HOMOGENEOUS FAMILIES AND THEIR AUTOMORPHISM GROUPS

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ABSTRACT. A homogeneous family of subsets over a given set is one with a very "rich" automorphism group. We prove the existence of a bi-universal element in the class of homogeneous families over a given infinite set and give an explicit construction of $2^{2^{\aleph_0}}$ isomorphism types of homogeneous families over a countable set.

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§0 Introduction

Homogeneous objects are often defined in terms of their automorphism groups. Rado's graph Γ, also known as the countable random graph, has the property that for any isomorphism f between two finite induced subgraphs of Γ there is an automorphism of Γ extending f. This property is the homogeneity of Rado's graph; and any graph whose automorphism group satisfied this condition is called homogeneous.

The automorphism group of Rado's graph was studied by Truss in [T2], and shown to be simple. Truss studied also the group $\mathrm{AAut}(\Gamma)$ of *almost automorphisms* of Rado's graph (see [T3] and also [MSST]). This is a *highly transitive* group extending $\mathrm{Aut}(\Gamma)$ (where "highly transitive" stands for "n-transitive for all n"; the group $\mathrm{Aut}(\Gamma)$ is not highly transitive).

In this paper we shall study *homogeneous families of sets* over infinite sets. Our definition of homogeneity of a family of sets implies that its automorphism group satisfies, among other conditions, that it is highly transitive. However, while all homogeneous graphs over a countable set are classified (see [LW]), this is not the case with homogeneous families over a countable set.

We shall show that there are $2^{2^{\aleph_0}}$ isomorphism types of homogeneous families over a countable set. This is done in Section 4. Fromthe proof we shall get $2^{2^{\aleph_0}}$ many permutation groups, each acting homogeneously on some family over ω, and each being isomorphic to the free group on $2^{\aleph_0}$ generators, but such that no two are conjugate in $\mathrm{Sym}(\omega)$.

In Section 3 we prove the existence of a bi-universal homogeneous family over any given infinite set. The definitions of bi-embedding and bi-universality are generalization of definitions made by Truss in his study of universal permutation groups [T1]. A short survey of results concerning the existence of universal objects can be found in the introduction to [KS1]. Results concerning abelian groups are in [KS2], and results on stable unsuperstable first order theories are in [KS3].

Homogeneous families were studied in [GGK] (where they were treated as bipartite graphs). There it was shown that the number of isomorphism types of homogeneous families over ω of size $\aleph_1$ is independent of ZFC and may be 1 as well as $2^{\aleph_1}$ in different models of set theory.

Model theorists will recognize that uncountable homogeneous families over a countable set are examples of two-cardinal models which are ω-homogeneous as well. Set theorists may be interested in the following

0.1 Problem: Is it consistent that $2^{\aleph_0}$ is large and that in some uncountable $\lambda < 2^{\aleph_0}$ there is a maximal homogeneous family (with respect to inclusion)?

We wish to remark finally that the existence of $2^{2^{\aleph_0}}$ isomorphism types of homogeneous families over ω follows from a general theorem about non-standard logics [Sh-c, VIII,§1] (for more details see also [Sh 266]). The virtue of the proof here (besides being elementary) is its explicitness and the information it gives about the embeddability of an arbitrary family in a homogeneous one.

NOTATION We denote disjoint unions by $\dot\cup$ and $\dot\bigcup$. A natural number n is the set $\{0, 1, \dots n-1\}$ of all smaller natural numbers.

§1 Getting started

Let $\mathcal{F} \subseteq \mathcal{P}(A)$ be a family of subsets of a given infinite set A. An *automorphism* of $\mathcal{F}$ is a permutation $\sigma \in \mathrm{Sym}(A)$ which satisfies that $X \in \mathcal{F} \Leftrightarrow \sigma[X] \in \mathcal{F}$ for every $X \subseteq A$. (By $\sigma[X]$ we denote $\{\sigma(x) : x \in X\}$ for $X \subseteq A$.) The group $\mathrm{Aut}(\mathcal{F}) \subseteq \mathrm{Sym}(A)$ is the group of all automorphisms of $\mathcal{F}$.

One way of defining when a family $\mathcal{F} \subseteq \mathcal{P}(A)$ is homogeneous is to demand that the bipartite graph $\langle A, \mathcal{F}, \in\rangle$ is homogeneous, namely that every finite partial automorphism of this graph which respects the sides extends to a total automorphism. We shall write a more complicated (though equivalent) definition. This will be needed in what follows.

1.1 Definition: *Suppose $\mathcal{F} \subseteq \mathcal{P}(A)$ is a given family of subsets of a set A. A demand on $\mathcal{F}$ is a pair $d = (h^d, f^d)$ such that h^d is a finite 1-1 function from A to A, f^d is a finite 1-1 function from $\mathcal{F}$ to $\mathcal{F}$ and $x \in X \Leftrightarrow h^d(x) \in f^d(X)$ for every $x \in \mathrm{dom} h^d, X \in \mathrm{dom} f^d$. We denote by $D = D(A, \mathcal{F})$ the set of all demands on $\mathcal{F}$. Let* $\mathrm{FG}(D)$ *be the free group over the set $D(A, \mathcal{F})$. We say that an automorphism $g \in \mathrm{Aut}(\mathcal{F})$* satisfies *a demand d if $g(x) = h^d(x)$ for $x \in \mathrm{dom} h^d$ and $g[X] = f^d(X)$ for $X \in \mathrm{dom} f^d$. We call a partial homomorphism $\varphi : \mathrm{FG}(D) \to \mathrm{Aut}(\mathcal{F})$ a* satisfying *homomorphism if $\varphi(d)$ satisfies d for $d \in \mathrm{dom}\varphi$. (By "partial" we mean that φ need not be defined on all generators of* $\mathrm{FG}(D)$.)

1.2 Definition: *A family $\mathcal{F}$ is* homogeneous *if and only if every $d \in D$ is satisfiable if and only if there is a (total) satisfying homomorphism $\varphi : \mathrm{FG}(D) \to \mathrm{Aut}(\mathcal{F})$. A group $G \subseteq \mathrm{Aut}\mathcal{F}$* acts homogeneously *on $\mathcal{F}$ if and only if G contains the image of a total satisfying homomorphism, or, equivalently if and only if every demand is satisfied by some element in G.*

When φ is a homomorphism as above, we say that φ *testifies* the homogeneity of $\mathcal{F}$.

When the set A is clear from the context, we write $D(\mathcal{F})$ instead of $D(A, \mathcal{F})$.

1.3 Examples:

(1) The family $\mathcal{F} = \{\{x\} : x \in A\}$ *of all singletons is homogeneous. The group* $\mathrm{Aut}(\mathcal{F})$ *is the group* $\mathrm{Sym}(A)$ *of all symmetries of* A*.*

(2) The family $\mathrm{Fin}(A)$ *of all finite subsets of* A *is not homogeneous, although* $\mathrm{Aut}(\mathrm{Fin}(A)) = \mathrm{Sym}(A)$*, because a demand* $d = (\emptyset, \{(X, Y)\})$ *cannot be satisfied when* X *and* Y *are finite sets of different cardinalities.*

(3) A countable family of random subsets of ω *is homogeneous in probability 1. The membership of a point in a random set is determined by flipping a coin.*

In [GGK] the following was proved:

1.4 Theorem: *Every homogeneous family of subsets of an infinite set* A *satisfies exactly one of the conditions below:*

(1) $\mathcal{F} = \{\emptyset\}$

(2) $\mathcal{F} = \{A\}$

(3) $\mathcal{F}$ *is the family of all singletons of* A

(4) $\mathcal{F}$ *is the family of all co-singletons of* A

(5) $\mathcal{F}$ *is an independent family, namely for every finite function* $\tau : \mathcal{F} \to \{+, -\}$*, the set* $B_\tau = \bigcap_{X \in \tau^{-1}(+)} X \;\cap\; \bigcap_{Y \in \tau^{-1}(-)} A \setminus Y$ *is infinite, and* $\mathcal{F}$ *is dually independent, namely for every function* $\tau : A \to \{+, -\}$ *there are infinitely many members of* $\mathcal{F}$ *containing* $\tau^{-1}(+)$ *and avoiding* $\tau^{-1}(-)$*. Equivalently, the first order theory of* $\langle A, \mathcal{F}, \in \rangle$ *is the first order theory of the random countable bipartite graph.*

§2 Direct limits and homogeneity

In this section we exhibit a method of constructing homogeneous families as direct limits. This method will be used in the following sections.

