ADDING ONE RANDOM REAL

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Abstract. We study the cardinal invariants of measure and category after adding one random real. In particular, we show that the number of measure zero subsets of the plane which are necessary to cover graphs of all continuous functions maybe large while the covering for measure is small.

1. Introduction

Let $\mathcal{J}$ be an ideal of subsets of the real line (where real line means $\mathbb{R}$, 2^ω or $[0,1]$). Define the following cardinal invariants:

(1) $\mathsf{add}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \ \& \ \bigcup \mathcal{A} \notin \mathcal{J}\}$,
(2) $\mathsf{cov}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \ \& \ \bigcup \mathcal{A} = \mathbb{R}\}$,
(3) $\mathsf{non}(\mathcal{J}) = \min\{|X| : X \subseteq \mathbb{R} \ \& \ \mathbb{X} \notin \mathcal{J}\}$,
(4) $\mathsf{cof}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \& \ \forall A \in \mathcal{J} \ \exists B \in \mathcal{A} \ A \subseteq B\}$.

Let $\mathcal{M}$ and $\mathcal{N}$ be the ideals of meager and of measure zero subsets of the real line respectively. Finally let $\mathfrak{b}$ be the size of the smallest unbounded family in ω^ω and $\mathfrak{d}$ the size of the smallest dominating family in ω^ω .

The relationship between these cardinals is described in the following diagram, where arrows means $\leq$:

$$\begin{array}{ccccccc}
\mathsf{cov}(\mathcal{N}) & \to & \mathsf{non}(\mathcal{M}) & \to & \mathsf{cof}(\mathcal{M}) & \to & \mathsf{cof}(\mathcal{N}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \mathfrak{b} & \to & \mathfrak{d} & & \\
 & & \uparrow & & \uparrow & & \\
\mathsf{add}(\mathcal{N}) & \to & \mathsf{add}(\mathcal{M}) & \to & \mathsf{cov}(\mathcal{M}) & \to & \mathsf{non}(\mathcal{N})
\end{array}$$

In addition $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$ and $\mathsf{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathsf{non}(\mathcal{M})\}$.

The proofs of those inequalities can be found in [1], [4] and [6]. In this paper we show that except for $\mathsf{cov}(\mathcal{N})$ and $\mathsf{non}(\mathcal{N})$ values of these invariants do not change when one random real is added. Let $\mathbf{B}$ be the measure algebra adding one random real.

Theorem 1.1 (Pawlikowski, Krawczyk)**.** *The following holds in* $\mathbf{V}^{\mathbf{B}}$*:*

(1) $\mathsf{add}(\mathcal{N}) = \mathsf{add}(\mathcal{N})^{\mathbf{V}}$ *and* $\mathsf{cof}(\mathcal{N}) = \mathsf{cof}(\mathcal{N})^{\mathbf{V}}$,
(2) $\mathsf{cov}(\mathcal{N}) \geq \mathsf{cov}(\mathcal{N})^{\mathbf{V}}$ *and* $\mathsf{non}(\mathcal{N}) \leq \mathsf{non}(\mathcal{N})^{\mathbf{V}}$,
(3) $\mathsf{cov}(\mathcal{N}) \geq \mathfrak{b}$ *and* $\mathsf{non}(\mathcal{N}) \leq \mathfrak{d}$,
(4) $\mathfrak{b} = \mathfrak{b}^{\mathbf{V}}$ *and* $\mathfrak{d} = \mathfrak{d}^{\mathbf{V}}$,
(5) $\mathsf{cov}(\mathcal{M}) \geq \mathsf{cov}(\mathcal{M})^{\mathbf{V}}$ *and* $\mathsf{non}(\mathcal{M}) \leq \mathsf{non}(\mathcal{M})^{\mathbf{V}}$,

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(6) $\mathsf{add}(\mathcal{M}) \geq \mathsf{add}(\mathcal{M})^{\mathbf{V}}$ *and* $\mathsf{cof}(\mathcal{M}) \leq \mathsf{cof}(\mathcal{M})^{\mathbf{V}}$*.*

Proof (1), (2) and (4) is folklore (see [7]). (3) is due to Krawczyk (see [7] or [5]), (5) is due to Pawlikowski ([7]) and (6) follows from (5), (4) and the remarks above. $\square$

For a set $H \subseteq \mathbb{R} \times \mathbb{R}$ and $x, y \in \mathbb{R}$ let $(H)_x = \{y : \langle x, y\rangle \in H$ and let $(H)^y = \{x : \langle x, y\rangle \in H\}$.

We will use the following classical lemma:

Lemma 1.2. *Suppose that r is a random real over $\mathbf{V}$. Then*

1. *for every $x \in \mathbf{V}[r] \cap \mathbb{R}$ there exists a Borel function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $x = f(r)$,*
2. *for every Borel measure zero set $F \in \mathbf{V}[r]$ there exists a Borel measure zero set $H \subseteq \mathbb{R} \times \mathbb{R}$, $H \in \mathbf{V}$ such that $F = (H)_r$.* $\square$

We will need the following characterization of $\mathsf{cov}(\mathcal{M})$ and $\mathsf{non}(\mathcal{M})$.

