Coding with ladders a well ordering of the reals

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September 15, 2020

Abstract

Any model of ZFC + GCH has a generic extension (made with a poset of size $\aleph_2$) in which the following hold: $MA + 2^{\aleph_0} = \aleph_2 +$ *there exists a* Δ^2_1*-well ordering of the reals.* The proof consists in iterating posets designed to change at will the guessing properties of ladder systems on ω_1. Therefore, the study of such ladders is a main concern of this article.

1 Preface

The character of possible well-orderings of the reals is a main theme in set theory, and the work on long projective well-orderings by L. Harrington [4] can be cited as an example. There, the relative consistency of ZFC + MA $+2^{\aleph_0} > \aleph_1$ with the existence of a Δ^1_3 well-ordering of the reals is shown. A different type of question is to ask about the impact of large cardinals on

*This research was supported by The Israel Science Foundation founded by the Israel Academy of Sciences and Humanities. Publication # 485.

definable well-orderings. Work of Shelah and Woodin [7], and Woodin [9] is relevant to this type of question. Assuming in V a cardinal which is both measurable and Woodin, Woodin [9] proved that if CH holds, then there is no Σ^2_1 well-ordering of the reals. This result raises two questions:

1. If large cardinals and CH are assumed in V, can the Σ^2_1 result be strengthen to Σ^2_2? That is, is there a proof that large cardinals and CH imply no Σ^2_2 well-orderings of the reals?

2. What happens if CH is not assumed?

Regarding the first question, Abraham and Shelah [2] describes a poset of size $\aleph_2$ (assuming GCH) which generically adds no reals and provides a Δ^2_2 well-ordering of the reals. Thus, if one starts with any universe with a large cardinal κ, one can extend this universe with a small size forcing and obtain a Δ^2_2 well-ordering of the reals. Since small forcings will not alter the assumed largeness of a cardinal in V, the answer to question 1 is negative.

Regarding the second question, Woodin (unpublished) uses an inaccessible cardinal κ to obtain a generic extension in which

1. MA for σ-centered posets + $2^{\aleph_0} = \kappa$, and

2. there is a Σ^2_1 well-ordering of the reals.

Solovay [8] shows that the inaccessible cardinal is dispensable: any model of ZFC has a small size forcing extension in which the following holds:

1. MA for σ-centered posets + $2^{\aleph_0} = \aleph_2$, and

2. there is a Σ^2_1 well-ordering of the reals.

In [3] we show how Woodin's result can be strengthened to obtain the full Martin's axiom. We prove there that if V satisfies the GCH and contains an inaccessible cardinal κ, then there is a poset of cardinality κ that gives generic extensions in which

1. MA + $2^{\aleph_0} = \kappa$, and

2. there is a Σ^2_1 well-ordering of the reals.

Our aim in this paper is to show that the inaccessible cardinal is not really necessary, even to get the full Martin's Axiom.

Theorem 1.1 *Assume $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. There is a forcing poset of size $\aleph_2$ that provides a cardinal preserving extension in which Martin's Axiom $+2^{\aleph_0} = \aleph_2$ holds, and there is a Σ^2_1 well-ordering of the reals. In fact, there is even a $\Sigma^{2[\aleph_1]}$ well-ordering of the reals there.*

The concepts Σ^2_1 and $\Sigma^{2[\aleph_1]}$ will soon be defined, but first we shall point to what we consider to be the main novelty of this paper, the use of ladder systems as coding devices. A ladder over $S \subseteq \omega_1$ is a sequence $\overline{\eta} = \langle \eta_\delta \mid \delta \in S \rangle$ where $\eta_\delta : \omega \to \delta$ is increasing and cofinal in δ. Two ladders over S, $\overline{\eta}'$ a subladder of $\overline{\eta}$, may encode a real (a subset of ω). Namely the coding of a real r is expressed by the relationship between η'_δ and η_δ (for every δ). Splitting ω_1 into $\aleph_2$ pairwise almost disjoint stationary sets, it is possible to encode $\aleph_2$ many reals (and hence a well-ordering) using $\aleph_2$ pairs of ladder sequences. Of course, we need some property that ensures uniqueness of these ladders, in order to make this well-ordering definable. Such a property will be obtained in relation with the guessing power of the ladders. A ladder system $\langle \eta_\delta \mid \delta \in S \rangle$ is said to be club (closed unbounded set) guessing if for every closed unbounded $C \subseteq \omega_1$, $[\eta_\delta] \subseteq^* C$ for some $\delta \in S$. It turns out that there is much freedom to manipulate the guessing properties of ladders, and, technically speaking, this shall be a main concern of the paper.

We now define the Σ^2_1 and $\Sigma^{2[\aleph_1]}$ relations. The structure with the membership relation on the collection of all hereditarily countable sets is denoted $H(\aleph_1)$. Second-order formulas over $H(\aleph_1)$ that contain n alternations of quantifiers are denoted Σ^2_n when the external quantifier is an existential class quantifier. Thus a Σ^2_n formula has the form

$$\exists X_1 \forall X_2 \ldots X_n \varphi(X_1, \ldots, X_n)$$

where φ may only contain first-order quantifiers over $H(\aleph_1)$ and predicates $X_1, \ldots, X_n$ are interpreted as subsets of $H(\aleph_1)$. (One can either write $X_i(s)$ treating X_i as a predicate, or $s \in X_i$ treating X_i as a class.) Σ^2 denotes the union of all Σ^2_n formulas.

If the second-order quantifiers only quantify classes (subsets of $H(\aleph_1)$) of cardinality $\leq \aleph_1$, then the resulting set of formulas is denoted $\Sigma^{2[\aleph_1]}_n$. So $\Sigma^{2[\aleph_1]}_1$ for example denotes second order formulas of the form "there exists a subset X of $H(\aleph_1)$ of size $\leq \aleph_1$ such that $\varphi(X)$" where φ is a first order formula. We write $\Sigma^{2[\aleph_1]}$, without a subscript, for $\bigcup_{n<\omega} \Sigma^{2[\aleph_1]}_n$.

In Theorem 1.1 above, we get a well-ordering which is $\Sigma^{2[\aleph_1]}$, and we will explain now why $MA + 2^{\aleph_0} > \aleph_1$ implies that such a relation is necessarily Σ^2_1. This transformation which replaces any number of quantifiers over sets of size $\aleph_1$ with a single existential quantifier over arbitrary subsets of $H(\aleph_1)$ is a trick of Solovay's that was used by him in [8]. The basic idea is to use the almost-disjoint-sets coding (Jensen and Solovay [5]) in a way which will be sketched here.

Theorem 1.2 (Solovay) *Assume MA+$2^{\aleph_0} > \aleph_1$. Any $\Sigma^{2[\aleph_1]}$ formula $\varphi(\overline{x})$ over $H(\aleph_1)$, with free variables $x_1, \ldots, x_n$, is equivalent to a Σ^2_1 formula $\psi(\overline{x})$.*

Proof. It seems easier to prove first that every $\Sigma^{2[<\mathfrak{c}]}$ formula is equivalent with a Σ^2_1 formula. (The $\Sigma^{2[<\mathfrak{c}]}$ formulas are second order formulas over $H(\aleph_1)$ in which class quantification occurs only for subset of $H(\aleph_1)$ of size less than continuum.) Then the theorem follows because the $\Sigma^{2[\aleph_1]}$ classes are a naturally characterized subclass of the $\Sigma^{2[<\mathfrak{c}]}$.

So let $\varphi(x)$ be any $\Sigma^{2[<\mathfrak{c}]}$ formula. The equivalent Σ^2_1 formula ψ begins as follows (with existential class quantifiers mixed with first-order quantifiers which do not change the complexity of the formula):

> *There is a set $\tau \subset \mathcal{P}(\omega)$ such that the relation*
>
> $$x <_\tau y \textit{ iff } y \setminus x \textit{ is finite}$$
>
> *is a well-order of τ such that there is no infinite $a \subseteq \omega$ with $a \subseteq^* x$ for all $x \in \tau$. There is also a map $\mu : \tau \longrightarrow H(\aleph_1)$, which is* onto *$H(\aleph_1)$, and there is a map $\rho : \tau \longrightarrow [\omega]^{\aleph_0}$ such that for distinct $x, y \in \tau$, $\tau(x)$ and $\tau(y)$ are almost disjoint. ($[\omega]^{\aleph_0}$ is the collection of infinite subsets of ω.)*

Then ψ continues with first-order quantifiers that replace the $\Sigma^{2[<\mathfrak{c}]}$ quantifiers of φ in the following manner. To represent any $X \subseteq H(\aleph_1)$ of size $< \mathfrak{c}$, look at the set $\mu^{-1}X \subseteq \tau$. Since its size is $< \mathfrak{c}$, there is by Martin's Axiom an infinite set $a \subseteq \omega$ almost included in every set in $\mu^{-1}X$. Hence $\mu^{-1}X$ is bounded in τ. So there is t_0 in τ so that $\mu^{-1}(x) <_\tau t_0$ for every $x \in X$. Now look at the collection $\{\rho(t) \mid t <_\tau t_0\}$ of almost-disjoint sets (its cardinality is $< \mathfrak{c}$) and use Martin's Axiom to encode with one r the set $\rho[\mu^{-1}X]$. That is find $r \subset \omega$ such that for $t <_\tau t_0$, $\rho(t) \cap r$ is finite iff $\mu(t) \in X$. Then r and t_0 represent X.

2 Ladder systems

The notation $A \subseteq^* B$ is used for "almost inclusion" on subsets of ω_1, meaning that $A \setminus B$ is finite. Similarly $A =^* B$ is defined if $A \subseteq^* B$ and $B \subseteq^* A$. $A \neq^* B$ is the negation of $A =^* B$.

Definition 2.1 *1. A* ladder system *over $S \subseteq \omega_1$ (consisting of limit ordinals) is a sequence $\overline{\eta} = \langle \eta_\delta \mid \delta \in S \rangle$, where η_δ is an increasing ω-sequence converging to δ. S is called "the domain" of $\overline{\eta}$, and is denoted* $\mathrm{dom}(\overline{\eta})$*. $\overline{\eta}$ is called "trivial" if* $\mathrm{dom}(\overline{\eta})$ *is non-stationary. The range of η_δ is denoted $[\eta_\delta]$ (so $[\eta_\delta] = \{\eta_\delta(i) \mid i \in \omega\}$), and $\bigcup_{\delta \in S} [\eta_\delta]$ is the "range" of $\overline{\eta}$. So, $[\eta_\delta] \subset^* C$ means that, except for finitely many k's, $\eta_\delta(k) \in C$ always holds.*

2. *Let $\overline{\eta}$ and $\overline{\mu}$ be two ladder systems. We say that $\overline{\eta}$ and $\overline{\mu}$ are* almost disjoint *iff, for some club $C \subseteq \omega_1$, for any $\delta \in C \cap \mathrm{dom}(\overline{\eta}) \cap \mathrm{dom}(\overline{\mu})$, $[\eta_\delta] \cap [\mu_\delta] =^* \emptyset$.*

3. *We say that $\overline{\eta}$ is a subladder of $\overline{\mu}$ iff the following holds for some club $C \subseteq \omega_1$:*

$$C \cap \mathrm{dom}(\overline{\eta}) \subseteq \mathrm{dom}(\overline{\mu}) \text{ and for } \delta \in C \cap \mathrm{dom}(\overline{\eta}),\ [\eta_\delta] \subseteq^* [\mu_\delta].$$

In such a case we write $\overline{\eta} \lhd \overline{\mu}$. Also, $\overline{\eta} =^* \overline{\mu}$ iff both $\overline{\eta} \lhd \overline{\mu}$ and $\overline{\mu} \lhd \overline{\eta}$. That is, $\overline{\eta} =^* \overline{\mu}$ iff there is a club set $C \subseteq \omega_1$ such that $\mathrm{dom}(\overline{\eta}) \cap C = \mathrm{dom}(\overline{\mu}) \cap C$*, and $[\eta_\delta] =^* [\mu_\delta]$ for $\delta \in \mathrm{dom}(\overline{\eta}) \cap C$.*

4. *The difference ladder $\overline{\rho} = \overline{\eta} \setminus \overline{\mu}$ is defined by*

$$[\rho_\delta] = \begin{cases} [\eta_\delta] & \text{if } \delta \in \ \mathrm{dom}(\overline{\eta}) \setminus \ \mathrm{dom}(\overline{\mu}) \\ [\eta_\delta] \setminus [\mu_\delta] & \text{if this set is infinite} \\ \text{undefined} & \text{otherwise} \end{cases}$$

It is the $\lhd$-maximal ladder included in $\overline{\eta}$ and (almost) disjoint from $\overline{\mu}$.

