Borel partitions of infinite sequences of reals

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Introduction

The starting point of our work is a Ramsey-type theorem of Galvin (unpublished) which asserts that if the unordered pairs of reals are partitioned into finitely many Borel classes (or even classes which have the property of Baire) then there is a perfect set P such that all pairs from P lie in the same class. The obvious generalization to n-tuples for $n \geq 3$ is false. For example, look at the coloring of triples where a triple $\{x, y, z\}$ with $x < y < z$ is colored *red* provided that $y - x < z - y$ and *blue* otherwise. Then any perfect set will contain triples of both colors. Galvin conjectured that this is the only bad thing that can happen. It will be simpler to state this if we identify the reals with 2^ω ordered by the lexicographical ordering and define for distinct $x, y \in 2^\omega$ $\Delta(x, y)$ to be the least n such that $x(n) \neq y(n)$. Let the *type* of an increasing n-tuple of reals $\{x_0, \ldots x_{n-1}\}_<$ be the ordering $\prec$ on $\{0, \ldots, n-2\}$ defined by $i \prec j$ iff $\Delta(x_i, x_{i+1}) < \Delta(x_j, x_{j+1})$. Galvin proved that for any Borel coloring of triples of reals there is a perfect set P such that the color of any triple from P depends only on its type and conjectured that an analogous result is true for any n. This conjecture has been proved by Blass ([Bl]). As a corollary it follows that if the unordered n-tuples of reals are colored into finitely many Borel classes there is a perfect set P such that the n-tuples from P meet at most $(n-1)!$ classes. The key ingredient in the proof is the well-known Halpern-Laüchli theorem ([HL]) on partitions of products of finitely many tree. In this paper we consider extensions of this result to partitions of infinite increasing sequences of reals. Define a type of an increasing sequence of reals as before and say that such a sequence $\{x_n : n < \omega\}$ is *strongly increasing* if its type is the standard ordering on ω, i.e. if $\Delta(x_n, x_{n+1}) < \Delta(x_m, x_{m+1})$ whenever $n < m$. We show, for example, that for any Borel or even analytic partition of all increasing sequences of

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reals there is a perfect set P such that all strongly increasing sequences from P lie in the same class. In fact, for any finite set $\mathcal{C}$ of types there is a perfect set P such that for any type in $\mathcal{C}$ all increasing sequence from P of that type have the same color. It should be pointed out that the same statement is false if $\mathcal{C}$ is an infinite set of types.

Our result stands in the same relation to Blass' theorem as the Galvin-Prikry theorem ([GP]) to the ordinary Ramsey's theorem and the proof again relies heavily on the Halpern-Laüchi theorem. There are known several extensions of the Halpern-Laüchli theorem that are relevant to this work. Milliken ([Mi]) considered partitions of nicely embedded infinite subtrees of a perfect tree and obtained a partition result in the spirit of Galvin-Prikry however in a different direction from ours, and Laver ([La]) proved a version of this theorem for products of infinitely many perfect trees.

The paper is organizes as follows. In §1 we introduce some notation and present some results on perfect trees which we will need later. In §2 we reduce the main theorem to two lemmas which are then proved in §§3 and 4. We shall present our result using the terminology of forcing. If $\mathcal{P}$ is a forcing notion we let, as usual, $RO(\mathcal{P})$ denote the regular open algebra of $\mathcal{P}$, i.e. a complete Boolean algebra in which $\mathcal{P}$ is densely embedded. If $\mathbf{b}$ is a Boolean value in $RO(\mathcal{P})$ and $p \in \mathcal{P}$ we shall say that p *decides* $\mathbf{b}$ if either $p \leq \mathbf{b}$ or $p \leq \mathbf{1} - \mathbf{b}$. For all undefined terminology of forcing see, for example, [Ku].