Let

$$\mathcal{S} = \{\mathcal{S} \in ([\omega]^{<\omega})^\omega : \forall \backslash\ |\mathcal{S}(\backslash)| \leq (\backslash + \infty)^{\in}\}.$$

Theorem 1.3 ([2])**.** *The following conditions are equivalent:*

1. $\mathsf{cov}(\mathcal{M}) \geq \kappa$*,*
2. *for every family $F \subseteq \omega^\omega$ of size $< \kappa$ there exists $g \in \omega^\omega$ such that*
$$\forall f \in F\ \exists^\infty n\ f(n) = g(n).$$
3. *for every family $F \subseteq \omega^\omega$ of size $< \kappa$ there exists $S \in \mathcal{S}$ such that*
$$\forall f \in F\ \exists^\infty n\ f(n) \in S(n).$$

Similarly,

1. $\mathsf{non}(\mathcal{M}) \geq \kappa$*,*
2. *for every family $F \subseteq \omega^\omega$ of size $< \kappa$ there exists $g \in \omega^\omega$ such that*
$$\forall f \in F\ \forall^\infty n\ f(n) \neq g(n).$$
3. *for every family $F \subseteq \mathcal{S}$ of size $< \kappa$ there exists $f \in \omega^\omega$ such that*
$$\forall S \in F\ \forall^\infty n\ f(n) \notin S(n). \quad \square$$

2. Cohen reals

In this section we will show that invariants $\mathsf{cov}(\mathcal{M})$ and $\mathsf{non}(\mathcal{M})$ do not change when random reals are added.

Theorem 2.1. *The following holds in $\mathbf{V}^{\mathbf{B}}$:*

1. $\mathsf{cov}(\mathcal{M}) = \mathsf{cov}(\mathcal{M})^{\mathbf{V}}$ *and* $\mathsf{non}(\mathcal{M}) = \mathsf{non}(\mathcal{M})^{\mathbf{V}}$*,*
2. $\mathsf{add}(\mathcal{M}) = \mathsf{add}(\mathcal{M})^{\mathbf{V}}$ *and* $\mathsf{cof}(\mathcal{M}) = \mathsf{cof}(\mathcal{M})^{\mathbf{V}}$*.*

Proof (1) By 1.1, it is enough to show that in $\mathbf{V}^{\mathbf{B}}$, $\mathsf{cov}(\mathcal{M}) \leq \mathsf{cov}(\mathcal{M})^{\mathbf{V}}$ and $\mathsf{non}(\mathcal{M}) \geq \mathsf{non}(\mathcal{M})^{\mathbf{V}}$.

By 1.3, there exists a family $F \subseteq \omega^\omega$ of size $\mathsf{cov}(\mathcal{M})^{\mathbf{V}}$ such that

$$\forall S \in \mathcal{S}\ \exists f \in F\ \forall^\infty n\ f(n) \notin S(n).$$

By 1.3, to finish the proof it is enough to show that

$$\mathbf{V}^{\mathbf{B}} \models \forall g \in \omega^\omega\ \exists f \in F\ \forall^\infty n\ f(n) \neq g(n).$$

Let $\dot{g}$ be a $\mathbf{B}$-name for an element of ω^ω. Define for $n \in \omega$,

$$S(n) = \left\{ k \in \omega : \mu\left(\llbracket \dot{g}(n) = k \rrbracket_{\mathbf{B}}\right) > \frac{1}{(n+1)^2} \right\},$$

where μ is the Lebesgue measure. It is clear that $|S(n)| < (n+1)^2$ for all n. Therefore there exists $f \in F$ and $N \in \omega$ such that $f(n) \notin S(n)$ for all $n \geq N$. We claim that

$$\Vdash_{\mathbf{B}} \forall^\infty n \ f(n) \neq \dot{g}(n).$$

Let $p \in \mathbf{B}$. Find $n > N$ such that $\sum_{k=n}^{\infty} k^{-2} < \mu(p)$. Then

$$q = p - \bigcup_{k=n}^{\infty} \llbracket \dot{g}(k) = f(k) \rrbracket_{\mathbf{B}} > 0$$

and

$$q \Vdash_{\mathbf{B}} \forall k > n \ f(k) \neq \dot{g}(k).$$

To show that $\mathsf{non}(\mathcal{M}) \geq \mathsf{non}(\mathcal{M})^{\mathbf{V}}$ holds in $\mathbf{V}^{\mathbf{B}}$, we "dualize" the above argument.

Suppose that $F \subseteq \omega^\omega$ is a family of size $\mathsf{non}(\mathcal{M})$ in $\mathbf{V}^{\mathbf{B}}$ such that

$$\mathbf{V}^{\mathbf{B}} \models \forall g \in \omega^\omega \ \exists f \in F \ \exists^\infty n \ f(n) = g(n).$$

Let $\dot{F} = \{\dot{f} : f \in F\}$ be a set of $\mathbf{B}$-names for elements of F. Without loss of generality we can assume that $\dot{F} \in \mathbf{V}$. For $\dot{f} \in \dot{F}$ let $S_f \in \mathcal{S}$ be defined as

$$S_f(n) = \left\{ k \in \omega : \mu\left(\llbracket \dot{f}(n) = k \rrbracket_{\mathbf{B}}\right) > \frac{1}{(n+1)^2} \right\}.$$

As before we show that

$$\forall g \in \omega^\omega \ \exists \dot{f} \in \dot{F} \ \exists^\infty n \ g(n) \in S_f(n),$$

which by 1.1, finishes the proof.