5. *Given any $A \subseteq \omega_1$, the restriction ladder $\overline{\eta} \restriction A$ is naturally defined, and its domain is $A \cap \mathrm{dom}(\overline{\eta})$. If $x \subseteq \omega$ is infinite, then $\overline{\eta} \restriction x$ means something else: it is obtained by enumerating $x = \{x_k \mid k \in \omega\}$ in increasing order, and setting $(\overline{\eta} \restriction x) = \overline{\rho}$ where $\rho_\delta(k) = \eta_\delta(x_k)$ for every $\delta \in \mathrm{dom}(\overline{\eta})$.*

We shall define some properties of ladders (in fact, of $=^*$ equivalence classes).

Definition 2.2 *Let $\overline{\eta}$ be a ladder over S.*

1. *We say that $\overline{\eta}$ is* club-guessing *iff for every club $C \subseteq \omega_1$ there is $\delta \in S$ such that $[\eta_\delta] \subseteq^* C$. (So, in this case, $\delta \in C$, and hence* $\mathrm{dom}(\overline{\eta})$ *is stationary if $\overline{\eta}$ is club guessing.) For brevity, we may use the term* guessing *instead of club-guessing.*

2. *We say that $\overline{\eta}$ is* strongly *club guessing (or just strongly guessing) iff for any club $C \subseteq \omega_1$ for some club D, if $\delta \in D \cap S$ then $[\eta_\delta] \subseteq^* C$. If $\overline{\eta}$ is strongly guessing and $\overline{\rho} \lhd \overline{\eta}$, then clearly $\overline{\rho}$ is also strongly guessing. (Be careful: if $\overline{\rho} \lhd \overline{\eta}$ and $\overline{\eta}$ is guessing, you cannot infer that $\overline{\rho}$ is guessing, unless $\overline{\rho}$ is non-trivial.) The trivial ladder is (trivially) strongly guessing, and hence we cannot say that a strongly guessing ladder is always guessing. A strongly guessing non-trivial ladder is, of course, guessing.*

3. *We say that a club set $C \subseteq \omega_1$* avoids *$\overline{\eta}$ iff for every $\delta \in S$ (except a non-stationary set), $[\eta_\delta] \cap C =^* \emptyset$.*

4. *We say that $\overline{\eta}$ is* avoidable *iff some club set avoids $\overline{\eta}$. If every ladder over S is avoidable, then we say that S itself is* avoidable*. Hence, in particular, if S is non-stationary, then S is avoidable. Remark that if $\overline{\eta}$ is avoidable, then $\overline{\eta}$ is non-guessing. So $\overline{\eta}$ is strongly guessing and avoidable iff $\overline{\eta}$ is trivial. The collection of all avoidable sets forms an ideal which will be shown to be normal in the following subsection.*

5. **Maximal ladders**. *Suppose that $\overline{\eta}$ is some strongly guessing ladder over S, and $X \supseteq S$ is a subset of ω_1. If every ladder over X and (almost) disjoint from $\overline{\eta}$ is avoidable, then we say that $\overline{\eta}$ is* maximal for *X. In such a case, for every $X' \subseteq X$, $\overline{\eta} \restriction X'$ is maximal for X'. The trivial ladder $\emptyset$ is trivially maximal for any avoidable set. Our terminology may be misleading because a maximal ladder for X is not necessarily defined over X, it is rather the maximality which is for X. Thus, if $\overline{\eta}$ is maximal for X, then $\overline{\mu} \lhd \overline{\eta}$ for every strongly guessing ladder $\overline{\mu}$ over a subset of X. (Because $\overline{\mu} \setminus \overline{\eta}$ is, in that case, strongly guessing and disjoint from $\overline{\eta}$, and is hence avoidable. Thus* $\mathrm{dom}(\overline{\mu} \setminus \overline{\eta})$

is not stationary, and hence $\overline{\mu} \lhd \overline{\eta}$.) Hence if both $\overline{\mu}$ and $\overline{\eta}$ are maximal for X, then $\overline{\mu} =^* \overline{\eta}$. We denote this unique ladder, maximal for X, by $\chi(X)$.

It is easy to see that if $\overline{\eta}$ is maximal for X and $X_0 \subseteq X$ then $\overline{\eta} \restriction X_0$ is maximal for X_0.

2.1 Ideals connected with ladders

We are going to define four ideals on ω_1: the ideal of non-guessing restrictions, denoted $I_{\overline{\eta}}$, the ideal of avoidable sets, denoted I_0, the ideal of maximal guesses, denoted I_1, and the ideal of bounded intersections, $I(\overline{S})$. Then we will prove that all are normal ideals.

Definition 2.3 The ideal of non-guessing restrictions. *Let $\overline{\eta}$ be a guessing ladder over X. The collection of all subsets $S \subseteq \omega_1$ for which $\overline{\eta} \restriction S$ is not guessing is a proper, normal ideal, denoted $I_{\overline{\eta}}$.*

The ideal of avoidable sets. *$S \in I_0$ iff every ladder system over S is avoidable.*

The ideal of maximal guesses. *The ideal I_1 is the collection of all sets $X \subseteq \omega_1$ such that there is a maximal ladder for X.*

So $S \in I_1$ iff there is a strongly guessing ladder system $\overline{\eta}$ such that $\mathrm{dom}(\overline{\eta}) \subseteq S$ and any ladder over S and disjoint from $\overline{\eta}$ is avoidable. As said above, this unique ladder $\overline{\eta}$ is denoted $\chi(S)$. (Uniqueness is up to $=^*$, where non-stationary sets and finite differences do not count.)

In case $S \in I_0$, then $S \in I_1$, and $\chi(S)$ is the trivial (empty) ladder $\emptyset$. So

$$I_0 \subseteq I_1. \tag{1}$$

The ideal of bounded intersections. *Let $\overline{S} = \langle S_i \mid i \in \omega_2 \rangle$ be a collection of $\aleph_2$ stationary subsets of ω_1 such that the intersection of any two is non-stationary (we say that $\overline{S}$ is a sequence of pairwise almost disjoint stationary sets). The ideal $I(\overline{S})$ consists of those sets $H \subseteq \omega_1$ for which*

$$|\{i \in \omega_2 \mid H \cap S_i \text{ is stationary }\}| \leq \aleph_1.$$

Sets in $I(\overline{S})$ will also be called $\overline{S}$-small sets. It may seem that $I(\overline{S})$ is not connected to ladders, but we will later show the consistency of $I(\overline{S}) = I_1$.

Lemma 2.4 *All four ideals are normal.*

Proof. An ideal on ω_1 is said to be normal if it is closed under diagonal unions.

The ideal of non-guessing restrictions. Let $\overline{\eta}$ be a guessing ladder over X. To prove normality of $I_{\overline{\eta}}$, suppose $A_\xi \in I_{\overline{\eta}}$, for $\xi < \omega_1$. Thus, for every ξ there is a club set C_ξ such that $\delta \in A_\xi \cap X \Rightarrow [\eta_\delta] \not\subseteq^* C_\xi$. Let

$$A = \nabla_{\xi \in \omega_1} A_\xi \stackrel{\text{def}}{=} \{\alpha \in \omega_1 \mid \exists \xi < \alpha(\alpha \in A_\xi)\}$$

be the diagonal union, and $C = \Delta_{\xi \in \omega_1} C_\xi$ be the diagonal intersection of the club sets. Then $A \in I_{\overline{\eta}}$ because for $\delta \in A \cap X$, $[\eta_\delta] \not\subseteq^* C$.

The ideal of avoidable sets. We check that I_0 is normal. Suppose $S_\xi \in I_0$ for $\xi \in \omega_1$, and let $S = \nabla_{\xi \in \omega_1} S_\xi$ be the diagonal union. Let $\overline{\eta} = \langle \eta_\delta \mid \delta \in S \rangle$ be any ladder over S, and we will show that $\overline{\eta}$ is avoidable and hence that $S \in I_0$. Indeed, a slightly more general fact will be used later:

> If $S_\xi \subseteq \omega_1$ are arbitrary sets, $S = \nabla_{\xi \in \omega_1} S_\xi$, and $\overline{\eta}$ is a ladder over S such that $\overline{\eta} \restriction S_\xi$ is avoidable for every $\xi \in \omega_1$, then $\overline{\eta}$ is avoidable.

To see this, let C_ξ for $\xi \in \omega_1$ be a club set that avoids $\overline{\eta} \restriction S_\xi$, and let $C = \Delta_{\xi \in \omega_1} C_\xi$ be their diagonal intersection. Then C avoids $\overline{\eta}$, as can easily be checked.

The ideal of maximal guesses. We prove that I_1 is normal. So suppose that $S_\xi \in I_1$ for $\xi \in \omega_1$ are given, and $S = \nabla_{\xi \in \omega_1} S_\xi$ is their diagonal union. We must prove that $S \in I_1$. First we claim that the sets $\{S_\xi \mid \xi \in \omega_1\}$ may be assumed to be pairwise disjoint. Indeed, define $S^*_\xi = S_\xi \setminus \bigcup \{S_{\xi'} \mid \xi' < \xi\}$. Then $S = \nabla S^*_\xi$, and the sets S^*_ξ are in I_1 and are pairwise disjoint. So we do assume now that the S_ξ's are pairwise disjoint. For every $\xi \in \omega_1$, $\chi(S_\xi)$ is a strongly guessing ladder over its domain $S^0_\xi \subseteq S_\xi$ (and $S^0_\xi = \emptyset$ when $S_\xi \in I_0$). Define $S^0 = \nabla_{\xi\in\omega_1} S^0_\xi$. Clearly $S^0 \subseteq S$. For $\delta \in S^0$ define η_δ to be $\chi(S_\xi)_\delta$ for the (unique) $\xi < \delta$ such that $\delta \in S^0_\xi$.

Claim: $\overline{\eta} = \langle \eta_\delta \mid \delta \in S^0 \rangle$ is maximal for S, and hence $S \in I_1$.