1   Basic properties of perfect trees

Perfect trees Let $2^{<\omega}$ denote the set of all finite $\{0, 1\}$-sequences ordered by extension. $T \subseteq 2^{<\omega}$ is called a *perfect tree* if it is an initial segment of $2^{<\omega}$ and every element of T has two incomparable extensions in T. Let $\mathcal{P}$ denote the poset of all perfect trees partially ordered by inclusion. Thus $\mathcal{P}$ is the well-known Sacks forcing ([Sa]). For a subset C of T let T_C be the set of all nodes in T which are comparable to an element of C. If $\{s\}$ is a singleton we shall simply write T_s instead of $T_{\{s\}}$. For a tree T let $T(n)$ denote the n-th level of T, i.e. the set of all $s \in T$ which have exactly n predecessors. We say that a node s in T is *splitting* if it has two immediate extensions. Given integers $m \leq k$ let us say that a set D is (m, k)-*dense* in T provided D is contained in $T(k)$ and every node in $T(m)$ has an extension in D. Given

trees $T_0, \ldots T_{d-1}$ and a subset A of ω let

$$\otimes_{i<d}^A T_i = \bigcup_{n \in A} \otimes_{i<d} T_i(n).$$

If A is ω we usually omit it. We are now ready to state a version of the Halpern-Laüchli theorem ([HL]).

Theorem 1 ([HL]) *For every integer $d < \omega$ given perfect trees T_i, for $i < d$, and a partition*

$$\otimes_{i<d} T_i = K_0 \cup K_1$$

for every infinite subset A of ω there are $(x_0, \ldots, x_{d-1}) \in \otimes_{i<d} T_i$ and $\epsilon \in \{0, 1\}$ such that for every m there is $k \in A$ and sets D_i, for $i < d$, such that D_i is (m, k)-dense in T_i and $\otimes_{i<d} D_i \subseteq K_\epsilon$.

The amoeba forcing $\mathcal{A}(\mathcal{P})$ To the poset $\mathcal{P}$ we associate the *amoeba poset* $\mathcal{A}(\mathcal{P})$. Elements of $\mathcal{A}(\mathcal{P})$ are pairs (T, n), where $T \in \mathcal{P}$ and $n \in \omega$. Say that $(T, n) \leq (S, m)$ iff $T \leq S, n \geq m$, and $T \restriction (m+1) = S \restriction (m+1)$. If in addition $n = m$ we shall say that (T, n) is a *pure extension* of (S, m). If G an $\mathcal{A}(\mathcal{P})$-generic filter over a model of set theory let

$$T(G) = \bigcup \{T \restriction (n+1) : (T, n) \in G\}.$$

Then, by genericity, $T(G)$ is a perfect tree and is called the $\mathcal{A}(\mathcal{P})$-*generic tree* derived from G.

Combs An *n-comb* C is a tree such that there is some strongly increasing sequence of reals $\{x_i : i < n\}$ and some $m > \Delta(x_{n-2}, x_{n-1})$ such that C is the set of all initial segments of length $< m$ of members of this sequence. An *infinite comb* is a tree such that there is some strongly increasing sequence $\{x_n : n < \omega\}$ such that C is the set of all finite initial segments of members of this sequence. Clearly there is a 1-1 correspondence between infinite combs and strongly increasing sequences and we shall in fact state our theorem in terms of infinite combs. For a tree T if $n < \omega$ is such that $T \restriction (n+1)$ is a comb let $\mathcal{C}_\omega(T, n)$ denote the set of all infinite combs contained in T and extending $T \restriction (n+1)$. Let $\mathcal{C}_\omega(T) = \mathcal{C}_\omega(T, 0)$. Note that $\mathcal{C}_\omega(T)$ has a natural topology as a subspace of $\mathcal{P}(T)$ with the Tychonoff topology. Thus we can speak about Borel, analytic, etc. subsets of $\mathcal{C}_\omega(T)$.

