# The Strength Of The Isomorphism Property[1]

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Abstract

In §1 of this paper, we characterize the isomorphism property of nonstandard universes in terms of the realization of some second–order types in model theory. In §2, several applications are given. One of the applications answers a question of D. Ross in [R] about infinite Loeb measure spaces.

## 0. Introduction

We always use $^*V$ for a nonstandard universe. We refer to [CK] or [SB] for the definition of nonstandard universes.

In the book [SB], there is an interesting example (see [SB, Theorem 1.2.12.(e)]) for illustrating the unusual behavior of infinite Loeb measure spaces. The example of [SB] says that in a nonstandard universe called a polyenlargement, the statement (†) is true, where the statement (†) is the following:

> Every infinite Loeb measure space has a subset $S$ such that $S$ has infinite Loeb outer measure, but the intersection of $S$ with any finite Loeb measure set has Loeb measure zero.

Under certain definition, the set $S$ is called measurable but has infinite outer measure and zero inner measure (see [SB]). The diagonal argument for constructing $S$ in [SB] depends on the construction of polyenlargements, say, an iterated ultrapower (or ultralimit) construction. During the preparation of the book [SB], K. D. Stroyan asked (see [R]) whether or not (†) can be proved by some nice general properties of nonstandard universes without mentioning any particular construction. The first natural candidate would be C. W. Henson's *isomorphism property* [H1].

Let $\mathcal{L}$ be a first–order language. An $\mathcal{L}$–model $\mathfrak{A}$ is called internally presented in $^*V$ if the base set $A$ and every interpretation under $\mathfrak{A}$ of a symbol in $\mathcal{L}$ are internal in $^*V$. (For any $\mathcal{L}$–model $\mathfrak{A}$ and symbol $P$ in $\mathcal{L}$ we write $P^{\mathfrak{A}}$ for the interpretation of $P$

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under $\mathfrak{A}$. We sometime use $\mathcal{L}_{\mathfrak{A}}$ for the language of $\mathfrak{A}$.) Let $\kappa$ be an infinite cardinal. A nonstandard universe ${}^*V$ is said to satisfy the $\kappa$–isomorphism property ($IP_\kappa$ for short) if

> for any two internally presented $\mathcal{L}$–models $\mathfrak{A}$ and $\mathfrak{B}$ with $|\mathcal{L}| < \kappa$, $\mathfrak{A} \equiv \mathfrak{B}$ implies $\mathfrak{A} \cong \mathfrak{B}$,

where "$\equiv$" means *to be elementarily equivalent to* and "$\cong$" means *to be isomorphic to*. It is easy to see that $IP_\kappa$ implies $IP_{\kappa'}$ when $\kappa' \leq \kappa$.

Instead of using the isomorphism property, D. Ross in [R] proved that a property called *the $\kappa$–special model axiom* for any infinite cardinal $\kappa$, which is stronger than $IP_\kappa$, implies (†). In [R], Ross showed also that $\kappa$–special model axiom has many new consequences, which hadn't been proved by $IP_\kappa$ then. In his paper, Ross asked which of those results can or cannot be proved by $IP_\kappa$. The most important question among them is that if we can or cannot prove (†) by $IP_\kappa$ for some infinite cardinal $\kappa$. Basically, it was not known back then whether or not the $\kappa$–special model axiom is strictly stronger that $IP_\kappa$ (see [R]).

The first author then answered the most of Ross's questions in [J]. In that paper, Jin showed that $IP_\kappa$ for arbitrary large $\kappa$ does not imply some consequences of the $\aleph_0$–special model axiom. As a corollary $IP_\kappa$ is strictly weaker than the $\kappa$–special model axiom. He also showed that many of the consequences of the $\kappa$–special model axiom in [R] are also the consequences of $IP_\kappa$. Unfortunately, [J] didn't answer Ross's question about (†).

In the another direction, the authors of [JK] proved that (†) is true in some ultrapowers of the standard universe. Since we need iterated ultrapower construction to build the nonstandard universes of the $\kappa$–special model axiom while we need only one–step ultrapower construction to build the nonstandard universes of $IP_\kappa$ (see [H2]), the result of [JK] seems to suggest that $IP_\kappa$ have the right strength to prove (†).

The main purpose of this paper is to solve Ross's question about (†). In §1, we characterize $IP_\kappa$ in terms of the realization of some second–order types. By applying Theorem 1 of §1, we show in §2 that Ross's question about (†) has a positive answer, *i.e.* (†) can be proved by $IP_{\aleph_0}$. In §2, we reprove also three known results in [J] by using the same method in a uniform way. The new method simplifies significantly the original proofs in [J].

Notation for model theory in this paper will be consistent with [CK].