Homogeneity is not, in general, preserved under usual direct limits of families. For example, an increasing union of homogeneous families need not be homogeneous itself. We therefore consider here a stronger relation of embeddability, called here "multi-embeddability", which, roughly speaking, preserves the satisfaction of previously satisfied demands. Direct limits of *this* relation can be made homogeneous, as we shall presently see.

2.1 Definition: *Let $T_i = \langle A_i, \mathcal{F}_i, D_i, G_i, \varphi_i \rangle$, $(i = 0, 1)$, be respectively a set A_i, a family of subsets $\mathcal{F}_i \subseteq \mathcal{P}(A_i)$, the collection of demands $D_i = D(\mathcal{F}_i)$, an automorphism group $G_i \subseteq \mathrm{Aut}(\mathcal{F}_i)$ and a partial satisfying homomorphism $\varphi_i : \mathrm{FG}(D_i) \to G_i$. Let $\bar{T}_i = A_i \cup \mathcal{F}_i \cup D_i \cup G_i$. We call a function $\Phi : \bar{T}_0 \to \bar{T}_1$ a multi-embedding of T_0 into T_1 (and write $\Phi : T_0 \to T_1$) if:*

(1) $\Phi \restriction A_0$ is a 1-1 function into A_1

(2) $\Phi \restriction \mathcal{F}_0$ is a 1-1 function into $\mathcal{F}_1$

(3) $\Phi \restriction D_0$ is a 1-1 function into D_1

(4) $\Phi \restriction G_0$ is a group monomorphism into G_1

And the following rules hold for $x \in A_0$, $X \in \mathcal{F}_0$, $d \in D_0$ and $g \in G_0$:

(a) $x \in X \Leftrightarrow \Phi(x) \in \Phi(X)$

(b) $\Phi[\mathrm{dom} h^d] = \mathrm{dom} h^{\Phi(d)}$, $\Phi[\mathrm{dom} f^d] = \mathrm{dom} f^{\Phi(d)}$, $\Phi((h^d(x)) = h^{\Phi(d)}(\Phi(x))$ and $\Phi((f^d(X)) = f^{\Phi(d)}(\Phi(X)$

(c) $\Phi(g(x)) = \Phi(g)(\Phi(x))$ and $\Phi(g[X]) = \Phi(g)[\Phi(X)]$

(d) $\Phi(d) \in \mathrm{dom}\varphi_1$ and $\Phi(\varphi_0(d)) = \varphi_1(\Phi(d))$ for every $d \in \mathrm{dom}\varphi_0$

We say that a multi-embedding Φ is successful if in addition to the conditions above also the following holds

(e) $\Phi(d) \in \mathrm{dom}\varphi_1$ for every $d \in D_0$.

2.2 Definition: *Suppose I is a directed set and $T_i = \langle A_i, \mathcal{F}_i, D_i, G_i, \varphi_i \rangle$ is as in definition 2.1 above for $i \in I$. Suppose that $\Phi_i^j : T_i \to T_j$ is a multi-embedding for $i \le j$, and*

(i) $\Phi_i^i = \mathrm{id}$

(ii) $\Phi_j^k \Phi_i^j = \Phi_i^k$ for $i \le j \le k$.

Then we call $\mathbf{T} = \langle T_i : (i \in I); (\Phi_i^j, \varphi_i^j) \rangle$ a direct system of multi-embeddings. We call $\mathbf{T}$ successful if in addition to (i) and (ii) the following condition holds:

(iii) for every $i \in I$ there is $j \ge i$ such that Φ_i^j is successful.

2.3 Theorem: *Suppose $\langle T_i : (i \in I);\ (\Phi_i^j, \varphi_i^j)\rangle$ is a successful direct system of embeddings. Let $T^* = \langle A^*, \mathcal{F}^*, D^*, G^*, \varphi^*\rangle := \lim_{\longrightarrow I} T_i$. Then $\mathcal{F}^*$ is homogeneous, with φ^* testifying homogeneity, and the canonical mapping $\Phi_i : T_i \to T^*$ is a successful multi-embedding.*

Proof: : We first recall the definition of a direct limit.

An equivalence relation $\sim$ is defined over $\dot{\bigcup}_{i\in I}\bar{T}_i$ as follows: $a{\sim}b \Leftrightarrow (\exists i \le j)(\Phi_i^j(a) = b \vee \Phi_i^j(b) = a)$. Conditions (i)–(ii) above imply that $\sim$ is indeed an equivalence relation. We define the canonical map $\Phi_i(a) = [a]_\sim$. Next we set $A^* = \dot{\bigcup}_{i\in I} A_i/{\sim}$ and observe the following:

2.4 Fact: *For every infinite cardinal κ, if I and every A_i are of cardinality $\le \kappa$, then $|A^*| \le \kappa$.*

We let $\mathcal{F}^* = \dot{\bigcup}_{i\in I}\mathcal{F}_i/{\sim}$, $G^* = \dot{\bigcup}_{i\in I} G_i/{\sim}$ and $D^* = \dot{\bigcup}_{i\in I} D_i/{\sim}$.

For $x^*, y^* \in A^*, X^* \in \mathcal{F}^*, d^* \in D^*$ and $g^* \in G^*$ we note:

(1) $x^* \in X^*$ iff there is some $i \in I$ and $x \in A_i, X \in \mathcal{F}_i$ such that $x \in X$ and $\Phi_i(x) = x^*$, $\Phi_i(X) = X^*$.

(2) $g^*(x^*) = y^*$ iff $g(x) = y$ for some $i \in I$ such that $x \in A_i, g \in G_i$ and $\Phi_i(x) = x^*$, $\Phi_i(y) = y^*$ and $\Phi_i(g) = g^*$.

(3) $\varphi^*(d^*) = g^*$ iff there is $i \in I$ such that $\varphi_i(d) = g$ and $\Phi_i(d) = d^*, \Phi_i(g) = g^*$.

We leave verification of this to the reader and that the following hold.

(a) $\mathcal{F}^* \subseteq \mathcal{P}(A^*)$

(b) $G^* \subseteq \mathrm{Aut}(\mathcal{F}^*)$

(c) $D^* = D(\mathcal{F}^*)$

(d) $\varphi^* : D^* \to G^*$ is a (total) satisfying homomorphism.

(e) $\Phi_j \Phi_i^j = \Phi_i$ for $i \le j$ in I

We conclude that $\Phi_i : T_i \to T^*$ is a successful embedding for every $i \in I$.

Homogeneity of $\mathcal{F}^*$ follows readily from (c) and (d) above. ☺ 2.3

§3 Bi-universal homogeneous families

The result proved in this section is the existence of a bi-universal member in the class of homogeneous families over a given infinite set.

Let us make the following definition:

3.1 Definition: *We call an embedding of structures $\Phi : M \to N$ a bi-embedding if for every automorphism $g \in \mathrm{Aut}(M)$ there is an automorphism $g' \in \mathrm{Aut}(N)$ such that $\Phi(g(x)) = g'(\Phi(x))$ for all $x \in M$.*

We observe that if $f : M \to N$ is a bi-embedding then f induces an embedding of $\mathrm{Aut}(M)$ into the group of all restrictions to $f[M]$ of elements in the set-wise stabilizer of $f[M]$ in $\mathrm{Aut}(N)$; that is, an embedding as permutation groups (see [T1]). We can think of a bi-embedding as a simultaneous embedding of both a structure and its automorphism group.

3.2 Definition: *A structure M^* in a class of structures K is bi-universal if for every structure $M \in K$ there is a bi-embedding $\Phi : M \to M^*$.*

3.3 Remarks:

(1) The definition of embedding of permutation grpups (see [T1]) is obtained by from this one by adding the condition that Φ is onto.