To show the second part use 1.1 and the fact that $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$ and $\mathsf{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathsf{non}(\mathcal{M})\}$. $\square$

Recall that a set $X \subseteq \mathbb{R}$ has strong measure zero if for every sequence of positive reals $\langle \varepsilon_n : n \in \omega \rangle$ there exists a sequence of intervals $\langle I_n : n \in \omega \rangle$ such that the length of I_n is $\leq \varepsilon_n$ and $X \subseteq \bigcup_m I_m$. Note that, equivalently we can request that $X \subseteq \bigcap_{n \in \omega} \bigcup_{m > n} I_m$.

Theorem 2.2. *Suppose that $X \subseteq \mathbb{R}$ and $X \in \mathbf{V}$. Then X has strong measure zero in $\mathbf{V}$ iff X has strong measure zero in $\mathbf{V}^{\mathbf{B}}$.*

Proof It is easy to see that for every sequence $\langle \varepsilon_n : n \in \omega \rangle \in \mathbf{V}^{\mathbf{B}}$ there exists a sequence $\langle \delta_n : n \in \omega \rangle \in \mathbf{V}$ such that $\delta_n \leq \varepsilon_n$ for all n. Therefore, if X has strong measure zero in $\mathbf{V}$ then X has strong measure zero in $\mathbf{V}^{\mathbf{B}}$.

Suppose that X does not have strong measure zero in $\mathbf{V}$ and let $\langle \varepsilon_n : n \in \omega \rangle$ be a sequence of positive reals witnessing that. Suppose that X has strong measure zero in $\mathbf{V}^{\mathbf{B}}$. Let $\langle \delta_n : n \in \omega \rangle$ be a decreasing sequence of positive reals such that $\delta_n < \varepsilon_k$ for all $k \leq n^3$. Let $\langle \delta'_n : n \in \omega \rangle$ be a decreasing sequence of positive rationals such that $\delta'_{2k} = \delta'_{2k+1}$ and $\delta'_n < \delta_n$. By the assumption we can find a sequence of intervals $\langle I_n : n \in \omega \rangle \in \mathbf{V}^{\mathbf{B}}$ such that $X \subseteq \bigcap_{n \in \omega} \bigcup_{m > n} I_m$ and the length of I_m is less than δ'_{2m}. Let $\langle I(n,k) : k \in \omega \rangle$ be a partition of $\mathbb{R}$ into rational intervals of the length δ'_n. Each interval I_m is covered by $I(2m,k) \cup I(2m,k+1)$ for some $k = k(m)$. Let

$\langle \dot{I}_n : n \in \omega \rangle$ be a $\mathbf{B}$-name for the sequence $\langle I(2m, k(m)), I(2m, k(m)+1) : m \in \omega \rangle$ (i.e. $\dot{I}_{2m}$ is a name for $I(2m, k(m))$ and $\dot{I}_{2m+1}$ is that for $I(2m, k(m)+1)$). Thus

$$\Vdash_{\mathbf{B}} \text{"the length of } \dot{I}_n \text{ is } \delta'_n \ \& \ X \subseteq \bigcap_{n \in \omega} \bigcup_{m > n} \dot{I}_m\text{"}.$$

Now define for $n = 2m + i$ $(i = 0, 1)$:

$$\mathcal{A}_n = \left\{ I(2m, k) : \mu\left([\![\dot{I}_n = I(2m, k)]\!]_{\mathbf{B}} \right) > \frac{1}{(n+1)^2} \right\}.$$

Note that $|\mathcal{A}_n| < (n+1)^2$ (some $\mathcal{A}_n$'s may be empty). By the choice of the sequence $\langle \delta_n : n \in \omega \rangle$ if we order lexicographically the intervals in $\bigcup_{n \in \omega} \mathcal{A}_n$ in a sequence $\langle J_n : n \in \omega \rangle$, then the length of J_n will be $\leq \varepsilon_n$. Let $x \in X$ be such that $x \notin \bigcup_{n \in \omega} J_n$. Note that then for each $n \in \omega$

$$\mu([\![x \in \dot{I}_n]\!]) \leq \frac{1}{(n+1)^2}$$

Let $p \in \mathbf{B}$. Find n such that $\sum_{k=n}^{\infty} k^{-2} < \mu(p)$. Then

$$q = p - \bigcup_{k=n}^{\infty} [\![x \in \dot{I}_n]\!]_{\mathbf{B}} > 0$$

and

$$q \Vdash_{\mathbf{B}} \forall k > n \ x \notin \dot{I}_n. \quad \square$$

The proof of 2.2 seems to suggest that a filter $\mathcal{F} \in \mathbf{V}$ on ω which cannot be extended to a rapid filter in $\mathbf{V}$ cannot be extended to a rapid filter in $\mathbf{V}^{\mathbf{B}}$. However, this is not the case. First, let us recall that a non-principal filter $\mathcal{F}$ on ω is called rapid if for every increasing function $f \in \omega^\omega$ there exists $X \in \mathcal{F}$ such that $|X \cap f(n)| \leq n$ for all n.