## 1. Characterization of the isomorphism property

We use always $\mathcal{L}$ for a first–order language. Let $X$ be an $n$–ary predicate symbol which is not in $\mathcal{L}$. We call $\Gamma(X)$ an $n$–$\Delta^1_0(\mathcal{L})$ type iff $\Gamma(X)$ is a consistent set of $\mathcal{L} \cup \{X\}$–sentences. Let $\mathfrak{A}$ be an $\mathcal{L}$–model with base set $A$ and let $\Gamma(X)$ be an $n$–$\Delta^1_0(\mathcal{L})$ type. We say that $\Gamma(X)$ is consistent with $\mathfrak{A}$ iff $\Gamma(X) \cup \mathit{Th}(\mathfrak{A})$ is consistent, where $\mathit{Th}(\mathfrak{A})$ is the set of all $\mathcal{L}$–sentences which are true in $\mathfrak{A}$. We say that $\mathfrak{A}$ realizes $\Gamma(X)$ iff there exists an $S \subseteq A^n$ such that the $\mathcal{L} \cup \{X\}$–model $\mathfrak{A}_S = (\mathfrak{A}, S)$, where $S$ is the interpretation of $X$ under $\mathfrak{A}_S$, is a model of $\Gamma(X)$.

Let ${}^*V$ be a nonstandard universe. Let $\mathfrak{A}$ be an $\mathcal{L}$–model with base set $A$ in the standard universe. We write ${}^*\mathfrak{A}$ for an internally presented $\mathcal{L}$–model in ${}^*V$ with base set ${}^*A$ and the interpretation $P^{{}^*\mathfrak{A}} = {}^*(P^{\mathfrak{A}})$ for every symbol $P \in \mathcal{L}$. It is not hard to see that $\mathfrak{A} \equiv {}^*\mathfrak{A}$. In fact, $\mathfrak{A}$ can be considered as an elementary submodel of ${}^*\mathfrak{A}$.

**Main Theorem** *Let $\kappa < \beth_\omega$ be a regular cardinal. Then the following are equivalent:*

*(1) $IP_\kappa$,*

*(2) For any first–order language $\mathcal{L}$ with fewer than $\kappa$ many symbols, for any $n$–$\Delta^1_0(\mathcal{L})$ type $\Gamma(X)$ and for any internally presented $\mathcal{L}$–model $\mathfrak{A}$ in ${}^*V$, if $\Gamma(X)$ is consistent with $\mathfrak{A}$, then $\mathfrak{A}$ realizes $\Gamma(X)$.*

We will break the main theorem into following two theorems.

**Theorem 1.** *Assume $\kappa < \beth_\omega$ is a regular cardinal. Let ${}^*V$ be a nonstandard universe which satisfies $IP_\kappa$. For any first–order language $\mathcal{L}$ with fewer than $\kappa$ many symbols, for any $n$–$\Delta^1_0(\mathcal{L})$ type $\Gamma(X)$ and for any internally presented $\mathcal{L}$–model $\mathfrak{A}$ in ${}^*V$, if $\Gamma(X)$ is consistent with $\mathfrak{A}$, then $\mathfrak{A}$ realizes $\Gamma(X)$.*

**Proof:** Let ${}^*V$, $\mathcal{L}$, $\Gamma(X)$ and $\mathfrak{A}$ are as described in the theorem. We want to show that $\mathfrak{A}$ realizes $\Gamma(X)$.

Since $\Gamma(X)$ is consistent with $\mathfrak{A}$, there exists an $\mathcal{L}$–model $\mathfrak{B}$ with base set $B$ and an $S' \subseteq B^n$ such that the $\mathcal{L} \cup \{X\}$–model $\mathfrak{B}_{S'} = (\mathfrak{B}, S')$ is a model of $Th(\mathfrak{A}) \cup \Gamma(X)$. We can assume $|B| \leq \kappa$ by the Downward Löwenheim–Skolem–Tarski theorem. Furthermore we can assume that $\mathfrak{B}$ is in the standard universe because $\kappa < \beth_\omega$. Let ${}^*\mathcal{B}_{S'} = ({}^*\mathcal{B},\ {}^*S')$ be the internally presented $\mathcal{L} \cup \{X\}$–model in ${}^*V$ defined above. It is easy to see now that $\mathfrak{A} \equiv\ {}^*\mathfrak{B}$. By $IP_\kappa$, there is an isomorphism $i$ from ${}^*\mathfrak{B}$ to $\mathfrak{A}$. Let

$$S = \{(i(b_1), i(b_2), \ldots, i(b_n)) : (b_1, b_2, \ldots, b_n) \in\ {}^*S'\}.$$

Then $i$ is an isomorphism from $({}^*\mathfrak{B},\ {}^*S')$ to $(\mathfrak{A}, S)$ in $\mathcal{L} \cup \{X\}$. Since $\mathfrak{B}_{S'} \models \Gamma(X)$, then ${}^*\mathfrak{B}_{S'} \models \Gamma(X)$. Since ${}^*\mathfrak{B}_{S'} \cong \mathfrak{A}_S$, we conclude that $\mathfrak{A}$ realizes $\Gamma(X)$. $\square$

**Remark:** If we replace the predicate symbol $X$ in the definition of $n$–$\Delta^1_0(\mathcal{L})$ types by a new constant symbol $c$, the proof of Theorem 1 can still go through. So, as a corollary of Theorem 1, $IP_\kappa$ implies $\kappa$–saturation.

