"On the Strong Equality between Supercompactness and Strong

by

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Abstract: We show that supercompactness and strong compactness can
properties of pairs of regular cardinals. Specifically, we show that if $V \models$
model (which in interesting cases contains instances of supercompactnes
cardinal and cofinality preserving generic extension $V[G] \models$ ZFC + GCH i
tion) for $\kappa \leq \lambda$ regular, if $V \models$ "κ is λ supercompact", then $V[G] \models$ "κ is
so that, (b) (equivalence) for $\kappa \leq \lambda$ regular, $V[G] \models$ "κ is λ strongly comp
supercompact", except possibly if κ is a measurable limit of cardinals whic

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§0 Introduction and Preliminaries

It is a well known fact that the notion of strongly compact cardi
singularity in the hierarchy of large cardinals. The work of Magidor [M
the least strongly compact cardinal and the least supercompact cardinal c
also, the least strongly compact cardinal and the least measurable cardin
The work of Kimchi and Magidor [KiM] generalizes this, showing that the
compact cardinals and the class of supercompact cardinals can coincide (e
of Menas [Me] and [A] at certain measurable limits of supercompact cardina
n strongly compact cardinals (for n a natural number) and the first n meas
can coincide. Thus, the precise identity of certain members of the class of st
cardinals cannot be ascertained vis à vis the class of measurable cardinal
supercompact cardinals.

An interesting aspect of the proofs of both [Ma1] and [KiM] is tha
all "bad" instances of strong compactness are not obliterated. Specifically
since the strategy employed in destroying strongly compact cardinals w
supercompact is to make them non-strongly compact after a certain point
a Prikry sequence or a non-reflecting stationary set of ordinals of the appro
there may be cardinals κ and λ so that κ is λ strongly compact yet κ isn't λ
Thus, whereas it was proven by Kimchi and Magidor that the classes of st
and supercompact cardinals can coincide (with the exceptions noted abo

The purpose of this paper is to answer the above question in the affir

cally, we prove the following

Theorem. *Suppose $V \models$ ZFC + GCH is a given model (which in interestin*

instances of supercompactness). There is then some cardinal and cofin

generic extension $V[G] \models$ ZFC + GCH in which:

(a) (Preservation) For $\kappa \le \lambda$ regular, if $V \models$ "κ is λ supercompact", then

supercompact". The converse implication holds except possibly when $\kappa =$

λ supercompact}.

(b) (Equivalence) For $\kappa \le \lambda$ regular, $V[G] \models$ "κ is λ strongly compact"

is λ supercompact", except possibly if κ is a measurable limit of cardin

supercompact.

Note that the limitation given in (b) above is reasonable, since trivia

surable, $\kappa < \lambda$, and $\kappa = \sup\{\delta < \kappa : \delta$ is either λ supercompact or λ stro

then κ is λ strongly compact. Further, it is a theorem of Menas [Me] th

for κ the first, second, third, or αth for $\alpha < \kappa$ measurable limit of cardina

strongly compact or κ^+ supercompact, κ is κ^+ strongly compact yet κ isn

pact. Thus, if there are sufficiently large cardinals in the universe, it will n

to have a complete coincidence between the notions of κ being λ strongly

being λ supercompact for λ a regular cardinal.

Specifically, if $\lambda > \kappa$ is so that $\mathrm{cof}(\lambda) < \kappa$, then by a theorem of Magido
supercompact iff κ is λ^+ supercompact, so automatically, by clause (a) of
supercompactness is preserved between V and $V[G]$. Also, if $\lambda > \kappa$ is so
then by a theorem of Solovay [SRK], κ is λ strongly compact iff κ is λ^+ st
so by clause (b) of our Theorem, it can never be the case that $V[G] \models$
compact" unless $V[G] \models$ "κ is λ supercompact" as well. Further, if $\lambda >$
$\mathrm{cof}(\lambda) \geq \kappa$, then it is not too difficult to see (and will be shown in Sect
is λ' strongly compact or λ' supercompact for all $\lambda' < \lambda$, then κ is λ st
and there is no reason to believe κ must be λ supercompact. In fact,
of Magidor [Ma4] (irrespective of GCH) that if μ is a supercompact car
always be many cardinals $\kappa, \lambda < \mu$ so that $\lambda > \kappa$ is a singular cardinal o
κ is λ strongly compact, κ is λ' supercompact for all $\lambda' < \lambda$, yet κ isn't λ
Thus, there can never be a complete coincidence between the notions of κ
compact and κ being λ supercompact if $\lambda > \kappa$ is an arbitrary cardinal, ass
supercompact cardinals in the universe.

The structure of this paper is as follows. Section 0 contains our introdu
and preliminary material concerning notation, terminology, etc. Sectio
discusses the basic properties of the forcing notion used in the iteratio
construct our final model. Section 2 gives a complete statement and proo
of Magidor mentioned in the above paragraph and proves our Theorem

Before beginning the material of Section 1, we briefly mention some pr

mation. Essentially, our notation and terminology are standard, and wher

case, this will be clearly noted. We take this opportunity to mention we w

GCH throughout the course of this paper. For $\alpha < \beta$ ordinals, $[\alpha, \beta], [\alpha$

(α, β) are as in standard interval notation. If f is the characteristic functio

then $x = \{\beta : f(\beta) = 1\}$.

When forcing, $q \geq p$ will mean that q is stronger than p. For P a part

formula in the forcing language with respect to P, and $p \in P$, $p \| \varphi$ will m

For G V-generic over P, we will use both $V[G]$ and V^P to indicate the u

by forcing with P. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We

to time, confuse terms with the sets they denote and write x when we a

especially when x is some variant of the generic set G.

If κ is a cardinal, then for P a partial ordering, P is (κ, ∞)-distrib

sequence $\langle D_\alpha : \alpha < \kappa \rangle$ of dense open subsets of P, $D = \bigcap_{\alpha < \kappa} D_\alpha$ is a de

of P. P is κ-closed if given a sequence $\langle p_\alpha : \alpha < \kappa \rangle$ of elements of P so

implies $p_\beta \leq p_\gamma$ (an increasing chain of length κ), then there is some p

bound to this chain) so that $p_\alpha \leq p$ for all $\alpha < \kappa$. P is $< \kappa$-closed if I

all cardinals $\delta < \kappa$. P is κ-directed closed if for every cardinal $\delta < \kappa$ an

set $\langle p_\alpha : \alpha < \delta \rangle$ of elements of P (where $\langle p_\alpha : \alpha < \delta \rangle$ is directed if for ev

elements $p_\rho, p_\nu \in \langle p_\alpha : \alpha < \delta \rangle$, p_ρ and p_ν have a common upper bound) th always be continued. P is $<\kappa$-strategically closed if P is δ-strategicall

cardinals $\delta < \kappa$. P is $\prec \kappa$-strategically closed if in the two person gan

players construct an increasing sequence $\langle p_\alpha : \alpha < \kappa \rangle$, where player I plays

player II plays even and limit stages, then player II has a strategy which e

can always be continued. Note that trivially, if P is κ-closed, then P i

closed and $\prec \kappa^+$-strategically closed. The converse of both of these facts i

For κ a regular cardinal, two partial orderings to which we will refe

the standard partial orderings Q^0_κ for adding a Cohen subset to κ^+ using c

support κ and Q^1_κ for adding κ^+ many Cohen subsets to κ using condition

$< \kappa$. The basic properties and explicit definitions of these partial orderin

in [J].

Finally, we mention that we are assuming complete familiarity wit

strong compactness and supercompactness. Interested readers may consult

for further details. We note only that all elementary embeddings witness

compactness of κ are presumed to come from some fine, κ-complete, nor

over $P_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$. Also, where appropriate, all ultrapowers

pact ultrafilter over $P_\kappa(\lambda)$ will be confused with their transitive isomorph

§1 The Forcing Conditions

In this section, we describe and prove the basic properties of the forcin

shall use in our later iteration. Let $\delta < \lambda$, $\lambda \geq \aleph_1$ be regular cardinals in o

S_p, a subset of α not stationary at its supremum nor having any initial se

stationary at its supremum, so that $\beta \in S_p$ implies $\beta > \delta$ and $\mathrm{cof}(\beta) =$

$q \geq p$ iff $q \supseteq p$ and $S_p = S_q \cap \sup(S_p)$, i.e., S_q is an end extension of S_p.

that for G V-generic over $P^0_{\delta,\lambda}$ (see [Bu] or [KiM]), in $V[G]$, a non-refle

set $S = S[G] = \cup\{S_p : p \in G\} \subseteq \lambda^+$ of ordinals of cofinality δ has been

bounded subsets of λ^+ are the same as those in V, and cardinals, cofina

have been preserved. It is also virtually immediate that $P^0_{\delta,\lambda}$ is δ-directed

Work now in $V_1 = V^{P^0_{\delta,\lambda}}$, letting $\dot{S}$ be a term always forced to denote

$P^2_{\delta,\lambda}[S]$ is the standard notion of forcing for introducing a club set C whi

S (and therefore makes S non-stationary). Specifically, $P^2_{\delta,\lambda}[S] = \{p :$ For

ordinal $\alpha < \lambda^+$, $p : \alpha \to \{0,1\}$ is a characteristic function of C_p, a club

that $C_p \cap S = \emptyset\}$, ordered by $q \geq p$ iff C_q is an end extension of C_p. It is a

(see [MS]) that for H V_1-generic over $P^2_{\delta,\lambda}[S]$, a club set $C = C[H] = \cup\{C$

which is disjoint to S has been introduced, the bounded subsets of λ^+

those in V_1, and cardinals, cofinalities, and GCH have been preserved.

Before defining in V_1 the partial ordering $P^1_{\delta,\lambda}[S]$ which will be used t

compactness, we first prove two preliminary lemmas.

Lemma 1. $\Vdash_{P^0_{\delta,\lambda}}$ "$\clubsuit(\dot{S})$", *i.e.*, $V_1 \models$ "*There is a sequence* $\langle x_\alpha : \alpha \in S\rangle$

$\alpha \in S$, $x_\alpha \subseteq \alpha$ *is cofinal in* α, *and for any* $A \in [\lambda^+]^{\lambda^+}$, $\{\alpha \in S : x_\alpha \subseteq A\}$ *i*

$x \in [\lambda^+]^{\delta}$, $\lambda^+ = \sup\{\alpha < \lambda^+ : \mathrm{cof}(\alpha) = \delta$ and $y_\alpha = x\}$. This then allo

$\langle x_\alpha : \alpha \in S\rangle$ by letting x_α be y_β for the least $\beta \in S - (\alpha + 1)$ so that $y_$

unbounded in α. By genericity, each x_α is well-defined.

Let now $p \in P^0_{\delta,\lambda}$ be so that $p \Vdash$ "$\dot{A} \in [\lambda^+]^{\lambda^+}$ and $\dot{K} \subseteq \lambda^+$ is club". W

some $r \geq p$ and some $\zeta < \lambda^+$, $r \Vdash$ "$\zeta \in \dot{K} \cap \dot{S}$ and $\dot{x}_\zeta \subseteq \dot{A}$". To do this, we in

an increasing sequence $\langle p_\alpha : \alpha < \delta\rangle$ of elements of $P^0_{\delta,\lambda}$ and increasing sequ

$\delta\rangle$ and $\langle \gamma_\alpha : \alpha < \delta\rangle$ of ordinals $< \lambda^+$ so that $\beta_0 \leq \gamma_0 \leq \beta_1 \leq \gamma_1 \leq \cdots \leq$

$(\alpha < \delta)$. We begin by letting $p_0 = p$ and $\beta_0 = \gamma_0 = 0$. For $\eta = \alpha + 1$ <

let $p_\eta \geq p_\alpha$ and $\beta_\eta \leq \gamma_\eta$, $\beta_\eta \geq \max(\beta_\alpha, \gamma_\alpha, \sup(\mathrm{dom}(p_\alpha))) + 1$ be so tha

and $\gamma_\eta \in \dot{K}$". For $\rho < \delta$ a limit, let $p_\rho = \bigcup_{\alpha<\rho} p_\alpha$, $\beta_\rho = \bigcup_{\alpha<\rho} \beta_\alpha$, and γ_ρ

that since $\rho < \delta$, p_ρ is well-defined, and since $\delta < \lambda^+$, $\beta_\rho, \gamma_\rho < \lambda^+$. Also,

$\bigcup_{\alpha<\delta} \beta_\alpha = \bigcup_{\alpha<\delta} \gamma_\alpha = \bigcup_{\alpha<\delta} \sup(\mathrm{dom}(p_\alpha)) < \lambda^+$. Call ζ this common sup. We

$q = \bigcup_{\alpha<\delta} p_\alpha \cup \{\zeta\}$ is a well-defined condition so that $q \Vdash$ "$\{\beta_\alpha : \alpha \in \delta -$

$\zeta \in \dot{K} \cap \dot{S}$".

To complete the proof of Lemma 1, we know that as $\langle \beta_\alpha : \alpha \in \delta - \{$

each $y \in \langle y_\alpha : \alpha < \lambda^+\rangle$ must appear λ^+ times at ordinals of cofinality δ, w

$\eta \in (\zeta, \lambda^+)$ so that $\mathrm{cof}(\eta) = \delta$ and $\langle \beta_\alpha : \alpha \in \delta - \{0\}\rangle = y_\eta$. If we let r

$r \Vdash$ "$\dot{S} \cap [\zeta, \eta] = \{\zeta, \eta\}$", then $r \Vdash$ "$\dot{x}_\zeta = y_\eta = \langle \beta_\alpha : \alpha \in \delta - \{0\}\rangle$". This pro

LEMMA 2. *Let S' be an initial segment of S so that S' is not stationary a*
nor has any initial segment which is stationary at its supremum. There is
$\langle y_\alpha : \alpha \in S' \rangle$ so that for every $\alpha \in S'$, $y_\alpha \subseteq x_\alpha$, $x_\alpha - y_\alpha$ is bound
$\alpha_1 \neq \alpha_2 \in S'$, then $y_{\alpha_1} \cap y_{\alpha_2} = \emptyset$.