The comb forcing $\mathcal{C}$ Let $\mathcal{C}$ be the subposet of $\mathcal{A}(\mathcal{P})$ consisting of all pairs (T, n) such that $T \upharpoonright (n + 1)$ is a comb, with the induced ordering. Let us say that (T, n) has width d if $T \upharpoonright (n + 1)$ is a d-comb. The notion of pure extension is defined as in the case of $\mathcal{A}(\mathcal{P})$. If $(R, m) \leq (T, n)$ and if these two conditions have the same width then we say that (R, m) is a *width preserving* extension of (T, n). Note that in this case (R, n) is a pure extension of (T, n) which is equivalent in terms of forcing with (R, m). Clearly, if G is a $\mathcal{C}$-generic filter over some model of set theory the set

$$C(G) = \bigcup \{T \upharpoonright (n + 1) : (T, n) \in G\}$$

is a infinite comb, we call it the *generic comb* derived from G.

2   The main theorem

The main result of this paper is the following partition theorem.

Theorem 2 *For every partition*

$$\mathcal{C}_\omega(2^{<\omega}) = K_0 \cup K_1$$

where K_0 is analytic and K_1 co-analytic there is a perfect tree T and $i \in \{0, 1\}$ such that $\mathcal{C}_\omega(T) \subseteq K_i$.

The proof of the theorem will consist of two lemmas which combined yield the desired result.

Lemma 1 *Let $\mathbf{b}$ be a Boolean value in $RO(\mathcal{C})$ and let $(S, n) \in \mathcal{C}$. Then there is a pure extension (T, n) of (S, n) which decides $\mathbf{b}$.*

Lemma 2 *Let T be an $\mathcal{A}(\mathcal{P})$-generic tree over a model of set theory M. Then every infinite comb contained in T is $\mathcal{C}$-generic over M.*

Given these two lemmas it is quite easy to prove the theorem. Take a countable transitive model M of ZFC^- containing the codes of K_0 and K_1. Consider forcing with $\mathcal{C}$ as defined in M. Note that if C is a generic comb the statement whether C belongs to K_0 is absolute between $M[C]$ and V. Let $\mathbf{b}$ be the Boolean value that this statement is true in $M[C]$. Then it follows

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from Lemma 1 that there is a pure extension $(S, 0)$ of the maximal condition which decides $\mathbf{b}$, let us say, for concreteness, that it forces $\mathbf{b}$. Now consider forcing over M with $\mathcal{A}(\mathcal{P})$ and take a generic filter G over M which contains $(S, 0)$. Let T be the generic tree derived from G. Then by Lemma 2 every infinite comb contained in T is $\mathcal{C}$-generic over M and, since it is contained in S as well, it follows that it is in K_0. Thus T is the homogeneous tree we seek. In the next two sections we prove Lemmas 1 and 2 and thus complete the proof.

3 Proof of Lemma 1

Unless otherwise stated in this section we work with the forcing notion $\mathcal{C}$ introduced in §1. Given a Boolean value $\mathbf{b}$ in the completion algebra $RO(\mathcal{C})$ let us say that a condition (T, n) *accepts* $\mathbf{b}$ if $(T, n) \leq \mathbf{b}$ and that it *rejects* $\mathbf{b}$ if $(T, n) \leq \mathbf{1} - \mathbf{b}$. We shall need the following auxiliary lemma.

Lemma 3 *Let (S, n) be a condition in $\mathcal{C}$ of width d and let $\mathbf{b} \in RO(\mathcal{C})$ be a Boolean value. Then there is a pure extension (T, n) of (S, n) such that either (T, n) accepts $\mathbf{b}$ or no extension of (T, n) of width $d + 1$ accepts $\mathbf{b}$.*