Next we are going to prove the converse of Theorem 1. Before going further, we need to introduce more notation. The first is about model pairs. Let $\mathcal{L}$ be a language. We call a language $\mathcal{L}'$ *the language for $\mathcal{L}$–model pairs* if

(1) $\mathcal{L}$ and $\mathcal{L}'$ have same function symbols,

(2) every relation symbol in $\mathcal{L}$ is in $\mathcal{L}'$ and $\mathcal{L}'$ contains two additional unary relation symbols $P$ and $Q$,

(3) for every constant symbol $c$ in $\mathcal{L}$, $\mathcal{L}'$ contains exactly two copies of $c$, say, $c_0$ and $c_1$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two $\mathcal{L}$–models. A model pair $\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}$ is an $\mathcal{L}'$–model with base set $A \cup B$ (we assume $A \cap B = \emptyset$) such that

(1) for every function symbol or relation symbol $R$ in $\mathcal{L}$, $R^{\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}} = R^{\mathfrak{A}} \cup R^{\mathfrak{B}}$,

(2) $P^{\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}} = A$ and $Q^{\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}} = B$,

(3) $c_0^{\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}} = c^{\mathfrak{A}}$ and $c_1^{\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}} = c^{\mathfrak{B}}$.

Let $\mathcal{L}$ be a language and let $R$ be an unary predicate symbol. For any $\mathcal{L}$–formula $\phi$, we write $\phi^R$, the relativization of $\phi$ under $R$, for the formula defined inductively by

(1) $\phi^R = \phi$ if $\phi$ is an atomic formula,

(2) if $\phi = \neg\psi$, then $\phi^R = \neg\psi^R$,

(3) if $\phi = \psi \wedge \chi$, then $\phi^R = \psi^R \wedge \chi^R$,

(4) if $\phi = \exists x\psi$, then $\phi^R = \exists x(R(x) \wedge \psi^R)$.

**Theorem 2.** *Let ${}^*V$ be a nonstandard universe. Let $\kappa$ be a regular cardinal. If for any language $\mathcal{L}$ with fewer than $\kappa$–many symbols, for any internally presented $\mathcal{L}$–model $\mathfrak{A}$ in ${}^*V$, and for any 2–$\Delta_0^1(\mathcal{L})$ type $\Gamma(X)$ which is consistent with $\mathfrak{A}$, the model $\mathfrak{A}$ realizes $\Gamma(X)$, then ${}^*V$ satisfies $IP_\kappa$.*

**Proof:** Let $\mathcal{L}$ be a language with fewer than $\kappa$–many symbols. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two in ternally presented $\mathcal{L}$–models in ${}^*V$ such that $\mathfrak{A} \equiv \mathfrak{B}$. We want to show that $\mathfrak{A} \cong \mathfrak{B}$.

Let $\mathcal{L}'$ be the language for $\mathcal{L}$–model pairs and let $\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}$ be the model pair of $\mathfrak{A}$ and $\mathfrak{B}$. We want now to define a 2–$\Delta_0^1(\mathcal{L}')$ type $\Gamma(X)$ which will be used to force an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. Let

$$\Gamma(X) = \{\phi_n(X) : n = 0, 1, 2, 3, 4\} \cup \{\psi_\varphi(X) : \varphi \text{ is an } \mathcal{L}\text{–formula.}\},$$

where

$$\phi_0(X) = \forall x \forall y (X(x, y) \to P(x) \wedge Q(y))$$

$$\phi_1(X) = \forall x (P(x) \to \exists y X(x, y))$$

$$\phi_2(X) = \forall y (Q(y) \to \exists x X(x, y))$$

$$\phi_3(X) = \forall x \forall y \forall z (X(x, z) \wedge X(y, z) \to x = y)$$

$$\phi_4(X) = \forall x \forall y \forall z (X(z, x) \wedge X(z, y) \to x = y)$$

$$\psi_\varphi(X) = \forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_n (\bigwedge_{k=1}^{n} X(x_k, y_k) \to (\varphi^P(x_1, \ldots, x_n) \leftrightarrow \varphi^Q(y_1, \ldots, y_n))).$$