PROOF OF LEMMA 2: We define by induction on $\alpha \leq \alpha_0 = \sup S' + 1$ a
that $\mathrm{dom}(h_\alpha) = S' \cap \alpha$, $h_\alpha(\beta) < \beta$, and $\langle x_\beta - h_\alpha(\beta) : \beta \in S' \cap \alpha \rangle$ is pairwi
sequence $\langle x_\beta - h_{\alpha_0}(\beta) : \beta \in S' \rangle$ will be our desired sequence.

If $\alpha = 0$, then we take h_α to be the empty function. If $\alpha = \beta + 1$ a
we take $h_\alpha = h_\beta$. If $\alpha = \beta + 1$ and $\beta \in S'$, then we notice that sinc
has order type δ and is cofinal in γ, for all $\gamma \in S' \cap \beta$, $x_\beta \cap \gamma$ is boun
allows us to define a function h_α having domain $S' \cap \alpha$ by $h_\alpha(\beta) = 0$, and
$h_\alpha(\gamma) = \min(\{\rho : \rho < \gamma, \rho \geq h_\beta(\gamma), \text{ and } x_\beta \cap \gamma \subseteq \rho\})$. By the next to la
the induction hypothesis on h_β, $h_\alpha(\gamma) < \gamma$. And, if $\gamma_1 < \gamma_2 \in S' \cap \alpha$,
$(x_{\gamma_1} - h_\alpha(\gamma_1)) \cap (x_{\gamma_2} - h_\alpha(\gamma_2)) \subseteq (x_{\gamma_1} - h_\beta(\gamma_1)) \cap (x_{\gamma_2} - h_\beta(\gamma_2)) = \emptyset$ b
hypothesis on h_β. If $\gamma_2 = \beta$, then $(x_{\gamma_1} - h_\alpha(\gamma_1)) \cap (x_{\gamma_2} - h_\alpha(\gamma_2)) = (x_{\gamma_1} - h$
by the definition of $h_\alpha(\gamma_1)$. The sequence $\langle x_\gamma - h_\alpha(\gamma) : \gamma \in S' \cap \alpha \rangle$ is thu

If α is a limit ordinal, then as S' is non-stationary at its supremu
initial segment which is stationary at its supremum, we can let $\langle \beta_\gamma : \gamma$
strictly increasing, continuous sequence having sup α so that for all $\gamma <$

And, if $\rho_1, \rho_2 \in (\beta_\gamma, \beta_{\gamma+1})$, then $(x_{\rho_1} - h_\alpha(\rho_1)) \cap (x_{\rho_2} - h_\alpha(\rho_2)) \subseteq (x_{\rho_1}$

$(x_{\rho_2} - h_{\beta_{\gamma+1}}(\rho_2)) = \emptyset$ by the definition of $h_{\beta_{\gamma+1}}$. If $\rho_1 \in (\beta_\gamma, \beta_{\gamma+1})$, $\rho_2 \in$

$\gamma < \sigma$, then $(x_{\rho_1} - h_\alpha(\rho_1)) \cap (x_{\rho_2} - h_\alpha(\rho_2)) \subseteq x_{\rho_1} \cap (x_{\rho_2} - \beta_\sigma) \subseteq \rho_1 - \beta_\sigma$

Thus, the sequence $\langle x_\rho - h_\alpha(\rho) : \rho \in S' \cap \alpha \rangle$ is again as desired. This pro

At this point, we are in a position to define in V_1 the partial orderin

will be used to destroy strong compactness. $P^1_{\delta,\lambda}[S]$ is now the set of all 4-tu

satisfying the following properties.

1. $w \in [\lambda^+]^{<\lambda}$.

2. $\alpha < \lambda$.

3. $\bar{r} = \langle r_i : i \in w \rangle$ is a sequence of functions from α to $\{0, 1\}$, i.e., a seq

   of α.

4. $Z \subseteq \{x_\beta : \beta \in S\}$ is a set so that if $z \in Z$, then for some $y \in [w]^\delta$, y

   bounded in the β so that $z = x_\beta$.

Note that the definition of Z implies $|Z| < \lambda$.

The ordering on $P^1_{\delta,\lambda}[S]$ is given by $\langle w^1, \alpha^1, \bar{r}^1, Z^1 \rangle \le \langle w^2, \alpha^2, \bar{r}^2, Z^2 \rangle$

hold.

1. $w^1 \subseteq w^2$.

2. $\alpha^1 \le \alpha^2$.

3. If $i \in w^1$, then $r^1_i \subseteq r^2_i$.

If $W = \langle\langle w^\beta, \alpha^\beta, \bar{r}^\beta, Z^\beta\rangle_{\beta<\gamma<\delta}\rangle$ is a directed set of elements of $P^1_{\delta,\lambda}[S$

the regularity of δ any δ sequence from $\bigcup_{\beta<\gamma} w^\beta$ must contain a δ sequen

some $\beta < \gamma$, it can easily be verified that $\langle \bigcup_{\beta<\gamma} w^\beta, \bigcup_{\beta<\gamma} \alpha^\beta, \bigcup_{\beta<\gamma} \bar{r}^\beta, \bigcup_{\beta<\gamma} Z^\beta\rangle$ is

for each element of W. (Here, if $\bar{r}^\beta = \langle r^\beta_i : i \in w^\beta\rangle$, then $r_i \in \bigcup_{\beta<\gamma} \bar{r}^\beta$ if

$r_i = \bigcup_{\beta<\gamma} r^\beta_i$, taking $r^\beta_i = \emptyset$ if $i \notin w^\beta$.) This means $P^1_{\delta,\lambda}[S]$ is δ-directed clos

At this point, a few intuitive remarks are in order. If κ is λ stron

$\lambda \geq \kappa$ regular, then it must be the case (see [SRK]) that λ carries a κ-a

ultrafilter. If $\delta < \kappa < \lambda$, the forcing $P^1_{\delta,\lambda}[S]$ has specifically been designed

fact. It has been designed, however, to destroy the λ strong compactness o

possible", making little damage. In the case of the argument of [KiM], th

stationary set S is added directly to λ in order to kill the λ strong compact

situation, the non-reflecting stationary set S, having been added to λ^+ an

not kill the λ strong compactness of κ by itself. The additional forcing $P^1_{\delta,\lambda}$

to do the job. The forcing $P^1_{\delta,\lambda}[S]$, however, has been designed so that if n

resurrect the λ supercompactness of κ by forcing further with $P^2_{\delta,\lambda}[S]$.

Lemma 3. *$V_1^{P^1_{\delta,\lambda}[S]} \models$ "κ is not λ strongly compact" if $\delta < \kappa < \lambda$.*

Remark: Since we will only be concerned in general when κ is stron

and $\delta < \kappa < \lambda$, we assume without loss of generality that this is the case

rest of the paper.

that p can be extended to a condition q so that for some ordinal α^q <

sequence $\langle s_i : i < \delta\rangle$ of $\mathcal{D}$ measure 1 sets, $q\Vdash$ "$\bigcap_{i<\delta} \dot{s}_i \subseteq \alpha^q$", an immediate

We use a Δ-system argument to establish this. First, for G_1 V_1-gene

and $i < \lambda^+$, let $r^*_i = \cup\{r^p_i : \exists p = \langle w^p, \alpha^p, \bar{r}^p, Z^p\rangle \in G_1[r^p_i \in \bar{r}^p]\}$. It

$\Vdash_{P^1_{\delta,\lambda}[S]}$ "$\dot{r}^*_i : \lambda \to \{0, 1\}$ is a function whose domain is all of λ". To s

$\langle w^p, \alpha^p, \bar{r}^p, Z^p\rangle$, since $|Z^p| < \lambda$, $w^p \in [\lambda^+]^{<\lambda}$, and $z \in Z^p$ implies $z \in [\lambda^+]$

$q = \langle w^q, \alpha^q, \bar{r}^q, Z^q\rangle$ given by $\alpha^q = \alpha^p$, $Z^q = Z^p$, $w^q = w^p \cup \bigcup\{z : z \in Z^p$

$i \in w^q\rangle$ defined by $r'_i = r_i$ if $i \in w^p$ and r'_i is the empty function if $i \in w^q$

defined condition. (This just means we may as well assume that for $p =$

$z \in Z^p$ implies $z \subseteq w^p$.) Further, since $|Z^q| < \lambda$, $\cup\{\beta : \exists z \in Z^q[z =$

Therefore, if $\gamma' \in (\gamma, \lambda^+)$ and $S' \subseteq \gamma'$ is so that $\sup S' = \gamma'$ and S' is an

of S so that S' is not stationary at its supremum nor has any initial se

stationary at its supremum, then by Lemma 2, there is a sequence $\langle y_\beta : \beta$

every $\beta \in S'$, $y_\beta \subseteq x_\beta$, $x_\beta - y_\beta$ is bounded in β, and if $\beta_1 \neq \beta_2 \in S'$, the

This means that if $z \in Z^q$ and $z = x_\beta$ for some β, then $y_\beta \subseteq w$.

Choose now for $\beta \in S'$ sets y^1_β and y^2_β so that $y_\beta = y^1_\beta \cup y^2_\beta$, y^1_β

$|y^1_\beta| = |y^2_\beta| = \delta$. If $\rho \in (\alpha^q, \lambda)$, then for each β so that $x_\beta \in Z^q$ and f

such that $i \in y_\beta$, we can extend r'_i to $r''_i : \rho \to \{0, 1\}$ by letting $r''_i|\alpha^q =$

$\alpha \in [\alpha^q, \rho)$, $r''_i(\alpha) = 0$ if $i \in y^1_\beta$ and $r''_i(\alpha) = 1$ if $i \in y^2_\beta$. For $i \in w^q$ so density that $\Vdash_{P^1_{\delta,\lambda}[S]}$ " $\dot r_i^* \to \{0,1\}$ is a function whose domain is all of λ"

let $r_i^\ell = \{\alpha < \lambda : r_i^*(\alpha) = \ell\}$ for $\ell \in \{0,1\}$.

For each $i < \lambda^+$, pick $p_i = \langle w^{p_i}, \alpha^{p_i}, \bar r^{p_i}, Z^{p_i}\rangle \geq p$ so that $p_i \Vdash$ "$\dot r_i^{\ell(i)}$

$\ell(i) \in \{0,1\}$. This is possible since $\Vdash_{P^1_{\delta,\lambda}[S]}$ "For each $i < \lambda^+$, $\dot r_i^0 \cup \dot r_i^1 = \lambda$

of generality, by extending p_i if necessary, we can assume that $i \in w^{p_i}$.

$w^{p_i} \in [\lambda^+]^{<\lambda}$, we can find some stationary $A \subseteq \{i < \lambda^+ : \mathrm{cof}(i) = \lambda\}$ so th

forms a Δ-system, i.e., so that for $i \neq j \in A$, $w^{p_i} \cap w^{p_j}$ is some constant

an initial segment of both. (Note we can assume that for $i \in A$, $w_i \cap i =$

fixed $\ell(*) \in \{0,1\}$, for every $i \in A$, $p_i \Vdash$ "$\dot r_i^{\ell(*)} \in \dot{\mathcal{D}}$".) Also, by clause 4)

of the forcing, $|Z^{p_i}| < \lambda$ for each $i < \lambda^+$. Therefore, $Z^{p_i} \in [[\lambda^+]^\delta]^{<\lambda}$, so

by GCH, the same sort of Δ-system argument allows us to assume in addi

$i \in A$, $Z^{p_i} \cap \mathcal{P}(w)$ is some constant value Z. Further, since each $\alpha^{p_i} < \lambda$,

that α^{p_i} is some constant α^0 for $i \in A$. Then, since any $\bar r^{p_i} = \langle r_j : j \in$

composed of a sequence of functions from α_0 to 2, $\alpha_0 < \lambda$, and $|w| < \lambda$,

to conclude that for $i \neq j \in A$, $\bar r^{p_i} | w = \bar r^{p_j} | w$. And, since $i \in w^{p_i}$, we kn

also assume (by thinning A if necessary) that $B = \{\sup(w^{p_i}) : i \in A\}$ is so

implies $i \leq \sup(w^{p_i}) < \min(w^{p_j} - w) \leq \sup(w^{p_j})$. We know in addition

$X = \langle x_\beta : \beta \in S\rangle$ that for some $\gamma \in S$, $x_\gamma \subseteq A$. Let $x_\gamma = \{i_\beta : \beta < \delta\}$.

We are now in a position to define the condition q referred to earli

by defining each of the four coordinates of q. First, let $w^q = \bigcup_{\beta<\delta} w^{p_{i_\beta}}$. A define Z^q, let $Z^q = \bigcup_{\beta<\delta} Z^{p_\beta} \cup \{\{i_\beta : \beta < \delta\}\}$. By the last three sentences i

paragraph and our construction, $\{i_\beta : \beta < \delta\}$ generates a new set which

in Z^q, and Z^q is well-defined.