(2) Example 1.3 (1) above indicates that if a bi-universal family $\mathcal{F}^*$ over a set A^* exists, then for some $A \subseteq A^*$ of cardinality $|A^*|$ the restrictions of automorphisms of $\mathcal{F}^*$ to A include the full symmetric group $\mathrm{Sym}(A)$.

3.4 Lemma: *For every infinite $T = \langle A, \mathcal{F}, D, G, \varphi\rangle$ there is a set B such that $|A| = |B|$ and a successful multi-embedding*

$$\Phi : T \to \langle A\dot{\cup}B, \mathcal{P}(A\dot{\cup}B), D(A\dot{\cup}B, \mathcal{P}(A\dot{\cup}B)), \mathrm{Sym}(A\dot{\cup}B), \varphi'\rangle$$

Proof: We specify the points of B. A point in B is a finite function from the power set of a finite subset of A to $\{0,1\}$, namely $f \in B \Leftrightarrow f : \mathcal{P}(D_f) \to \{0,1\}$ and $D_f \subseteq A$ is finite. We let $\Phi\restriction A = \mathrm{id}$. For $X \in \mathcal{F}$ we define $\Phi(X)$ as follows: $\Phi(X) = X \cup \{f \in B : f(X\cap D_f) = 1\}$. We let $\Phi(\sigma)\restriction A = \sigma$ and let $\Phi(\sigma)(f) = g \Leftrightarrow \sigma[D_f] = D_g \wedge f(X) = g(\sigma[X])$ for all $X \subseteq D_f$. It is straightforward to verify that $\Phi\restriction\mathrm{Sym}(A)$ is a group monomorphism. We verify condition (c) in the definition of successful embedding (definition 2.1 above). Suppose $X \subseteq A$ and $\sigma \in \mathrm{Sym}(A)$ are given.

$\Phi(\sigma)[\Phi(X)] =$
$\sigma[X]\dot\cup\Phi(\sigma)[\{g \in B : g(X \cap D_g) = 1\}] =$
$\sigma[X]\dot\cup\{\Phi(\sigma)(g) : g \in B \wedge g(X \cap D_g) = 1\} =$
$\sigma[X]\dot\cup\{f \in B : f(\sigma[X] \cap D_f) = 1\} =$
$\Phi(\sigma[X])$

The definition of $\Phi\restriction D(\mathcal{P}(A)$ is determined uniquely by condition (b) in 2.1 above. We need to specify φ' and prove that (d) holds. For this we notice that:

3.5 Claim: *The family $\mathcal{F} = \{\Phi(X) : X \subseteq A\}$ satisfies that for every finite function $\tau : \mathcal{F} \to \{+,-\}$ the set $B_\tau = \bigcap_{X\in\tau^{-1}(+)} \Phi(X) \cap \bigcap_{Y\in\tau^{-1}(-)}(A\dot\cup B) \setminus \Phi(Y)$ has the same cardinality as $A\dot\cup B$.*

Proof: The proof of this is well known. ☺ 3.5

3.6 Corollary: *For every demand d on $\mathcal{F}$ there is a permutation $\sigma \in \mathrm{Sym}(A\dot\cup B)$ such that $\sigma(x) = h^d(x)$ and $\Phi(\sigma[X]) = \Phi[f^d(X)]$ for every $x \in \mathrm{dom}h^d$ and $X \in \mathrm{dom}f^d$.*

Proof: For every $\tau : \mathrm{dom}f^d \to \{+,-\}$ it holds that $|B_\tau| = |A\dot\cup B| = |B'_\tau|$ where $B'_\tau = \bigcap_{X\in\tau^{-1}(+)} \Phi(f^d(X))\cap\bigcap_{X\in\tau^{-1}(-)} \Phi(A\setminus f^d(X))$. (This means, informally, that every "cell" in the Venn diagram of $\mathrm{dom}f^{\Phi(d)}$ and every "cell" of the Venn diagram of $\mathrm{ran}f^{\Phi(d)}$ is of cardinality $|A\dot\cup B|$). Therefore it is trivial to extend h^d to a permutation that carries B_τ onto B'_τ for every τ. ☺ 3.6

Now let us define $\varphi'(\Phi(d)) = \Phi(\varphi(d))$ for every $d \in \mathrm{dom}\varphi$ and for all $d \in D \setminus \mathrm{dom}\varphi$ let us pick by claims 3.5 and 3.6 above a permutation $\varphi'(\Phi(d))$ that extends $\Phi(d)$. ☺ 3.4

3.7 Theorem: *Suppose A_0 is a given infinite set. There is a successful direct system of embeddings $\mathbf{T} = \langle T_n : (n \in \omega); (\Phi^n_m, \varphi^n_m)\rangle$ such that:*
(1) A_n is of cardinality $|A_0|$
(2) $\mathcal{F}_n = \mathcal{P}(A_n)$
(3) $G_n = \mathrm{Sym}(A_n)$.

Proof: Let $T_0 = \langle A_0, \mathcal{P}(A_0), D(\mathcal{F}_0), \mathrm{Sym}(A_0), \varphi_0 : \{e\} \to \{\mathrm{id}_{A_0}\}\rangle$. Now use Lemma 3.4 inductively. ☺ 3.7

3.8 Theorem: *For every infinite set A^* there is a homogeneous family $\mathcal{F}^* \subseteq \mathcal{P}(A^*)$, and an infinite subset $A \subseteq A^*$ of cardinality $|A^*|$ such that $\mathcal{P}(A) = \{X \cap A : X \in \mathcal{F}^*\}$ and $\mathrm{Sym}(A) \subseteq \{g\restriction A : g \in \mathrm{Aut}(\mathcal{F}^*)\}$. Therefore any injection $f : A^* \to A$ induces* *a bi-embedding of every family $\mathcal{F} \subseteq \mathcal{P}(A^*)$ (not necessarily homogeneous) into $\mathcal{F}^*$. In particular, $\mathcal{F}^*$ is bi-universal in the class of all homogeneous families over A^*.*

Proof: By Theorem 3.7 there is a successful direct system of embeddings $\mathbf{T} = \langle T_n : (n \in \omega); (\Phi^n_m, \varphi^n_m)\rangle$ such that:

(1) $|A_n| = |A^*|$

(2) $\mathcal{F}_n = \mathcal{P}(A_n)$

(3) $G_n = \mathrm{Sym}(A_n)$.

By Theorem 2.3 and the side remark 2.4 it follows that the family $\mathcal{F}^*$ obtained by the direct limit is a homogeneous family of subsets of a set A^{**} of size $|A^*|$, and we may assume that $A^{**} = A^*$. The canonical map Φ_0 is a successful multi-embedding, and therefore in particular a bi-embedding. Let A be the image of A_0 under Φ_0. As $\mathcal{F}_0 = \mathcal{P}(A_0)$ and $G_0 = \mathrm{Sym}(A_0)$, we conclude that $\mathcal{P}(A) = \{X \cap A : X \in \mathcal{F}^*\}$ and $\mathrm{Sym}(A) \subseteq \{g\restriction A : g \in \mathrm{Aut}(\mathcal{F}^*)\}$. The Theorem is now obvious. ☺ 3.8

§4 The number of isomorphism types of homogeneous families over ω

In this section we make a second use of the method of direct limits as introduced in Section 2 to determine the number of isomorphism types of homogeneous families over a countable set. It was conjectured in [GGK] that this number is the maximal possible, namely $2^{2^{\aleph_0}}$. An isomorphism between two families $\mathcal{F}_0 \subseteq \mathcal{P}(A_0)$ and $\mathcal{F}_1 \subseteq \mathcal{P}(A_1)$ is, of course, a 1-1 onto function $f : A_0 \to A_1$ which satisfies $X \in \mathcal{F}_0 \Leftrightarrow f[X] \in \mathcal{F}_1$.

To obtain $2^{2^{\aleph_0}}$ non isomorphic homogeneous families over a countable set, it is enough to obtain $2^{2^{\aleph_0}}$ *different* such families; for then dividing by isomorphism, the size of each class is $2^{\aleph_0}$, and therefore there are $2^{\aleph_0}$ classes (see below).