We can see that the sentences $\{\phi_n(X) : n = 0, 1, 2, 3, 4\}$ say that $X$ is a one to one correspondence between $P$ and $Q$. Hence $\Gamma(X)$ says that the one to one correspondence $X$ is actually an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. It is easy to check that for any two $\mathcal{L}$–models $\mathfrak{A}'$ and $\mathfrak{B}'$, the model pair $\mathfrak{C}_{\mathfrak{A}',\mathfrak{B}'}$ realizes $\Gamma(X)$ if and only if $\mathfrak{A}' \cong \mathfrak{B}'$. We need now only to show that $\Gamma(X)$ is consistent with $\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}$. Since $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent, there exists an ultrafilter $\mathcal{F}$ on some cardinal $\lambda$ such that the ultrapower of $\mathfrak{A}$ and the ultrapower of $\mathfrak{B}$ modulo $\mathcal{F}$ are isomorphic (see [S]). Hence the ultrapower of $\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}$ modulo $\mathcal{F}$, which is the model pair of the ultrapower of $\mathfrak{A}$ and the ultrapower of $\mathfrak{B}$ modulo $\mathcal{F}$, realizes $\Gamma(X)$. On the other hand, the ultrapower of $\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}$ is elementarily equivalent to $\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}$. So $\Gamma(X)$ is consistent with $\mathfrak{C}_{\mathfrak{A},\mathfrak{B}}$. $\square$

**Remarks:** (1) As a corollary we have that in a nonstandard universe, the realizability for all $2$–$\Delta^1_0(\mathcal{L})$ types is equivalent to the realizability of all $n$–$\Delta^1_0(\mathcal{L})$ types for every $n$. (2) We didn't required that $\kappa < \beth_\omega$ in Theorem 2.

## 2. The applications

The first application will give an answer to Ross's question about (†). In order to avoid dealing with the lengthy definition of Loeb measure we are going to express (†) in an internal version as Ross did (see [R]).

We use the words *finite* or *infinite* for externally finite or externally infinite, respectively. We use *finite or *infinite for internally finite or internally infinite, respectively. For example, if $n \in {}^*\mathbb{N} \setminus \mathbb{N}$, where $\mathbb{N}$ is the set of all standard natural numbers, then the set $\{0, 1, \ldots, n\}$ is both *finite and infinite. We use $\mathbb{R}$ for the set of all standard reals.

Let ${}^*V$ be a nonstandard universe. Let $r \in {}^*\mathbb{R}$. We say that $r$ is finite if there is a standard $n \in \mathbb{N}$ such that $|r| < n$. Otherwise we call $r$ infinite. We say that $r$ is an infinitesimal if $|r| < \frac{1}{n}$ for every standard $n \in \mathbb{N}$.

**Application 1.** *($IP_{\aleph_0}$) Suppose $\Omega$ is an infinite internal set and $\mathcal{B}$ is an internal subalgebra of ${}^*\mathcal{P}(\Omega)$ which contains all singletons. Let $\mu : \mathcal{B} \to^* [0, \infty)$ be an internal, finitely additive measure with $\mu(\Omega)$ infinite and $\mu(\{x\})$ infinitesimal for every $x \in \Omega$. Then there exists a subset $S \subseteq \Omega$ such that*

*(1) for any $D \in \mathcal{B}$ with $\mu(D)$ finite, for any $n \in \mathbb{N}$, there exists an $E \in \mathcal{B}$ such that $D \cap S \subseteq E$ and $\mu(E) < \frac{1}{n}$,*

*(2) for any $D \in \mathcal{B}$, if $S \subseteq D$, then $\mu(D)$ is infinite.*

**Proof:** Let $\Omega$, $\mathcal{B}$ and $\mu$ be as described in the Application 1. Let ${}^*\mathbb{R}$ be the set of hyperreal numbers. Assume that $\Omega$, $\mathcal{B}$ and ${}^*\mathbb{R}$ are all disjoint.

We form first an internally presented $\mathcal{L}_{\mathfrak{A}}$–model $\mathfrak{A}$ with base set $A = \Omega \cup \mathcal{B} \cup^* \mathbb{R}$ such that

$$\mathfrak{A} = (A\,; \Omega, \mathcal{B}, ^*\mathbb{R}, \in, \mu, \cap, \smallsetminus, +, *, \leq, 0, 1),$$

where $\Omega$, $\mathcal{B}$ and $^*\mathbb{R}$ are three unary relations on $A$, $\in \subseteq \Omega \times \mathcal{B}$ is the membership relation, $\mu : \mathcal{B} \mapsto {}^*\mathbb{R}$ is the finite additive measure, $\cap$ is the set intersection and $\smallsetminus$ is the set subtraction on $\mathcal{B}$, and $(^*\mathbb{R}\,; +, *, \leq, 0, 1)$ is the usual hyperreal ordered field. For simplicity, we do not distinguish a symbol in $\mathcal{L}_{\mathfrak{A}}$ from its interpretation under $\mathfrak{A}$.