Proof. We first prove that $\overline{\eta}$ is strongly guessing. Well, if $C \subseteq \omega_1$ is club, find for each $\xi \in \omega_1$ a club set D_ξ such that for $\delta \in D_\xi \cap S^0_\xi$, $(\chi(S_\xi))_\delta \subseteq^* C$. Now define $D = \Delta_{\xi\in\omega_1} D_\xi$ to be the diagonal intersection. It follows that for every $\delta \in S^0 \cap D$, $[\eta_\delta] \subset^* C$.

To prove maximality, assume $\overline{\mu}$ is defined on S and is disjoint from $\overline{\eta}$. Then $\overline{\mu} \restriction S_\xi$ is disjoint from $\chi(S_\xi) \restriction S_\xi \setminus (\xi+1)$. Hence $\overline{\mu} \restriction S_\xi$ is avoidable for every $\xi < \omega_1$, and by the proof of normality of I_0, $\overline{\mu}$ is avoidable.

The ideal of bounded intersections. Let $\overline{S} = \langle S_i \mid i \in \omega_2 \rangle$ be a collection of pairwise almost disjoint stationary subsets of ω_1 defining $I(\overline{S})$. If H_ξ for $\xi \in \omega_1$ are in $I(\overline{S})$, then there is a bound $j_0 < \omega_2$ such that for every $j_0 \leq j < \omega_2$ $S_j \cap H_\xi$ is non-stationary. Hence $\nabla_\xi S_j \cap H_\xi = S_j \cap \nabla H_\xi$ is non-stationary, and thus $\nabla_\xi H_\xi \in I(\overline{S})$.

2.2 $A(\overline{S}, \overline{\eta})$

In this subsection we formulate a statement, $A(\overline{S}, \overline{\eta})$, and show that it implies $I(\overline{S}) = I_1$. The consistency of $A(\overline{S}, \overline{\eta})$ will be proved in the subsequent sections.

Definition 2.5 *$A(\overline{S}, \overline{\eta})$ is the conjunction of the following six statements:*

A1 $\overline{S} = \langle S_i \mid i \in \omega_2 \rangle$ is a sequence of pairwise almost disjoint stationary subsets of ω_1. $\overline{\eta}$ is a ladder system, and $\bigcup_{i<\omega_2} S_i \subseteq \mathrm{dom}(\overline{\eta})$.

A2 Every ladder disjoint from $\overline{\eta}$ is avoidable. (It immediately follows that if $\overline{\mu}$ is strongly guessing, then $\overline{\mu} \setminus \overline{\eta}$ is both avoidable and strongly guessing and thus $\overline{\mu} \setminus \overline{\eta} =^* \emptyset$, so that $\overline{\mu} \lhd \overline{\eta}$.)

A3 For every $i < \omega_2$, $S_i \in I_1$. In fact, $\chi(S_i)$ is defined over S_i (and it is a non-trivial strongly guessing ladder over S_i such that any ladder over a subset of S_i and disjoint from $\chi(S_i)$ is avoidable). It follows by A2 that $\chi(S_i) \lhd \overline{\eta}$.

A4 *If* $X \subseteq \omega_1$ *is such that* $X \cap S_i$ *is non-stationary for every* $i < \omega_2$*, then* X *is avoidable (equivalently, in view of (A2),* $\overline{\eta} \restriction X$ *is avoidable).*

A5 *If* $X \subseteq \omega_1$ *is not* $\overline{S}$*-small,* $\overline{\rho}$ *is a ladder over* X *and* $\overline{\rho} \lhd \overline{\eta}$*, then there exists* $i < \omega_2$ *such that* $S_i \subseteq X$ *and* $\chi(S_i) \lhd \overline{\rho}$*.*

A6 *For every* $i \in \omega_2$ *either* $(\chi(S_i), \overline{\eta} \restriction S_i)$ *is clearly not encoding, or else* $r = d(\chi(S_i), \overline{\eta} \restriction S_i)$ *is defined, and in this case* $r = d(\chi(S_j), \overline{\eta} \restriction S_j)$ *for unboundedly many* j*'s. The meaning of this statement is clarified later in this subsection.*

$A'(\overline{T}, \overline{\eta})$ is the following statement: $\overline{T} = \langle T_i \mid i < \omega_2 \rangle$ is a sequence of pairwise almost disjoint stationary subsets of ω_1. For every $i < \omega_2$, $T_i \in I_1$, and if S_i denotes $\mathrm{dom}(\chi(T_i))$, then $A(\overline{S}, \overline{\eta})$ holds for $\overline{S} = \langle S_i \mid i < \omega_2 \rangle$.

We first collect some simple consequences of the first five statements of $A(\overline{S}, \overline{\eta})$.

Lemma 2.6 *The first five statements of* $A(\overline{S}, \overline{\eta})$ *imply that:*

1. *If* $\overline{\rho} \lhd \overline{\eta}$ *is avoidable, then* $\mathrm{dom}(\overline{\rho})$ *is* $\overline{S}$*-small.*
2. $I_0 \subseteq I(\overline{S})$.
3. *If* $\overline{\mu} \lhd \overline{\eta}$ *is strongly guessing, then* $\mathrm{dom}(\overline{\mu})$ *is* $\overline{S}$*-small.*
4. *Actually: If* $\overline{\mu}$ *is strongly guessing, then* $\mathrm{dom}(\overline{\mu})$ *is* $\overline{S}$*-small.*
5. $I_1 = I(\overline{S})$.

Proof. To prove *1*, assume $\overline{\rho} \lhd \overline{\eta}$ but $X = \mathrm{dom}(\overline{\rho})$ is not $\overline{S}$-small. Then (*A5*) implies that, for some $i < \omega_2$, $\chi(S_i) \lhd \overline{\rho}$. Hence $\overline{\rho}$ is not avoidable (by (*A3*) which says that $\chi(S_i)$ is (strongly) guessing).

We prove *2*. If $X \in I_0$ (X is avoidable) then any ladder system over X, and in particular $\overline{\eta} \restriction X$, is avoidable. Hence (by item *1*) $\mathrm{dom}(\overline{\eta} \restriction X)$ is $\overline{S}$-small. Thus X is $\overline{S}$-small (because $X = X_0 \cup X_1$ where $X_0 = X \cap \bigcup_i S_i$ and $X_1 = X \setminus X_0$. X_1 is clearly $\overline{S}$-small, and $X_0 = \mathrm{dom}(\overline{\eta} \restriction X)$).

To prove *3*, assume that $\mathrm{dom}(\overline{\mu})$ is not $\overline{S}$-small. Split $\overline{\mu}$ into $\overline{\mu}^1$ and $\overline{\mu}^2$, two "halves" defined by taking $(\mu^1)_\delta$ to be an infinite co-infinite subset of

μ_δ (for every $\delta \in \mathrm{dom}(\overline{\mu})$), and letting $\overline{\mu}^2 = \overline{\mu} \setminus \overline{\mu}^1$. If $X = \mathrm{dom}(\overline{\mu})$ is *not* $\overline{S}$-small, then, by (*A5*) applied to $\overline{\mu}^1$, there is i such that $S_i \subseteq X$ and

$$\chi(S_i) \lhd \overline{\mu}^1. \tag{2}$$

Since $\overline{\mu}$ is strongly guessing, $\overline{\mu}^2 \restriction S_i$ is strongly guessing (and non-trivial as its domain is the stationary set S_i), but formula (2) shows that $\overline{\mu}^2 \restriction S_i$ is disjoint from $\chi(S_i)$, and this contradicts the maximality of $\chi(S_i)$ for S_i.

To prove *4*, suppose that $\overline{\rho}$ is a strongly guessing ladder over X. To show that $X \in I(\overline{S})$, we reduce this claim to the case that $\overline{\rho} \lhd \overline{\eta}$. Look at $\overline{\rho} \setminus \overline{\eta}$ and its domain

$$X_1 = \{\delta \in X \mid [\rho_\delta] \setminus [\eta_\delta] \text{ is infinite}\}.$$

By (*A2*), $\overline{\rho} \setminus \overline{\eta}$ is avoidable. But, as $\overline{\rho}$ is strongly guessing, any subladder of $\overline{\rho}$ is also strongly guessing, and hence $\overline{\rho} \setminus \overline{\eta}$ is strongly guessing and avoidable, which could only be if X_1 is non-stationary.

Now set $X_2 = X \setminus X_1$, and $\overline{\mu} = \overline{\rho} \restriction X_2$. Then $\overline{\mu} \lhd \overline{\eta}$ is strongly guessing, and hence by the previous item $\mathrm{dom}(\overline{\mu})$ is $\overline{S}$-small.

Finally we prove *5*. If $X \in I_1$ then $X = X_0 \cup X_1$, where $X_0 \in I_0$ and X_1 is the domain of a strongly guessing ladder—namely $\chi(X)$. Hence $X \in I(\overline{S})$ by items *2* and *4*.

Suppose now that $X \in I(\overline{S})$. By definition, there is $\gamma < \omega_2$ such that, for $i \geq \gamma$, $X \cap X_i$ is non-stationary. Let $\langle T_j \mid j \in \omega_1 \rangle$ be an ω_1-enumeration of the collection $\{S_i \mid i < \gamma\}$. Then each $T_j \in I_1$ by (*A3*). Let $T = \nabla_{j \in \omega_1} T_j$ be the diagonal union. By normality of $I_1, T \in I_1$. Hence $X \cap T \in I_1$. But $X \setminus T$ has only countable intersections with each T_j (for in fact $(X \setminus T) \cap T_j \subseteq j+1$), and hence, certainly, has non-stationary intersections with *every* S_i, and is thus in I_0 (by (*A4*)). As $I_0 \subseteq I_1$ (by formula (1) in Definition 2.3), $X \in I_1$. ∎

We will prove next that if $A(\overline{S}, \overline{\eta})$ holds, then $\overline{\eta}$ is determined, up to an I_1 set, as that ladder $\overline{\eta}$ for which $(\exists \overline{S}) A(\overline{S}, \overline{\eta})$.

Lemma 2.7 *If the first five statements hold for $A(\overline{S}, \overline{\eta}^1)$ and $A(\overline{T}, \overline{\eta}^2)$, then $I(\overline{S}) = I(\overline{T}) = I_1$, and $\{\delta \in \omega_1 \mid [\eta^1_\delta] \neq^* [\eta^2_\delta]\} \in I_1$.*

Proof. Define $S^1 = \mathrm{dom}(\overline{\eta}^1 \setminus \overline{\eta}^2)$, and $S^2 = \mathrm{dom}(\overline{\eta}^2 \setminus \overline{\eta}^1)$. We claim that $S^1, S^2 \in I_1$. This implies the lemma because $S^1 \cup S^2 \in I_1$ follows. By symmetry, it suffices to deal with only one of these sets, for example with S^1.