The technique used to achieve this is embedding a family $\mathcal{F} \subseteq \mathcal{P}(A)$ in a homogeneous family $\mathcal{F}' \subseteq \mathcal{P}(A^*)$ for $A^* \supseteq A$ in such a way that $\{X \cap A : X \in \mathcal{F}'\} = \mathcal{F}$. In other words, we will "homogenize" a family $\mathcal{F}$ "without adding sets" to $\mathcal{F}$. Thus, starting with distinct $\mathcal{F}$-s we obtain distinct homogeneous $\mathcal{F}'$-s.

4.1 Lemma: *There is a pair of countable sets $A_0 \subseteq A^*$ (in fact, for every pair $A_0 \subseteq A^*$ of countable sets satisfying $A^* \setminus A_0$ infinite) such that for every family $\mathcal{F} \subseteq \mathcal{P}(A_0)$ satisfying* $\mathrm{Fin}(A_0) \subseteq F$ *there is a homogeneous family $\mathcal{F}' \subseteq \mathcal{P}(A^*)$ satisfying $\{X \cap A_0 : X \in \mathcal{F}'\} = \mathcal{F}$*

This lemma determines the number of isomorphism types of homogeneous families over a countable set:

4.2 Corollary: *There are $2^{2^{\aleph_0}}$ isomorphism types of homogeneous families over a countable set.*

Proof: There are $2^{2^{\aleph_0}}$ different families $\{\mathcal{F}_\alpha : \alpha < 2^{2^{\aleph_0}}\}$, such that $\mathrm{Fin}(A_0) \subseteq \mathcal{F}_\alpha \subseteq \mathcal{P}(A_0)$. For each $\mathcal{F}_\alpha$ there is, by 4.1, a homogeneous family $\mathcal{F}'_\alpha \subseteq \mathcal{P}(A^*)$ that satisfies $\{X \cap A_0 : X \in \mathcal{F}'_\alpha\} = \mathcal{F}_\alpha$. Therefore, $\alpha \neq \beta$ implies that $\mathcal{F}'_\alpha \neq \mathcal{F}'_\beta$. Let us define an equivalence relation over $2^{2^{\aleph_0}}$: $\alpha \sim \beta \Leftrightarrow$ there is an isomorphism between $\mathcal{F}'_\alpha$ and $\mathcal{F}'_\beta$. There are at most $2^{\aleph_0}$ many members in an equivalence class $[\alpha]_\sim$, as there are $2^{\aleph_0}$ many permutations of A^*, and therefore at most $2^{\aleph_0}$ many different isomorphic images of $\mathcal{F}'_\alpha$. As $2^{\aleph_0} \times 2^{\aleph_0} = 2^{\aleph_0}$, while $2^{2^{\aleph_0}} > 2^{\aleph_0}$, there must be $2^{2^{\aleph_0}}$ many equivalence classes over $\sim$, and therefore $2^{2^{\aleph_0}}$ many isomorphism types of homogeneous families over A^*. ☺ 4.2

We prepare for the proof lemma 4.1. Before plunging into the formalism, let us state the idea behind the proof. We use the set of demands over a family and the free group associated with this set to construct a successful extention in which the automorphisms act freely. Thus, we can control sets in the orbit of an "old" set so that their intersections with the "old" set is either finite or "old".

We need some notation: Let $\mathrm{FG}(D)$ be the free group over the set $D = D(\mathcal{F})$ for some family $\mathcal{F}$. If $\mathcal{F}$ is countable, this group is also countable. We view $\mathrm{FG}(D)$ as the collection of all reduced words in the alphabet $C = D \cup \{d^{-1} : d \in D\}$ (a word is reduced if there is no occurrence of dd^{-1} or $d^{-1}d$ in it) and the group operation, denoted by $\circ$, is juxtaposition and cancellation (so $w_1 \circ w_2$ is a reduced word, and its length may be strictly smaller than $\lg w_1 + \lg w_2$). We let c range over the alphabet C, and let c^{-1} denote d^{-1} if $c = d$ or d if $c = d^{-1}$. We denote by e the unit of the free group, which is the empty sequence $\langle\rangle$. For convenient discussion we also adopt the notation h^c and f^c, by which we mean h^d and f^d if $c = d$ and the respective inverses $(h^d)^{-1}$ and $(f^d)^{-1}$ otherwise. Now we can define:

4.3 Definition: *Suppose that $\mathbf{T} = \langle T_i : (i \in I); \Phi_i^j \rangle$ is a successful direct system of multi-embeddings. For every $j \in I$:*

(1) A homomorphism $\xi_j : \mathrm{FG}(D_i) \to G_j$ is defined by $\xi_j(d) := \varphi_j(\Phi(d))$.

(2) We call a word $w = c_0 \dots c_k \in \mathrm{FG}(D_j)$ new if c_l is not in the range of Φ_i^j for all $l \leq k$ and all $i < j$. A word $w \in \mathrm{FG}(D_j)$ is old if it is in the range of Φ_i^j for some $i < j$.

(3) For a word $w \in \mathrm{FG}(D_j)$ and $X \in \mathcal{F}_i$ we define what $f^w(X)$ is. Let $w = w_0 w_1 \dots w_l$ where for each $k \leq l$ the word w_k is either new or old. For a new word $w_k = c_k^0 \dots c_k^{l(k)}$ we denote by f^{w_k} the composition $f^{c_k^{l(k)}} \dots f^{c_k^0}$. If this composition is empty, we say

that f^{w_k} is not defined. If w_k is old, then $\xi_j(w) \in \mathrm{Aut}(\mathcal{F}_j)$ and induces a 1-1 function $f^{w_k} : \mathcal{F}_j \to \mathcal{F}_j$. Let f^w be the composition $f^{w_l} \dots f^{w_0}$. If this composition is empty, we say that f^w is not defined.

(4) *Analogously to the definition in* (3), *we define h^w.*

To prove lemma 4.1 we need an expansion of the technique of direct limits by some more structure. This is needed to enable us to handle uncountably many demands by adding just countably many points. We first define (a particular case of) inverse systems. Then we form direct limits of inverse systems to obtain a pair of sets as required by the lemma.

4.4 Definition: *a sequence $\mathbf{T} = \langle T^m : m < \omega \rangle$, where $T^m = \langle A^m, \mathcal{F}^m, D^m, G^m, \varphi^m \rangle$, is called an inverse system if:*

(1) *$\mathcal{F}^m \subseteq \mathcal{P}(A^m)$, $D = D(A^m, \mathcal{F}^m)$, $G^m \subseteq \mathrm{Aut}(\mathcal{F}^m)$ and $\varphi^m : \mathrm{FG}(D^m) \to G^m$ is a partial satisfying homomorphism.*

(2) *A^m and $\mathcal{F}^m$ are* **countable**

For $m \le m'$

(3) *$A^m \subseteq A^{m'}$*

(4) *$\mathcal{F}^m \subseteq \{X \cap A^m : X \in \mathcal{F}^{m'}\}$*

(5) *$G^m \subseteq \{g \restriction A^m : g \in G^{m'}, g \restriction A^m \in \mathrm{Sym}(A^m)\}$*

For a demand $d \in D^{m'}$ we define $d \restriction A^m$ iff $\mathrm{dom} h^d \cup \mathrm{ran} h^d \subseteq A^m$ and for every distinct $X, Y \in \mathrm{dom} f^d \cup \mathrm{ran} f^d$ the sets $X \cap A^m$ and $Y \cap A^m$ are distinct. When $d \restriction A^m$ is defined, $h^{d \restriction A^m} = h^d$ and $f^{d \restriction A^m}$ is obtained from f^d by replacing every $X \in \mathrm{dom} f^d \cup \mathrm{ran} f^d$ by $X \cap A^m$. Clearly, when $d \restriction A^m$ is defined, it belongs to D^m, and every $d \in D^m$ equals $d' \restriction A^m$ for some $d' \in D^{m'}$ by (3) and (4).