We claim now that $q \geq p$ is so that $q \Vdash$ " $\bigcap_{\beta<\delta} \dot{r}^{\ell(*)}_{i_\beta} \subseteq \alpha^q$". To see t

claim fails. This means that for some $q^1 \geq q$ and some $\alpha^q \leq \eta < \lambda$, $q^1 \Vdash$

Without loss of generality, since q^1 can always be extended if necessary,

that $\eta < \alpha^{q^1}$. But then, by the definition of $\leq$, for δ many $\beta < \delta$, $q^1 \Vdash$

immediate contradiction. Thus, $q \Vdash$ " $\bigcap_{\beta<\delta} \dot{r}^{\ell(*)}_{i_\beta} \subseteq \alpha^q$", which, since $\delta < \kappa$,

$q \Vdash$ " $\bigcap_{\beta<\delta} \dot{r}^{\ell(*)}_{i_\beta} \in \dot{\mathcal{D}}$ and $\dot{\mathcal{D}}$ is a κ-additive uniform ultrafilter over λ". This p

Recall we mentioned prior to the proof of Lemma 3 that $P^1_{\delta,\lambda}[S]$ is de

further forcing with $P^2_{\delta,\lambda}[S]$ will resurrect the λ supercompactness of κ, assu

iteration has been done. That this is so will be shown in the next section. I

we give an idea of why this will happen by showing that the forcing $P^0_{\delta,\lambda} * (I$

is rather nice. Specifically, we have the following lemma.

Lemma 4. *$P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$ is equivalent to $Q^0_\lambda * \dot{Q}^1_\lambda$.*

Proof of Lemma 4: Let G be V-generic over $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$, with

$G^2_{\delta,\lambda}$ the projections onto $P^0_{\delta,\lambda}$, $P^1_{\delta,\lambda}[S]$, and $P^2_{\delta,\lambda}[S]$ respectively. Each $G^i_{\delta,\lambda}$

generic. So, since $P^1_{\delta,\lambda}[S] \times P^2_{\delta,\lambda}[S]$ is a product in $V[G^0_{\delta,\lambda}]$, we can rewri such that $D = \{\langle p, q\rangle \in P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot S]$: For some α, $\mathrm{dom}(p) = \mathrm{dom}(q) = \alpha$

and $q \Vdash$ "$\alpha \in \dot C$"$\}$ is dense in $P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot S]$ and is λ-closed. This easily im

equivalence. Thus, V and $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$ have the same cardinals and cofi

proof of Lemma 4 will be complete once we show that in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$, $P^1_{\delta,\lambda}$

to Q^1_λ.

To this end, working in $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$, we first note that as $S \subseteq \lambda$

stationary set all of whose initial segments are non-stationary, by Lemma 2,

$\langle x_\beta : \beta \in S\rangle$, there must be a sequence $\langle y_\beta : \beta \in S\rangle$ so that for every β

$x_\beta - y_\beta$ is bounded in β, and if $\beta_1 \neq \beta_2 \in S$, then $y_{\beta_1} \cap y_{\beta_2} = \emptyset$. Give

easy to observe that $P^1 = \{\langle w, \alpha, \bar r, Z\rangle \in P^1_{\delta,\lambda}[S]$: For every $\beta \in S$, ei

$y_\beta \cap w = \emptyset\}$ is dense in $P^1_{\delta,\lambda}[S]$. To show this, given $\langle w, \alpha, \bar r, Z\rangle \in P^1_{\delta,\lambda}[S]$,

let $Y_w = \{y \in \langle y_\beta : \beta \in S\rangle : y \cap w \neq \emptyset\}$. As $|w| < \lambda$ and $y_{\beta_1} \cap y_{\beta_2} = \emptyset$ f

$|Y_w| < \lambda$. Hence, as $|y| = \delta < \lambda$ for $y \in Y_w$, $|w'| < \lambda$ for $w' = w \cup (\cup Y$

$\langle w', \alpha, \bar r', Z\rangle$ for $\bar r' = \langle r'_i : i \in w'\rangle$ defined by $r'_i = r_i$ if $i \in w$ and r'_i is the e

$i \in w' - w$ is a well-defined condition extending $\langle w, \alpha, \bar r, Z\rangle$. Thus, P^1 is

so to analyze the forcing properties of $P^1_{\delta,\lambda}[S]$, it suffices to analyze the fo

of P^1.

For $\beta \in S$, let $Q_\beta = \{\langle w, \alpha, \bar r, Z\rangle \in P^1 : w = y_\beta\}$, and let $Q' = \{\langle w$

$w \subseteq \lambda^+ - \bigcup_{\beta\in S} y_\beta\}$. Let Q'' be those elements of $\prod_{\beta\in S} Q_\beta \times Q'$ of support

product ordering. Adopting the notation of Lemma 3, given $p = \langle\langle q_\beta :$

p and p' different coordinates of $p\}$ is dense in Q', so Q and Q'' are for

Then, for $p = \langle\langle q_\beta : \beta \in A\rangle, q\rangle \in Q$ where $A \subseteq S$ and $|A| < \lambda$, as w

for $\beta_1 \neq \beta_2 \in A$ $(y_{\beta_1} \cap y_{\beta_2} = \emptyset)$, $w^{q_{\beta_1}} \cap w^q = \emptyset$, $\alpha^{q_{\beta_1}} = \alpha^{q_{\beta_2}} = \alpha^q$ fo

the domains of any two $\bar{r}^{q_{\beta_1}}$, $\bar{r}^{q_{\beta_2}}$ are disjoint for $\beta_1 \neq \beta_2 \in A$, $Z^{q_{\beta_1}}$

$\beta_1 \neq \beta_2 \in A$, the domains of $\bar{r}^{q_\beta}$ and $\bar{r}^q$ are disjoint for $\beta \in A$, and $Z^{q_\beta} \cap Z$

the function $F(p) = \langle \bigcup_{\beta \in A} w^{q_\beta} \cup w^q, \alpha, \bigcup_{\beta \in A} \bar{r}^{q_\beta} \cup \bar{r}^q, \bigcup_{\beta \in A} Z^{q_\beta} \cup Z^q\rangle$ can easily

an isomorphism between Q and P^1. Thus, over $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$, forcing with

and Q'' are all equivalent.

We examine now in more detail the exact nature of Q''. For $\beta \in$

$|Q_\beta| = \lambda$. It quickly follows from the definition of Q_β that Q_β is $< \lambda$-

forcing equivalent to adding a Cohen subset to λ. Since the definitions of

ensure that for $\langle w, \alpha, \bar{r}, Z\rangle \in Q'$, $Z = \emptyset$ (for every $\beta \in S$, $w \cap y_\beta = \emptyset$

$x_\beta - y_\beta$ is bounded in δ), Q' can easily be seen to be a re-representatio

forcing where instead of working with functions whose domains have card

are subsets of $\lambda \times \lambda^+$, we work with functions whose domains have cardina

subsets of $\lambda \times (\lambda^+ - \bigcup_{\beta \in S} y_\beta)$. Thus, Q'' is isomorphic to a Cohen forcing

having domains of cardinality $< \lambda$ which adds λ^+ many Cohen subsets t

sentence of the last paragraph, this means that over $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$, the forc

Q^1_λ are equivalent. This proves Lemma 4.

strong compactness of κ when forcing with $P^2_{\delta,\lambda}[S]$. Once the fact S is
been destroyed by forcing with $P^2_{\delta,\lambda}[S]$, Lemma 4 shows that this last coo
condition $p \in P^1_{\delta,\lambda}[S]$ and change in the ordering in a sense become irrelev

It is clear from Lemma 4 that $P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$, being equival
preserves GCH, cardinals, and cofinalities, and has a dense subset which is
satisfies λ^{++}-c.c. Our next lemma shows that the forcing $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ is

LEMMA 5. $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ *preserves GCH, cardinals, and cofinalities, is <*
closed, and is λ^{++}-c.c.

PROOF OF LEMMA 5: Let $G' = G^0_{\delta,\lambda} * G^1_{\delta,\lambda}$ be V-generic over $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$
$V[G']$-generic over $P^2_{\delta,\lambda}[S]$. Thus, $G' * G^2_{\delta,\lambda} = G$ is V-generic over $P^0_{\delta,\lambda} * (P^1_{\delta,}$
$P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] \times P^2_{\delta,\lambda}[\dot{S}])$. By Lemma 4, $V[G] \models$ GCH and has the sam
cofinalities as V, so since $V[G'] \subseteq V[G]$, forcing with $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ over V
cardinals, and cofinalities.

We next show the $< \lambda$-strategic closure of $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$. We first note
$P^1_{\delta,\lambda}[\dot{S}]) * P^2_{\delta,\lambda}[\dot{S}] = P^0_{\delta,\lambda} * (P^1_{\delta,\lambda}[\dot{S}] * P^2_{\delta,\lambda}[\dot{S}])$ has by Lemma 4 a dense subs
closed, the desired fact follows from the more general fact that if $P * \dot{Q}$ is a
with a dense subset R so that R is $< \lambda$-closed, then P is $< \lambda$-strategically
this more general fact, let $\gamma < \lambda$ be a cardinal. Suppose I and II play to bui
chain of elements of P, with $\langle p_\beta : \beta \leq \alpha + 1 \rangle$ enumerating all plays by I

$R \subseteq P * \dot{Q}$. At stage $\alpha + 2$, II chooses $\langle p_{\alpha+2}, \dot{q}_{\alpha+2}\rangle$ so that $\langle p_{\alpha+2}, \dot{q}_{\alpha+2}\rangle \in$

$\langle p_{\alpha+2}, \dot{q}_{\alpha+2}\rangle \geq \langle p_{\alpha+1}, \dot{q_\alpha}\rangle$; this makes sense, since inductively, $\langle p_\alpha, \dot{q}_\alpha\rangle \in$

as I has chosen $p_{\alpha+1} \geq p_\alpha$, $\langle p_{\alpha+1}, \dot{q}_\alpha\rangle \in P * \dot{Q}$. By the $< \lambda$-closure of

stage $\eta \leq \gamma$, II can choose $\langle p_\eta, \dot{q_\eta}\rangle$ so that $\langle p_\eta, \dot{q_\eta}\rangle$ is an upper bound to

and β is even or a limit ordinal$\rangle$. The preceding yields a winning strateg

$< \lambda$-strategically closed.

Finally, to show $P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ is λ^{++}-c.c., we simply note that this f

general fact about iterated forcing (see [Ba]) that if $P * \dot{Q}$ satisfies λ^{++}-c.c.,

λ^{++}-c.c. (Here, $P = P^0_{\delta,\lambda} * P^1_{\delta,\lambda}[\dot{S}]$ and $Q = P^2_{\delta,\lambda}[\dot{S}]$.) This proves Lemma

We remark that $\Vdash_{P^0_{\delta,\lambda}}$ "$P^1_{\delta,\lambda}[\dot{S}]$ is λ^+-c.c.", for if $\mathcal{A} = \langle p_\alpha : \alpha < \lambda^+\rangle$

antichain of elements of $P^1_{\delta,\lambda}[S]$ in $V[G^0_{\delta,\lambda}]$, then as $V[G^0_{\delta,\lambda}]$ and $V[G^0_{\delta,\lambda}$

same cardinals, $\mathcal{A}$ would be a size λ^+ antichain of elements of $P^1_{\delta,\lambda}[S]$ i

By Lemma 4, in this model, a dense subset of $P^1_{\delta,\lambda}[S]$ is isomorphic to Q^1_λ

same definition in either $V[G^0_{\delta,\lambda}]$ or $V[G^0_{\delta,\lambda}][G^2_{\delta,\lambda}]$ (since $P^0_{\delta,\lambda}$ is λ-strategi

$P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]$ is λ-closed) and so is λ^+-c.c. in either model.

We conclude this section with a lemma which will be used later in

LEMMA 6. *For* $V_1 = V^{P^0_{\delta,\lambda}}$, *the models* $V_1^{[illegible]}$ $[illegible]$ *and* $V_1^{P^1_{\delta,\lambda}[S]}$ *con*

sequences of elements of V_1.

PROOF OF LEMMA 6: By Lemma 4, since $P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]$ is equivalent to

and $V \subseteq V^{P^0_{\delta,\lambda}} \subseteq V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$, the models V, $V^{P^0_{\delta,\lambda}}$, and $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$ all c

λ sequences of elements of V. Thus, since a λ sequence of elements of V_1

represented by a V-term which is actually a function $h : \lambda \to V$, it imm

that $V^{P^0_{\delta,\lambda}}$ and $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$ contain the same λ sequences of elements of

Let now $f : \lambda \to V_1$ be so that $f \in (V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]})^{P^1_{\delta,\lambda}[S]} = V_1^{P^1_{\delta,\lambda}[S]}$

$g : \lambda \to V_1$, $g \in V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$ be a term for f. By the previous paragraph,

Lemma 4 shows that $P^1_{\delta,\lambda}[S]$ is λ^+-c.c. in $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$, for each $\alpha < \lambda$, t

defined in $V^{P^0_{\delta,\lambda} * P^2_{\delta,\lambda}[\dot{S}]}$ by $\{p \in P^1_{\delta,\lambda}[S] : p$ decides a value for $g(\alpha)\}$ is so tha

"$|\mathcal{A}_\alpha| \le \lambda$". Hence, by the preceding paragraph, since $\mathcal{A}_\alpha$ is a set of ele

$\mathcal{A}_\alpha \in V^{P^0_{\delta,\lambda}}$ for each $\alpha < \lambda$. Therefore, again by the preceding paragrap

$\langle \mathcal{A}_\alpha : \alpha < \lambda \rangle \in V^{P^0_{\delta,\lambda}}$. This just means that the term $g \in V^{P^0_{\delta,\lambda}}$ can

$V_1^{P^1_{\delta,\lambda}[S]}$, i.e., $f \in V_1^{P^1_{\delta,\lambda}[S]}$. This proves Lemma 6.