Set $\overline{\rho} = \overline{\eta}^1 \setminus \overline{\eta}^2$ (so $S^1 = \mathrm{dom}(\overline{\rho})$). Since it is disjoint from $\overline{\eta}^2, \overline{\rho}$ is avoidable (by item (*A2*) of $A(\overline{T}, \overline{\eta}^2)$). Yet, $\overline{\rho} \lhd \overline{\eta}^1$, and so, by Lemma 2.6 (1), $\mathrm{dom}(\overline{\rho})$ is $\overline{S}$-small, which, in view of Lemma 2.6(*5*), implies that $S^1 \in I_1$. ∎

Whenever $A(\overline{S}, \overline{\eta})$ holds, a set of reals can be decoded which we denote $\mathrm{code}(\overline{S}, \overline{\eta})$. We will encode reals (subsets of ω) by taking subladders of $\overline{\eta}$ appropriately chosen. Suppose that σ is a cofinal subset of order-type ω of some $\delta < \omega_1$. Identifying σ with ω, any $\sigma' \subseteq \sigma$ corresponds to a subset of ω. This encoding of reals as subsets of σ is too crude, because if we take end segments of σ and σ' then a different real may be decoded. Since we shall be able to recover the ladder $\overline{\eta}$ only up to finite changes we must have a more stable decoding procedure. So we look for a function d that associates with every pair (σ', σ) as above some real $d(\sigma', \sigma)$ so that:

$$\text{If } \sigma_1 =^* \sigma_2 \text{ and } \sigma_1' =^* \sigma_2', \text{ then } d(\sigma_1', \sigma_1) = d(\sigma_2', \sigma_2).$$

The range of d should be all subsets of ω, i.e., for every σ for every $x \subseteq \omega$ there is $\sigma' \subseteq \sigma$ such that $d(\sigma', \sigma) = x$. It is not difficult to find such a function d, and we assume that the reader has picked one. (For example, you may look at the intervals of σ formed by successive members of σ' and take those cardinalities that appear infinitely often.)

Now let $\overline{\sigma}' \lhd \overline{\sigma}$ be two ladders; we say that $(\overline{\sigma}', \overline{\sigma})$ encodes the real $r \subseteq \omega$ if, for every $\delta \in \mathrm{dom}(\overline{\sigma}')$, $d([\sigma_\delta'], [\sigma_\delta]) = r$. We may just write $d(\overline{\sigma}', \overline{\sigma}) = r$ in such a case.

Not every pair $\overline{\sigma}' \lhd \overline{\sigma}$ encodes a real. An extreme case is when, for every $\delta_1 \neq \delta_2$ in $\mathrm{dom}(\overline{\sigma}')$, $d(\sigma_{\delta_1}', \sigma_{\delta_1}) \neq d(\sigma_{\delta_2}', \sigma_{\delta_2})$. We shall say in such a case that $(\overline{\sigma}', \overline{\sigma})$ are "clearly" not encoding.

Now we can understand the meaning of *A6*. If $A(\overline{S}, \overline{\eta})$ holds, we define

$$\mathrm{code}(\overline{S}, \overline{\eta}) = \{ r \subseteq \omega \mid r = d(\chi(S_i), \overline{\eta} \restriction S_i) \text{ for some } i \in \omega_2 \}.$$

Clearly if $r \in \mathrm{code}(\overline{S}, \overline{\eta})$, then $r = d(\chi(S_i), \overline{\eta} \restriction S_i)$ for an unbounded set of $i \in \omega_2$.

Lemma 2.8 *If* $A(\overline{S}, \overline{\eta}^1)$ *and* $A(\overline{T}, \overline{\eta}^2)$, *then* $\mathrm{code}(\overline{S}, \overline{\eta}^1) = \mathrm{code}(\overline{T}, \overline{\eta}^2)$.

Proof. Suppose that $r \in \mathrm{code}(\overline{S}, \overline{\eta}^1)$ and let $U \subset \omega_2$ be the unbounded set of indices i such that $r = d(\chi(S_i), \overline{\eta}^1 \restriction S_i)$. We must check that for some (and hence for unboundedly many) $j \in \omega_2$, $r = d(\chi(T_j), \overline{\eta}^2 \restriction T_j)$ We know that $[\overline{\eta}^1_\delta] =^* [\overline{\eta}^2_\delta]$ except for an I_1 set, and $I_1 = I(\overline{S}) = I(\overline{T})$ (Lemma 2.7). That is, if $H = \{\delta \in \omega_2 \mid [\overline{\eta}^1_\delta] \neq^* [\overline{\eta}^2_\delta]\}$, then $H \in I_1$, and hence $H \in I(\overline{S})$. Thus there is an index $i \in U$ such that

$$H \cap S_i \ \text{ is non-stationary.} \tag{3}$$

That is,

1. $\overline{\eta}^1 \restriction S_i =^* \overline{\eta}^2 \restriction S_i$ (that is, $[\overline{\eta}^1_\delta] =^* [\overline{\eta}^2_\delta]$ for all $\delta \in S_i$, except for a non-stationary set),
2. $d(\chi(S_i), \overline{\eta}^1 \restriction S_i) = r$.

Now $\chi(S_i)$ is maximal for S_i (a stationary set) and hence its domain S_i is not avoidable. So by *(A4)* of $A(\overline{T}, \overline{\eta}^2)$, for some $j \in \omega_2$, $X = S_i \cap T_j$ is stationary. Hence $\chi(S_i) \restriction X$ is maximally guessing (and non-trivial). Similarly $\chi(T_j) \restriction X$ is maximally guessing, and thus

$$\chi(S_i) \restriction X =^* \chi(T_j) \restriction X$$

by the uniqueness of the maximal ladder over X (namely $\chi(X)$). Since $X \cap H$ is non-stationary (by (3) above),

$$\overline{\eta}^1 \restriction X =^* \overline{\eta}^2 \restriction X$$

and thus $(\chi(T_j), \overline{\eta}^2 \restriction T_j)$ encodes a real, and $d(\chi(T_j), \overline{\eta}^2 \restriction T_j) = r$.

3 The consistency of $A(\overline{S}, \overline{\eta})$

Our aim in this section is to prove the following
Theorem. Assume that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Suppose that $\overline{T} = \langle T_i \mid i < \omega_2 \rangle$ is a collection of $\aleph_2$ pairwise almost disjoint stationary subsets of ω_1, and $\overline{\eta}$ is a ladder system such that

(1) $\overline{\eta} \restriction T_i$ is guessing (but not necessarily strongly guessing) for every $i < \omega_2$.

(2) range$(\overline{\eta}) \cap T_i$ is empty for every i.

Then there is a generic extension in which $A'(\overline{T}, \overline{\eta})$ and Martin's Axiom hold.

The extension is an iteration of the posets $R(\overline{\mu})$, and $P(\overline{\eta}, C)$ described below. Before proving this theorem, however, we review some notions from proper forcing theory.

3.1 Some proper forcing theory

This short subsection assembles some known definitions and results on proper forcing, such as α-properness and S-properness for a stationary set S. Our notations and terms are taken (with some minor changes) from Shelah's book [6] (see also [1]).

Recall that if P is a forcing poset and $N \prec H_\lambda$ a countable elementary substructure, then a condition $q \in P$ is N generic iff for every $D \in N$, dense in P, every extension of q is compatible with some condition in $D \cap N$. A forcing poset P is *proper* is for some cardinal λ, for every countable $N \prec H_\lambda$ such that $P \in N$, every $p \in P \cap N$ has an extension that is N generic.

Definition 3.1 (of α-properness.) *Let α be a countable ordinal. A poset P is said to be α-proper iff for every large enough cardinal λ,* **if** *$\langle N_i \mid i \leq \alpha \rangle$ is an increasing, continuous sequence of countable elementary submodels of H_λ such that $P \in N_0$ and $\langle N_j \mid j \leq i \rangle \in N_{i+1}$ for every $i < \alpha$,* **then** *any $p_0 \in P \cap N_0$ can be extended to $q \in P$ that is N_i-generic for every $i \leq \alpha$.*

Definition 3.2 *Let $S \subseteq \omega_1$ be stationary. A forcing poset P is S-proper if it is proper for structures M such that $M \cap \omega_1 \in S$. That is, P is S-proper iff for sufficiently large λ, if $M \prec H_\lambda$ is countable, $S, P \in M$, and $M \cap \omega_1 \in S$ then any $p \in P \cap M$ can be extended to an M-generic condition.*

A stronger property is that of a poset being S-complete. It means that whenever $M \prec H_\lambda$ is countable, with $P, S \in M$, and $M \cap \omega_1 \in S$, then every increasing and generic ω-sequence of conditions in $P \cap M$ has an upper bound in P. (A sequence of conditions is *generic* if it intersects every dense set of P in M.)

The notion (E, α)-properness is defined in Shelah ([6] (Chapter V). Just as properness is equivalent to the preservation of stationarity of $S_{\aleph_0}(\mu)$, so is (E, α)-properness equivalent to the preservation of an appropriate notion of stationarity defined there. However, for our article, a notion of somewhat less generality suffices.

Let I^ω be the collection of all increasing sequences of countable ordinals. We write $\overline{\alpha} = \langle \alpha_i \mid i < \omega \rangle$ for $\overline{\alpha} \in I^\omega$. The club guessing property can be regarded as a notion of non-triviality of subsets of I^ω.

Definition 3.3 *1. A family $E \subseteq I^\omega$ is* stationary *if for every club $C \subseteq \omega_1$ there is $\overline{\alpha} \in E$ such that $[\overline{\alpha}] = \{\alpha_i \mid i \in \omega\} \subset C$.*

2. *Let $E \subseteq I^\omega$ be stationary. We say that the poset P is E-proper (or (E, ω)-proper, to emphasize that this notion is related to ω-properness) iff for every sufficiently large cardinal λ, whenever $M_i \prec H_\lambda$, for $i < \omega$, are countable with $E, P \in M_0$ and are such that $M_i \in M_{i+1}$ for all $i < \omega$, if*

$$\langle M_i \cap \omega_1 \mid i < \omega \rangle \in E$$

then any $p \in P \cap M_0$ can be extended to a condition which is M_i-generic for every $i < \omega$.

In Shelah [6] it is proved that the countable support iteration of posets that are S-proper (α-proper or E-proper) is again S-proper (α-proper or E-proper, respectively). Also, if P is S-proper (E-proper), then, in V^P, S (respectively E) remains stationary.

Lemma 3.4 *If $E \subseteq I^\omega$ is stationary and P is an E-proper poset, then E remains stationary in V^P.*

Proof. Let D be a name in V^P forced by some $p \in P$ to be a club subset of ω_1. Define an ω_1 sequence $\langle M_i \mid i \in \omega_1 \rangle$ where $M_i \prec H_\lambda$ are countable with $\langle M_i \mid i \leq j \rangle \in M_{j+1}$, and such that $p, P, D \in M_0$. The set $C = \{M_i \cap \omega_1 \mid i \in \omega_1\rangle$ is closed unbounded in ω_1. Since E is stationary, there is $\overline{\alpha} \in E$ such that $\{\alpha_n \mid n \in \omega\} \subset C$. Then $\alpha_n = M_{i(n)} \cap \omega_1$ and $N_n = M_{i(n)}$ is an increasing sequence of structures with $N_n \in N_{n+1}$ and such that $\langle N_i \cap \omega_1 \mid i < \omega \rangle \in E$. So there is an extension $q \in P$ that is N_i-generic for every $i < \omega$. So for every i $q \Vdash \alpha_i \in D$. (Because q forces that D is unbounded below $\alpha_1 = \omega_1^{N_i}$.) Thus $q \Vdash [\overline{\alpha}] \subseteq D$, as required.

We shall define now two subsets of I^ω, $E_{\overline{\eta}}$ and $D_{\overline{\eta}}$, which will be used later.