Theorem 2.3. *Suppose that $\mathcal{D}$ is a rapid filter on ω. Then there exists a filter $\mathcal{F}$ such that:*

(1) *$\mathcal{F}$ cannot be extended to a rapid filter in $\mathbf{V}$,*
(2) *$\mathbf{V}^{\mathbf{B}} \models$ "$\mathcal{F}$ can be extended to a rapid filter".*

Proof Let $\mathcal{F}$ be the family of all subsets A of ω such that for some set $X \in \mathcal{D}$ the sequence

$$\frac{|A \cap [n^2, (n+1)^2)|}{2n+1} \xrightarrow{n \in X} 1.$$

It should be clear that if $A \subseteq B$, $A \in \mathcal{F}$ then $B \in \mathcal{F}$ and the same set $X \in \mathcal{D}$ witnesses it. Moreover if $A, B \in \mathcal{F}$ is witnessed by $X_A, X_B \in \mathcal{D}$ then the intersection $X_A \cap X_B \in \mathcal{D}$ witnesses that $A \cap B \in \mathcal{F}$. Consequenlty $\mathcal{F}$ is a non-principal filter on ω. We claim that $\mathcal{F}$ cannot be extended to a rapid filter. Suppose that a set $A \subseteq \omega$ is such that $|A \cap n^3| \leq n$ for $n \in \omega$. Then for each $m \in \omega$ we have

$$\frac{|A \cap [m^2, (m+1)^2)|}{2m+1} \leq \frac{[(m+1)^{2/3}] + 1}{2m+1} \leq \frac{(m+1)^{2/3} + 1}{2m+1}$$

and hence

$$\lim_{m \to \infty} \frac{|A \cap [m^2, (m+1)^2)|}{2m+1} = 0.$$

Consequently the complement $\omega \setminus A$ of the set A belongs to $\mathcal{F}$ and A cannot be in any filter extending $\mathcal{F}$.

To prove the assertion (2) we work with the measure algebra on the space $\prod\limits_{n\in\omega}[n^2,(n+1)^2)$ equipped with the natural product measure μ (we use the same symbol as for the Lebesgue measure since this measure corresponds to Lebesgue measure under cannonical mapping of underlying space onto the interval $[0,1]$). Suppose that $r \in \prod\limits_{n\in\omega}[n^2,(n+1)^2)$ is a random real over $\mathbf{V}$ and work in $\mathbf{V}[r]$. First note that for a set $A \in \mathcal{F}$ and $X \in \mathcal{D}$,

$$\mu\left(\left\{x \in \prod_{n\in\omega}[n^2,(n+1)^2) : \forall^\infty m \in X\ x(m) \notin A\right\}\right) \leq \sum_{n=0}^{\infty} \prod_{\substack{m=n\\ m\in X}}^{\infty} \left(1 - \frac{|A \cap [m^2,(m+1)^2)|}{2m+1}\right) = 0.$$

In particular, since r is a random real,

$$\forall X \in \mathcal{D}\ \forall \mathcal{A} \in \mathcal{F}\ \mathcal{A} \cap \mathsf{range}(\nabla\restriction\mathcal{X}) \neq \emptyset.$$

Consequently $\mathcal{F} \cup \{\mathsf{range}(r\restriction X) : X \in \mathcal{D}\}$ generates a filter $\mathcal{F}^\star$. We are going to show that it is a rapid filter. Suppose that $f \in \omega^\omega \cap \mathbf{V}[r]$ is an increasing function. Since random real forcing is ω^ω-bounding we can assume that $f \in \mathbf{V}$. Since $\mathcal{D}$ was a rapid filter in $\mathbf{V}$ we find a set $X \in D$ such that $|X \cap f(n)| \leq n$ for $n \in \omega$. Look at the set $A = \{r(n) : n \in X\}$. For every $n \in \omega$ we have:

$$|A \cap f(n)| \leq |X \cap f(n)| \leq n.$$

The theorem is proved. $\square$

3. Random reals

Theorem 1.1 shows that in $\mathbf{V}^{\mathbf{B}}$, $\mathsf{cov}(\mathcal{N}) \geq \max\{\mathsf{cov}(\mathcal{N})^{\mathbf{V}}, \mathfrak{b}^{\mathbf{V}}\}$. In this section we will show that it is consistent that $\mathsf{cov}(\mathcal{N})^{\mathbf{V}^{\mathbf{B}}} > \max\{\mathsf{cov}(\mathcal{N})^{\mathbf{V}}, \mathfrak{b}^{\mathbf{V}}\}$.

We will need the following notation:

Definition 3.1. Let $\mathcal{N}_2$ be the ideal of measure zero subsets of $\mathbb{R} \times \mathbb{R}$ and let $\mathsf{Borel}(\mathbb{R})$ be the collection of all Borel mappings from $\mathbb{R}$ into $\mathbb{R}$. Define

$$\mathsf{cov}^\star(\mathcal{N}) = \min\Big\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{N}_2\ \&\ \forall f \in \mathsf{Borel}(\mathbb{R})\ \forall \mathbb{B} \in \mathsf{Borel} \setminus \mathcal{N}\ \exists \mathbb{H} \in \mathcal{A}\ \big\{x \in B : \langle x, f(x)\rangle \in H\big\} \notin \mathcal{N}\Big\}$$

and

$$\mathsf{non}^\star(\mathcal{N}) = \min\Big\{|X| : X \subseteq \mathsf{Borel}(\mathbb{R})\ \&\ \forall \mathbb{H} \in \mathcal{N}_{\not\models}\ \forall \mathbb{B} \in \mathsf{Borel} \setminus \mathcal{N}\ \exists \mho \in \mathbb{X}\ \big\{x \in B : \langle x, f(x)\rangle \notin H\big\} \notin \mathcal{N}\Big\}.$$