§2 The Case of One Supercompact Cardinal with no Larger Inaccessi

In this section, we give a proof of our Theorem, starting from a mo a supercompact cardinal, then there always must be cardinals $\delta < \lambda$ so tha

compact yet δ isn't λ supercompact.

Lemma 7. *(Magidor [Ma4]): Suppose κ is a supercompact cardinal. Then*

is λ_δ strongly compact for λ_δ the least singular strong limit cardinal $> \delta$

is not λ_δ supercompact, yet δ is α supercompact for all $\alpha < \lambda_\delta\}$ is unbou

Proof of Lemma 7: Let $\lambda_\kappa > \kappa$ be the least singular strong limit cardi

κ, and let $j : V \to M$ be an elementary embedding witnessing the λ_κ su

of κ with $j(\kappa)$ minimal. As $j(\kappa)$ is least, $M \models$ "κ is not λ_κ supercompact"

and λ_κ is a strong limit cardinal, $M \models$ "κ is α supercompact for all $\alpha <$

Let $\mu \in V$ be a κ-additive measure over κ, and let $\langle \lambda_\alpha : \alpha < \lambda_\kappa \rangle$ b

cardinals cofinal in λ_κ in both V and M. As $M^{\lambda_\kappa} \subseteq M$ and λ_κ is a stron

$\mu \in M$. Also, as $M \models$ "κ is α supercompact for all $\alpha < \lambda_\kappa$", the closure

allow us to find a sequence $\langle \mu_\alpha : \alpha < \kappa \rangle \in M$ so that $M \models$ "μ_α is a fine, no

ultrafilter over $P_\kappa(\lambda_\alpha)$". Thus, we can define in M the collection μ^* of subs

$A \in \mu^*$ iff $\{\alpha < \kappa$: $A|\lambda_\alpha \in \mu_\alpha\} \in \mu$, where for $A \subseteq P_\kappa(\lambda_\kappa)$, $A|\lambda_\alpha = \{p \cap$

It is easily checked that μ^* defines in M a κ-additive fine ultrafilter over

$M \models$ "κ is α supercompact for all $\alpha < \lambda_\kappa$, κ is not λ_κ supercompact, yet

We note that the proof of Lemma 7 goes through if λ_δ becomes th

strong limit cardinal $> \delta$ of cofinality δ^+, of cofinality δ^{++}, etc. To see th

the closure properties of M and the strong compactness of κ ensure tha

each carry κ-additive measures μ_{κ^+}, $\mu_{\kappa^{++}}$, etc. which are elements of M.

may then be used in place of the μ of Lemma 7 to define the strongly co

μ^* over $P_\kappa(\lambda_\kappa)$.

We return now to the proof of our Theorem. Let $\bar{\delta} = \langle \delta_\alpha : \alpha \leq \kappa \rangle$

inaccessibles $\leq \kappa$, with $\delta_\kappa = \kappa$. Note that since we are in the simple cas

the only supercompact cardinal in the universe and has no inaccessibles a

assume each δ_α isn't $\delta_{\alpha+1}$ supercompact and for the least regular cardinal

$V \models$ "δ_α isn't λ_α supercompact", $\lambda_\alpha < \delta_{\alpha+1}$. (If δ were the least cardinal

supercompact for β the least inaccessible $> \delta$ yet δ isn't β supercompact

provide the desired model.)

We are now in a position to define the partial ordering P used in t

Theorem. We define a κ stage Easton support iteration $P_\kappa = \langle \langle P_\alpha, \dot{Q}_\alpha \rangle : \alpha$

define $P = P_{\kappa+1} = P_\kappa * \dot{Q}_\kappa$ for a certain class partial ordering Q_κ definab

definition is as follows:

1. P_0 is trivial.

subset of λ^+ introduced by $P^0_{\omega,\lambda}$ for $\lambda < \lambda_\alpha$ and $\dot S_{\lambda_\alpha}$ is a term for th

stationary subset of λ^+_α introduced by $P^0_{\omega,\lambda_\alpha}$.

3. $\dot Q_\kappa$ is a term for the Easton support iteration of $\langle P^0_{\omega,\lambda} * (P^1_{\omega,\lambda}[\dot S_\lambda] \times$

is a regular cardinal$\rangle$, where as before, $\dot S_\lambda$ is a term for the non-refle

subset of λ^+ introduced by $P^0_{\omega,\lambda}$.

The intuitive motivation behind the above definition is that below κ a

ble, we must first destroy and then resurrect all "good" instances of stro

i.e., those which also witness supercompactness, but then destroy the leas

instance of strong compactness, thus destroying all "bad" instances of s

ness beyond the least "bad" instance. Since κ is supercompact, it has no

of strong compactness, so all instances of κ's supercompactness are dest

resurrected.

Lemma 8. *For G a V-generic class over P, V and $V[G]$ have the sam*

cofinalities, and $V[G] \models$ ZFC + GCH.

Proof of Lemma 8: Write $G = G_\kappa * H$, where G_κ is V-generic over

$V[G_\kappa]$-generic class over Q_κ. We show $V[G_\kappa][H] \models$ ZFC, and by assumi

being that $V[G_\kappa] \models$ GCH and has the same cardinals and cofinalities

$V[G_\kappa][H] \models$ GCH and has the same cardinals and cofinalities as $V[G_\kappa]$ (a

To do this, note that Q_κ is equivalent in $V[G_\kappa] = V_1$ to the Easton s cardinal). We first show inductively that for any successor cardinal $\delta^+ >$ V_1 with the iteration of $\langle Q^0_\lambda * \dot{Q}^1_\lambda : \kappa < \lambda < \delta^+$ and λ is a successor car cardinals, cofinalities, and GCH. If δ is regular (meaning δ is a successo κ has no inaccessibles above it), then this iteration can be written as Q where $Q_{<\delta}$ is the iteration of $\langle Q^0_\lambda * \dot{Q}^1_\lambda : \kappa < \lambda < \delta$ and λ is a successo induction, forcing over V_1 with $Q_{<\delta}$ preserves cardinals, cofinalities, and forcing over $V_1^{Q_{<\delta}}$ with $\dot{Q}^0_\delta * \dot{Q}^1_\delta$ will preserve GCH and the cardinals an $V_1^{Q_{<\delta}}$, forcing over V_1 with $Q_{<\delta} * (\dot{Q}^0_\delta * \dot{Q}^1_\delta)$ preserves cardinals, cofinalities is singular, let $\gamma < \delta$ be a cardinal in V_1, and write the iteration of $\langle Q^0_\lambda *$ and λ is a successor cardinal$\rangle$ as $Q_{<\gamma^+} * \dot{Q}^{\geq\gamma^+}$, where $Q_{<\gamma^+}$ is as above term in V_1 for the rest of the iteration; if $\gamma < \kappa$, then $Q_{<\gamma^+}$ is trivial term for the whole iteration. By induction, $V_1^{Q_{<\gamma^+}} \models$ "γ is a cardinal $\mathrm{cof}(\gamma) = \mathrm{cof}^{V_1}(\gamma)$", so as $V_1^{Q_{<\gamma^+}} \models$ "$Q^{\geq\gamma^+}$ is γ-closed", $V_1^{Q_{<\gamma^+} * \dot{Q}^{\geq\gamma^+}} \models$ $2^\gamma = \gamma^+$, and $\mathrm{cof}(\gamma) = \mathrm{cof}^{V_1}(\gamma)$" , i.e., GCH, cardinals, and cofinalit preserved when forcing over V_1 with $Q_{<\gamma^+} * \dot{Q}^{\geq\gamma^+}$. In addition, since t shows any $f : \gamma \to \delta$ or $f : \gamma \to \delta^+$, $f \in V^{Q_{<\gamma^+} * \dot{Q}^{\geq\gamma^+}}$ is so that $f \in V_1^{Q_<}$ $\gamma < \delta$, the fact $V_1^{Q_{<\gamma^+}}$ and V_1 have the same cardinals and cofinalities, to fact $V_1^{Q_{<\gamma^+} * \dot{Q}^{\geq\gamma^+}} \models$ "δ is a singular limit of cardinals satisfying GCH" yi over V_1 with $Q_{<\gamma^+} * \dot{Q}^{\geq\gamma^+}$ preserves δ is a singular cardinal of the same co

It is now easy to show $V_2 = V[G_\kappa][H] \models$ ZFC + GCH and has the sam

cofinalities as $V[G_\kappa] = V_1$. To show $V_2 \models$ GCH and has the same cardinals

as V_1, let again γ be a cardinal in V_1, and write $Q_\kappa = Q_{<\gamma^+} * \dot{Q}$, where

V_1 for the rest of Q_κ. As before, $V_1^{Q_{<\gamma^+}} \models$ "$2^\gamma = \gamma^+$ and $\mathrm{cof}(\gamma) = \mathrm{cof}$

$V_1^{Q_{<\gamma^+}} \models$ "Q is γ-closed", $V_2 \models$ "$2^\gamma = \gamma^+$ and $\mathrm{cof}(\gamma) = \mathrm{cof}^{V_1}(\gamma)$", i.e., by t

of γ, $V_2 \models$ GCH, and all cardinals of V_1 are cardinals of the same cofinalit

as all functions $f : \gamma \to \delta$, $\delta \in V_1$ some ordinal, $f \in V_2$ are so that $f \in V_1$

sentence, it is the case $V_2 \models$ Power Set, and since $V_2 \models AC$ and Q_κ is an

iteration, by the usual arguments, the aforementioned fact implies V_2

Thus, $V_2 \models$ ZFC.

It remains to show that $V[G_\kappa] \models$ GCH and has the same cardinals

as V. To do this, we first note that Easton support iterations of δ-stra

partial orderings are δ-strategically closed for δ any regular cardinal. T

induction. If R_1 is δ-strategically closed and $\Vdash_{R_1}$ "$\dot{R}_2$ is δ-strategically c

$p \in R_1$ be so that $p \Vdash$ "$\dot{g}$ is a strategy for player II ensuring that the game

an increasing chain of elements of $\dot{R}_2$ of length δ can always be continue

II begins by picking $r_0 = \langle p_0, \dot{q}_0 \rangle \in R_1 * \dot{R}_2$ so that $p_0 \geq p$ has been chos

the strategy f for R_1 and $p_0 \Vdash$ "$\dot{q}_0$ has been chosen according to $\dot{g}$", and

$\alpha + 2$ picks $r_{\alpha+2} = \langle p_{\alpha+2}, \dot{q}_{\alpha+2} \rangle$ so that $p_{\alpha+2}$ has been chosen according

that $p_{\alpha+2} \Vdash$ "$\dot{q}_{\alpha+2}$ has been chosen according to $\dot{g}$", then at limit stages λ

bound to $\langle q_\alpha : \alpha < \lambda \rangle$, meaning the condition $\langle p_\lambda, q_\lambda \rangle$ is as desired.

together with the usual proof at limit stages (see [Ba], Theorem 2.5) t

support iteration of δ-closed partial orderings is δ-closed, yield that δ-str

preserved at limit stages of all of our Easton support iterations of δ-stra

partial orderings. In addition, the ideas of this paragraph will also sh

support iterations of $\prec \delta^+$-strategically closed partial orderings are $\prec$

closed for δ any regular cardinal.

For $\alpha < \kappa$ and $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$, since $\lambda_\alpha < \delta_{\alpha+1}$, the definition of Q_α

$V^{P_\alpha} \models$ "$|Q_\alpha| < \delta_{\alpha+1}$". This fact, together with Lemma 5 and the definitio

now yield the proof that $V^{P_{\alpha+1}} \models$ GCH and has the same cardinals and

is virtually identical to the proof given in the first part of this lemma that

has the same cardinals and cofinalities as V_1, replacing γ-closure with γ-s

which also implies that the forcing adds no new functions from γ to the g

If λ is a limit ordinal so that $\bar{\lambda} = \sup(\{\delta_\alpha : \alpha < \lambda\})$ is singular, then

that $V^{P_\lambda} \models$ GCH and has the same cardinals and cofinalities as V is vir

as the just referred to proof of the first part of this lemma for virtually i

as in the previous sentence, keeping in mind that since $|P_\alpha| < \delta_\alpha$ induct

$|P_\lambda| = \bar{\lambda}^+$. If $\lambda \leq \kappa$ is a limit ordinal so that $\bar{\lambda} = \lambda$, then for cardinals $\gamma \leq$

$V^{P_\lambda} \models$ "γ is a cardinal and $\mathrm{cof}(\gamma) = \mathrm{cof}^V(\gamma)$" is once more as before, as i

We now show that the intuitive motivation for the definition of P as paragraph immediately preceding the statement of Lemma 8 actually wor

Lemma 9. *If $\delta < \gamma$ and $V \models$ "δ is γ supercompact and γ is regular", then over P, $V[G] \models$ "δ is γ supercompact".*

Proof of Lemma 9: Let $j : V \to M$ be an elementary embedding w supercompactness of δ so that $M \models$ "δ is not γ supercompact". For $\delta = \delta_{\alpha_0}$, let $P = P_{\alpha_0} * \dot{Q}'_{\alpha_0} * \dot{T}_{\alpha_0} * \dot{R}$, where $\dot{Q}'_{\alpha_0}$ is a term for the full sup $\langle P^0_{\omega,\lambda} * (P^1_{\omega,\lambda}[\dot{S}_\lambda] \times P^2_{\omega,\lambda}[\dot{S}_\lambda]) : \delta^+ \leq \lambda \leq \gamma$ and λ is regular$\rangle$, $\dot{T}_{\alpha_0}$ is a term for and $\dot{R}$ is a term for the rest of P. We show that $V^{P_{\alpha_0} * \dot{Q}'_{\alpha_0}} \models$ "δ is γ super will suffice, since $\Vdash_{P_{\alpha_0} * \dot{Q}'_{\alpha}}$ "$\dot{T}_{\alpha_0} * \dot{R}$ is γ-strategically closed", so as the reg GCH in $V^{P_{\alpha_0} * \dot{Q}'_{\alpha_0}}$ imply $V^{P_{\alpha_0} * \dot{Q}'_{\alpha_0}} \models$ "$|[\gamma]^{<\delta}| = \gamma$", if $V^{P_{\alpha_0} * \dot{Q}'_{\alpha_0}} \models$ "δ is γ then $V^{P_{\alpha_0} * \dot{Q}'_{\alpha_0} * \dot{T}_{\alpha_0} * \dot{R}} = V^P \models$ "δ is γ supercompact via any ultrafilter $\mathcal{U}$

To this end, we first note we will actually show that for $G_{\alpha_0} * G'_{\alpha_0}$ t V-generic over $P_{\alpha_0} * \dot{Q}'_{\alpha_0}$, the embedding j extends to $k : V[G_{\alpha_0} * G'_{\alpha_0}] \to$ $H \subseteq j(P)$. As $\langle j(\alpha) : \alpha < \gamma \rangle \in M$, this will be enough to allow the ultrafilter $x \in \mathcal{U}$ iff $\langle j(\alpha) : \alpha < \gamma \rangle \in k(x)$ to be given in $V[G_{\alpha_0} * G'_{\alpha_0}]$.