PROOF: Let $\{t_0, \ldots, t_{d-1}\}_<$ be the increasing enumeration of $S(n)$ in the lexicographical ordering. We first find an infinite set A and a perfect subtree S^* of S such that for any $m \in A$ and $z_0, \ldots, z_d \in S^*(m)$ such that $z_i \geq t_i$ for $i < d$ and $z_d \geq t_{d-1}$, letting $Z = \{z_i : i \leq d\}$, if there is a pure extension of (S_Z^*, m) deciding $\mathbf{b}$ then already (S_Z^*, m) decides $\mathbf{b}$. This can be done by a standard fusion argument. Moreover we can arrange that between any two consecutive levels in A there is at most one splitting node. We now define a coloring:

$$\otimes_{i<d} S_{t_i}^* = K_0 \cup K_1 \cup K_2$$

as follows. Given $(x_0, \ldots, x_{d-1}) \in \otimes_{i<d} S_{t_i}^*$ let $m \in A$ be the least such that x_{d-1} has two extensions z_{d-1} and z_d in $S^*(m)$. For $i < d - 1$ let z_i be the lexicographically least extension of x_i in $S^*(m)$. Let $Z = \{z_i : i \leq d\}$ and put $(x_0, \ldots, x_{d-1})$ in K_0 if (S_Z^*, m) accepts $\mathbf{b}$, in K_1 if it rejects $\mathbf{b}$, and in K_2 otherwise. By the Halpern-Laüchli theorem we can find $(x_0, \ldots, x_{d-1}) \in \otimes_{i<d} S_{t_i}^*$ and $\epsilon \in \{0, 1, 2\}$ such that for every m there is $k \in A$ and sets D_i,

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for $i < d$, such that D_i is (m, k)-dense in $S^*_{x_i}$ and $\otimes_{i<d} D_i \subseteq K_\epsilon$. We may assume that $(x_0, \ldots, x_{d-1}) \in K_\epsilon$, as well.

We now build an increasing sequence $(b_k)_{k<\omega}$ of elements of A and a perfect subtree T of S^* which will have one splitting node on levels between b_k and b_{k+1}. To begin let b_0 be the level of the x_i and let $T(b_0) = \{x_i : i < d\}$. This uniquely determines $T \upharpoonright (b_0 + 1)$ as the set of all initial segments of elements of $T(b_0)$. Suppose now we have defined b_k and $T \upharpoonright (b_k + 1)$. We choose one node y in $T(b_k)$ and we will arrange so that the only splitting node of T on levels between b_k and b_{k+1} is above y. Let m be the least level which is in A and such that y has two extensions, say y' and y'' in $S^*(m)$. Now find some $b \in A$ and sets D_i, for $i < d$ such that D_i is (m, b)-dense in $S^*_{x_i}$ and such that $\otimes_{i<d} D_i \subseteq K_\epsilon$. Set $b_{k+1} = b$ and let $D = \bigcup_{i<d} D_i$. For each element in $T(b_k) \cup \{y', y''\}$ pick a lexicographically least point in D above it. Let $T(b_{k+1})$ be the set of points thus chosen. This uniquely defines $T \upharpoonright (b_{k+1} + 1)$. During our construction we arrange the choice of the points y in such a way that the final tree T is perfect. Let $B = \{b_k : k < \omega\}$. It follows that $\otimes^B_{i<d} T_{t_i} \subseteq K_\epsilon$.

We now show that (T, n) is the required condition. First note that if (R, l) is any extension of (T, n) then there there is $m \in A$ such that R has no splitting nodes on levels between l and m and hence (R, l) and (R, m) are equivalent condition. Suppose now that some condition of width $d + 1$ below (T, n) accepts $\mathbf{b}$ and let (R, m) be such a condition with m minimal such that $m \in A$. Let $Z = R(m) = \{z_0, \ldots, z_d\}_<$ be the increasing enumeration in the lexicographical order and let k be the largest such that $b_k < m$. Then since on levels between b_k and b_{k+1} there is at most one splitting node it follows that $R(b_k)$ has size d. Let $R(b_k) = \{y_0, \ldots, y_{d-1}\}_<$ be the increasing enumeration. By the construction of T it follows that y_{d-1} was the point chosen at stage k, that z_{d-1} and z_d are the only extensions of y_{d-1} in T on level m, and that z_i is the lexicographically least extension of y_i in $S^*(m)$ for $i < d - 1$. Thus $(y_0, \ldots, y_{d-1})$ is colored according to whether (S^*_Z, m) accepts $\mathbf{b}$, rejects $\mathbf{b}$, or cannot decide. Since (R, m) is a pure extension of (S^*_Z, m) which accepts $\mathbf{b}$ and $m \in A$ by the property of S^* it follows that (S^*_Z, m) also accepts $\mathbf{b}$ and thus $(y_0, \ldots, y_{d-1}) \in K_0$. Hence we must have $\epsilon = 0$.