We form next a 1–$\Delta^1_0(\mathcal{L}_{\mathfrak{A}})$ type $\Gamma(X)$ such that

$$\Gamma(X) = \{\phi(X), \psi_n(X), \chi_n(X) : n = 1, 2, \ldots\},$$

where

$$\phi(X) = \forall x (X(x) \rightarrow \Omega(x))$$

$$\psi_n(X) = \forall U (\mathcal{B}(U) \wedge \mu(U) < n \rightarrow \exists V (\mathcal{B}(V) \wedge \forall x (X(x) \wedge x \in U \rightarrow x \in V) \wedge \mu(V) < \frac{1}{n}))$$

$$\chi_n(X) = \forall U (\mathcal{B}(U) \wedge \forall x (X(x) \rightarrow x \in U) \rightarrow \mu(U) > n).$$

Notice that in $\mathfrak{A}$, the element 1 is definable, so do $n$ and $\frac{1}{n}$ for every $n \in \mathbb{N}$.

The sentence $\phi(X)$ says that $X$ is a subset of $\Omega$. The sentence $\psi_n(X)$ says that the intersection of $X$ with any $U$ in $\mathcal{B}$ with measure less than $n$ has outer measure less than $\frac{1}{n}$. The sentence $\chi_n(X)$ says that $X$ has outer measure greater than $n$. So the application 1 is true if and only if $\mathfrak{A}$ realizes $\Gamma(X)$. Hence, by the Theorem 1, it suffices to show that $\mathfrak{A}$ is consistent with $\Gamma(X)$.

Let $T = \mathit{Th}(\mathfrak{A})$.

**Claim:** $T \cup \Gamma(X)$ is consistent.

Proof of Claim: By Downward Löwenheim-Skolem Theorem we can find a countable model $\mathfrak{A}_0 \preccurlyeq \mathfrak{A}$ with base set $A_0 = \Omega_0 \cup \mathcal{B}_0 \cup \mathbb{R}_0$. Since

$$\exists U (\mathcal{B}(U) \wedge \forall x (\Omega(x) \rightarrow x \in U))$$

is true in $\mathfrak{A}$, it is true in $\mathfrak{A}_0$. Hence $\Omega_0 \in \mathcal{B}_0$. Since $\mu(\Omega) > n$ for all $n \in \mathbb{N}$ are true in $\mathfrak{A}$, they are also true in $\mathfrak{A}_0$. Hence $\mu(\Omega_0)$ is infinite in $\mathfrak{A}_0$. Since

$$\forall U \forall x \forall y (\mathcal{B}(U) \wedge \Omega(x) \wedge \Omega(y) \wedge (x \in U \wedge y \in U \rightarrow x = y) \rightarrow \mu(U) < \frac{1}{n}$$

for all $n \in \mathbb{N}$ are true in $\mathfrak{A}$, they are also true in $\mathfrak{A}_0$. Hence the measure of every singleton is infinitesimal in $\mathfrak{A}_0$. Let

$$\{B \in \mathcal{B}_0 : \mu(B) \text{ is finite }\} = \{B_n : n \in \mathbb{N}\}.$$

It is now easy to pick

$$x_n \in \Omega_0 \setminus (\bigcup_{k=0}^{n-1} B_k \cup \{x_k : k < n\})$$

because $\Omega_0$ has infinite measure and the measure of $\bigcup_{k=0}^{n-1} B_k \cup \{x_k : k < n\}$ is finite. Also notice that the measure of a finite set $\{x_k : k < n\}$ for $n \in \mathbb{N}$ is infinitesimal because the sum of finitely many infinitesimals is an infinitesimal and $\mathcal{B}_0$ is closed under finite union.

Let $S_0 = \{x_n : n \in \mathbb{N}\}$. It is obvious that $(\mathfrak{A}_0, S_0)$ is a model of $T \cup \Gamma(X)$. $\square$

Next three applications are also the questions of [R] and were proved in [J]. The purpose of including them here with simplified proofs is to illustrate that $IP_\kappa$ is an "easy to use" tool in nonstandard analysis.

**Application 2.** *($IP_{\aleph_0}$) Suppose that $(P, <_P)$ and $(Q, <_Q)$ are two internal linear orders without endpoints. There is an order–preserving map $f : P \mapsto Q$ such that $f[P]$ is cofinal in $Q$.*

**Proof:** Without loss of generality, we can assume that $P \cap Q = \emptyset$. Let $\mathfrak{A}$ be an internally presented $\mathcal{L}_\mathfrak{A}$–model with base set $A = P \cup Q$ such that

$$\mathfrak{A} = (A\,; P, Q, \leq_P, \leq_Q),$$

where $P$ and $Q$ are two binary relations on $A$, and $<_P$ and $<_Q$ are the correspondent orders on $P$ and $Q$. We define a 2–$\Delta^1_0(\mathcal{L}_\mathfrak{A})$ type $\Gamma(X)$ such that

$$\Gamma(X) = \{\phi(X), \psi(X), \chi(X), \pi(X)\}$$

where

$$\phi(X) = \forall x \forall y (X(x,y) \to P(x) \wedge Q(y))$$

$$\psi(X) = \forall x \exists! y (P(x) \to X(x,y))$$

$$\chi(X) = \forall x_0 \forall x_1 \forall y_0 \forall y_1 (x_0 <_P x_1 \wedge X(x_0, y_0) \wedge X(x_1, y_1) \to y_0 <_Q y_1)$$

$$\pi(X) = \forall y_0 \exists x \exists y_1 (Q(y_0) \to P(x) \wedge X(x, y_1) \wedge y_0 <_Q y_1).$$