We construct H in stages. In M, as $\delta = \delta_{\alpha_0}$ is the critical poin $\dot{Q}'_{\alpha_0}) = P_{\alpha_0} * \dot{R}'_{\alpha_0} * \dot{R}''_{\alpha_0} * \dot{R}'''_{\alpha_0}$, where $\dot{R}'_{\alpha_0}$ will be a term for the full su term for the rest of the portion of $j(P_{\alpha_0})$ defined below $j(\delta)$, and $\dot{R}'''_{\alpha_0}$

for $j(\dot{Q}'_{\alpha_0})$. This will allow us to define H as $H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H'''_{\alpha_0}$. F

$\langle G^0_{\omega,\lambda} * (G^1_{\omega,\lambda} \times G^2_{\omega,\lambda}) : \delta^+ \leq \lambda \leq \gamma$ and λ is regular$\rangle$, we let $H_{\alpha_0} =$ G

$\langle G^0_{\omega,\lambda} * (G^1_{\omega,\lambda} \times G^2_{\omega,\lambda}) : \delta^+ \leq \lambda < \gamma$ and λ is regular$\rangle * \langle G^0_{\omega,\gamma} * G^1_{\omega,\gamma}\rangle$. T

same as G'_{α_0}, except, since $M \models$ "δ is not γ supercompact", we omit th

$G^2_{\omega,\gamma}$.

To construct H''_{α_0}, we first note that the definition of P ensures $|P_{\alpha_0}$

δ is necessarily Mahlo, P_{α_0} is δ-c.c. As $V[G_{\alpha_0}]$ and $M[G_{\alpha_0}]$ are both mod

definition of R'_{α_0} in $M[H_{\alpha_0}]$, Lemmas 4, 5, and 8, and the remark immed

Lemma 5 then ensure that $M[H_{\alpha_0}] \models$ "The portion of R'_{α_0} below γ is

portion of R'_{α_0} at γ is a γ-strategically closed partial ordering followed by

ordering". Since $M^\gamma \subseteq M$ implies $(\gamma^+)^V = (\gamma^+)^M$ and P_{α_0} is δ-c.c., Le

shows $V[G_{\alpha_0}]$ satisfies these facts as well. This means applying the argu

6.4 of [Ba] twice, in concert with an application of the fact a portion of

strategically closed, shows $M[H_{\alpha_0} * H'_{\alpha_0}] = M[G_{\alpha_0} * H'_{\alpha_0}]$ is closed un

with respect to $V[G_{\alpha_0} * H'_{\alpha_0}]$, i.e., if $f : \gamma \to M[H_{\alpha_0} * H'_{\alpha_0}]$, $f \in V[G$

$f \in M[H_{\alpha_0} * H'_{\alpha_0}]$. Therefore, as $M[H_{\alpha_0} * H'_{\alpha_0}] \models$ "R''_{α_0} is both γ-strategi

$\prec \gamma^+$-strategically closed", these facts are true in $V[G_{\alpha_0} * H'_{\alpha_0}]$ as well.

Observe now that GCH allows us to assume $\gamma^+ < j(\delta) < j(\delta^+)$

$M[H_{\alpha_0} * H'_{\alpha_0}] \models$ "$|R''_{a_0}| = j(\delta)$ and $|\mathcal{P}(R''_{\alpha_0})| = j(\delta^+)$" (this last fact follo

Work in $V[G_{\alpha_0} * H'_{\alpha_0}]$. Player I picks $p_\alpha \in D_\alpha$ extending $\sup(\langle q_\beta : \beta$

q_{-1} is the trivial condition), and player II responds by picking $q_\alpha \geq p_\alpha$ (s

the $\prec \gamma^+$-strategic closure of R''_{α_0} in $V[G_{\alpha_0} * H'_{\alpha_0}]$, player II has a winn

this game, so $\langle q_\alpha : \alpha < \gamma^+ \rangle$ can be taken as an increasing sequence of

$q_\alpha \in D_\alpha$ for $\alpha < \gamma^+$. Clearly, $H''_{\alpha_0} = \{p \in R''_{\alpha_0} : \exists \alpha < \gamma^+ [q_\alpha \geq p]\}$ is our

generic object over R''_{α_0} which has been constructed in $V[G_{\alpha_0} * H'_{\alpha_0}] \subseteq V$

$H''_{\alpha_0} \in V[G_{\alpha_0} * G'_{\alpha_0}]$.

To construct H'''_{α_0}, we note first that as in our remarks in Lemma 8,

below the least inaccessible $> \delta$ and γ is regular, $\gamma = \sigma^+$ for some σ. This a

in $V[G_{\alpha_0}]$ $Q'_{\alpha_0} = Q''_{\alpha_0} * \dot{Q}'''_{\alpha_0}$, where Q''_{a_0} is the full support iteration of $\langle P^0_{\omega}$

$P^2_{\omega,\lambda}[\dot{S}_\lambda] : \delta^+ \leq \lambda \leq \sigma$ and λ is regular$\rangle$ and $\dot{Q}'''_{\alpha_0}$ is a term for $P^0_{\omega,\gamma} * (P^1_{\omega,\gamma}$

This factorization of Q'_{α_0} induces through j in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ a factoriz

$R^4_{a_0} * \dot{R}^5_{\alpha_0} = \langle$ the full support iteration of $\langle P^0_{\omega,\lambda} * (P^1_{\omega,\lambda}[\dot{S}_\lambda] \times P^2_{\omega,\lambda}[\dot{S}_\lambda]) : j$

and λ is regular $\rangle * \langle \dot{P}^0_{\omega,j(\gamma)} * (P^1_{\omega,j(\gamma)}[\dot{S}_{j(\gamma)}] \times P^2_{\omega,j(\gamma)}[\dot{S}_{j(\gamma)}])\rangle$.

Work now in $V[G_{\alpha_0} * H'_{\alpha_0}]$. In $M[H_{\alpha_0} * H'_{\alpha_0}]$, as previously noted, R''_{α_0} i

closed. Since $M[H_{\alpha_0} * H'_{\alpha_0}]$ has already been observed to be closed under γ

respect to $V[G_{\alpha_0} * H'_{\alpha_0}]$, and since any γ sequence of elements of $M[H_{\alpha_0} * H$

represented, in $M[H_{\alpha_0} * H'_{\alpha_0}]$, by a term which is actually a function $f : \gamma -$

$M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ is closed under γ sequences with respect to $V[G_{\alpha}$

$f : \gamma \rightarrow M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$, $f \in V[G_{\alpha_0} * H'_{\alpha_0}]$, then $f \in M[H_{\alpha_0} * H'_{\alpha_0} *$

$V[G_{\alpha_0}]$ and $G_{\alpha_0} \in V[G_{\alpha_0} * H_{\alpha_0}]$. Also, our construction to this point gu

$V[G_{\alpha_0} * H'_{\alpha_0}]$, the embedding j extends to j^*: $V[G_{\alpha_0}] \to M[H_{\alpha_0} * H'_{\alpha_0} *$

GCH in $V[G_{\alpha_0} * H'_{\alpha_0}]$ implies $V[G_{\alpha_0} * H'_{\alpha_0}] \models$ "$|Q''_{\alpha_0}| = |G''_{\alpha_0}| = \gamma$", the

implies $\{j^*(p) : p \in G''_{\alpha_0}\} \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$. Since $\{j^*(p) : p \in$

$M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}] \models$ "$R^4_{\alpha_0}$ is equivalent to a $j^*(\delta) = j(\delta)$-directed closed p

and $j(\delta) > \gamma$, $q = \sup\{j^*(p) : p \in G''_{\alpha_0}\}$ can be taken as a condition in R^4_{α}

Note that GCH in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ implies $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}] \models$

and by choice of $j : V \to M$, $V[G_{\alpha_0} * H'_{\alpha_0}] \models$ "$|j(\gamma)| = \gamma^+$ and $|j(\gamma^+)| =$

the number of dense open subsets of $R^4_{\alpha_0}$ in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ is $(2^{j(\gamma)})^{M}$

$(j(\gamma)^+)^{M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]}$ which has cardinality $(\gamma^+)^V = (\gamma^+)^{V[G_{\alpha_0} * H'_{\alpha_0}]}$, w

$\alpha < \gamma^+\rangle \in V[G_{\alpha_0} * H'_{\alpha_0}]$ enumerate all dense open subsets of $R^4_{\alpha_0}$ in $M[H$

The γ-closure of $R^4_{\alpha_0}$ in $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ and hence in $V[G_{\alpha_0} * H'_{\alpha_0}$

$M[H_{a_0} * H'_{\alpha_0} * H''_{\alpha_0}]$-generic object $H^4_{\alpha_0}$ over $R^4_{\alpha_0}$ containing q to be con

standard way in $V[G_{\alpha_0} * H'_{\alpha_0}]$, namely let $q_0 \in D_0$ be so that $q_0 \geq q$, and a

by the γ-closure of $R^4_{\alpha_0}$ in $V[G_{\alpha_0} * H'_{\alpha_0}]$, let $q_\alpha \in D_\alpha$ be so that $q_\alpha \geq$ su

As before, $H^4_{\alpha_0} = \{p \in R^4_{\alpha_0} : \exists \alpha < \gamma^+ [q_\alpha \geq p]\} \in V[G_{\alpha_0} * H'_{\alpha_0}] \subseteq V[G_{\alpha_0}$

our desired generic object.

By the above construction, in $V[G_{\alpha_0} * G'_{\alpha_0}]$, the embedding $j^* : V[G$

$H'_{\alpha_0} * H''_{\alpha_0}]$ extends to an embedding $j^{**} : V[G_{\alpha_0} * G''_{\alpha_0}] \to M[H_{\alpha_0} * H'_{\alpha_0} *$

will be done once we have constructed in $V[G_{\alpha_0} * G'_{\alpha_0}]$ the appropriate ge

general fact about transference of generics via elementary embeddings (
1.2, Fact 2, pp. 5-6), since $j^{**} : V[G_{\alpha_0} * G''_{\alpha_0}] \to M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} *$
every element of $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ can be written $j^{**}(F)(a)$ with
cardinality γ, $j^{**\prime\prime}G^0_{\omega,\gamma} * G^2_{\omega,\gamma}$ generates an $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$-ge

It remains to construct $H^6_{\alpha_0}$, our $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}]$-ger
$P^1_{\omega,j(\gamma)}[S_{j(\gamma)}]$. To do this, first note that $H^4_{\alpha_0}$ (which was constructed in
$M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$-generic over $R^4_{\alpha_0}$, a partial ordering which in $M[H_{\alpha_0}$
$j(\delta)$-closed. Since $j(\delta) > \gamma$ and $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}]$ is closed under γ sequen
to $V[G_{\alpha_0} * H'_{\alpha_0}]$, we can apply earlier reasoning to infer $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0}$
under γ sequences with respect to $V[G_{\alpha_0} * H'_{\alpha_0}]$, i.e., if $f : \gamma \to M[H_{\alpha_0} * H$
$f \in V[G_{\alpha_0} * H'_{\alpha_0}]$ then $f \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$.