If $w = c_0 \dots c_k \in FG(D^{m'})$ and $c_i \restriction A^m$ is defined for every $i \le k$, we define $w \restriction A^m$ as $c_0 \restriction A^m \dots c_k \restriction A^m$ (it is obvious what $c \restriction A^m$ is). The restriction $\restriction$ is a partial homomorphism from $\mathrm{FG}(D^{m'})$ onto $\mathrm{FG}(D^m)$. The last condition is

(6) *If $d \in \mathrm{dom}\varphi^{m'}$ and $d \restriction A^m$ is defined, then $d \restriction A^m \in \mathrm{dom}\varphi^m$ and $\varphi^m(d \restriction A^m) = \varphi^{m'}(d) \restriction A^m$ (the operation of $\varphi^{m'}(d)$ on A^m depends only on $d \restriction A^m$ when $d \restriction a^m$ is defined).*

4.5 Definition: *Given an inverse system $\mathbf{T} = \langle T^m : m < \omega \rangle$ we define the inverse limit $\varprojlim \mathbf{T} = T^* = \langle A^*, \mathcal{F}^*, D^*, G^*, \varphi^* \rangle$ as follows:*

(a) $A^* = \bigcup_m A^m$.

For every $x \in A^*$ let $m(x)$ be the least m such that $x \in A^m$.

(b) $\mathcal{F}^* = \{X \subseteq A^* : (X \cap A^m \in \mathcal{F}^m)$ *for all but finitely many* $m\}$. *For* $X \in \mathcal{F}^*$ *we let* $m(X)$ *be the least such that* $X \cap A^m \in \mathcal{F}^m$ *for every* $m \geq m(X)$.

We call $X \in \mathcal{F}^*$ *bounded if* $X \subseteq A^m$ *for some* m.

(c) $G^* = \{g \in \mathrm{Sym}(A^*) : (g \restriction A^m \in G^m)$ *for all but finitely many* $m\}$. *Let* $m(g)$ *be the least such that* $g \restriction A^m \in G^m$ *for every* $m \geq m_g$.

It is easy to verify that $G^* \subseteq \mathrm{Aut}(\mathcal{F}^*)$.

(d) $D^* = D(\mathcal{F}^*)$

It is easy to verify that for every $d^* \in D^*$ *there is some* $m(d^*)$ *such that for all* $m \geq m(d^*)$ *it is true that* $d^* \restriction A^m$ *is defined, and* $d^* \restriction A^m \in D^m$.

(e) $\varphi^*(d^*) = \bigcup_{m \geq m_{d^*}} \varphi^m(d^* \restriction A^m)$ *and is defined iff* $d^* \restriction A^m \in \mathrm{dom}\varphi^m$ *for all* $m \geq m_{d^*}$

4.6 Definition: *Suppose that* $\mathbf{T}_0 = \langle T_0^m : m < \omega \rangle$ *and* $\mathbf{T}_1 = \langle T_1^m : m < \omega \rangle$ *are inverse systems, and let* $\varprojlim \mathbf{T}_0 = T_0 = \langle A_0, \mathcal{F}_0, D_0, G_0, \varphi_1 \rangle$ *and* $\varprojlim \mathbf{T}_1 = T_1 = \langle A_1, \mathcal{F}_1, D_1, G_1, \varphi_1 \rangle$ *be their respective inverse limits. We call a sequence* $\langle \overset{m}{\Phi} : T_0^m \to T_1^m : m < \omega \rangle$ *of multi-embeddings an* inverse system of multi-embeddings *if for* $m \leq m'$ *we have:*

(1) $\overset{m'}{\Phi} \restriction A_0^m = \overset{m}{\Phi} \restriction A_0^m$

(2) $\overset{m'}{\Phi}(X) \restriction A_1^m = \overset{m}{\Phi}(X \cap A_0^m)$ *for every* $X \in \mathcal{F}_0^{m'}$ *for which* $X \cap A_0^m \in \mathcal{F}_0^m$

(3) $\overset{m'}{\Phi}(g) \restriction A_1^m = \overset{m}{\Phi}(g \restriction A_0^m)$ *for every* $g \in G_0^m$ *for which* $g \restriction A_0^m \in G^m$

When $\langle \overset{m}{\Phi} : T_0^m \to T_1^m : m < \omega \rangle$ *is an inverse system of multi-embeddings we define a multi-embedding* $\Phi = \varprojlim \overset{m}{\Phi} : T_0 \to T_1$ *as follows:*

$\Phi \restriction A_0^* = \bigcup \overset{m}{\Phi} \restriction A_0^m$

$\Phi(X) = \bigcup_{m \geq m(X)} \overset{m}{\Phi}(X \cap A_0^m)$ *for* $X \in F_0^*$

$\Phi(g) = \bigcup_{m \geq m_g} \overset{m}{\Phi}(g \cap A_0^m)$ *for* $g \in G_0^*$

Call $\Phi = \varprojlim \overset{m}{\Phi}$ *a* multi-embedding of inverse systems.

4.7 Claim: *If* $\mathbf{T}_0 = \langle T_0^m : m < \omega\}$ *and* $\mathbf{T}_1 = \langle T_1^m : m < \omega\}$ *are inverse system and* $\langle \overset{m}{\Phi} : T_0^m \to T_1^m : m < \omega \rangle$ *is an inverse system of multi-embeddings such that every* $\overset{m}{\Phi}$ *is successful, then* $\Phi = \varprojlim_m \overset{m}{\Phi}$ *is also successful.*

Proof: Suppose that $d \in D_0$ and we shall show that $\Phi(d) \in \mathrm{dom}\varphi_1$. There is some m_d such that for all $m \geq m_d$ the restriction $d \restriction A_m$ is defined. As $\overset{m}{\Phi}$ is successful, $\overset{m}{\Phi}(d \restriction A_m)$ belongs to $\mathrm{dom}\varphi_1^m$ for $m \geq m_d$. Therefore $\varphi_1(\bigcup_{m \geq m_d} \overset{m}{\Phi}(d \restriction A_m) = \varphi_1(\Phi(d))$ exists and belongs to G_1.

☺ 4.7

We shall construct a two dimensional system $\mathbf{T} = \langle T^m_n : n, m < \omega \rangle$ and successful multi-embeddings $\overset{m}{\Phi}{}^{n+1}_n : T^m_n \to T^m_{n+1}$ such that for every n,

(1) $\mathbf{T}_n = \langle T^m_n : m < \omega \rangle$ is an inverse system.

(2) $\langle \overset{m}{\Phi}{}^{n+1}_n : m < \omega \rangle$ is an inverse system of successful multi-embeddings.

Then a direct system will result: $T_n = \underset{\longleftarrow}{\lim} \mathbf{T}_n$ and $\Phi^{n+1}_n = \underset{\longleftarrow}{\lim} \overset{m}{\Phi}{}^{n+1}_n$.

Let $T^m_0 = \langle m+1, \mathcal{P}(m+1), D(\mathcal{P}(m+1)), \{\mathrm{id}\}, \{(e, \mathrm{id})\} \rangle$. Clearly, $T_0 = \underset{\longleftarrow}{\lim} \mathbf{T}_0 = \langle \omega, \mathcal{P}(\omega), D(\mathcal{P}(\omega)), \{\mathrm{id}\}, \{(e, \mathrm{id})\} \rangle$.

Suppose now that $T_n = \underset{\longleftarrow}{\lim} T^m_n$ is defined, where $T^m_n = \langle A^m_n, \mathcal{F}^m_n, D^m_n, G^m_n, \varphi^m_n \rangle$, and that $\Phi^n_{n-1} = \underset{\longleftarrow}{\lim} \overset{m}{\Phi}{}^n_{n-1}$ is also defined (when $n > 0$)

We assume, for simplicity, that $\Phi^n_{n-1} \restriction A_{n-1} = \mathrm{id}$ (if $n > 0$) and, furthermore, identify $\mathrm{FG}(D^m_{n-1})$ with its image under $\overset{m}{\Phi}{}^n_{n-1}$, and write $\mathrm{FG}(D^m_{n-1}) \subseteq \mathrm{FG}(D^m_n)$ as well as $\mathrm{FG}(D_{n-1}) \subseteq \mathrm{FG}(D_n)$. Thus, the new words of $\mathrm{FG}(D_n)$ coincide with $\mathrm{FG}(D_n \setminus D_{n-1})$, and similarly for $\mathrm{FG}(D^m_n)$.