In $\Gamma(X)$ the sentence $\phi(X)$ says that $X$ is a relation between $P$ and $Q$, the sentence $\psi(X)$ says that $X$ is the graph of a function from $P$ to $Q$, the sentence $\chi(X)$ says that the function is order–preserving, and $\pi(X)$ says that the function is a cofinal embedding. If there exists an $S \subseteq P \times Q$ such that $(\mathfrak{A}, S) \models \Gamma(X)$, then it is easy to see that the map $f$ defined by its graph $S$ is the order–preserving map we are looking for. By Theorem 1, we need only to show that $T \cup \Gamma(X)$ is consistent, where $T = Th(\mathfrak{A})$.

**Claim:** $T \cup \Gamma(X)$ is consistent.

Proof of Claim: Let $\mathfrak{A}_0 = (A_0; P_0, Q_0, \leq_{P_0} \leq_{Q_0})$ be a countable elementary submodel of $\mathfrak{A}$. By the Compactness Theorem and Löwenheim–Skolem Theorem $\mathfrak{A}_0$ can be elementarily extended to a countable model $\mathfrak{A}_1 = (A_1, P_1, Q_1, \leq_{P_1}, \leq_{Q_1})$ such that the set of all rational numbers in $[0, 1)$, together with the usual order, can be order-isomorphically embedded into the set $\{q \in Q_1 : \forall x \in Q_0\ (x \leq_{Q_1} q)\}$. Let $i_0$ be that embedding. By the same argument we can find an elementary chain of length $\omega$ of countable models $\mathfrak{A}_0 \preccurlyeq \mathfrak{A}_1 \preccurlyeq \cdots$ and a sequence of maps $\{i_n : n \in \omega\}$ such that $i_n$ is an order–preserving map from the set of all rational numbers in $[n, n+1)$ with the usual order to $\{q \in Q_{n+1} : \forall x \in Q_n\ (x \leq_{Q_{n+1}} q)\}$. Let $\mathfrak{A}_\omega = \bigcup_{n\in\omega} \mathfrak{A}_n$ and let $i_\omega = \bigcup_{n\in\omega} i_n$. Since $\mathfrak{A}_0$ is elementarily equivalent to both $\mathfrak{A}$ and $\mathfrak{A}_\omega$, then $\mathfrak{A}_\omega$ is a model of $T$. It is easy to see that $i_\omega$ is an order-preserving map from the set of all positive rational numbers cofinally into $Q_\omega$. Since every countable order without a right endpoint can be cofinally embedded into the set of all positive rational numbers, then $P_\omega$ can be cofinally embedded into $Q_\omega$. Hence $\mathfrak{A}_\omega$ realizes $\Gamma(X)$. This proves the consistency of $T \cup \Gamma(X)$. $\square$

**Application 3.** *($IP_{\aleph_0}$) Let $(P, <_P)$ be an internal partial order with no right endpoints. There is an external subset $S \subseteq P$ such that $\{s \in S : s <_P p\}$ is internal for every $p \in P$.*

**Proof:** Let $\mathcal{Q} = {}^*\mathcal{P}(P)$. Let $\mathfrak{A}$ be an internally presented $\mathcal{L}_{\mathfrak{A}}$–model with base set $A = P \cup \mathcal{Q}$ such that

$$\mathfrak{A} = (A\,; P, \mathcal{Q}, \in, <_P, \cap, \setminus),$$

where $P$ and $\mathcal{Q}$ are two unary relations, $\in \subseteq P \times \mathcal{Q}$ is the membership relation, $<_P$ is the order on $P$, $\cap$ is the set intersection on $\mathcal{Q}$ and $\setminus$ is the set subtraction on $\mathcal{Q}$. We define a 1–$\Delta^1_0(\mathcal{L}_{\mathfrak{A}})$ type $\Gamma(X)$ such that

$$\Gamma(X) = \{\phi(X), \psi(X), \chi(X)\}$$

where

$$\phi(X) = \forall x(X(x) \to P(x))$$

$$\psi(X) = \forall x \exists U(P(x) \to \mathcal{Q}(U) \wedge \forall y(y <_P x \wedge X(y) \leftrightarrow y \in U))$$

$$\chi(X) = \forall U \exists x(\mathcal{Q}(U) \to (x \in U \wedge \neg X(x)) \vee (X(x) \wedge \neg x \in U)).$$

The sentence $\phi(X)$ says that $X$ is a subset of $P$, the sentence $\psi(X)$ says that for every $x$ in $P$ there exists a $U$ in ${}^*\mathcal{P}(P)$ such that $U = \{y \in X : y <_P x\}$, and the sentence $\chi(X)$ says that for all $U$ in ${}^*\mathcal{P}(P)$, the set $X$ is different from $U$. It is easy to see that if there is an $S \subseteq P$ such that $(\mathfrak{A}, S) \models \Gamma(X)$, then $S$ the set we are looking for. By Theorem 1, it suffices to show that $T \cup \Gamma(X)$ is consistent, where $T = Th(\mathfrak{A})$.