Now since then $\otimes^B_{i<d} T_{t_i} \subseteq K_0$, a similar analysis shows that any other extension of (T, n) of width $d + 1$ accepts $\mathbf{b}$. But then it follows that (T, n) also accepts $\mathbf{b}$. $\square$

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PROOF OF LEMMA 1: Let (S, n) be a condition in $\mathcal{C}$ and let $\mathbf{b}$ be a Boolean value. Assume that there is no pure extension of (S, n) which accepts $\mathbf{b}$. We find a pure extension (T, n) of (S, n) which rejects $\mathbf{b}$. We shall build the tree T by a fusion argument. Along the way we shall construct a decreasing sequence $(T^{(0)}, a_0) \geq (T^{(1)}, a_1) \geq \ldots$ of conditions in $\mathcal{A}(\mathcal{P})$.

To begin let $(T^{(0)}, a_0) = (S, n)$. Suppose now $(T^{(k)}, a_k)$, has been defined. Let $\{Z_i : i < l\}$ be an enumeration of all subsets Z of $T^{(k)}(a_k)$ which generate a comb extending $S \upharpoonright (n + 1)$. The inductive assumption is that for each such Z the condition $(T_Z^{(k)}, a_k)$ does not have a pure extension accepting $\mathbf{b}$. To avoid excessive notation let R be a variable denoting a perfect subtree of $T^{(k)}$. We initially set R to be equal $T^{(k)}$ and then trim it down in l steps as follows. At step i consider Z_i. Since (R_{Z_i}, a_k) is a pure extension of $(T_{Z_i}^{(k)}, a_k)$ from the inductive assumption it follows that it does not have a pure extension accepting $\mathbf{b}$. If the size of Z_i is d_i then by Lemma 3 there is a pure extension (Q, a_k) of (R_{Z_i}, a_k) such that no extension of (Q, a_k) of width $d_i + 1$ accepts $\mathbf{b}$. We now shrink R as follows. For every $s \in Z_i$ replace R_s by Q_s and for $s \in T^{(k)}(a_k) \setminus Z_i$ keep R_s the same. After all the l steps have been completed pick a node y in $T^{(k)}(a_k)$. Let a_{k+1} be the least a such that y has two extensions in $R(a)$. Keep those two extensions of y and for every other node in $T^{(k)}(a_k)$ pick exactly one extension on level a_{k+1}. Let then $T^{(k+1)}$ be the set of all nodes of R comparable to one of these nodes. If now Z is any subset of $T^{(k+1)}(a_{k+1})$ which generates a comb extending $S \upharpoonright (n + 1)$ we claim that there is no pure extension of $(T^{(k+1)}, a_{k+1})$ accepting $\mathbf{b}$. Notice that the set of all predecessors of members of Z on level a_k is listed as one of the Z_i. Since between levels a_k and a_{k+1} there is at most one splitting of $T^{(k+1)}$ it follows that $\operatorname{card}(Z) \leq d_i + 1$. If the size of Z is d_i then every pure extension of $(T^{(k+1)}, a_{k+1})$ is equivalent to a pure extension of $(T^{(k)}, a_k)$, but by the inductive hypothesis such a condition cannot accept $\mathbf{b}$. On the other hand if the size of Z is $d_i + 1$ at stage i of the construction of $T^{(k+1)}$ we have ensured that no such condition accepts $\mathbf{b}$. This shows that the inductive hypothesis is preserved.