Let $\dot{\bigcup}_m D^m_n$ be the disjoint union of D^m_n. We view A_n as a subset of the following set $B_{n+1} = \{xw : x \in A_n, w \in \mathrm{FG}(\dot{\bigcup}_m D^m_n)\}$. The expression xw is the formal string $xc_0 \ldots c_w$ where $w = c_0 \ldots, c_k$, and x is identified with xe (where e is the empty string).

4.8 Fact: *B_{n+1} is countable.*

The fact holds because each D^m_n is countable.

Now define $B^m_{n+1} = \{xw : x \in A^m_n, w \in \mathrm{FG}(\dot{\bigcup}_{m' \le m} D^{m'}_n)\}$. Clearly, $A^m_n \subseteq B^m_{n+1}$.

Next we define an operation $\xi_{n+1}(c) : B_{n+1} \to B_{n+1}$ for every $c \in D_n$ (there are, of course, uncountably many c-s!).

We want that $\xi_{n+1}(c) \restriction B^n_{n+1}$ to depend only on $c \restriction A^n_n$ whenever $c \restriction A^m_n$ is defined.

If $x \in \mathrm{dom} h^c$, we let $\xi_{n+1}(c)(x) = h^c(x)$.

For all other points in B_{n+1}, we let $\xi_{n+1}(c)(xw) = xw \circ (c \restriction A^m_0)$ if m is the least such that $xw \in B^m_n$ **and** $c \restriction A^m_0 (\in C^m_1)$ is defined.

There is a unique extension of ξ_{n+1} to a homomorphism from $\mathrm{FG}(D_n)$ to $\mathrm{Sym}(B_{n+1})$, which we also call ξ_{n+1}.

4.9 Claim: *For every $w \in \mathrm{FG}(D_n)$ there is some $m(w)$ such that:*

(1) B^m_{n+1} is invariant under $\xi_{n+1}(w)$ for all $m \ge m_w$.

(2) If $w \ne e$ then for every $xv \in B_{n+1} \setminus A^{m_w}_n$, we have $\xi_{n+1}(w)(xv) = xv \circ w \ne xv$.

Proof: (1) is clear from the definition. For (2) notice that if $c \restriction A^m_0$ is defined then the finitely many points in $\mathrm{dom} h^c$ belong to A^m_0. Then $\xi_{n+1}(c)(xv) = xv \circ (w \restriction A^m_0)$.

From4.9 (2) it follows readily that ξ_{n+1} is, in fact a *monomorphism*, as for every $w \in \mathrm{FG}(d_0)$ there is some m_w for which $w \restriction A_0^m$ is defined.

Let $\overset{m}{\xi}_{n+1}(c \restriction A_n^m) = \xi_{n+1}(c) \restriction B_{n+1}^n$ for all $c \in D_n$ for which $c \restriction A_n^m$ is defined.

Now we can define $A_{n+1} = \{\xi_{n+1}(w)(x) : x \in A_n, w \in \mathrm{FG}(D_n)\}$ and $A_{n+1}^m = A_n \cap B_{n+1}^m = \{\overset{m}{\xi}(w)(x) : x \in A_0^m, w \in \mathrm{FG}(D_0^m)$. (We remark that $A_{n+1} \neq B_{n+1}$, because when $x \in \mathrm{dom} h^c$, the point $xc \notin A_{n+1}$).

Clearly, A_{n+1} is invariant under $\xi_{n+1}(w)$ for every $w \in \mathrm{FG}(D_n)$, and also $A_{n+1}^{m_w}$ is, if $w \restriction A_n^{m_w}$ is defined.

Having defined A_{n+1} we let $\overset{m}{\Phi}{}_n^{n+1} : A_n^m \to A_{n+1}^m$ be the identity. Therefore also $\Phi_n^{n+1} \restriction A_n$ is the identity.

Now let us define $\overset{m}{\Phi}{}_n^{n+1} \restriction \mathcal{F}_n^m$. For every $X \in \mathcal{F}_n^m$ and $xw \in A_{n+1}^m$ we determine whether $xw \in \overset{m}{\Phi}{}_n^{n+1}(X)$ by induction on the length of w.

If $\lg w = 0$ then necessarily $xw = x$, and we let $x \in \overset{m}{\Phi}{}_0^1(X) \Leftrightarrow x \in X$ for every $X \in \mathcal{F}_n^m$ and $x \in A_n^m$.

Suppose that this is done for all words of length k and that $\lg wc = k+1$.

Distinguish two cases: when c is old and when c is new.

First case: c is old, namely $c \in C_{n-1}^m$ (this case does not exist when $n = 0$). Here we have that $\overset{m}{\xi}_n(c) = \overset{m}{\varphi}_n(c)$ is defined, and is an automorphism of $\mathcal{F}_n^m$. Let $xwc \in \overset{m}{\Phi}{}_0^1(X) \Leftrightarrow xw \in \overset{m}{\Phi}{}_n^{n+1}(\overset{m}{\xi}_{n-1}(c^{-1})[X])$.

Second case: c is new. Let $xwc \in \overset{m}{\Phi}{}_n^{n+1}(X) \Leftrightarrow xw \in \overset{m}{\Phi}{}_n^{n+1}(f^{c^{-1}}(X))$. In the right hand side we mean that $f^{c^{-1}}(X)$ is defined and $xw \in \overset{m}{\Phi}{}_n^{n+1}(f^{c^{-1}}(X))$.

Now we can set $\Phi_n^{m+1}(X) = \bigcup_{m \geq m(X)} \overset{m}{\Phi}{}_n^{n+1} X \cap A_n^m$.

4.10 Fact: *For every old $w \in \mathrm{FG}(D_n)$ and every $X \in F_n$ it holds that $\Phi_n^{n+1}(\varphi_n(w)[X]) = \xi_{n+1}(w)[\Phi_n^{n+1}(X)]$ (rule (c) in 2.1).*

The proof of the fact is straightforward using induction on word length.

4.11 Claim: *For every $w \in \mathrm{FG}(D_n)$ and every $X \in \mathcal{F}_n$ there is $m \geq m_w$ such that*

(1) $f^w(X)$ is defined iff $f^{w \restriction A_n^m}(X \cap A_n^m)$ is defined

(2) $f^{w^{-1}}(X)$ is defined iff $f^{w^{-1} \restriction A_n^m}(X \cap A_n^m)$ is defined

(3) for every $x \in A_n$ with $m(x) \geq m$, $\xi_{n+1}(w)(x) = x(w \restriction A_n^{m(x)}) \in \Phi_n^{n+1}(X) \Leftrightarrow x \in f^{w^{-1}}(X)$ (where by $x \in f^{w^{-1}}(X)$ we mean that $f^{w^{-1}}(X)$ is defined **and** *$x \in f^{w^{-1}}(X)$).*

Proof: If $f^w(X)$ is defined, then $f^{w \restriction A_n^m}(X \cap A_n^m)$ is defined whenever $w \restriction A_n^m$ is defined and equals $f^w(X) \cap A_n^m$. Conversely, if $f^w(X)$ is not defined, then there is some $m \geq m_w$

such that $X \cap A_n^m \neq Y \cap A_n^m$ for all $Y \in \mathrm{dom}^w$ (if there is one X for which $f^w(X)$ is not defined, then $\mathrm{dom} f^w$ is necessarily finite) and therefore $f^{w \restriction A_n^m}(X \cap A_n^m)$ is not defined.

Fromthe definition of ξ_{n+1} and $m(x) \geq m_w$ it follows that $\xi_{n+1}(x) = x(w \restriction A_n^m)$. From the definition of $\Phi_n^{n+1} \restriction \mathcal{F}_n$ it is immediate that $x(w \restriction A_n^m) \in \Phi_n^{n+1}(X) \Leftrightarrow x \in f^{w^{-1}}(X)$. ☺ 4.11

4.12 Fact: *$\Phi_n^{n+1}(X) \cap A_{n+1}^m$ depends only on $X \cap A_n^m$ whenever $X \cap A_n^m \in \mathcal{F}_n^m$.* ☺ 4.12

Now we can define $\mathcal{F}_{n+1} = \{\xi_{n+1}(w)[\Phi_n^{n+1}(X)] : X \in \mathcal{F}_n, w \in \mathrm{FG}(D_n)\}$.