As a consequence of 1.2, we get:

Lemma 3.2. $\mathsf{cov}^\star(\mathcal{N}) = \mathsf{cov}(\mathcal{N})^{\mathbf{V}^{\mathbf{B}}}$ *and* $\mathsf{non}^\star(\mathcal{N}) = \mathsf{non}(\mathcal{N})^{\mathbf{V}^{\mathbf{B}}}$. $\square$

The goal of this section is to show that the coefficient $\mathsf{cov}^\star(\mathcal{N})$ can be large while both $\mathfrak{b}$ and $\mathsf{cov}(\mathcal{N})$ are small and that $\mathsf{non}^\star(\mathcal{N})$ can be small while both $\mathsf{non}(\mathcal{N})$ and $\mathfrak{d}$ are large.

The key to our construction is the following theorem:

Theorem 3.3. *There exists a forcing notion $\mathcal{P}$, adding generically a continuous function $h_G : \mathbb{R} \longrightarrow \mathbb{R}$, such that*

1. *$\mathcal{P}$ is σ-centered,*
2. $\forall f \in \omega^\omega \cap \mathbf{V}^{\mathcal{P}}\ \exists g \in \mathbf{V} \cap \omega^\omega\ \exists^\infty n\ f(n) \leq g(n)$,
3. *for every $H \in \mathcal{N}_2 \cap \mathbf{V}$, $\{x : \langle x, h_G(x)\rangle \in H\}$ has measure zero.*

PROOF Let $\mathcal{T}$ consists of all pairs $\langle \varepsilon, \phi\rangle$ where ε is a rational number in $(0,1)$ and $\phi : 2^{<\omega} \times 2^{<\omega} \longrightarrow [0,1]$ is a function such that for $s, t \in 2^{<\omega}$:

1. $\phi(\emptyset, \emptyset) > 0$,
2. $\phi(s,t) \leq 2^{-(|s|+|t|)}$,
3. $\phi(s^\frown 0, t) + \phi(s^\frown 1, t) = \phi(s,t) = \phi(s, t^\frown 0) + \phi(s, t^\frown 1)$.

We define the partial order $\mathcal{P}$. Conditions are pairs $p = \langle h, u\rangle$ such that

1. $u \in [\mathcal{T}]^{<\omega}$
2. $h : 2^{\leq m} \longrightarrow 2^{<\omega}$ for some $m = m(p)$,
3. if $s \subseteq t \in 2^{\leq m}$ then $h(s) \subseteq h(t)$,
4. if $\langle \varepsilon, \phi\rangle \in u$ then
$$\sum_{s \in 2^m} 2^{|h(s)|} \phi\big(s, h(s)\big) > \varepsilon.$$

The order $\leq$ on $\mathcal{P}$ is the natural one:
$$(h,u) \geq (h', u') \iff h \supseteq h' \ \&\ u \supseteq u'.$$

Lemma 3.4. *Suppose that $p = (h,u) \in \mathcal{P}$. Then there is $q = (h', u) \in \mathcal{P}$ such that $q \geq p$, $m(q) > m(p)$ and if $s \in 2^{m(q)}$ then $\big|h'(s)\big| > \Big|h'\big(s \restriction m(p)\big)\Big|$.*

PROOF What we have to do is to extend h. Note that if we put $h'(s^\frown i) = h(s)$ (for $s \in 2^{m(p)}$) then (h', u) is a condition stronger than p. So the only problem is to extend the "values" of h.

Take $\delta > 0$ such that for every $\langle \varepsilon, \phi\rangle \in u$
$$\delta \cdot \sum_{s \in 2^{m(p)}} 2^{1+|h(s)|} < \sum_{s \in 2^{m}(p)} 2^{|h(s)|} \phi\big(s, h(s)\big) - \varepsilon.$$

Lemma 3.5. *There are $m' > m(p)$ and $e : 2^{m'} \longrightarrow 2$ such that for each $\langle \varepsilon, \phi\rangle \in u$, $s \in 2^{m(p)}$:*
$$(\otimes) \qquad \frac{1}{2}\phi\big(s, h(s)\big) - \delta < \sum_{s \subseteq t \in 2^{m'}} \phi\big(t, h(s)^\frown e(t)\big).$$

PROOF Let $n = |u|$ and let $m' > m(p)$ be such that $2^{-m'}/\delta^2 < 1/n$. Fix $s \in 2^{m(p)}$. We are going to find a function $e_s : \{t \in 2^{m'} : s \subseteq t\} \longrightarrow 2$ satisfying the condition $(\otimes)$ for each $\langle \varepsilon, \phi\rangle \in u$. Consider the space Ω of all functions from $\{t \in 2^{m'} : s \subseteq t\}$ to 2. The space carries the natural (product) probability measure P. For $\langle \varepsilon, \phi\rangle \in u$ define a random variable $Y_\phi : \Omega \longrightarrow [0,1]$ by
$$Y_\phi(e) = \sum_{s \subseteq t \in 2^{m'}} \phi\big(t, h(s)^\frown e(t)\big).$$