**Claim:** $T \cup \Gamma(X)$ is consistent.

Proof of Claim: Let $\mathfrak{A}_0 = (A_0; P_0, \mathcal{Q}_0, \dots)$ be a countable elementary submodel of $\mathfrak{A}$. It suffices to construct a set $S = \{s_n : n \in \omega\} \subseteq P_0$ such that $(\mathfrak{A}_0, S) \models \Gamma(X)$. Let $P_0 = \{p_n : n \in \omega\}$ and let $\mathcal{Q}_0 = \{Q_n : n \in \omega\}$. Since $P$ has no right endpoints, then $P_0$ has no right endpoints. Now we can pick the elements $s_k$ and $t_k$ from $P_0$ for every $k \in \omega$ such that

(1) $s_0 < t_0 < s_1 < t_1 < \cdots$,

and for every $k \in \omega$

(2) we have $s_k \not< p_k$ and

(3) either both $s_k$ and $t_k$ are in $Q_k$ or both $s_k$ and $t_k$ are not in $Q_k$.

Let $S = \{s_n : n \in \omega\}$. Then $S$ differs from every element in $\mathcal{Q}_0$. For every $p \in P_0$ the set $\{s \in S : s \leq p\}$ is finite, and hence is in $\mathcal{Q}_0$ because for every finite set

$\{a_1, \ldots, a_n\} \subseteq P_0$ the sentence

$$\exists U(\mathcal{Q}(U) \wedge \forall x(x \in U \leftrightarrow \bigvee_{i=1}^{n} x = a_i))$$

is true in $\mathfrak{A}$ and therefore, it is true in $\mathfrak{A}_0$.

The arguments above showed that $(\mathfrak{A}_0, S) \models \Gamma(X)$. $\square$

**Application 4.** *($IP_{\aleph_0}$) Let $P$ and $Q$ be two $^*$infinite internal sets There is a bijection $f : P \mapsto Q$ such that for every $^*$finite $b \subseteq P$ and for every $^*$finite $c \subseteq Q$, the restriction of $f$ to $b$ and the restriction of $f^{-1}$ to $c$ are internal.*

**Proof:** Without loss of generality, we can assume that $P \cap Q = \emptyset$. Let $\mathfrak{A}$ be an internally presented $\mathcal{L}_{\mathfrak{A}}$–model with base set $A = P \cup Q \cup F$, where $F = \{f : f$ is an internal bijection from some $^*$finite subset of $P$ to $Q\}$, such that

$$\mathfrak{A} = (A\,; P, Q, F, R)$$

where $P$, $Q$ and $F$ are three unary relations on $A$ and $R \subseteq P \times Q \times F$ is defined by

$$(a, b, f) \in R \quad \text{iff} \quad (a, b) \in f.$$

We now define a $2$–$\Delta_0^1(\mathcal{L}_{\mathfrak{A}})$ type $\Gamma(X)$ such that

$$\Gamma(X) = \{\phi_n(X) : n = 0, 1, 2, 3, 4, 5\},$$

where

$$\phi_0(X) = \forall x \forall y(X(x, y) \to P(x) \wedge Q(y))$$

$$\phi_1(X) = \forall x \forall y \forall z(X(x, z) \wedge X(y, z) \to x = y)$$

$$\phi_2(X) = \forall x \forall y \forall z(X(z, x) \wedge X(z, y) \to x = y)$$

$$\phi_3(X) = \forall x \exists y((P(x) \to X(x, y)) \wedge (Q(x) \to X(y, x)))$$

$$\phi_4(X) =$$

$$\forall g \exists f(F(g) \to F(f) \wedge \forall x(\exists y R(x, y, g) \leftrightarrow \exists y R(x, y, f)) \wedge \forall x \forall y(R(x, y, f) \to X(x, y)))$$

$$\phi_5(X) =$$

$$\forall g \exists f(F(g) \to F(f) \wedge \forall y(\exists x R(x, y, g) \leftrightarrow \exists x R(x, y, f)) \wedge \forall x \forall y(R(x, y, f) \to X(x, y))).$$