Finally let $T = \bigcap T^{(k)}$. Throughout the construction we make the choice of the points y above which we keep a splitting node carefully to ensure that the final tree T is perfect. It follows that no condition (R, m) extending (T, n) accepts $\mathbf{b}$ and hence (T, n) rejects $\mathbf{b}$, as desired. $\square$

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4 Proof of Lemma 2

In the proof of Lemma 2 we need the following lemma whose proof is almost identical to the proof of Lemma 3 and is thus omitted.

Lemma 4 *Let $(S, n) \in \mathcal{C}$ be a condition of width d and let U be a set of infinite combs. Then there is pure extension (T, n) of (S, n) such that either $\mathcal{C}_\omega(T, n)$ is contained in U or there is no extension (R, m) of (T, n) of width $d + 1$ such that $\mathcal{C}_\omega(R, m)$ is contained in U.*

Now note that to complete the proof of Lemma 2 and Theorem 2 it suffices to prove the following.

Lemma 5 *Let (S, n) be a condition in $\mathcal{A}(\mathcal{P})$ and let D be a dense open subset of $\mathcal{C}$. Then there is a pure extension (T, n) of (S, n) such that for every infinite comb C in $\mathcal{C}_\omega(T)$ there is m such that $(T_{C(m)}, m) \in D$.*

PROOF: We first show that if $(S, n) \in \mathcal{C}$ there is a pure extension (T, n) of (S, n) such that for every $C \in \mathcal{C}_\omega(T, n)$ there is $m \geq n$ such that $(T_{C(m)}, m) \in D$. To begin find an infinite subset A of ω and a pure extension (S^*, n) of (S, n) such that for every $m \in A$ and every subset Z of $S^*(m)$ which generates a comb extending $S \restriction (n + 1)$ if there is a pure extension of (S_Z^*, m) which is in D then already (S_Z^*, m) is in D. Let then

$$U = \{C \in \mathcal{C}_\omega(S^*, n) : \text{ there is } m \text{ such that } (S_{C(m)}^*, m) \in D\}$$

Assume now towards contradiction that there is no pure extension (T, n) of (S^*, n) such that $\mathcal{C}_\omega(T, n)$ is contained in U. As in the proof of Lemma 1 we build a decreasing sequence $(T^{(0)}, a_0) \geq (T^{(1)}, a_1) \geq \ldots$ of conditions in $\mathcal{A}(\mathcal{P})$. To begin set $(T^{(0)}, a_0) = (S^*, n)$. Suppose now $(T^{(k)}, a_k)$ has been defined. Our inductive assumption is that for any subset Z of $T^{(k)}(a_k)$ which generates a comb extending $S \restriction (n + 1)$ there is no pure extension (Q, a_k) of $(T_Z^{(k)}, a_k)$ such that $\mathcal{C}_\omega(Q, a_k)$ is contained in U. Let $\{Z_i : i < l\}$ be an enumeration of all such Z. To avoid excessive notation let, as before, R be a variable denoting a perfect subtree of $T^{(k)}$. To begin set R equal to $T^{(k)}$. We then successively trim down R in l steps as follows. Suppose that step i has been completed. Since (R_{Z_i}, a_k) is a pure extension of $(T_{Z_i}^{(k)}, a_k)$, by the inductive hypothesis it has no pure extension (Q, a_k) such that $\mathcal{C}_\omega(Q, a_k)$