Definition 3.5 1. *Let $\overline{\eta}$ be a ladder system and $S = \mathrm{dom}(\overline{\eta})$. Define $E_{\overline{\eta}} \subseteq I^\omega$ by*

$$\overline{\alpha} = \langle \alpha_i \mid i < \omega \rangle \in E_{\overline{\eta}}$$

iff

> *$\overline{\alpha} \in I^\omega$ and, for $\delta = sup\{\alpha_i \mid i < \omega\}$, $\delta \in S$ and $\overline{\alpha}$ is an end segment of η_δ (i.e., for some k, $\eta_\delta(k+i) = \alpha_i$ for all is).*

It is obvious that $E_{\overline{\eta}}$ is stationary iff $\overline{\eta}$ is club guessing. Thus, if $\overline{\eta}$ is club guessing and P is $E_{\overline{\eta}}$-proper, then $\overline{\eta}$ remains a guessing ladder in V^P.

2. *The set $D_{\overline{\mu}} \subseteq I^\omega$ (D is for disjoint) is defined for any ladder $\overline{\mu}$ as follows: $\overline{\alpha} \in D_{\overline{\mu}}$ iff for $\delta = sup\{\alpha_i \mid i < \omega\}$, either $\delta \notin \mathrm{dom}(\overline{\mu})$ or $[\overline{\alpha}] \cap [\mu_\delta] =^* \emptyset$. If $\overline{\mu}$ is disjoint from $\overline{\eta}$, then $E_{\overline{\eta}} \subseteq D_{\overline{\mu}}$. Thus, in this case, if P is $(D_{\overline{\mu}}, \omega)$-proper, then P is $(E_{\overline{\eta}}, \omega)$-proper as well.*

3.2 The building blocks

Two families of posets are described in this subsection: $R(\overline{\mu})$ and $P(\overline{\eta}, C)$.

The poset $R(\overline{\mu})$. Let $\overline{\mu}$ be a ladder over a set $S \subseteq \omega_1$. The poset $R(\overline{\mu})$ introduces a generic club to ω_1 that avoids $\overline{\mu}$. So, naturally,

$$c \in R(\overline{\mu})$$

iff

> $c \subseteq \omega_1$ is countable, closed (in particular $\max(c) \in c$), and for every $\delta \in S$, $[\mu_\delta] \cap c$ is finite.

The ordering on $R(\overline{\mu})$ is end-extension.

The cardinality of $R(\overline{\mu})$ is the continuum. It is clear that $R(\overline{\mu})$ is $\omega_1 \setminus S$ complete. A short argument is needed in order to prove that it is proper.

Observe first that for any condition $q \in R(\overline{\mu})$ and dense set $D \subseteq R(\overline{\mu})$, if $\alpha_0 = \max(q)$ then there is a closed unbounded set of ordinals $\gamma < \omega_1$, $\alpha_0 < \gamma$, such that for every α_1 with $\alpha_0 < \alpha_1 < \gamma$ there is an extension $q' \in D$ such that $q' \subset \gamma$ and $\alpha_1 \in q'$ is the successor of α_0 in q'. For example, the club set can be obtained by defining a continuous, increasing chain $\langle N_\alpha \mid \alpha \in \omega_1 \rangle$ of countable elementary substructures of some H_λ with $\overline{\mu}$ and the dense set D in N_0. Then $\langle \omega_1 \cap N_\alpha \mid \alpha \in \omega_1 \rangle$ is as required.

Suppose that a countable $M \prec H_\lambda$ and a condition $p_0 \in R(\overline{\mu}) \cap M$ are given. We want to define an increasing, generic sequence of conditions p_i extending p_0 so that for $\delta = M \cap \omega_1$, $p = \bigcup_{i\in\omega} p_i \cup \{\delta\}$ is a condition. The case $M \cap \omega_1 \notin S$ is trivial and so assume that $\delta = M \cap \omega_1 \in S$. The problem is that we may decide infinitely often to put $\mu_\delta(n)$ in $\bigcup_i p_i$, and then p is not a condition. The preliminary observation enables the construction of the sequence p_i in such a way that $p \cap [\mu_\delta] \subseteq p_0$ is finite. The point is that when we need to extend a condition p_i into a dense set D, we first consider the club set formulated above (do it in the substructure M) and find a limit ordinal γ in the club that is in M. Now $\alpha_1 < \gamma$ is chosen so that the interval $[\alpha_1, \gamma]$ is disjoint to $[\mu_\delta]$. (The fact that $[\mu_\delta]$ is only an ω sequence implies the existence of such an ordinal.

$R(\overline{\mu})$ is not ω-proper. For suppose $M_i,\ i < \omega$ is an increasing sequence of elementary submodels such that $\alpha_i = M_i \cap \omega_1 \in [\mu_\delta]$ for infinitely many i's, where $\delta = \sup\{\alpha_i \mid i < \omega\}$. Then no condition can be generic for all of the M_is. However, if $\delta \notin S$ or $\langle M_i \cap \omega_1 \mid i < \omega \rangle$ is disjoint from $[\mu_\delta]$ (or has only a finite intersection) then there is no problem in finding such a generic condition. That is, $R(\overline{\mu})$ is $(D_{\overline{\mu}}, \omega)$-proper. In fact, if p_i is any sequence of increasing conditions where $p_i \in M_i$ is M_{i-1} generic, then $\bigcup_i p_i$ gives a condition. This property is stronger than $(D_{\overline{\mu}}, \omega)$-properness, but in application we shall mix proper forcings with $R(\overline{\mu})$ forcings and hence the iteration itself is $(D_{\overline{\mu}}, \omega)$- proper.

Hence we have the following which will be used in Lemma 3.9.

Lemma 3.6 *Suppose that $\overline{\mu}$ is a ladder system and $A, B \subseteq \mathrm{dom}(\overline{\mu})$ are such that $\overline{\mu} \restriction A \cap B$ is not guessing. Then $R(\overline{\mu} \restriction A)$ is $E_{\overline{\mu} \restriction B}$-proper.*

Proof. Suppose that $A, B \subseteq \mathrm{dom}(\overline{\mu})$ are such that $\overline{\mu} \restriction A \cap B$ is not guessing. Let $C \subseteq \omega_1$ be a club set such that, for every $\delta \in A \cap B$, $[\mu_\delta] \not\subseteq^* C$. Suppose that $M_i \prec H_\lambda$ for $i < \omega$ are as in the definition of $E_{\overline{\mu} \restriction B}$ properness and $\delta = \sup(M_i \cap \omega_1 \mid i < \omega)$. So, $E_{\overline{\mu} \restriction B},\ R(\overline{\mu} \restriction A) \in M_0$. Hence $A, B \in M_0$ and thus $C \in M_0$ can be assumed. Then $M_i \cap \omega_1 \in C$ for every i. Since $\langle M_i \cap \omega_1 \mid i < \omega \rangle \in E_{\overline{\mu} \restriction B}$, $\delta \in B$ and $[\mu_\delta] =^* \{M_i \cap \omega_1 \mid i < \omega\}$. Thus $[\mu_\delta] \subseteq^* C$ and hence $\delta \notin A \cap B$. So $\delta \notin A$ and as $R(\overline{\mu} \restriction A)$ is $\omega_1 \setminus A$ complete, there is no problem in finding a condition that is M_i-generic for every i. ■

The poset $P(\overline{\eta}, c)$. Let $\overline{\eta}$ be a guessing ladder over a stationary co-stationary set S, such that

$$S \cap \ \mathrm{range}(\overline{\eta}) \text{ is non-stationary.}$$

(See Definition 2.1 for range($\overline{\eta}$). Then, for any club set $c \subseteq \omega_1$, the poset $P(\overline{\eta}, c)$ introduces a generic club set $D \subset \omega_1$, such that for every $\delta \in D \cap S$, $[\eta_\delta] \subset^* c$. This may be viewed as forcing a club subset to the stationary set $\{\delta \in S \mid [\eta_\delta] \subseteq^* c\} \cup (\omega_1 \setminus S)$.

Accordingly, we define $d \in P(\overline{\eta}, c)$ iff $d \subseteq \omega_1$ is countable, closed (with $\max(d) \in d$), and for every $\delta \in d \cap S$, $[\eta_\delta] \subseteq^* c$.

The order is end-extension.

It is easy to check that any condition has extensions to arbitrary heights (as there are no restrictions on $\omega_1 \setminus S$). The cardinality of $P(\overline{\eta}, c)$ is the continuum.

$P(\overline{\eta}, c)$ is not necessarily proper, because if, for $\delta = M \cap \omega_1$, $[\eta_\delta] \not\subseteq^* c$, then no M-generic condition can be found. Still, $P(\overline{\eta}, c)$ possesses two good properties which allows its usage:

1. $P(\overline{\eta}, c)$ is $(\omega_1 \setminus S)$-complete (the proof of this is obvious).

2. $P(\overline{\eta}, c)$ is $(E_{\overline{\eta}}, \omega)$-proper. ($E_{\overline{\eta}}$ is stationary since $\overline{\eta}$ is guessing.)

We check the second property — it is for its sake that the requirement that $\mathrm{dom}(\overline{\eta}) \cap \mathrm{range}(\overline{\eta})$ is non-stationary was made. So let $\langle M_i \mid i < \omega \rangle$ be an increasing sequence of countable elementary submodels of H_λ, with $M_i \in M_{i+1}$, and such that $P(\overline{\eta}, c), \overline{\eta}, c \in M_0$. Denote $\delta_i = M_i \cap \omega_1$, and $\delta = \sup\{\delta_i \mid i < \omega\}$.

The assumption is that $\langle \delta_i \mid i < \omega \rangle \in E_{\overline{\eta}}$, and the desired conclusion is that any $p_0 \in P \cap M_0$ can be extended to a condition that is generic for every M_i. So the assumption is that $\delta \in S$ and $\langle \delta_i \mid i < \omega \rangle$ is an end segment of η_δ. Since $\mathrm{dom}(\overline{\eta}) \cap \mathrm{range}(\overline{\eta})$ is non-stationary, $\delta_i \notin S$ (becauseM_0 contains a club that is disjoint from this intersection), and it is easy to find (in M_{i+1}) an M_i-generic condition extending any given condition (using the $(\omega_1 \setminus S)$-completeness). Thus, given $p_0 \in P(\overline{\eta}, c) \cap M_0$, we may construct an increasing sequence of conditions $p_i \in M_i$, such that p_{i+1} is M_i-generic. Then $p = \{\delta\} \cup \bigcup_{i<\omega} p_i$ is in $P(\overline{\eta}, c)$ because $[\eta_\delta] \subseteq^* c$ follows from the fact that $\delta_i \in c$ for every i (as $c \in M_i$).

As a warm-up we shall present some simple models obtained by countable support iteration of the posets $R(\overline{\mu})$ and $P(\overline{\eta}, c)$ just described. We assume $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ in the ground model.

1. A model in which $MA + 2^{\aleph_0} = \aleph_2 + \omega_1$ *is avoidable*. This is achieved by iterating *c.c.c* posets to obtain Martin's Axiom, and posets of the form $R(\overline{\mu})$ (varying over all possible ladders $\overline{\mu}$ over ω_1). Countable support

is used in this iteration of proper forcing posets, and hence the final poset is proper. The final poset satisfies the $\aleph_2$-chain condition (see [6], Chapter VIII, or [1]). The length of the iteration is ω_2 so that each possible *c.c.c* poset of size $\aleph_1$ and each ladder $\overline{\mu}$ are taken care of at some stage.