The sentences $\phi_0(X), \phi_1(X), \phi_2(X), \phi_3(X)$ say that $X$ is a one to one onto correspondence between $P$ and $Q$. The sentence $\phi_4(X)$ says that the restriction of $X$ on any *finite set of $P$ (as the domain of an element $g$ in $F$) coincides with an element $f$ in $F$. The sentence $\phi_5(X)$ says that the restriction of $X$ on any *finite subset of $Q$ (as the range of an element $g$ in $F$) coincides also with an element $f$ in $F$. It is easy to see that if there exists an $S \subseteq P \times Q$ such that $(\mathfrak{A}, S) \models \Gamma(X)$, then the bijection induced by $S$ is the map we are looking for. Let $T = Th(\mathfrak{A})$. By Theorem 1, we need only to show that

**Claim:** $T \cup \Gamma(X)$ is consistent.

Proof of Claim: Let $\mathfrak{A}_0 = (A_0; P_0, Q_0, F_0, R_0)$ be a countable elementary submodel of $\mathfrak{A}$. It suffices to find a relation $S \subseteq P_0 \times Q_0$ such that $S$ is the graph of a bijection $i$ from $P_0$ to $Q_0$ and for every $C$ which is the domain of a function in $F_0$ and for every $D$ which is the range of a function in $F_0$, both $i \restriction C$ and $(i^{-1} \restriction D)^{-1}$ are functions in $F_0$.

Let $F_0 = \{f_n : n \in \omega\}$. We want to construct a sequence $\{i_m : m \in \omega\} \subseteq F_0$ such that

(1) $i_0 \subseteq i_1 \subseteq \imath_2 \subseteq \cdots$,

(2) for every $f \in F_0$ there is an $m \in \omega$ such that $dom(f) \subseteq dom(i_m)$ and $range(f) \subseteq range(i_m)$.

The claim follows from the construction because we can let $i = \bigcup_{m\in\omega} i_m$. It is easy to check that

(a) $i$ is one to one function,

(b) $dom(i) = P_0$ and $range(i) = Q_0$,

(c) for every $f \in F_0$ there exists an $m \in \omega$ such that $i \restriction dom(f) = i_m \restriction dom(f) \in F_0$ and $(i^{-1} \restriction range(f))^{-1} = (i_m^{-1} \restriction range(f))^{-1} \in F_0$.

(c) is true because both sentences

$$\forall g \in F \forall f \in F(dom(g) \subseteq dom(f) \to \exists h \in F(h = f \restriction dom(g)))$$

and

$$\forall g \in F \forall f \in F(range(g) \subseteq range(f) \to \exists h \in F(h = (f^{-1} \restriction range(g))^{-1}))$$

are true in $\mathfrak{A}$. Then let $S$ be the graph of $i$ and we have now $(\mathfrak{A}_0, S) \models \Gamma(X)$.

Before constructing $\{i_m : m \in \omega\}$ let's observe two facts.

*Fact 1.* Suppose $f, g \in F_0$. If $dom(f) \cap dom(g) = \emptyset$ and $range(f) \cap range(g) = \emptyset$, then $f \cup g \in F_0$.

*Fact 2.* For any $f, g \in F_0$ there are $h, j \in F_0$ such that $dom(f) = dom(h)$, $range(g) \cap range(h) = \emptyset$ and $range(f) = range(j)$, $dom(g) \cap dom(j) = \emptyset$.

*Fact 1* and *Fact 2* above are true in $\mathfrak{A}_0$ because they are true in $\mathfrak{A}$.

Let $i_0 = f_0$. Assume that we have constructed $\{i_m : m < k\} \subseteq F_0$ such that

$$i_0 \subseteq i_1 \subseteq \cdots \subseteq i_{k-1},$$

$$dom(f_m) \subseteq dom(i_{2m}) \text{ when } 2m < k$$

and

$$range(f_m) \subseteq range(i_{2m+1}) \text{ when } 2m + 1 < k.$$

Case 1: $k = 2n$.

Let $C = dom(f_n) \smallsetminus dom(i_{k-1})$. If $C = \emptyset$, then let $i_k = i_{k-1}$. Otherwise let $h \in F_0$ such that $dom(h) = C$ and $range(h) \cap range(i_{k-1}) = \emptyset$. The function $h$ exists by *Fact 2.* Let $i_k = i_{k-1} \cup h$. The function $i_k \in F_0$ by *Fact 1.*

Case 2: $k = 2n + 1$.

Let $D = range(f_n) \smallsetminus range(i_{k-1})$. If $D = \emptyset$, then let $i_k = i_{k-1}$. Otherwise let $h \in F_0$ such that $range(h) = D$ and $dom(h) \cap dom(i_{k-1}) = \emptyset$. Again the function $h$ exists by *Fact 2* and $i_k = i_{k-1} \cup h \in F_0$ by *Fact 1.* $\square$

**Remark**: The claims in above four applications could also be shown by quoting simply four results from [R] or other papers. We present our own proofs here because these proofs use only countable models so that the reader can read the paper without knowing special models.

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