Let $\mathcal{F}_{n+1}^m = \{\overset{m}{\xi}_{n+1}(w)(X) : X \in \mathcal{F}_{n+1}^m, w \in \mathrm{FG}(D_n^m)\}$.

4.13 Claim: *$\mathcal{F}_{n+1}^m$ is countable for every m.*

Proof: The fact follows by the countability of $\mathrm{FG}(D_n^m)$ *and 4.12.* ☺ 4.13

We finished defining $\mathbf{T}_{n+1}$ and $\langle \overset{m}{\Phi}{}_n^{n+1} : m < \omega \rangle$, and verified that $\mathbf{T}_{n+1}$ is an inverse system, that $\langle \overset{m}{\Phi}{}_n^{n+1} : m < \omega \rangle$ is an inverse system of successful multi-embedding and that, consequently, $\Phi_n^{n+1} : T_n \to T_{n+1}$ is a multi-embedding of inverse systems.

Let $T^* = \langle A^*, \mathcal{F}^*, D^*, G^*, \varphi^* \rangle = \varinjlim T_n$. We show that the conclusion of lemma 4.1 holds for the pair of sets A_0 and A^*. Clearly, these sets are countable and $A_0 \subseteq A^*$. So all we need is:

4.14 Claim: *For every family $\mathcal{F} \subseteq \mathcal{P}(A_0)$ which includes* $\mathrm{Fin}(A_0)$ *there is a homogeneous family $\mathcal{F}' \subseteq \mathcal{F}^*$ such that $\mathcal{F}' \restriction A_0 = \mathcal{F}$.*

Proof: Suppose that $\mathcal{F} \subseteq \mathcal{P}(A_0)$ is a family which includes $\mathrm{Fin}(A_0)$. We work by induction on n and define $\mathcal{F}_n' \subseteq \mathcal{P}(A_n)$ for every n:

(1) $\mathcal{F}_0' = \mathcal{F}$.

(2) $\mathcal{F}_{n+1}' = \{\xi_n(w)[\Phi_n^{n+1}(X)] : w \in \mathrm{FG}(D(\mathcal{F}_n')), X \in \mathcal{F}_n'\}$

Let $\mathcal{F}' = \{\Phi_n(X) : X \in \mathcal{F}_n'\}$.

We claim that

(a) $\mathcal{F}' \subseteq \mathcal{F}^*$ and $\varphi^* \restriction D(\mathcal{F}')$ testifies that $\mathcal{F}'$ is homogeneous.

(b) $\{X \cap A_0 : X \in \mathcal{F}'\} = \mathcal{F}$.

To prove (a) suppose that $d \in D(\mathcal{F}')$ is a demand. Then there is some n and a demand $d_n \in D(\mathcal{F}_n')$ such that $\Phi_n(d_n) = d$. As Φ_n^{n+1} is successful, $\xi_n(d) = \varphi_{n+1}(\Phi_n^{n+1}(d_n)) =: g$ is defined. Now $\Phi_{n+1}(g) = \varphi^*(d)$ satisfies d and is an automorphism of $\mathcal{F}^*$. Why is it also an automorphism of $\mathcal{F}'$? Because of (2) above.

To prove (b) we notice that it is enough to prove by induction that for every n and $X \in \mathcal{F}_{n+1}'$, we have

$(*)_n$ $X \cap A_n \in \mathcal{F}'_n$ or is bounded.

For then it follows by induction that $X \cap A_0 \in \mathcal{F}$ for every n and $X \in \mathcal{F}_n$: if $X \cap A_n \in \mathcal{F}_n$ we have that $X \cap A_0 \in \mathcal{F}$ by the induction; if $X \cap A_n$ is bounded, then $X \cap A_0$ is finite and again in $\mathcal{F}$.

So let us prove $(*)_n$. We have to show that for every $w \in \mathrm{FG}(D_n)$ and every $X \in \mathcal{F}_n$ the set $\xi_{n+1}[\Phi_n^{n+1}(X)] \cap A_n$ belongs to $\mathcal{F}'_n$ or is bounded. We show something stronger.

$(**)_n$ For every $X \in \mathcal{F}'_n$ and $w \in \mathrm{FG}(D(\mathcal{F}'_n))$ if $f^w(X)$ is defined then $\xi_{n+1}[\Phi_n^{n+1}(X)] = \Phi_n^{n+1}(f^w(X))$ (and therefore $\xi_{n+1}[\Phi_n^{n+1}(X)] \cap A_n = \Phi_n^{n+1}(f^w(X)) \cap A_n = f^w(X) \in \mathcal{F}'_n$). If $f^w(X)$ is not defined, then $\xi_{n+1}[\Phi_n^{n+1}(X)] \cap A_n$ is bounded.

Suppose first that $f^w(X)$ is defined. Then obviously it belongs to $\mathcal{F}'_n$, because $w \in \mathrm{FG}(D(\mathcal{F}'_n))$. It is easy to check that $\xi_{n+1}(w)(xv) \in \Phi_n^{n+1}(f^w(X)) \Leftrightarrow xv \in \Phi_n^{n+1}(X)$.

So assume that $f^w(X)$ is not defined, and we want to prove that $\xi_{n+1}(w)[\Phi_n^{n+1}(X)] \cap A_n$ is bounded.

If $f^w(X)$ is not defined, then $X \notin \mathrm{ran} f^{w^{-1}}$. It is sufficient to see that the set

$$\{x \in A_n : \xi_{n+1}(w^{-1})(x) \in \Phi_n^{n+1}(X)\}$$

is bounded, because this set equals $\xi_{n+1}(w)[\Phi_n^{n+1}(X)] \cap A_n$. By 4.11 there is a large enough $m > m(w)$ such that for all $x \in A_n$ with $m(x) \geq m$ we have that

$$\xi_{n+1}(w)(x) = \overset{m(x)}{\xi}_{n+1}(x) = x(w \restriction A_n^{m(x)}) \in \overset{m(x)}{\Phi}{}_n^{n+1}(X \cap A_n^{m(x)}) \Leftrightarrow x \in f^{w^{-1}}(X)$$

But $f^{w^{-1}}(X)$ is not defined, and therefore $\xi_{n+1}(w)(x) \notin \Phi_n^{n+1}(X)$ for all $x \in A_n$ with $m(x) > m$, which is what we wanted. ☺ 4.1

We give a corollary of this proof.

4.15 Corollary: *There is a collection of $2^{2^{\aleph_0}}$ permutation groups over ω, $\langle G_\alpha : \alpha < 2^{2^{\aleph_0}} \rangle$ such that:*

(1) Every G_α is isomorphic to the free group on $2^{\aleph_0}$ generators.

(2) Every G_α testifies the homogeneity of some family $\mathcal{F}_\alpha \subseteq \mathcal{P}(\omega)$

(3) If $\alpha < \beta < 2^{\aleph_0}$, then G_α and G_β are not isomorphic as permuatatio groups.

Proof: We have shown that there are $2^{2^{\aleph_0}}$ many homogeneous sub-families of F^*, F'_α for $\alpha < 2^{2^{\aleph_0}}$. The restriction of φ^* to $\mathrm{FG}(D(A^*, \mathcal{F}_\alpha))$ is a monomorphism of the free group over a set of cardinality $2^{\aleph_0}$ into G^* which testifies homogeneity of $\mathcal{F}_\alpha$. This gives us

$2^{2^{\aleph_0}}$ *different* groups satisfying (1) and (2) in the corollary. To obtain (3), divide by the relation "isomorphic via a permutation of ω", and pick a member from every equivalence class. As in each class there are $2^{\aleph_0}$ many members at the most, we get that there are $2^{2^{\aleph_0}}$ classes. ☺ 4.15

We now wish to show that there is no homogeneous family over ω such that every homogeneous family over ω is isomorphic to one of its subfamilies. This will follow from the next lemma about the number of pairwise incompatible homogeneous families over a countable set. Two families over ω are *incompatible* if for some $X \subseteq \omega$ the set X belongs to one family while the set $\omega \setminus X$ belongs to the other. For every $X \subseteq \omega$ let us denote $X^0 := X$ and $X^1 := \omega \setminus X$.