By the Tchebyshev inequality we know that

$$P\left(\left|Y_\phi - \int Y_\phi \, d\Omega\right| \geq \delta\right) \leq \frac{\mathbf{D^2 Y}_\phi}{\boldsymbol{\delta^2}}.$$

If we put $X^t_\phi(e) = \phi\big(t, h(s)^\frown e(t)\big)$ (for $t \in 2^{m'}$, $s \subseteq t$) then X^t_ϕ's are independent random variables on Ω and $Y_\phi = \sum\limits_{s\subseteq t\in 2^{m'}} X^t_\phi$. Now,

$$\mathbf{D^2 Y}_\phi = \int \left(\mathbf{Y}_\phi - \int \mathbf{Y}_\phi \, \mathbf{d}\Omega\right)^{\mathbf{2}} \mathbf{d}\Omega = \int \left(\sum_{\mathbf{s}\subseteq\mathbf{t}\in\mathbf{2^{m'}}} \left(\mathbf{X}^{\mathbf{t}}_\phi - \int \mathbf{X}^{\mathbf{t}}_\phi \, \mathbf{d}\Omega\right)\right)^{\mathbf{2}} \mathbf{d}\Omega =$$

$$\sum_{s\subseteq t\in 2^{m'}} \int \left(X^t_\phi - \int X^t_\phi \, d\Omega\right)^2 d\Omega$$

(for the last equality we use the independence of X^t_ϕ's). Since $\left|X^t_\phi - \int X^t_\phi \, d\Omega\right| \leq 2^{-(m'+|h(s)|+1)}$ we get

$$\mathbf{D^2 Y}_\phi \leq \mathbf{2^{m'-m(p)}} \cdot \mathbf{2^{-2m'-2|h(s)|-2}} < \mathbf{2^{-m'}}.$$

Hence

$$P\left(\left|Y_\phi - \int Y_\phi \, d\Omega\right| \geq \delta\right) < 2^{-m'}/\delta^2 < 1/n$$

and therefore we can find $e_s \in \Omega$ such that for each $(\varepsilon, \phi) \in u$ we have $\int Y_\phi - \delta \, d\Omega \leq Y_\phi(e_s)$. Since

$$\int Y_\phi \, d\Omega = \sum_{s\subseteq t\in 2^{m'}} \int X^t_\phi \, d\Omega = \frac{1}{2} \sum_{s\subseteq t\in 2^{m'}} \phi(t, h(s)) = \frac{1}{2}\phi(s, h(s))$$

we get that e_s is as required. $\square$

Define $h' : 2^{\leq m'} \longrightarrow 2^{<\omega}$ by the following conditions:

(1) $h' \restriction 2^{\leq m(p)} = h$,
(2) if $s \in 2^{<m'} \setminus 2^{\leq m(p)}$ then $h'(s) = h(s \restriction m(p))$,
(3) if $s \in 2^{m'}$ then $h'(s) = h\big(s\restriction m(p)\big)^\frown e(s)$.

Thus $m(q) = m'$, but we have to prove that $q = (h', u)$ is a condition.

Note that for $\langle \varepsilon, \phi\rangle \in u$ we have then:

$$\sum_{t\in 2^{m'}} 2^{|h'(t)|}\phi\big(t, h'(t)\big) = \sum_{s\in 2^{m(p)}} \sum_{s\subseteq t\in 2^{m'}} 2^{1+|h(s)|} \cdot \phi(t, h(s)^\frown e(t)) >$$

$$> \sum_{s\in 2^{m(p)}} 2^{|h(s)|}\phi\big(s, h(s)\big) - \delta \cdot \sum_{s\in 2^{m(p)}} 2^{1+|h(s)|} > \varepsilon. \quad \square$$

Suppose that $G \subseteq \mathcal{P}$ is generic over $\mathbf{V}$. Let $\widetilde{h}_G = \bigcup\{h : \langle h, u\rangle \in G\}$ and for every $x \in 2^\omega$, let $h_G(x) = \bigcup_{n\in\omega} \widetilde{h}_G(x\restriction n)$. It follows immediately from 3.4 that

Lemma 3.6. *$h_G(x) : 2^\omega \longrightarrow 2^\omega$ is a continuous function in $\mathbf{V}[G]$.* $\square$

Lemma 3.7. *For every measure zero set $H \subseteq 2^\omega \times 2^\omega$ which is coded in $\mathbf{V}$, the set*

$$\{x \in 2^\omega : \langle x, h_G(x)\rangle \notin H\}$$

has measure one.

PROOF Fix H as above. Suppose that $p = \langle h, u\rangle \in \mathcal{P}$ and $\varepsilon > 0$ are given.

It is enough to show that $\mu\Big(\{x \in 2^\omega : \langle x, h_G(x)\rangle \notin H\}\Big) > 1-\varepsilon$ holds for every rational $\varepsilon > 0$. Suppose that $p = \langle h, u\rangle \in \mathcal{P}$ and $m = m(p)$.

