{Y} \in [\lambda]^\lambda$ (and $p \in \underset{\sim}{G}_P$)” and by the assumption of the case:

$$p \Vdash \text{“}\Big\{\bigcap_{i\in u'_\beta} x_i : \beta \in Y\Big\} \text{ is a set of non-zero members of } \underset{\sim}{B}$$

any two having zero intersection in $\underset{\sim}{B}$”.

This contradicts an assumption on B. $\square_{1.4}$

* * *

We can phrase the consistency result as one on colouring.

1.5 Lemma. *1) In 1.4 we can add:*

(e) *if c is a symmetric function from $\left[2^{\aleph_0}\right]^{<\omega}$ to $\{0,1\}$ then at least one of the following holds:*

(α) *we can find pairwise disjoint $w_i \subseteq 2^{\aleph_0}$ for $i < 2^{\aleph_0}$ such that: $c \restriction [w_i] < \aleph_0$ is constantly zero but*
$$\bigwedge_{i<j} (\exists u \subseteq w_i, \exists v \subseteq w_j) \left[c[u \cup v] = 1 \right]$$

(β) *we can find an unbounded $B \subseteq 2^{\aleph_0}$ such that $c \restriction [B]^{<\omega}$ is constantly* 0.

It is natural to ask:

1.6 Question. Can we replace $2^{\aleph_0}$ by $\lambda < 2^{\aleph_0}$? $\aleph_1$ by $\mu < \lambda$? What is the consistency strength of the statements we prove consistent? (see later). Does λ strongly inaccessible k_2^2-Mahlo (see [Sh289]) suffice?

1.7 Discussion. Of course, 1.5(e) $\Rightarrow$ 1.4(c) $\Rightarrow$ 1.4(d). Starting with λ weakly compact we can get a c.c.c. forcing notion P of cardinality λ, such that in $V^P, 2^{\aleph_0} = \lambda$ and (e) of 1.5 holds for $c : \left[2^{\aleph_0}\right]^2 \rightarrow \{0,1\}$ (so $c(u) = 0$ if $|u| \neq 2$) and this suffices for the result. Also we can generalize to higher cardinals. We shall deal with this elsewhere.

1.8 Theorem. *Concerning the consistency strength, in 1.4 it suffices to assume*

($*$) *λ is strongly inaccessible and for every $F : [\lambda]^{<\aleph_0} \rightarrow \mu$ and club C we can find $B \subseteq C$, (or just $B \subseteq \lambda$) $otp(B) = \omega_1$ such that*

(a) B is F-indiscernible i.e. if $n < \omega, u, v \in [B]^n$ then $F(u) = F(v)$

(b) for every $n < \omega$ there is $B' \in [C]^\lambda$ such that:
$$\text{if } u \in [B']^n \text{ and } v \in [B]^n \text{ then } F(u) = F(v)$$

Proof. Let $R = \{\bar{Q} : \bar{Q} \in H(\lambda), \bar{Q} \in \mathcal{K}^n_{\mu,\kappa}\}$ ordered by $\bar{Q}^1 < \bar{Q}^2$ if $\bar{Q}^1 = \bar{Q}^2 \restriction \ell g(\bar{Q}^1)$. Clause (b) takes care also of "the end extension" clause and for 1.3(A)(4), Clause (b) the proof is the same.
A somewhat less natural property though suffices.
(Note: Clause (b) also helps to get rid of the club C).

1.9 Claim. *In 1.4 it suffices to assume*

$(*)'$ *if* $F : [\lambda]^{<\aleph_0} \to \mu$ *then there is* $B \subseteq \lambda$, $otp(B) = \omega$ *such that*

(a) $F \restriction [B]^n$ *is constant for* $n < \omega$

(b) *if* $u \triangleleft v^\ell \in [B]^{<\aleph_0}$ *for* $\ell = 1, 2$ *then we can find* $v_i \in [\lambda]^n$ *for* $i < \lambda, u \subseteq v_i$, $min(v_i \backslash u) \geq i$, *and* $i < j \Rightarrow F(v^1 \cup v^2) = F(v_i \cup v_j)$.

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