2. Given a guessing ladder $\overline{\eta}$ such that $\mathrm{dom}(\overline{\eta}) \cap \mathrm{range}(\overline{\eta}) = \emptyset$ a model of $MA + 2^{\aleph_0} = \aleph_2$ can be obtained in which $\overline{\eta}$ is strongly guessing. This time posets of type $P(\overline{\eta}, c)$ are iterated (varying club sets $c \subseteq \omega_1$) as well as *c.c.c.* posets. The iteration is with countable support and of length ω_2 as before. Put $S = \mathrm{dom}(\overline{\eta})$. Then S is stationary (as $\overline{\eta}$ is guessing) and co-stationary (as $S \cap \mathrm{range}(\overline{\eta}) = \emptyset$). Since each poset $P(\overline{\eta}, c)$ is $\omega_1 \setminus S$ complete, and each *c.c.c.* poset is obviously $\omega_1 \setminus S$ proper, we have here an iteration of $\omega_1 \setminus S$ proper posets. Thus the final poset itself is $\omega_1 \setminus S$ proper and ω_1 is not collapsed. Moreover, since the iterands (both $P(\overline{\eta}, c)$ and the *c.c.c.* posets) are $E_{\overline{\eta}}$ proper, the final iteration is $E_{\overline{\eta}}$ proper. Hence $\overline{\eta}$ remains guessing at each stage and in the final extension. It is strongly guessing since we took explicit steps to ensure this.

3. Now we want to combine 1 and 2. We are given a guessing ladder system $\overline{\eta}$ defined over a stationary co-stationary set T, such that $T \cap \mathrm{range}(\overline{\eta}) = \emptyset$, and we want a generic extension in which $\overline{\eta}$ is maximal for ω_1. For the iteration, decompose ω_2 into three sets $\omega_2 = J \cup K \cup I$ of cardinality $\aleph_2$ each. At stage $\alpha < \omega_2$ of the iteration, supposing that P_α has been defined, define the poset Q_α in V^{P_α} as follows:

   (a) If $\alpha \in J$, then Q_α is a *c.c.c* poset, and the iteration of all posets along J guarantees Martin's Axiom.

   (b) For $\alpha \in K$, Q_α will be of type $R(\overline{\mu})$ where $\overline{\mu} \in V^{P_\alpha}$ is a ladder system disjoint from $\overline{\eta}$. $R(\overline{\mu})$ is proper and it is $(D_{\overline{\mu}}, \omega)$ proper. Hence as $E_{\overline{\eta}} \subseteq D_{\overline{\mu}}$, $R(\overline{\mu})$ is $E_{\overline{\eta}}$ proper.

   (c) For $\alpha \in I$, Q_α will be of type $P(\overline{\eta}, c)$ where c is a club set in V^{P_α}. These posets are $\omega_2 \setminus T$ complete, and $E_{\overline{\eta}}$ proper.

Any of the posets along the iteration is either proper or $\omega_2 \setminus T$ proper (namely, the $P(\overline{\eta}, c)$ posets which are $\omega_2 \setminus T$ complete). So the iteration itself is $\omega_1 \setminus T$ proper, and thus ω_1 is not collapsed. Moreover, the posets are $E_{\overline{\eta}}$ proper, and hence $\overline{\eta}$ retains its guessing property in the extension.

3.3 The iteration scheme

Recall that our aim is to prove the following theorem.

Theorem 3.7 *Assume $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Suppose*

(1) A sequence $\overline{T} = \langle T_i \mid i \in \omega_2 \rangle$ of pairwise almost disjoint stationary subsets of ω_1. (Almost disjoint in the sense that $T_i \cap T_j$ is non-stationary.)

(2) A ladder system $\overline{\eta}$ such that

1. *$\bigcup\{T_i \mid i \in \omega_2\} \subseteq \mathrm{dom}(\overline{\eta})$, and $\omega_1 \setminus \mathrm{dom}(\overline{\eta})$ is stationary.*
2. *For every i, $\overline{\eta} \restriction T_i$ is club guessing.*
3. *$\mathrm{dom}(\overline{\eta}) \cap$ range$(\overline{\eta}) = \emptyset$.*

Then there is cofinality preserving generic extension in which $A'(\overline{T}, \overline{\eta})$ and $MA + 2^{\aleph_0} = \aleph_2$ hold. (The definition of $A'(\overline{T}, \overline{\eta})$ is immediately after Definition 2.5.)

Proof. It is not difficult to get $\overline{T}$ and $\overline{\eta}$ as in the theorem, and the following section contains a generic construction of such objects. Here we just assume their existence and prove the theorem. The generic extension is made via $P = P_{\omega_2}$, obtained as an iteration, $\langle P_\alpha \mid \alpha \leq \omega_2 \rangle$, with countable support of posets of cardinality $\aleph_1$. At successor stages, $P_{\alpha+1} \cong P_\alpha * Q_\alpha$, where $Q_\alpha \in V^{P_\alpha}$ is one of the following three types.

(1) A c.c.c. poset. (To finally obtain Martin's Axiom.)

(2) A $P(\overline{\sigma}, c)$ poset, where $\overline{\sigma} \in V^{P_\alpha}$ is a guessing ladder such that $\overline{\sigma} \lhd \overline{\eta}$, and $c \in V^{P_\alpha}$ is a club set. Recall that $P(\overline{\sigma}, c)$ introduces a generic club subset D such that $\delta \in D \cap \mathrm{dom}(\overline{\sigma})$ implies $\sigma_\delta \subseteq^* c$. We have checked that this poset is $(\omega_1 \setminus \mathrm{dom}(\overline{\sigma}))$-complete, and $(E_{\overline{\sigma}}, \omega)$-proper (as $\mathrm{dom}(\overline{\eta}) \cap \mathrm{range}(\overline{\eta}) = \emptyset$).

(3) The third type of iterated posets is $R(\overline{\mu})$ where $\overline{\mu} \in V^{P_\alpha}$ is a ladder system. This forcing makes $\overline{\mu}$ avoidable. We have seen that $R(\overline{\mu})$ is proper, $(D_{\overline{\mu}}, \omega)$-proper, and $(\omega_1 \setminus \mathrm{dom}(\overline{\mu}))$-complete.

Each iterated poset Q_α is $(\omega_1 \setminus \mathrm{dom}(\overline{\eta}))$ proper in V^{P_α}. Hence the iteration itself is $(\omega_1 \setminus \mathrm{dom}(\overline{\eta}))$ proper, and it satisfies the α_2-c.c.

We must specify how to choose the posets Q_α for the iteration. Every P_α will have cardinality $\leq \aleph_2$ and will satisfy the $\aleph_2$-c.c. When we say that a name in V^{P_α} satisfies property ϕ, we mean that it is forced by every condition in P_α to satisfy ϕ. We say that a V^{P_α} name of a subset of ω_1 is *standard* iff it associates with every $\beta \in \omega_1$ a maximal antichain of conditions that decide whether β is in this subset or not. Every subset of ω_1 in V^{P_α} has (an equivalent) standard name. For every poset P of size $\aleph_2$ that satisfies the $\aleph_2$-c.c., Fix an enumeration $\{E(P, \gamma) \mid \gamma < \omega_2\}$ of all standard names in V^P of subsets of ω_1 and of ladder systems. Thus any ladder or subset of ω_1 in V^{P_α} has a name of the form $E(P_\alpha, \gamma)$ for some $\gamma < \omega_2$. Fix a natural well-ordering of the pairs $\{\langle \alpha, \gamma \rangle \mid \alpha, \gamma < \omega_2\}$ that has order-type ω_2. So each $\langle \alpha, \gamma \rangle$ has its "place" in ω_2. This will serve in the choice of Q_α.

To define the iteration, we partition ω_2 (in V):

$$\omega_2 = J \cup K \cup L \cup \bigcup \{I_i \mid i < \omega_2\},$$

where each set in this partition has cardinality $\aleph_2$. The type of Q_α depends on the set in this partition that contains α.

For $\alpha \in J$, Q_α is a c.c.c. poset of cardinality $\aleph_1$, and the iteration of these posets in J shall provide Martin's Axiom. By now this is so standard that no further details will be given.

For $\alpha \in K$, Q_α will be of type $R(\overline{\mu})$, where $\overline{\mu} \in V^{P_\alpha}$ is a ladder system disjoint from $\overline{\eta}$ (namely, $\overline{\mu}$ is a name forced by every condition to be a ladder-system disjoint from $\overline{\eta}$). The final result of iterating these posets along K is that, every $\overline{\mu} \in V^P$ disjoint from $\overline{\eta}$ is avoidable in V^P. Thus property (*A2*) of $A'(\overline{T}, \overline{\eta})$ can be assured.

Before going on, let's discuss the problem involved in the direct approach to obtain (*A5*) and why we do not get $A(\overline{T}, \overline{\eta})$ but rather $A'(\overline{T}, \overline{\eta})$ (namely $A(\overline{S}, \overline{\eta})$ where $S_i \subseteq T_i$). A possible approach to (*A5*) is to consider each possible ladder $\overline{\rho} \lhd \overline{\eta}$ such that $X = \mathrm{dom}(\overline{\rho})$ is not $\overline{T}$-small, and to find for this $\overline{\rho}$ some $i \in \omega_2$ such that $X \cap T_i$ is stationary. Then, if possible, to transform $\overline{\rho} \restriction X \cap T_i$ into a maximally guessing ladder. For this to have any chance, it must be the case that $\overline{\rho} \restriction T_i$ is guessing. Yet it is possible that $\overline{\rho} \restriction T_i$ is non-guessing for every i. In this case we must shrink the T_i's so as to make X $\overline{S}$-small. This shows the need for defining subsets $S_i \subseteq T_i$. But now $(A4)$ causes a problem because, if $X \subseteq \omega_1$ is such that in V^P $X \cap S_i$ is non-stationary for every $i < \omega_2$, then we must be able to identify this X at some intermediary stage of the iteration so as to make $\overline{\eta} \restriction X$ avoidable. Yet, as the S_i are not yet all defined in any intermediate stage, it is not clear how to identify these X's.

We describe now in general terms how the sets I_i from the partition will be used in the iteration. For every $i < \omega_2$ let $\alpha(i)$ be the first ordinal in I_i. A stationary subset $S_i \subseteq T_i$ and a guessing ladder $\overline{\sigma}^i$ over S_i will be defined in $V^{P_{\alpha(i)}}$. The iteration of the posets Q_α for $\alpha \in I_i$ will make $\overline{\sigma}^i$ maximal for T_i, and will achieve $(A3)$ by establishing $\chi(T_i) = \overline{\sigma}^i$. Finally, in V^P, $A(\overline{S}, \overline{\eta})$ will hold for $\overline{S} = \langle S_i \mid i \in \omega_2 \rangle$.

To define S_i and $\overline{\sigma}^i$ we assume a function, ρ, which assigns to any α of the form $\alpha = \alpha(i)$ a name $\rho(\alpha) \in V^{P_\alpha}$ that is one of the following.

1. If i is an even ordinal then $\rho(\alpha(i))$ is a name of a real in V^{P_α}. The complete definition of ρ is given in the following section where it is used to define the encoding of the well-ordering of reals. Here we only assume that $\rho(\alpha)$ is defined.