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is contained in U. Let the size Z_i be d_i. Then by Lemma 4 there is a pure extension (Q, a_k) of (R_{Z_i}, a_k) in $\mathcal{C}$ such that if (Q^*, m) is an extension (Q, a_k) in $\mathcal{C}$ of width $d_i + 1$ then $\mathcal{C}_\omega(Q^*, m)$ is not contained in U. Now trim down R as follows. For nodes s in Z_i replace R_s by Q_s and for nodes s in $T^{(k)}(a_k) \setminus Z_i$ keep R_s the same. Finally when all the stages are completed and we have taken care of all the Z_i we choose a node y in $T^{(k)}(a_k)$ and let a_{k+1} be the least member of A above a_k such that y has two successors in $R(a_{k+1})$. Then $T^{(k+1)}$ is obtained from R by keeping those two successors of y and by keeping for every other node in $T^{(k)}(a_k)$ one successors and throwing away the remaining ones. Then $T^{(k+1)}$ is set to be the set of all nodes of the final R comparable to one of the chosen points. Note that in this way we arrange that for every subset Z of $T^{(k+1)}(a_{k+1})$ which generates a comb extending $S \upharpoonright (n+1)$ the set of all predecessors of members of Z on level a_k is listed as one of the Z_i and since between a_k and a_{k+1} there is at most one splitting node it follows that $\text{card}(Z) \leq d_i + 1$. Thus it follows that if (Q, a_{k+1}) is a pure extension of $(T_Z^{(k+1)}, a_{k+1})$ then $\mathcal{C}_\omega(Q, a_{k+1}) \setminus U \neq \emptyset$.

In then end we let $T = \bigcap T_k$. We make the choice of the nodes y above we choose a splitting at each stage judiciously so that the final tree T is perfect. It follows that if (R, m) is any extension of (T, n) in $\mathcal{C}$ then $\mathcal{C}_\omega(R, m) \setminus U \neq \emptyset$. Now since D is dense open we can find k and a condition $(R, a_k) \in D$ extending (T, n). Let $Z = R(a_k)$. By the property of S^* it follows that (S_Z^*, a_k) is also in D. But then $\mathcal{C}_\omega(S_Z^*, a_k) \subseteq U$, a contradiction.

Now to deal with the general case assume that only $(S, n) \in \mathcal{A}(\mathcal{P})$. We then proceed as in the successor stage of the previous case. We enumerate all subset Z of $S(n)$ which generate a comb as $\{Z_i : i < l\}$. Let, as before, R be a variable denoting a perfect subtree of S. To begin set R to be equal to S. We then trim down R successively in l stages. At stage i look at Z_i and apply the special case of the lemma to find a pure extension (Q, n) of (R_{Z_i}, n) such that for every infinite comb C extending $Q \upharpoonright (n+1)$ there is $m \geq n$ such that $(Q_{C(m)}, m) \in D$. Trim down R by replacing R_s by Q_s for every node $s \in Z_i$ and keeping R_s the same for evert $s \in S(n) \setminus Z_i$. We let T be equal to R after all the stages have been completed. It follows that $(T, n) \leq (S, n)$ and for every comb $C \in \mathcal{C}_\omega(T)$ there is m such that $(T_{C(m)}, m) \in D$. This finishes the proof of Lemma 5 and Theorem 2. $\qquad \square$

Remarks

In this paper we have only considered partitions of strongly increasing sequences of reals and have shown that every such partition into an analytic and a co-analytic piece has a perfect homogeneous set. A similar result can be obtained for any other type $\prec$ of increasing sequences. All we have to do is modify the forcing notion $\mathcal{C}$ so that the generic sequence produced has type $\prec$. Consequently, if $\mathcal{I}$ is a finite set of types of infinite increasing sequences of reals for every analytic partitions of infinite increasing sequences of reals we can find a perfect set P such that for every $\prec$ in $\mathcal{I}$ all the sequences from P which have type $\prec$ have the same color. On the other hand it is easy to see that if $\mathcal{I}$ is any infinite set of types there is a partitions such that no perfect set is homogeneous for all types in $\mathcal{I}$ simultaneously. Namely, choose for each $s \in 2^{<\omega}$ a type $\prec_s$ in $\mathcal{I}$ such that the function which maps s to $\prec_s$ is 1-1. Now given a sequence $\{x_n : n < \omega\}$ of type $\prec_s$ color it *red* if $\Delta(x_0, x_1) = s$ and *blue* otherwise. Let now P be a perfect set and let T be the tree of all finite initial segments of elements of P. Then for any s which is a splitting node of T there are sequences from P of type $\prec_s$ which are colored by either color.

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