Choose a perfect set F disjoint with H of measure so close to one that

$$\sum_{s\in 2^m} 2^{|h(s)|}\mu([s]\times[h(s)]\cap F) > 1-\varepsilon.$$

Define the function $\phi_F : 2^{<\omega}\times 2^{<\omega} \to [0,1]$ by

$$\phi_F(s,t) = \mu\big([s]\times[t]\cap F\big)$$

and note that $\langle 1-\varepsilon, \phi_F\rangle \in \mathcal{T}$. Moreover, $q = \langle h, u \cup \{\langle 1-\varepsilon, \phi_F\rangle\}\rangle$ is a condition. We show that

$$q \Vdash_{\mathcal{P}} \mu\Big(\{x \in 2^\omega : \langle x, h_{\dot G}(x)\rangle \notin H\}\Big) \geq 1-\varepsilon.$$

Let $F_n = \bigcup\{[s]\times[t] : s,t \in 2^n \ \&\ \big([s]\times[t]\big)\cap F \neq \emptyset\}$. Obviously $F = \bigcap_{n\in\omega} F_n$. Given $n \in \omega$ there is $q' = \langle h', u'\rangle$ stronger thatn q such that $m' = m(q') \geq n$ and $|h'(s)| \geq n$ for $s \in 2^{m'}$. Since $\langle 1-\varepsilon, \phi_F\rangle \in u \subseteq u'$ also $q'' = \langle h', u' \cup \{\langle 1-\varepsilon, \phi_{F_n}\rangle\}\rangle$ is a condition stronger than p and for $s \in 2^{m'}$, $\big([s]\times[h'(s)]\big)\cap F_n \neq \emptyset$ if and only if $[s]\times[h'(s)] \subseteq F_n$. Hence

$$\mu\left(\bigcup\{[s] : s \in 2^{m'} \ \&\ [s]\times[h'(s)] \subseteq F_n\}\right) = \sum_{s\in 2^{m'}} 2^{|h'(s)|}\phi_{F_n}\big(s, h'(s)\big) > 1-\varepsilon$$

and

$$q'' \Vdash_{\mathcal{P}} \mu\Big(\{x : \langle x, h_G(x)\rangle \in F_n\}\Big) \geq 1-\varepsilon.$$

Using density argument and passing to the limit we get

$$\mu\Big(\{x : \langle x, h_G(x)\rangle \in F\}\Big) \geq 1-\varepsilon. \quad \square$$

Lemma 3.8. *There exist centered families $\{\mathcal{P}_i : i \in I\}$, I countable, such that $\bigcup_{i\in I}\mathcal{P}_i$ is dense in $\mathcal{P}$ and for every maximal antichain $\{p_n : n \in \omega\}$ in $\mathcal{P}$ there exists a natural number $M(i)$ such that for every condition $q \in \mathcal{P}_i$ there exists $n \leq M(i)$ such that q and p_n are compatible.*

In particular, $\mathcal{P}$ does not add dominating reals.

PROOF For simplicity we will think of the second coordinates of conditions in $\mathcal{P}$ as finite sequences from $\mathcal{T}$.

Let

$$I = \{(N,k,h,\langle\varepsilon_i : i < N\rangle) : k, N \in \omega, h : 2^m \to 2^{<\omega}, \varepsilon_j \in (0,1)\cap\mathbb{Q} \text{ for } \mathbb{J} < \mathbb{N}.\}$$

For $i = (N,k,h,\langle\varepsilon_i : i < N\rangle) \in I$ let

$$\mathcal{P}_i = \Big\{\langle h, \langle\varepsilon_i,\phi_i\rangle_{i<N}\rangle \in \mathcal{P} : \forall i < N \ \Big(\phi_i(\emptyset,\emptyset) \geq 1/k \ \& \sum_{s\in 2^m} 2^{|h(s)|}\phi_i\big(s,h(s)\big) \geq \varepsilon_i + 1/k\Big)\Big\}.$$

Clearly each $\mathcal{P}_i$ is centered (conditions in $\mathcal{P}$ with the same h can be put together) and they cover $\mathcal{P}$.

We want to show that the families $\mathcal{P}_i$ have the required property. Assume not. Thus we have a maximal antichain $\langle p_k : k \in \omega\rangle$ in $\mathcal{P}$ and a sequence $\langle q_n : n \in \omega\rangle \subseteq \mathcal{P}_i$ (for some $i = (N,k,h,\langle\varepsilon_i : i < N\rangle)$) such that $q_n \perp_{\mathcal{P}} p_k$ for $k \leq n$.

Let $q_n = \langle h, \langle \varepsilon_i, \phi_i^n \rangle_{i<N} \rangle$, $\phi_i^n(\emptyset,\emptyset) \geq 1/k$, $\sum_{s\in 2^m} 2^{|h(s)|}\phi_i^n\big(s,h(s)\big) \geq \varepsilon_i + 1/k$. Passing to a subsequence we may assume that for each $i < N$ the sequence $\langle \phi_i^n : n \in \omega\rangle$ is pointwise converging (note that the space $[0,1]^\omega$ is compact).