2. If i is an odd ordinal, then $\rho(\alpha(i))$ is determined as a name of the ladder system $\rho(\alpha)$, defined as follows in V^{P_α}. With respect to the well-ordering of names in V^{P_α}, $\rho(\alpha)$ is the least ladder $\overline{\rho} \lhd \overline{\eta}$ that is not of the form $\rho(\alpha')$ for $\alpha' < \alpha$, and is such that for $D = \mathrm{dom}(\overline{\rho})$
$$\overline{\eta} \restriction D \cap T_i \text{ is guessing.}$$

Suppose that $\alpha = \alpha(i)$. Instead of defining the names S_i and Q_α directly in V^{P_α}, we let $G \subseteq P_\alpha$ be V-generic and we shall describe the interpretations of S_i and Q_α. We will later see (Lemma 3.9) that $\overline{\eta} \restriction T_i$ remains guessing in $V[G]$. In $V[G]$, collect all sets $X \subseteq \omega_1$ such that

1. a standard name of X appeared before α in the well-ordering of the names (i.e., for some $\alpha' \leq \alpha$ and $\gamma < \omega_2$, $\langle \alpha', \gamma \rangle$ is placed before α in the well-ordering of $\omega_2 \times \omega_2$, and $E(P_{\alpha'}, \gamma)$, the γth name in $V^{P_{\alpha'}}$, gives X), and

2. X is such that $\overline{\eta} \restriction (T_i \cap X)$ is not guessing (i.e., $T_i \cap X \in I_{\overline{\eta}}$).

Let $\langle X_\xi \mid \xi < \omega_1 \rangle$ be an enumeration of these sets. Take their diagonal union

$$A = \nabla_{\xi \in \omega_1} (X_\xi \cap T_i). \tag{4}$$

Then $A \in I_{\overline{\eta}}$. Since $\overline{\eta} \restriction T_i$ is guessing in $V[G]$, $T'_i = T_i \setminus A \notin I_{\overline{\eta}}$ (that is, $\overline{\eta} \restriction T'_i$ is guessing). (The reason for this specific definition of A and T'_i will only be apparent in the proof of item ($A5$) in V^P.)

Now $\rho(\alpha)$ is either a real or a ladder system in $V[G]$. Accordingly the definition of S_i and $\overline{\sigma}^i$ is split in two. Suppose that $\rho(\alpha)$ is a real $r \subseteq \omega$ in $V[G]$. We want to encode r. Define $S_i = T'_i$, and let $\overline{\sigma}^i \lhd \overline{\eta} \restriction S_i$ be a ladder system over S_i such that

$$d(\overline{\sigma}^i, \overline{\eta} \restriction S_i) = r.$$

Since $\overline{\eta} \restriction S_i$ is guessing and $\overline{\sigma}^i \lhd \overline{\eta} \restriction S_i$ has domain S_i, $\overline{\sigma}^i$ is also guessing.

Suppose next that $\alpha = \alpha(i)$ for i an odd ordinal and $\overline{\rho} = \rho(\alpha)$ is (in $V[G]$) a ladder over $X = \mathrm{dom}(\overline{\rho})$ (such that $\overline{\rho} \lhd \overline{\eta}$, and $\overline{\eta} \restriction X \cap T_i$ is guessing). Then $\overline{\eta} \restriction X \cap T'_i$ is guessing, because $T_i \setminus T'_i \in I_{\overline{\eta}}$. It follows that $\overline{\rho} \restriction X \cap T'_i$ is guessing as well, because $\overline{\rho} \lhd \overline{\eta}$ and $X \cap T'_i \subseteq \mathrm{dom}(\overline{\rho})$. In this case define

$$S_i = X \cap T'_i,$$

and define

$$\overline{\sigma}^i \lhd \overline{\rho} \restriction S_i \text{ so that } (\overline{\sigma}^i, \overline{\eta} \restriction S_i) \text{ is clearly not encoding.}$$

The iteration along I_i builds up the properties of $\overline{\sigma}^i$ and establishes $\chi(T_i) = \overline{\sigma}^i$ in V^P. For this, the posets Q_ξ, for $\xi \in I_i$, are of two types:

(1) $P(\overline{\sigma}^i, c)$, where c "runs" over all possible clubs. This ensures that $\overline{\sigma}^i$ becomes strongly club guessing in V^P. To enable the use of $P(\overline{\sigma}^i, c)$ we rely on the assumption, proved later to hold, that $\overline{\sigma}^i$ remains club guessing at each stage.

(2) $R(\overline{\mu})$, where $\overline{\mu}$ "runs" over all possible ladders over S_i that are disjoint from $\overline{\sigma}^i$. This ensures the maximality of $\overline{\sigma}^i$.

To satisfy item ($A4$) (in the definition of $A(\overline{S}, \overline{\eta})$), every X must be made avoidable whenever all the intersections $X \cap S_i$ are non-stationary. It suffices to show in such a case that the ladder $\overline{\eta} \restriction X$ is avoidable to conclude that X is avoidable, because any ladder disjoint from $\overline{\eta}$ is necessarily avoidable in V^P. It is the iteration along L that achieves this, by forcing with posets of type $R(\overline{\eta} \restriction X)$ as follows.

Given $\zeta \in L$ and a generic filter $G \subseteq P_\zeta$, we will define Q_ζ in $V[G]$. For $i < \omega_2$ such that $\alpha(i) < \zeta$, the sets S_i have been defined. For *every* $i < \omega_2$ define

$$S_i^* = \begin{cases} S_i & \text{if } \alpha(i) < \zeta \\ T_i & \text{otherwise} \end{cases}$$

Using the well-ordering of standard names, take the least set $X \subseteq \omega_1$ (if there is one) that was not taken before at a stage in L, such that

$$\forall i < \omega_2 \;\; \overline{\eta} \restriction X \cap S_i^* \text{ is not guessing.} \tag{5}$$

Then define Q_ζ to be $R(\overline{\eta} \restriction X)$ (or a trivial poset if no such X exists).

This ends the definition of the iteration, but it is not yet clear why items $(A4)$ and $(A5)$ hold in V^P. To prove $(A4)$ we shall first prove that if

$$(\forall i < \omega_2) X \cap S_i \text{ is non-stationary in } V^P,$$

then (5) holds at some stage V^{P_ξ}, $\xi \in L$, and hence $\overline{\eta} \restriction X$ is avoidable in the next step of the iteration. To see that this is indeed the case, we need the following pivotal observation.

Lemma 3.8 *Suppose $G \subseteq P_{\omega_2}$ is V-generic. If $\gamma < \omega_2$ and $\overline{\rho} \in V[G \restriction \gamma]$ is a ladder over X such that, in $V[G \restriction \gamma]$ $\overline{\rho} \lhd \overline{\eta}$ and*

$$|\{i \in \omega_2 \mid \overline{\eta} \restriction X \cap T_i \textit{ is guessing}\}| = \aleph_2. \tag{6}$$

Then there is i such that $S_i \subseteq X \cap T_i$ and $\overline{\sigma}^i \lhd \overline{\rho}$.

Proof. The proof of this lemma depends on the fact that for any i (with $\alpha(i) \geq \gamma$) such that $\overline{\eta} \restriction X \cap T_i$ is guessing in $V[G \restriction \gamma]$, $\overline{\eta} \restriction X \cap T_i$ remains guessing in $V[G \restriction \alpha(i)]$ as well. Thus, as the turn of $\overline{\rho}$ cannot be delayed ω_2 many times, at some stage $\alpha = \alpha(i)$, $\overline{\rho} = \rho(\alpha)$ holds, and then $S_i = X \cap T_i'$ and $\overline{\sigma}^i \lhd \overline{\rho} \restriction S_i$ were defined in $V[G \restriction \alpha]$. ∎

Now we can prove item $(A4)$ in $V[G]$. For this, let $X \subseteq \omega_1$ be such that $(\forall i < \omega_2)$ $X \cap S_i$ is non-stationary. We will show that, at some stage $\zeta \in L$, the poset $R(\overline{\eta} \restriction X)$ was taken as Q_ζ. If, for some $\gamma < \omega_2$, (6) of Lemma 3.8 holds in $V[G \restriction \gamma]$, then $S_i \subseteq X$ contradicts the fact that S_i is stationary. Hence formula (6) never holds, and for γ such that $X \in V[G \restriction \gamma]$ there are only boundedly many js for which $\overline{\eta} \restriction X \cap T_j$ is guessing. So let $\gamma < j_0 < \omega_2$ be such that if $\overline{\eta} \restriction X \cap T_j$ is guessing, then $j < j_0$. Since, in $V[G]$, $X \cap S_j$ is non-stationary for every $j < \omega_2$, there is a stage j_1 such that for every $j < j_0$, $X \cap S_j$ is non-stationary in $V[G \restriction j_1]$. Thus, for $\zeta \geq j_1$, in $V[G \restriction \zeta]$, for every $i < \omega_2$, if $i < j_0$ then S_i is defined (that is, $\alpha(i) \leq \zeta$) and $X \cap S_i$ is non-stationary, and hence $\overline{\eta} \restriction X \cap S_i$ is non-guessing, and if $i \geq j_0$, then $\overline{\eta} \restriction X \cap T_i$ is non-guessing. But this is exactly the condition required at stages $\zeta \in L$ to force with $R(\overline{\eta} \restriction X)$.

Finally, we turn to prove item $(A5)$. So let $\overline{\rho} \lhd \overline{\eta}$ with $X = \mathrm{dom}(\overline{\rho})$ be given in the generic extension $V[G]$. Then for some $\gamma < \omega_2$, $\overline{\rho} \in V[G \restriction \gamma]$, and a name of X appeared before γ in the well-ordering of names.

Case 1: In $V[G \restriction \gamma]$: There is $i_0 < \omega_2$ such that, for every $i \geq i_0$, $\alpha(i) > \gamma$ and

$$\overline{\eta} \restriction X \cap T_i \text{ is not guessing.}$$

Then, in defining S_i for $i \geq i_0$, $\alpha(i) > \gamma$, and the set X appears as some X_ξ (in equation (4)), and hence $X \cap S_i$ is at most countable (it is included in $\xi + 1$). Thus, in Case 1, $X \cap S_i$ is non-stationary (and even countable) for a co-bounded set of indices. That is, X is $\overline{S}$-small. (It is for this argument that, in defining S_i, we asked $S_i \cap A = \emptyset$)

Case 2: Not Case 1. Hence (6) holds in $V[G \restriction \gamma]$. So, by Lemma 3.8 there is i such that $S_i \subseteq X \cap T_i$ and $\overline{\sigma}^i \lhd \overline{\rho}$, which establishes $(A5)$. ∎

Our proof relied on preservation claims that some ladders retain their guessing property, and we intend now to prove these claims. First, set $T = \cup T_i$. Then $\omega_1 \setminus T$ is stationary by assumption, and all the posets used are $(\omega_1 \setminus T)$-proper. (The c.c.c. posets are certainly proper. The $P(\overline{\sigma}, c)$ posets (defined for $\overline{\sigma} \lhd \overline{\eta}$) are $(\omega_1 \setminus \mathrm{dom}(\overline{\sigma}))$-complete, and hence $(\omega_1 \setminus T)$-proper. The $R(\overline{\mu})$ posets are proper.) This secures the preservation of $\aleph_1$.