4.16 Lemma: *There is a collection $\{F_\alpha : \alpha < 2^{2^{\aleph_0}}\}$ of pairwise incompatible homogeneous families over ω.*

4.17 Corollary: There is no homogeneous family over ω such that every homogeneous family over ω is isomorphic to one of its subfamilies.

Proof: (of Corollary) Suppose to the contrary that $\mathcal{F}^*$ is a homogeneous family over ω with this property. By Lemma 4.16 pick a collection $\{F_\alpha : \alpha < 2^{2^{\aleph_0}}\}$ of pairwise incompatible homogeneous families over ω. For each $\alpha < 2^{2^{\aleph_0}}$ fix a permutation σ_α which embeds $\mathcal{F}_\alpha$ in $\mathcal{F}^*$. By the pigeon hole principle there are $\alpha < \beta < 2^{2^{\aleph_0}}$ and a permutation σ such that $\sigma_\alpha = \sigma_\beta = \sigma$. As $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are incompatible, let us find a set $X \subseteq \omega$ such that $X^0 \in F_\alpha$ and $X^1 \in \mathcal{F}_\beta$. Now $\sigma_\alpha(X^0) = \sigma(X^0) \in \mathcal{F}^*$, and $\sigma_\beta(X^1) = \sigma(X^1) \in \mathcal{F}^*$. This means that in $\mathcal{F}^*$ there is a set and its complement. This contradicts Theorem 1.4 that states that there is no homogeneous family over ω that contains a set and its complement. ☺ 4.17

We prove now lemma 4.16.

Proof: We use the direct system of inverse systems from the proof of lemma 4.1. The pairwise disjoint families will be over A^* rather than over ω, but as this is a countable set this makes no difference.

Let the variable η range over the set of all functions $\eta : \mathcal{P}(A_0) \to 2$ which satisfy $\eta(X) + \eta(A_0 \setminus X) = 1$ for all $X \subseteq A_0$. These are functions that select exactly one element from each pair of a set and its complement (for example, characteristic functions of ultra filters). There are $2^{2^{\aleph_0}}$ such functions.

For every function $\eta : \mathcal{P}(A_0) \to 2$ as above let $\mathcal{F}^0_\eta = \{X \subseteq A_0 : \eta(X) = 1\}$. The collection $\{\mathcal{F}_\eta : \eta : \mathcal{P}(A_0) \to 2\}$ is a collection of $2^{2^{\aleph_0}}$ pairwise incompatible families over

A_0. For every $m < \omega$ let $\mathcal{F}_\eta^{0,m}$ be the projection of $\mathcal{F}_\eta$ on A_0^m.

We know that for every $\mathcal{F}_\eta$ there is a homogeneous family $\mathcal{F}'\eta$ over A^* whose projection on A_0 equals $\mathcal{F}_\eta$ (modulo finite sets). However, it is NOT true that $\{\mathcal{F}'_\eta : \ \eta : \mathcal{P}(A_0) \to 2\}$ is a collection of pairwise incompatible families. In fact, $\Phi_0^1(X^0) \cap \Phi_0^1(X^1)$ is not empty for every $X \subseteq A_0$.

What we shall do now is refine the extension operation is such a way that not only the projection on A_0 is preserved, but also the disjointness of X^0 and X^1. This will be achieved by removing some of the points of A^*.

We define by induction on n a subset $\bar{D}_n \subseteq D_n$ and a subset $E_n \subseteq A_n$. Restricting ourselves to the points of $E = \bigcup_n E_n$ will provide the desired conservation property.

Let $E_0 = A_0$. Let $\bar{D}_0 = \bigcup_\eta D(E_0, \mathcal{F}_\eta^0)$.

4.18 Fact: *If $d \in \bar{D}_0$ then for no $X \subseteq A_0$ is it true that both X^0, X^1 belong to* $\operatorname{ran} f^d$.

We remove, thus, from the collection of demands all demands which mention simultaneously a set and its complement in their range.

Let us now define E_1 as follows:

$$E_1 = \{xw : x \in E_0, w = c_0 \dots c_k \in \mathrm{FG}(\dot{\bigcup_m} \bar{D}_0^m) \ \&\ x \notin \operatorname{dom} f^{c_0}\}$$

The variation on to the proof of 4.1 is that only a proper subset of words is being used. Hence, $E_1 \subseteq A_1$

4.19 Claim: *For every $X \subseteq A_0$ it holds that $\Phi_0^1(X^0) \cap \Phi_0^1(X^1) \cap E_1 = \emptyset$.*

Proof: By induction on the length of $w \in FG(\dot{\bigcup}_m \bar{D}_0^m)$ we shall see that $xw \notin \Phi_0^1(X^0) \cap \Phi_0^1(X^1)$.

If $\lg w = 0$ then $xw = x \in E_0 = A_0$. As $\Phi_0^1(X) \cap A_0 = X$ for all X, it follows that $x \notin \Phi_0^1(X^0) \cap \Phi_0^1(X^1)$.

Now suppose that $\lg wc = k + 1$. By the definition of the $\in$ relation over the set A_1 we know that $xwc \in \Phi_0^1(X^0)$ iff there is some Y such that $xw \in \Phi_0^1(Y)$ and $f^c(Y) = X^0$. Similarly, $xwc \in \Phi_0^1(X^1)$ iff there is some Z such that $xw \in \Phi_0^1(Z)$ and $f^c(Z) = X^1$. But X^0 and X^1 cannot both appear in $\operatorname{ran} f^c$ because $c \in \bar{D}_0^m$. Therefore xwc is not in the intersection. ☺ 4.19

Now we should notice that E_1 is invariant under $\xi_1(w)$ for all $w \in \mathrm{FG}(\bar{D}_0)$. Also, for every $w \in \mathrm{FG}(\bar{D}_0)$ and every $X \subseteq E_0$ it holds that $\xi_1(w)[\Phi_0^1(X^0)] \cap \xi_1(w)[\Phi_0^1(X^1) \cap E_1 = \emptyset$.

Let $\bar{\mathcal{F}}_1 = \{\xi_1(w)[\Phi_0^1(X)] : X \in \bar{F}_0, w \in \bar{D}_0\}$.

We proceed by induction on n, defining $\bar{D}_n$ and E_{n+1} for all $n > 0$.

First, let us view each $\eta : \mathcal{P}(E_0) \to 2$ as a partial function $\eta : \bar{\mathcal{F}}_1 \to 2$ by replacing every $X \subseteq E_0$ by $\Phi_0^1(X)$. Next extend each η to contain $\bar{\mathcal{F}}_1$ in its domain, demanding that

$$\eta(\xi_1(w)[\Phi_0^1(X)]) = \eta(X)$$

We refer to the resulting extended function also as η to avoid cumbersome notation. For every η let $\bar{\mathcal{F}}_{\eta,1} = \{X \in \bar{\mathcal{F}}_1 : \eta(X) = 1\}$.

Now define $\bar{D}_1 = \bigcup_\eta D(\bar{F}_{\eta,1})$.

Define E_{n+1} e'z $\bar{\mathcal{F}}_{n+1}$ as before. We should check the following:

4.20 Claim: *For all $X \in \bar{\mathcal{F}}_n$ it holds that $\Phi_n^{n+1}(X^0) \cap \Phi_n^{n+1}(X^1) \cap E_{n+1} = \emptyset$.*

Proof: By induction of word length. The case which should be added to the proof of 4.19 is the case when c as old, and is easily verified.

Having done the induction, we set $E = \bigcup_n E_n$. For every $\eta : \mathcal{P}(E_0) \to 2$ let $\mathcal{F}'_\eta$ be the homogeneous family obtained from $\mathcal{F}_\eta$ as in the proof of 4.1. The reader will verify that

(1) For every $X \subseteq E_0$ it holds that $\Phi_0(X^0) \cap \Phi_0(X^1) \cap E = \emptyset$

(2) For every $\eta : \mathcal{P}(E_0) \to 2$ the family $\mathcal{F}'_\eta \restriction E$ is homogeneous.

This completes the proof. ☺ 4.16

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