Choose in $V[G_{\alpha_0} * G'_{\alpha_0}]$ an enumeration $\langle p_\alpha : \alpha < \gamma^+\rangle$ of $G^1_{\omega,\gamma}$.
$V[G_{\alpha_0} * G'_{\alpha_0}]$, let f be an isomorphism between (a dense subset of) $P^1_{\omega,\gamma}[S$
gives us a sequence $\langle f(p_\alpha) : \alpha < \gamma^+\rangle$ of γ^+ many compatible elements
$p'_\alpha = f(p_\alpha)$, we may hence assume that $I = \langle p'_\alpha : \alpha < \gamma^+\rangle$ is an appro
object for Q^1_γ. By Lemma 6, $V[G_{\alpha_0} * G''_{\alpha_0} * G^0_{\omega,\gamma} * G^1_{\omega,\gamma} * G^2_{\omega,\gamma}] = V[G_{\alpha_0} * G$
$G''_{\alpha_0} * G^0_{\omega,\gamma} * G^1_{\omega,\gamma}] = V[G_{\alpha_0} * H'_{\alpha_0}]$ have the same γ sequences of elements
and hence of $V[G_{\alpha_0} * H'_{\alpha_0}]$. Thus, any γ sequence of elements of $M[H_{\alpha_0} *$
present in $V[G_{\alpha_0} * G'_{\alpha_0}]$ is actually an element of $V[G_{\alpha_0} * H'_{\alpha_0}]$ (so $M[H_{\alpha_0} *$
is really closed under γ sequences with respect to $V[G_{\alpha_0} * G'_{\alpha_0}]$).

closed", the facts $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is closed under γ sequences

$V[G_{\alpha_0} * G'_{\alpha_0}]$ and I is compatible imply that $q_\alpha = \cup\{j^{**}(p) : p \in I|\alpha\}$

is well-defined and is an element of $Q^1_{j(\gamma)}$. Further, if $\langle \rho, \sigma \rangle \in \mathrm{dom}(q_\alpha$

$(\underset{\beta<\alpha}{\cup} q_\beta \in Q^1_{j(\gamma)}$ as $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is closed under γ sequenc

to $V[G_{\alpha_0} * G'_{\alpha_0}]$), then $\sigma \in [\underset{\beta<\alpha}{\cup} j(\beta), j(\alpha))$. (If $\sigma < \underset{\beta<\alpha}{\cup} j(\beta)$, then let β

that $\sigma < j(\beta)$, and let ρ and σ be so that $\langle \rho, \sigma \rangle \in \mathrm{dom}(q_\alpha)$. It must t

that for some $p \in I|\alpha$, $\langle \rho, \sigma \rangle \in \mathrm{dom}(j^{**}(p))$. Since by elementarity and t

$I|\beta$ and $I|\alpha$, for $p|\beta = q \in I|\beta$, $j^{**}(q) = j^{**}(p)|j(\beta) = j^{**}(p|\beta)$, it must b

$\langle \rho, \sigma \rangle \in \mathrm{dom}(j^{**}(q))$. This means $\langle \rho, \sigma \rangle \in \mathrm{dom}(q_\beta)$, a contradiction.)

We define now an $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}]$-generic object H^6_{α}

that $p \in f''G^1_{\omega,\gamma}$ implies $j^{**}(p) \in H^{6,0}_{\alpha_0}$. First, for $\beta \in (j(\gamma), j(\gamma^+))$, let Q

$H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ be the forcing for adding β many Cohen subsets to $j(\gamma)$, i

$j(\gamma) \times \beta \to \{0,1\} : g$ is a function so that $|\mathrm{dom}(g)| < j(\gamma)\}$, ordered by i

note that since $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}] \models$ GCH, $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha}$

"$Q^1_{j(\gamma)}$ is $j(\gamma^+)$-c.c. and $Q^1_{j(\gamma)}$ has $j(\gamma^+)$ many maximal antichains". Th

$\mathcal{A} \in M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}]$ is a maximal antichain of $Q^1_{j(\gamma)}$, the

some $\beta \in (j(\gamma), j(\gamma^+))$. Also, since $V \subseteq V[G_{\alpha_0} * G''_{\alpha_0}] \subseteq V[G_{\alpha_0} * H'_{\alpha_0}]$

are all models of GCH containing the same cardinals and cofinalities, V

"$|j(\gamma^+)| = \gamma^+$". The preceding thus means we can let $\langle \mathcal{A}_\alpha : \alpha < \gamma^+ \rangle \in V[$

enumeration of the maximal antichains of $Q^1_{j(\gamma)}$ present in $M[H_{\alpha_0} * H'_{\alpha_0} * H$

$H^{6,0}_{\alpha_0} = \{p \in Q^1_{j(\gamma)} : \exists r \in \langle r_\alpha : \alpha \in (\gamma, \gamma^+)\rangle [r \le p]\}$ is our desired generic o

$\langle r_\alpha : \alpha \in (\gamma, \gamma^+)\rangle$, if α is a limit, we let $r_\alpha = \bigcup\limits_{\beta<\alpha} r_\beta$. By the facts $\langle q_\beta$:

(strictly) increasing and $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is closed under γ sequen

to $V[G_{\alpha_0} * G'_{\alpha_0}]$, this definition is valid. Assuming now r_α has been defined

define $r_{\alpha+1}$, let $\langle \mathcal{B}_\beta : \beta < \eta \le \gamma\rangle$ be the subsequence of $\langle \mathcal{A}_\beta : \beta \le \alpha + 1\rangle$

antichain $\mathcal{A}$ so that $\mathcal{A} \subseteq Q^{1,j(\alpha+1)}_{j(\gamma)}$. Since $q_\alpha, r_\alpha \in Q^{1,j(\alpha)}_{j(\gamma)}$, $q_{\alpha+1} \in Q^{1,j(\alpha}_{j(\gamma)}$

$j(\alpha + 1)$, the condition $r'_{\alpha+1} = r_\alpha \cup q_{\alpha+1}$ is well-defined, as by our earli

any new elements of $\mathrm{dom}(q_{\alpha+1})$ won't be present in either $\mathrm{dom}(q_\alpha)$ or do

thus using the fact $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0}]$ is closed under γ sequences

$V[G_{\alpha_0} * G'_{\alpha_0}]$ define by induction an increasing sequence $\langle s_\beta : \beta < \eta\rangle$ so

$s_\rho = \bigcup\limits_{\beta<\rho} s_\beta$ if ρ is a limit, and $s_{\beta+1} \ge s_\beta$ is so that $s_{\beta+1}$ extends some elem

just mentioned closure fact implies $r_{\alpha+1} = \bigcup\limits_{\beta<\eta} s_\beta$ is a well-defined conditio

In order to show $H^{6,0}_{\alpha_0}$ is $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}]$-generic over

show that $\forall \mathcal{A} \in \langle \mathcal{A}_\alpha : \alpha \in (\gamma, \gamma^+)\rangle \exists \beta \in (\gamma, \gamma^+) \exists r \in \mathcal{A}[r_\beta \ge r]$. To do th

that $\langle j(\alpha) : \alpha < \gamma^+\rangle$ is unbounded in $j(\gamma^+)$. To see this, if $\beta < j(\gamma^+)$ is a

for some $g : \gamma \to M$ representing β, we can assume that for $\lambda < \gamma$, $g(\lambda$

by the regularity of γ^+ in V, $\beta_0 = \bigcup\limits_{\lambda<\gamma} g(\lambda) < \gamma^+$, and $j(\beta_0) > \beta$. Thi

earlier remarks that if $\mathcal{A} \in \langle \mathcal{A}_\alpha : \alpha < \gamma^+\rangle$, $\mathcal{A} = \mathcal{A}_\rho$, then we can let β

that $\mathcal{A} \subseteq Q^{1,j(\beta)}_{j(\gamma)}$. By construction, for $\eta > \max(\beta, \rho)$, there is some $r \in \mathcal{A}$

Finally, since any $p \in Q^1_\gamma$ is so that for some $\alpha \in (\gamma, \gamma^+)$, $p = p|\alpha$, H

$V[G_{\alpha_0} * G_{\alpha_0} * G_{\omega,\gamma} * G_{\omega,\gamma}]$. This means the notions j (f) and $j^{***}(f^{-1}$

$j^{***}(f)$ is a definable isomorphism over $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H^4_{\alpha_0} * H^5_{\alpha_0}]$ b

subset of) $P^1_{\omega,j(\gamma)}[S_{j(\gamma)}]$ and $Q^1_{j(\gamma)}$, and $j^{***}(f^{-1})$ is its inverse. If $H^6_{\alpha_0} =$

$p \in H^{6,0}_{\alpha_0}\}$, then it is now easy to verify that $H^6_{\alpha_0}$ is an $M[H_{\alpha_0} * H'_{\alpha_0} * H''_{\alpha_0} * H$

object over (a dense subset of) $P^1_{\omega,j(\gamma)}[S_{j(\gamma)}]$ so that $p \in$ (a dense subset of)

$j^{***}(p) \in H^6_{\alpha_0}$. Therefore, for $H''' = H^4_{\alpha_0} * H^5_{\alpha_0} * H^6_{\alpha_0}$ and $H = H_{\alpha_0} * H$

$j : V \to M$ extends to $k : V[G_{\alpha_0} * G'_{\alpha_0}] \to M[H]$, so $V[G] \models$ "δ is γ super

regular. This proves Lemma 9.

Lemma 10. *For γ regular, $V[G] \models$ "δ is γ strongly compact iff δ is γ sup*

Proof of Lemma 10: Assume towards a contradiction the lemma is fals

be so that $V[G] \models$ "δ is γ strongly compact, δ isn't γ supercompact, γ is

the least such cardinal". As before, let $\delta = \delta_\alpha$, i.e., δ is the αth inaccess

$V \models$ "δ_α is γ supercompact", then Lemma 9 implies $V[G] \models$ "δ_α is γ su

it must be the case that $V \models$ "δ_α isn't γ supercompact". We therefore

λ_α the least regular cardinal so that $V \models$ "δ_α isn't λ_α supercompact".

In the manner of Lemma 9, write $P = P_\alpha * \dot{Q}_\alpha * \dot{Q}'_\alpha$, where P_α is the it

stage α, $\dot{Q}_\alpha$ is a term for the full support iteration of $\langle P^0_{\omega,\lambda} * (P^1_{\omega,\lambda}[\dot{S}_\lambda] \times$

case that $V \ldots = V \ldots \models$ "δ_α isn't λ_α strongly compact", so of cou

$V[G] \models$ "δ_a isn't γ strongly compact". This proves Lemma 10.

LEMMA 11. *For γ regular, $V[G] \models$ "δ is γ supercompact" iff $V \models$ "δ is γ*

PROOF OF LEMMA 11: By Lemma 9, if $V \models$ "δ is γ supercompact and γ

$V[G] \models$ "δ is γ supercompact". If $V[G] \models$ "δ is γ supercompact and γ

$V \models$ "δ is not γ supercompact", then as in Lemma 10, for the α so that

for λ_α the least regular cardinal so that $V \models$ "δ_α isn't λ_α supercompact

Lemma 10 then immediately yields that $V[G] \models$ "δ_α isn't $\lambda_\alpha \leq \gamma$ strongly

proves Lemma 11.

The proof of Lemma 11 completes the proof of our Theorem in the cas

supercompact cardinal in the universe and has no inaccessibles above it.

the Theorem to hold non-trivially.

§3 The General Case

We will now prove our Theorem under the assumption that there ma

one supercompact cardinal in the universe (including a proper class of su

general paradigm as in the last section, namely iterating the forcings of
Easton supports so as to destroy those "bad" instances of strong compac
be destroyed and so as to resurrect and preserve all instances of superco
each inaccessible δ_i, a certain coding ordinal $\theta_i < \delta_i$ will be chosen wher
we will use to define $P^0_{\theta_i,\lambda}$, $P^1_{\theta_i,\lambda}[S_{\theta_i,\lambda}]$, and $P^2_{\theta_i,\lambda}[S_{\theta_i,\lambda}]$, where $S_{\theta_i,\lambda}$ is th
stationary set of ordinals of cofinality θ_i added to λ^+ by $P^0_{\theta_i,\lambda}$. We wi
different values of θ_i, instead of having $\theta_i = \omega$ as in the last section, so as
strong compactness of some δ and yet preserve the λ supercompactness o
necessary. When θ_i can't be defined, we won't necessarily be able to destr
compactness of δ_i, although we will be able to preserve the λ supercomp
appropriate. This will happen when instances of the results of [Me] and
when there are certain limits of supercompactness.

Getting specific, let $\langle \delta_i : i \in \text{Ord} \rangle$ enumerate the inaccessibles of $V \models$
$\lambda_i > \delta_i$ be the least regular cardinal so that $V \models$ "δ_i isn't λ_i supercompa
exists. If no such λ_i exists, i.e., if δ_i is supercompact, then let $\lambda_i = \Omega$, wh
Ω as some giant "ordinal" larger than any $\alpha \in \text{Ord}$. If possible, choose θ_i
regular cardinal so that $\theta_i < \delta_j < \delta_i$ implies $\lambda_j < \delta_i$ (whenever $j < i$).
undefined for δ_i iff δ_i is a limit of cardinals which are $< \delta_i$ supercompact b
if δ_j is $< \delta_i$ supercompact, then $\lambda_j \geq \delta_i$.

We define now a class Easton support iteration $P = \langle\langle P_\alpha, \dot{Q}_\alpha\rangle : \alpha \in$ O either δ_i is α supercompact or $\alpha = \lambda_i$. Under these circumstances $\dot{Q}$

$(\prod_{\{i<\alpha:\delta_i \text{is } \alpha \text{ supercompact}\}} (P^0_{\theta_i,\alpha} * P^2_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}]) * \prod_{\{i<\alpha:\delta_i \text{ is } \alpha \text{ supercompac}}$
$(\prod_{\{i<\alpha:\alpha=\lambda_i\}} P^0_{\theta_i,\alpha} * \prod_{\{i<\alpha:\alpha=\lambda_i\}} P^1_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}]) = (\dot{P}^0_\alpha * \dot{P}^1_\alpha) \times (\dot{P}^2_\alpha * \dot{P}^3_\alpha)$, with t

elements of $\dot{P}^0_\alpha$ and $\dot{P}^2_\alpha$ will have full support, and elements of $\dot{P}^1_\alpha$ and $\dot{P}^3_\alpha$ w

$< \alpha$.