Lemma 3.9 *$\overline{\eta} \restriction T_i$ remains guessing in V^{P_α} for $\alpha = \alpha(i)$.*

Proof. This follows from the fact that the posets iterated at stages $\zeta < \alpha$ are all $(E_{\overline{\eta} \restriction T_i}, \omega)$-proper:

1. The c.c.c. posets are always ω-proper.
2. The $R(\overline{\mu})$ posets iterated at stages in K are defined for $\overline{\mu}$'s that are disjoint from $\overline{\eta}$. In such a case $E_{\overline{\eta}} \subseteq D_{\overline{\mu}}$. But we remarked that $R(\overline{\mu})$ is $(D_{\overline{\mu}}, \omega)$-proper.
3. The $P(\overline{\sigma}, c)$ posets introduced for $\zeta < \alpha$ are defined along I_j only for js such that $\alpha(j) < \alpha$, and thence for $\overline{\sigma}$'s such that $\overline{\sigma} \lhd \overline{\eta} \restriction T_j$, implying the $(E_{\overline{\eta}\restriction T_i}, \omega)$-properness. (Since $T_j \cap T_i$ is non-stationary, and range$(\overline{\eta}) \cap T_j = \emptyset$.)
4. The $R(\overline{\mu})$ posets defined along I_j for $\alpha(j) < \alpha$ are defined for ladders $\overline{\mu}$ over T_j. As T_j is almost disjoint from T_i, these $R(\overline{\mu})$'s are $(E_{\overline{\eta}\restriction T_i}, \omega)$-proper.
5. The $R(\overline{\eta} \restriction X)$ posets defined for $\zeta \in L$, $\zeta < \alpha$, are such that $\overline{\eta} \restriction X \cap T_i$ is non-guessing and the poset is thence $(E_{\overline{\eta}\restriction T_i}, \omega)$-proper (by Lemma 3.6).

Then, we must also show that the guessing ladder $\overline{\sigma}^i \lhd \overline{\eta} \restriction S_i$ defined in $V^{P_{\alpha(i)}}$ remains guessing at every stage in I_i (and thus the posets $P(\overline{\sigma}^i, c)$ can be applied). This is basically the same proof, done in $V^{P_{\alpha(i)}}$ for the quotient poset $P/P_{\alpha(i)}$ which is again a countable support iteration of posets as above that are $E_{\overline{\sigma}^i}$ proper.

4 The $\Sigma^{2[\aleph_1]}$ well-ordering

The main theorem, Theorem 1.1, is proved in this section. So $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ are assumed in the ground model V. We need a sequence $\overline{T}$ of pairwise almost disjoint stationary sets, and a guessing ladder system $\overline{\eta}$; and we are going to define them first.

Since we want to describe the $\aleph_2$ stationary sets in the language $\Sigma^{2[\aleph_1]}$, we need a compact form of generation for such sets. This is provided by the following definition.

Definition 4.1 *Let* ${}^{\leq\alpha}\{0,1\}$ *denote the set of all functions* $f : \beta \to \{0,1\}$ *for* $\beta \leq \alpha$. *Ordered by function extension, this forms a tree. Define* ${}^{<\alpha}\{0,1\}$ *similarly.*

A stationarity tree *is a subtree* $T \subseteq {}^{<\omega_1}\{0,1\}$ *of cardinality* $\aleph_1$ *such that:*

1. *If $f, g \in T$ and α are such that $f \restriction \alpha = g \restriction \alpha$ but $f(\alpha) \neq g(\alpha)$, then $f^{-1}\{1\} \cap g^{-1}\{1\} \subseteq \alpha$.*

2. *T has $\aleph_2$ branches of length ω_1 and each gives a stationary set (that is, the union of the nodes along any ω_1-branch forms a function f and $f^{-1}\{1\}$ is a stationary subset of ω_1). It follows from item 1 that the intersection of any pair of these stationary sets is countable. Thus the branches of T give $\aleph_2$ pairwise disjoint stationary sets enumerated as T_i for $i < \omega_2$.*

3. *We also require that $U = \cup_{i<\omega_2} T_i$ is a co-stationary set.*

The poset S, defined below, will produce a stationarity tree by forcing.

Conditions in S will be countable trees, together with countable information on the family of branches. Define $p \in S$ iff $p = (T_p, f_p)$ where:

1. For some countable ordinal β (called the "height" of p) $T_p \subseteq^{\leq \beta} \{0, 1\}$ is a countable tree of functions ordered by inclusion, and satisfying property *1*, and such that $T_p \cap {}^{\beta}\{0, 1\} \neq \emptyset$.

2. f_p is a countable (partial) map defined on ω_2 that assigns to ζ in its domain a node $f_p(\zeta) \in T_p \cap {}^{\beta}\{0, 1\}$. ($\mathrm{dom}(f_p)$ is called the "domain" of p.)

The extension relation $p_2 \geq p_1$ on S is defined by requiring that $T_{p_1} = T_{p_2} \cap {}^{\leq \alpha_1}\{0, 1\}$ where $\alpha_1 =$ height (T_{p_1}), and that $f_{p_2}(\zeta)$ extends $f_{p_1}(\zeta)$ for every $\zeta \in \mathrm{dom}(f_{p_1})$.

If $p_1, p_2 \in S$ are such that $T_{p_1} = T_{p_2}$, and f_{p_1} agrees with f_{p_2} on the intersection of the domains, then p_1 and p_2 are compatible. Hence, CH implies the $\aleph_2$-c.c. for S.

It is not difficult to prove that every condition has arbitrarily high extensions, and that for every $\xi \in \omega_2$ the set of conditions p with $\xi \in \mathrm{dom}(f_p)$ is dense in S. Clearly, S is countably closed. If $\langle p_n \mid n \in \omega \rangle$ is an increasing sequence of conditions, let $p = \sup\{p_n \mid n < \omega\}$ be defined as follows. $T_p = \bigcup\{T_{p_n} \mid n \in \omega\} \cup \{f_p(\xi) \mid \xi \in \mathrm{dom}(f_{p_n})$ for some $n \in \omega\}$ where $f_p(\xi) = \bigcup\{f_{p_n}(\xi) \mid n \in \omega \ \& \ \xi \in \mathrm{dom}(p_n)\}$. That is, if α is the height of p, then $T_p \cap {}^{\alpha}\{0, 1\}$ consists only of the functions $f_p(\xi)$ for $\xi \in \mathrm{dom}(p)$. If $G \subseteq S$ is V-generic, define $T = T(G) = \bigcup\{T_p \mid p \in G\}$. Then T is a stationarity tree. Define $f(\zeta) = \bigcup\{f_p(\zeta) \mid p \in G\}$. Then $f(\zeta)$ is an ω_1-branch of T, and for $\zeta_1 \neq \zeta_2$ $f(\zeta_1) \neq f(\zeta_2)$. Thus T has $\aleph_2$ many ω_1-branches. The fact that every $f(\xi)$ gives a stationary set requires a simple density argument. We will check now the following:

Claim 4.2 *Any ω_1-branch of T in $V[G]$ is some $f(\zeta)$.*

Proof: Suppose, toward a contradiction, that p forces that τ is a branch of T which is not $f(\zeta)$ for any $\zeta \in \omega_2$. Observe first that since S is σ-closed, any condition p can be extended to a condition q that describes τ up to height(p). Then, every condition p and $\xi \in \mathrm{dom}(p)$ have an extension q such that $f_q(\zeta)$ diverges from the value of τ determined by q. Repeating this procedure ω^2 times, we finally get an extension q of p with $\delta = \mathrm{height}(q)$ limit, and such that q determines τ as a branch of T_q of height δ which is different from each of the branches $f_q(\zeta)$. Since $T_q \cap {}^{\delta}\{0,1\}$ consists only of the branches of the form $f_q(\xi)$, the branch of τ is not in T_q. Then q forces $\tau \subseteq T \restriction \delta$. ∎

We denote with $\overline{T} = \langle T_i \mid i < \omega_2 \rangle$ the collection of stationary sets thus obtained from the branches $f(\zeta)$ of T. Let $U = \cup_{i<\omega_2} T_i$.

It is not difficult to show that $\omega_1 \setminus U$ is also stationary. If C is a name of a closed unbounded subset of ω_1, find a countable $M \prec H_\lambda$ with $C \in M$, and define an M-generic condition p that puts $\delta = M \cap \omega_1$ in $\omega_1 \setminus U$.

Next, we obtain a ladder system $\overline{\eta}$ over U such that range$(\overline{\eta}) \cap U = \emptyset$, and $\overline{\eta} \restriction T_i$ is guessing for every i. It is possible to get this $\overline{\eta}$ by forcing with the natural (countable) conditions. This forcing notion is countably closed, and, assuming CH, it has cardinality $\aleph_1$.

Now comes the main stage of the iteration.

Using the construction of the previous section we obtain an extension in which $A'(\overline{T}, \overline{\eta}) + MA + 2^{\aleph_0} = \aleph_2$ hold, and such that for every $i < \omega_2$ either $(\overline{\sigma}_i, \overline{\eta})$ is clearly not encoding (where $\overline{\sigma}_i = \chi(T_i)$ is the maximal ladder for T_i), or else it encodes a real r_i, and in that case $r_i = r_j$ for $\aleph_2$ many js (any encoded real is encoded unboundedly often). The set of encoded reals, $\mathcal{E} = \{r_i \mid i < \omega_2\}$, is (in some natural encoding of pairs) our well-ordering of the reals. We must prove that this well-ordering is $\Sigma^{2[\aleph_1]}$. After the extension, Claim 4.2 may no longer be true because new branches were added to T. However, the stationary sets T_i are $\Sigma^{2[\aleph_1]}$ definable. They are exactly those stationary sets X obtained from a branch of T and such that X is not avoidable (any ω_1-branch of T that is not one of the original $f(\xi)$

branches is almost disjoint to any original branch and hence by (*A4*) its stationary set is avoidable).

We describe the $\Sigma^{2[\aleph_1]}$ formula $\psi(x)$ that decodes this well-ordering ($\psi(x)$ iff $x \in \mathcal{E}$). First consider the $\Sigma^{2[\aleph_1]}$ formula $\varphi(T_0, \overline{\eta}_0)$(with class variables T_0 and $\overline{\eta}_0$) which says that T_0 is a stationarity tree, and $\overline{\eta}_0$ is a ladder system such that $A'(\overline{T}_0, \overline{\eta}_0)$ holds (where $\overline{T}_0$ is the collection of non-avoidable stationary sets derived from the branches of T_0). (The statement "there are $\aleph_2$ indices such that..." can be expressed by saying "there is no $\aleph_1$-class containing all the indices such that...").

This enables a $\Sigma^{2[\aleph_1]}$ rendering of the formula $x \in \mathcal{E}$:

$$\begin{aligned} \psi(x) \quad \equiv \quad & \text{there exists } T^0 \text{ and } \overline{\eta}^0 \text{ such that } \varphi(T^0, \overline{\eta}^0) \text{ and} \\ & x \in \mathrm{code}(S^0, \overline{\eta}^0) \end{aligned}$$

Clearly, $\psi(x)$ holds for every $x \in \mathcal{E}$ (by virtue of the "real" T and $\overline{\eta}$), and we must also prove that if $\psi(x)$ then $x \in \mathcal{E}$. But this follow from Lemma 2.8.

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