Let $\phi_i : 2^{<\omega} \times 2^{<\omega} \longrightarrow [0,1]$ be the limit functions, i.e.

$$\phi_i(s,t) = \lim_{n\to\infty} \phi_i^n(s,t).$$

The functions ϕ_i satisfy conditions (1)–(3) of the definition of $\mathcal{T}$ (for the first condition remember that $\phi_i^n(\emptyset,\emptyset) \geq 1/k$). Moreover

$$\sum_{s\in 2^m} 2^{|h(s)|}\phi_i\big(s,h(s)\big) = \lim_{n\to\infty} \sum_{s\in 2^m} 2^{|h(s)|}\phi_i^n\big(s,h(s)\big) \geq \varepsilon_i + 1/k.$$

Consequently $\langle h, \langle \varepsilon_i, \phi_i\rangle_{i<N}\rangle \in \mathcal{P}$. We find $k_0 \in \omega$ such that the conditions p_{k_0} and $\langle h, \langle \varepsilon_i, \phi_i\rangle_{i<N}\rangle$ are compatible. Let $\langle h^\star, \langle \varepsilon_i, \phi_i\rangle_{i<N^\star}\rangle \geq \langle h, \langle \varepsilon_i, \phi_i\rangle_{i<N}\rangle, p_{k_0}$ where $h^\star : 2^{\leq m^\star} \longrightarrow 2^{<\omega}$, $N^\star > N$. Then we have for $i < N$:

$$\varepsilon_i < \sum_{s\in 2^{m^\star}} 2^{|h^\star(s)|}\phi_i\big(s,h^\star(s)\big) = \lim_{n\to\infty} \sum_{s\in 2^{m^\star}} 2^{|h^\star(s)|}\phi_i^n\big(s,h^\star(s)\big).$$

Consequently for sufficiently large n we will have

$$\varepsilon_i < \sum_{s\in 2^{m^\star}} 2^{|h^\star(s)|}\phi_i^n\big(s,h^\star(s)\big).$$

So take $n > k_0$ such that the above holds for each $i < N$. Then $\langle h^\star, \langle \varepsilon_i, \phi_i^n\rangle_{i<N}\rangle$ is a condition in $\mathcal{P}$. Since it is stronger than q_n and compatible with $\langle h^\star, \langle \varepsilon_i, \phi_i\rangle_{i<N^\star}\rangle$ we conclude that $q_n \not\perp_{\mathcal{P}} p_{k_0}$, which contradicts the choice of q_n ($n > k_0$!!!).

Let $\{i_n : n \in \omega\}$ be an enumeration of I with infinitely many repetitions. Suppose that $\Vdash_{\mathcal{P}} \dot{f} \in \omega^\omega$.

Define a function $g \in \mathbf{V} \cap \omega^\omega$ as $g(k) = M(i_k)$, where $M(i_k)$ is the number obtained by applying the first part of the lemma to $\mathcal{P}_{i_k}$ and to the antichain $p_n = [\![\dot{f}(k) = n]\!]_{\mathcal{P}}$, $n \in \omega$. It is clear that

$$\Vdash_{\mathcal{P}} \exists^\infty n\ \dot{f}(k) \leq g(k). \quad \square$$

Theorem 3.9. *It is consistent with* ZFC *that* $\mathsf{cov}^\star(\mathcal{N}) > \max\{\mathsf{cov}(\mathcal{N}), \mathfrak{b}\}$ *and that* $\mathsf{non}^\star(\mathcal{N}) < \min\{\mathsf{non}(\mathcal{N}), \mathfrak{d}\}$.

Proof To construct the first model let $\mathcal{P}_{\omega_2}$ be the finite support iteration of $\mathcal{P}$ of length ω_2.

Let $\mathbf{V} \models 2^{\aleph_0} = \aleph_1$. It is clear that $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathsf{cov}^\star(\mathcal{N}) = \aleph_2$. Since $\mathcal{P}$ is σ-centered neither $\mathcal{P}$ nor a finite support iteration of $\mathcal{P}$ adds random reals (see [8] or [3]). Similarly, property stated in 3.8 implies that finite support iteration of $\mathcal{P}$ does not add dominating reals. Thus $\mathsf{cov}(\mathcal{N})$ and $\mathfrak{b}$ are both equal to $\aleph_1$ in $\mathbf{V}^{\mathcal{P}_{\omega_2}}$.

The second part of the theorem is proved similarly. Let $\mathbf{V} \models \mathbf{MA}$ & $2^{\aleph_0} = \aleph_2$ and let $\mathcal{P}_{\omega_1}$ be the finite support iteration of $\mathcal{P}$ of length $\aleph_1$.

By "dualizing" the above argument we show that

$$\mathbf{V}^{\mathcal{P}_{\omega_1}} \models \mathsf{non}^\star(\mathcal{N}) = \aleph_1 \ \&\ \mathsf{non}(\mathcal{N}) = \mathfrak{d} = \aleph_2. \quad \square$$

Theorem 3.10. *Any of the inequalities* $\mathsf{cov}^\star(\mathcal{N}) > \mathfrak{b}$, $\mathsf{non}^\star(\mathcal{N}) < \mathfrak{b}$ *is consistent with* ZFC.

PROOF For the first model add $\aleph_2$ random reals (simultanously) to a model of CH. Then, in the extension we will have $\mathfrak{d} = \aleph_1$ and $\mathsf{cov}^\star(\mathcal{N}) = \aleph_2$ (for the last note that if r is a random real over $\mathbf{V}$ then the constant function $h(x) = r$ "omits" all measure zero subsets of the plane coded in $\mathbf{V}$). The second model can be obtained by adding $\aleph_1$ random reals to a model of $\mathbf{MA} + 2^{\aleph_0} = \aleph_2$. $\square$

For the sake of completeness of the picture let us mention the following result which will appear in [9] (the forcing notion applied for it is a special case of the scheme presented there):

Theorem 3.11. *It is consistent with* ZFC *that* $\mathsf{cov}^\star(\mathcal{N}) < \mathsf{non}(\mathcal{M})$.

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