Note that unless $|\{i < \alpha : \delta_i \text{ is } < \alpha \text{ supercompact}\}| = \alpha$, the elements of

support for $i = 0, 1, 2, 3$.

The following lemma is the natural analogue to Lemma 8.

LEMMA 12. *For G a V-generic class over P, V and $V[G]$ have the sam*

cofinalities, and $V[G] \models ZFC + GCH$.

PROOF OF LEMMA 12: We show inductively that for any α, V and V^{P_α}

cardinals and cofinalities, and $V^{P_\alpha} \models$ GCH. This will suffice to show $V[$

has the same cardinals and cofinalities as V, since if $\dot{R}$ is a term so that P

$\Vdash_{P_\alpha}$ "The iteration $\dot{R}$ is $< \alpha$-strategically closed", meaning $V^{P_\alpha * \dot{R}}$ and V^{P}

cardinals and cofinalities $\leq \alpha$ and GCH holds in both of these models for

Assume now V and V^{P_α} have the same cardinals and cofinalities, an

We show V and $V^{P_{\alpha+1}} = V^{P_\alpha * \dot{Q}_\alpha}$ have the same cardinals and cofinalitie

GCH. If $\dot{Q}_\alpha$ is a term for the trivial partial ordering, this is clearly the cas

$\dot{Q}_\alpha$ is not a term for the trivial partial ordering. Let then $\dot{Q}'_\alpha$ be a term

$(\prod (\dot{P}^0 * P^2 [\dot{S}]) * \dot{P}^3) = (\dot{P}^0 * \dot{P}^1) \times (\dot{P}^4 * \dot{P}^3)$ where as earlier

$\big(\prod_{\{i<\alpha:\delta_i \text{ is } \alpha \text{ supercompact or } \alpha=\lambda_i\}} P_{\theta_i,\alpha}[\dot S_{\theta_i,\alpha}]\big) = P_\alpha * \dot P_\alpha$, where the ele

have full support, and the elements of $\dot P^6_\alpha$ will have support $< \alpha$. By Le

each $P^0_{\theta_i,\alpha} * (P^1_{\theta_i,\alpha}[\dot S_{\theta_i,\alpha}] \times P^2_{\theta_i,\alpha}[\dot S_{\theta_i,\alpha}])$ is equivalent to $Q^0_\alpha * \dot Q^1_\alpha$. We theref

V^{P_α}, Q'_α is equivalent to $(\prod_{\beta<\gamma} Q^0_\alpha) * (\prod_{\beta<\gamma} \dot Q^1_\alpha)$, where $\gamma = |\{i < \alpha : \delta_i$ is α s

$\alpha = \lambda_i\}|$ (γ is a cardinal in both V and V^{P_α} by induction), i.e., the full s

of γ copies of Q^0_α followed by the $< \alpha$ support product of γ copies of Q^1_α

$\prod_{\beta<\gamma} Q^0_\alpha$ is isomorphic to the usual ordering for adding γ many Cohen subs

conditions of support $< \alpha^+$, and since $\prod_{\beta<\gamma} Q^1_\alpha$ is composed of elements

$< \alpha$, $\prod_{\beta<\gamma} Q^1_\alpha$ is isomorphic to a single partial ordering for adding α^+ many

to α using conditions of support $< \alpha$. Hence, $V^{P_\alpha * \dot Q'_\alpha}$ and V^{P_α} have the

and cofinalities, and $V^{P_\alpha * \dot Q'_\alpha} \models$ GCH, so $V^{P_\alpha * \dot Q'_\alpha}$ and V have the sam

cofinalities. And, for G_α the projection of G onto P_α, if H is $V[G_\alpha]$-gene

any $i < \alpha$ so that $\alpha = \lambda_i$, we can omit the portion of H generic over $P^2_{\theta_i,\alpha}$

obtain a $V[G_\alpha]$-generic object H' for Q_α. Since $V \subseteq V[G_\alpha][H'] \subseteq V[G_\alpha][H$

5, it must therefore be the case that V, $V^{P_\alpha * \dot Q_\alpha} = V^{P_{\alpha+1}}$, and $V^{P_\alpha * \dot Q'_\alpha}$ al

cardinals and cofinalities and satisfy GCH.

To complete the proof of Lemma 12, if now α is a limit ordinal, th

and V^{P_α} have the same cardinals and cofinalities and $V^{P_\alpha} \models$ GCH is t

proof given in the last paragraph of Lemma 8, since the iteration still has e

closure and can easily be seen by GCH to be so that for any $\beta < \alpha$, $|P_\beta|$

We remark that if we rewrite $\dot{Q}_\alpha$ as $(\dot{P}^0_\alpha \times \dot{P}^2_\alpha) * (\dot{P}^1_\alpha \times \dot{P}^3_\alpha)$, then th

the proof of Lemma 12 combined with an argument analogous to the on

following the proof of Lemma 5 show $\Vdash_{P_\alpha * (\dot{P}^0_\alpha \times \dot{P}^2_\alpha)}$ "$\dot{P}^1_\alpha \times \dot{P}^3_\alpha$ is α^+-c.c."

definitions, $\Vdash_{P_\alpha}$ "$\dot{P}^0_\alpha \times \dot{P}^2_\alpha$ is α-strategically closed". These observations wi

proof of the following lemma, which is the natural analogue to Lemma 9.

Lemma 13. *If $\delta < \gamma$ and $V \models$ "δ is γ supercompact and γ is regular", then*

over P, $V[G] \models$ "δ is γ supercompact".

Proof of Lemma 13: We mimic the proof of Lemma 9. Let $j : V \to M$ b

embedding witnessing the γ supercompactness of δ so that $M \models$ "δ is not γ

and let α_0 be so that $\delta = \delta_{\alpha_0}$.

Let $P = P_\delta * \dot{Q}'_\delta * \dot{R}$, where P_δ is the iteration through stage δ, $\dot{Q}$

the iteration $\langle\langle P_\alpha / P_\delta, \dot{Q}_\alpha\rangle : \delta \le \alpha \le \gamma\rangle$, and $\dot{R}$ is a term for the rest o

since $\Vdash_{P_\delta * \dot{Q}'_\delta}$ "$\dot{R}$ is γ-strategically closed", the regularity of γ and GCH i

it suffices to show $V^{P_\delta * \dot{Q}'_\delta} \models$ "δ is γ supercompact".

We will again show that $j : V \to M$ extends to $k : V[G_\delta * G'_\delta] \to$

$H \subseteq j(P)$. In M, $j(P_\delta * \dot{Q}'_\delta) = P_\delta * \dot{R}'_\delta * \dot{R}''_\delta * \dot{R}'''_\delta$, where $\dot{R}'_\delta$ will be a term for

defined in M^{P_δ}) $\langle\langle P_\alpha / P_\delta, \dot{Q}_\alpha\rangle : \delta \le \alpha \le \gamma\rangle$, $\dot{R}''_\delta$ will be a term for the itera

in $M^{P_\delta * \dot{R}'_\delta}$) $\langle\langle P_\alpha / P_{\gamma+1}, \dot{Q}_\alpha\rangle : \gamma+1 \le \alpha < j(\delta)\rangle$, and $\dot{R}'''_\delta$ will be a term for where the term for the iteration $\langle\langle P_\alpha/P_\delta, \dot{Q}_\alpha\rangle : \delta \le \alpha < \gamma\rangle$ is the same as in

the form $(\dot{P}^0_{\theta_i,\gamma} * P^2_{\theta_i,\gamma}[\dot{S}_{\theta_i,\gamma}]) * P^1_{\theta_i,\gamma}[\dot{S}_{\theta_i,\gamma}]$ appearing in $\dot{R}'_\delta$ (more specificall

identical to one appearing in $\dot{Q}'_\delta$, and if $\dot{P}^0_{\theta_i,\gamma} * P^1_{\theta_i,\gamma}[\dot{S}_{\theta_i,\gamma}]$ appears in $\dot{R}'_\delta$ (n

in $\dot{P}^2_\gamma * \dot{P}^3_\gamma$), then either it appears as an identical term in $\dot{Q}'_\delta$, or (as is the

$i = \alpha_0$ and θ_i is defined) it appears as the term $(\dot{P}^0_{\theta_i,\gamma} * P^2_{\theta_i,\gamma}[\dot{S}_{\theta_i,\gamma}]) * P$

This allows us to define $H_\delta = G_\delta$, where G_δ is the portion of G V-gener

$H'_\delta = K * K'$, where K is the projection of G onto $\langle\langle P_\alpha/P_\delta, \dot{Q}_\alpha\rangle$: $\delta \le \alpha$

the projection of G onto $(P^0_\gamma * \dot{P}^1_\gamma) \times (P^2_\gamma * \dot{P}^3_\gamma)$ as defined in M.

To construct the next portion of the generic object H''_δ, note that

the definition of P_δ ensures $|P_\delta| = \delta$ and P_δ is δ-c.c. Thus, as before, GC

$M[G_\delta]$, the definition of $\dot{R}'_\delta$, the fact $M^\gamma \subseteq M$, and some applications o

[Ba] allow us to conclude that $M[H_\delta * H'_\delta] = M[G_\delta * H'_\delta]$ is closed under γ

respect to $V[G_\delta * H'_\delta]$. Thus, any partial ordering which is $\prec \gamma^+$-strateg

$M[H_\delta * H'_\delta]$ is actually $\prec \gamma^+$-strategically closed in $V[G_\delta * H'_\delta]$.

Observe now that if $\langle T_\alpha : \alpha < \eta\rangle$ is so that each T_α is $\prec \rho^+$-strateg

some cardinal ρ, then $\prod\limits_{\alpha<\eta} T_\alpha$ is also $\prec \rho^+$-strategically closed, for if $\langle f_\alpha : \alpha$

each f_α is a winning strategy for player II for T_α, then $\prod\limits_{\alpha<\eta} f_\alpha$, i.e., pick the

according to f_α, is a winning strategy for player II for $\prod\limits_{\alpha<\eta} T_\alpha$. This ob

implies $\Vdash_{P_\delta * \dot{R}'_\delta}$ "$\dot{R}''_\delta$ is $\prec \gamma^+$-strategically closed" in either $V[G_\delta * H'_\delta]$ or M

Write $\dot{R}_\delta^4$ as $\dot{R}_\delta^4 * \dot{R}_\delta^5$, where $\dot{R}_\delta^4$ is a term for the iteration $\langle\langle P_\alpha/P_{j(\delta)}, \dot{Q}_\alpha\rangle :$

$\alpha < j(\gamma)\rangle$ and $\dot{R}_\delta^5$ is a term for $\dot{Q}_{j(\gamma)}$. Also, write in V $\dot{Q}'_\delta = \dot{Q}''_\delta *$

is a term for the iteration $\langle\langle P_\alpha/P_\delta, \dot{Q}_\alpha\rangle : \delta \le \alpha < \gamma\rangle$ and $\dot{Q}'''_\delta$ is a te

let $G'_\delta = G''_\delta * G'''_\delta$ be the corresponding factorization of G'_δ. For any

$\dot{Q}_\alpha = (\dot{P}_\alpha^0 * \dot{P}_\alpha^1) \times (\dot{P}_\alpha^2 * \dot{P}_\alpha^3)$ appearing in $\dot{R}_\delta^4$, Lemma 4 and the fact

will have full support and elements of $\dot{P}_\alpha^1$ will have support $< \alpha$ imply

$T = P_\delta * \dot{R}'_\delta * \dot{R}''_\delta * \langle\langle P_\beta/P_{j(\delta)}, \dot{Q}_\beta\rangle : j(\delta) \le \beta < \alpha\rangle$, $\Vdash_T$ "(a dense sub

is γ^+-directed closed". Further, if $\alpha \in [j(\delta), j(\gamma)]$ is so that for some i,

must be the case that $j(\delta) < \delta_i$, for if $\delta_i \le j(\delta)$, then by a theorem of

since $M \models$ "δ_i is $< j(\delta)$ supercompact and $j(\delta)$ is $j(\gamma)$ supercompact

$j(\gamma)$ supercompact", a contradiction to the fact $M \models$ "$\alpha = \lambda_i < j(\gamma)$".

definition of θ_i, it must be the case that $j(\delta) \le \theta_i$, i.e., since $j(\delta) > \gamma$

means $\Vdash_T$ "$\dot{P}^0_{\theta_i,\alpha}$ and $P^1_{\theta_i,\alpha}[\dot{S}_{\theta_i,\alpha}]$ are γ^+-directed closed", so as elements

full support and elements of $\dot{P}_\alpha^3$ will have support $< \alpha$, $\Vdash_T$ "$\dot{P}_\alpha^2 * \dot{P}_\alpha^3$ is γ^+-d

i.e., $\Vdash_T$ "(A dense subset of) $(\dot{P}_\alpha^0 * \dot{P}_\alpha^1) \times (\dot{P}_\alpha^2 * \dot{P}_\alpha^3)$ is γ^+-directed closed

$\Vdash_{P_\delta * \dot{R}'_\delta * \dot{R}''_\delta}$ "(A dense subset of) $\dot{R}_\delta^4$ is γ^+-directed closed". Therefore, usi

of j, $j^* : V[G_\delta] \to M[H_\delta * H'_\delta * H''_\delta]$ which we have produced in $V[G_\delta * H$

GCH in $M[H_\delta * H'_\delta * H''_\delta]$ implies $M[H_\delta * H'_\delta * H''_\delta] \models$ "$|R_\delta^4| = j(\gamma)$ and

$V[G_\delta * H'_\delta] \models$ "$|j(\gamma^+)| = (\gamma^+)^V = \gamma^+$", and the closure properties of M

the remarks after the proof of Lemma 12 and the proof of Lemma 6, $M[H_\delta$

is closed under γ-sequences with respect to $V[G_\delta * G'_\delta]$.

Rewrite $\dot{R}^5_\delta$ as $(\prod_{\{i<j(\gamma):\delta_i \text{ is } j(\gamma) \text{ supercompact}\}} (\dot{P}^0_{\theta_i,j(\gamma)} *$

$\times \prod_{\{i<j(\gamma):j(\gamma)=\lambda_i\}} \dot{P}^0_{\theta_i,j(\gamma)}) * (\prod_{\{i<j(\gamma):\delta_i \text{ is } j(\gamma) \text{ supercompact or } j(\gamma)=\lambda_i\}}$

$= \dot{R}^6_\delta * \dot{R}^7_\delta$, where all elements of $\dot{R}^6_\delta$ will have full support, and all elements

support $< j(\gamma)$. By our earlier observation that products of (appropriate

closed partial orderings retain the same amount of strategic closure, it is

that Q^*_γ, the portion of Q_γ corresponding to R^6_δ, i.e., $Q^*_\gamma = {}_{\{i<\gamma:\delta_i \text{ is } \gamma}$

$(P^0_{\theta_i,\gamma} * P^2_{\theta_i,\gamma}[\dot{S}_{\theta_i,\gamma}]) \times \prod_{\{i<\gamma:\gamma=\lambda_i\}} P^0_{\theta_i,\gamma}$, is γ-strategically closed and ther

distributive. Hence, as we again have that in $V[G_\delta * H'_\delta]$, j^* extends to j^{**}

$M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$, we can use j^{**} as in the proof of Lemma 9 to t

projection of G'''_δ onto Q^*_γ, via the general transference principle of [C], S

2, pp. 5-6 to an $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$-generic object H^5_δ over R^6_δ.

By its construction, since $p \in G^4_\delta$ implies $j^{**}(p) \in H^5_\delta$, j^{**} extends i

$j^{***} : V[G_\delta * G''_\delta * G^4_\delta] \to M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$. And, since R^6_δ is γ-stra

$M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$ and $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$ contain the same

elements of $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$ with respect to $V[G_\delta * G'_\delta]$. As any γ sequ

of $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$ can be represented, in $M[H_\delta * H'_\delta * H''_\delta *$

which is actually a function $f : \gamma \to M[H_\delta * H'_\delta * H''_\delta * H^4_\delta]$, and as $M[H_\delta *$

closed under γ sequences with respect to $V[G_\delta * G'_\delta]$, $M[H_\delta * H'_\delta * H''_\delta * H$

support product $\prod_{\{i<\gamma:\delta_i \text{ is } \gamma \text{ supercompact or } \gamma=\lambda_i\}} P_{\theta_i,\gamma}[S_{\theta_i,\gamma}]$, with G^5_δ t

G'''_δ onto Q^{**}_γ. Next, for the purpose of the remainder of the proof of this l

and $i < j(\gamma)$ is an ordinal, say that $i \in \text{support}(p)$ iff for some non-trivi

of p, $\bar{p} \in P^0_{\theta_i,j(\gamma)}$. Analogously, it is clear what $i \in \text{support}(p)$ for $p \in F$

let $A = \{i < j(\gamma) :$ For some $p \in j^{**\prime\prime}G^4_\delta$, $i \in \text{support}(p)\}$, and let $B =$

some $q \in R^7_\delta$, $i \in \text{support}(q)$ but $i \notin \text{support}(p)$ for any $p \in j^{**\prime\prime}G^4_\delta\}$. Wri

where $A_0 = \{i \in A : j(\gamma) = \lambda_i\}$ and $A_1 = \{i \in A : j(\gamma) \neq \lambda_i\}$.

$H^5_\delta = \{q \in R^6_\delta : \exists p \in j^{**\prime\prime}G^4_\delta[q \leq p]\}$, $A, A_0, A_1, B \in M[H_\delta * H'_\delta * H''_\delta * H^4_\delta$

If $i \in A_1$, then by the genericity of H^5_δ, $P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$ contains a de

P^*_i given by Lemma 4 which is isomorphic to $Q^1_{j(\gamma)}$. Hence, we can infer t

support) product $\prod_{i\in A_1} P^*_i$ is dense in the ($< j(\gamma)$ support) product $\prod_{i\in A_1} P^1_{\theta_i,j}$

thus without loss of generality consider $\prod_{i\in A_1} P^*_i$ instead of $\prod_{i\in A_1} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,}$

$i \in A_0$, then since $j(\gamma) = \lambda_i$, by our earlier remarks, $\theta_i > \gamma$. This means

is γ^+-directed closed.

As we observed in the proof of Lemma 4, for any $i \in A$ and any $\langle$

$P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$, the first three coordinates $\langle w^i, \alpha^i, \bar{r}^i\rangle$ are a re-represent

ment of $Q^1_{j(\gamma)}$. Since the $< j(\gamma)$ support product of $j(\gamma)$ many copies of

phic to $Q^1_{j(\gamma)}$, for any condition $p = \langle\langle w^i, \alpha^i, \bar{r}^i, Z^i\rangle_{i<\ell_0<j(\gamma)}, \langle w^i, \alpha^i, \bar{r}^i,$

$\prod_{i\in A_0} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}] \times \prod_{i\in A_1} P^*_i$, we can in a unique and canonical way wi

where $\bar{r} \in Q^1$ and $\bar{Z} = \langle\langle Z^i : i < \ell_0 < j(\gamma)\rangle, \langle Z^i : i < \ell_0 < j(\gamma)\rangle\rangle$. Fo

γ^+-directed closed, the fact $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$ is closed under γ se

spect to $V[G_\delta * G'_\delta]$ means that we can in essence ignore each sequence $\bar{Z}$ as

the arguments used in Lemma 9 to construct the generic object for $Q^1_{j(\gamma)}$

$M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta]$-generic object $H^{6,0}_\delta$ for $\prod_{i\in A_0} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}] \times \prod_{i\in A}$

since $\prod_{i\in A_0} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}] \times \prod_{i\in A_1} P^*_i$ is γ^+-directed closed, $M[H_\delta * H'_\delta * H''_\delta *$

is closed under γ sequences with respect to $V[G_\delta * G'_\delta]$.

By our remarks following the proof of Lemma 12 and the ideas use

following the proof of Lemma 5, $\prod_{i\in B} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$ is $j(\gamma^+)$-c.c. in $M[H_\delta *$

and $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta * H^{6,0}_\delta]$. Since $\prod_{i\in B} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$ is a $< j(\gamma)$

and $P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$ has cardinality $j(\gamma^+)$ in $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta$

$i < j(\gamma)$, $\prod_{i\in B} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$ has cardinality $j(\gamma^+)$ in $M[H_\delta * H'_\delta * H''_\delta *$

We can thus as in Lemma 9 let $\langle \mathcal{A}_\alpha : \alpha < \gamma^+\rangle$ enumerate in $V[G_\delta * G$

antichains of $\prod_{i\in B} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma)}]$ with respect to $M[H_\delta * H'_\delta * H''_\delta * H^4_\delta *$

we can once more mimic the construction in Lemma 9 of H''_{α_0} to produce i

$M[H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta * H^{6,0}_\delta]$-generic object $H^{6,1}_\delta$ over $\prod_{i\in B} P^1_{\theta_i,j(\gamma)}[S_{\theta_i,j(\gamma}$

$H^6_\delta = H^{6,0}_\delta * H^{6,1}_\delta$ and $H = H_\delta * H'_\delta * H''_\delta * H^4_\delta * H^5_\delta * H^6_\delta$, then our construc

$j : V \to M$ extends to $k : V[G_\delta * G'_\delta] \to M[H]$, so $V[G] \models$ "δ is γ super

proves Lemma 13.

LEMMA 14: *For γ regular, $V[G] \models$ "δ is γ strongly compact iff δ is γ super*

possibly if for the i so that $\delta = \delta_i$, θ_i is undefined".

PROOF OF LEMMA 14: As in Lemma 10, we assume towards a contradic

is false, and let $\delta = \delta_{i_0} < \gamma$ be so that $V[G] \models$ "δ is γ strongly com

supercompact, θ_{i_0} is defined, γ is regular, and γ is the least such cardinal"

13 implies that if $V \models$ "δ is γ supercompact", then $V[G] \models$ "δ is γ super

Lemma 10, it must be the case that $\lambda_{i_0} \le \gamma$.

Write $P = P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}} * \dot{R}$, where $P_{\lambda_{i_0}}$ is the forcing through stag

term for the forcing at stage λ_{i_0}, and $\dot{R}$ is a term for the rest of the for

since $V \models$ "$\delta = \delta_{i_0}$ isn't λ_{i_0} supercompact", we can write $Q_{\lambda_{i_0}}$ as T_0

is $P^0_{\theta_{i_0},\lambda_{i_0}} * P^1_{\theta_{i_0},\lambda_{i_0}}[\dot{S}_{\theta_{i_0},\lambda_{i_0}}]$, and T_0 is the rest of $Q_{\lambda_{i_0}}$. Since $V^{P_{\lambda_{i_0}}} \models$

is $< \lambda_{i_0}$-strategically closed" (and hence adds no new bounded subse

forcing over $V^{P_{\lambda_{i_0}}}$), the arguments of Lemma 3 apply in $V^{P_{\lambda_{i_0}} * (\dot{T}_0 \times \dot{P}$

$V^{(P_{\lambda_{i_0}} * (\dot{T}_0 \times \dot{P}^0_{\theta_{i_0},\lambda_{i_0}})) * P^1_{\theta_{i_0},\lambda_{i_0}}[\dot{S}_{\theta_{i_0},\lambda_{i_0}}]} = V^{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}}} \models$ "δ_{i_0} isn't λ_{i_0} strongl

λ_{i_0} doesn't carry a δ_{i_0}-additive uniform ultrafilter".

It remains to show that $V^{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}} * \dot{R}} = V^P \models$ "δ_{i_0} isn't λ_{i_0} strongly c

weren't the case, then let $\dot{\mathcal{U}}$ be a term in $V^{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}}}$ so that $\Vdash_R$ "$\dot{\mathcal{U}}$ is a δ_{i_0}-a

ultrafilter over λ_{i_0}". Since $\Vdash_{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}}}$ "$\dot{R}$ is $\prec \lambda^+_{i_0}$-strategically closed" an

GCH, if we let $\langle x_\alpha : \alpha < \lambda^+_{i_0} \rangle$ be in $V^{P_{\lambda_{i_0}} * \dot{Q}_{\lambda_{i_0}}}$ a listing of all of the over λ_{i_0} in $V^{P_{\lambda_{i_0}}}$, which contradicts that there is no such ultrafilter

Thus, $V^P \models$ "δ_{i_0} isn't λ_{i_0} strongly compact", a contradiction to $V[G] \models$

compact". This proves Lemma 14.

Note that the analogue to Lemma 11 holds if $\delta = \delta_i$ and θ_i is def

regular, $V[G] \models$ "δ is γ supercompact" iff $V \models$ "δ is γ supercompact" if

defined. The proof uses Lemmas 13 and 14 and is exactly the same as the

11.

Lemmas 12–14 complete the proof of our Theorem in the general cas

§4 Concluding Remarks

In conclusion, we would like to mention that it is possible to use gener

methods of this paper to answer some further questions concerning the p

ships amongst strongly compact, supercompact, and measurable cardinals

it is possible to show, using generalizations of the methods of this paper,

of [Me] which states that the least measurable cardinal κ which is the l

compact or supercompact cardinals is not 2^κ supercompact is best possib

if $V \models$ "ZFC + GCH + κ is the least supercompact limit of supercomp

$\lambda > \kappa^+$ is a regular cardinal which is either inaccessible or is the succe of supercompact cardinals and every cardinal $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$ + dinal $\gamma \in [\kappa, \lambda)$, $2^\gamma = \lambda$ + κ is $< \lambda$ supercompact + κ is the least mea supercompact cardinals".

It is also possible to show using generalizations of the methods of th $V \models$ "ZFC + GCH + $\kappa < \lambda$ are such that κ is $< \lambda$ supercompact, $\lambda >$ cardinal which is either inaccessible or is the successor of a cardinal of c $h : \kappa \to \kappa$ is a function so that for some elementary embedding $j : V$ - the $< \lambda$ supercompactness of κ, $j(h)(\kappa) = \lambda$", then there is some cardina preserving generic extension $V[G] \models$ "ZFC + For every inaccessible δ cardinal $\gamma \in [\delta, h(\delta))$, $2^\gamma = h(\delta)$ + For every cardinal $\gamma \in [\kappa, \lambda)$, $2^\gamma =$ supercompact + κ is the least measurable cardinal". This generalizes a r (see [CW]), who showed, in response to a question posed to him by the fi it was possible to start from a model for "ZFC + GCH + $\kappa < \lambda$ are su supercompact and λ is regular" and use Radin forcing to produce a mo $2^\kappa = \lambda$ + κ is δ supercompact for all regular $\delta < \lambda$ + κ is the least measu In addition, it is possible to iterate the forcing used in the construction of t to show, for instance, that if $V \models$ "ZFC + GCH + There is a proper cl κ so that κ is κ^+ supercompact", then there is some cardinal and cofin generic extension $V[G] \models$ "ZFC + $2^\kappa = \kappa^{++}$ iff κ is inaccessible + Th

λ essentially any "reasonable" value below 2 . The proof of these result

[AS].

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