

**TOWARD CLASSIFYING  
UNSTABLE THEORIES  
SH500**

SAHARON SHELAH

Institute of Mathematics  
The Hebrew University  
Jerusalem, Israel

Rutgers University  
Department of Mathematics  
New Brunswick, NJ USA

ABSTRACT. We prove a consistency results saying, that for a simple (first order) theory, it is easier to have a universal model in some cardinalities, than for the theory of linear order. We define additional properties of first order theories, the  $n$ -strong order property ( $SOP_n$  in short). The main result is that a first order theory with the 4-strong order property behaves like linear orders concerning existence of universal models.

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## §0 INTRODUCTION

Having finished [Sh:a], an important direction seems to me to try to classify unstable theories; i.e. to find meaningful dividing lines. In [Sh 10] two such were the strict order property and the independence property, their disjunction is equivalent to instability (see [Sh:a],4.7 = [Sh:c],4.7,p.70). For theories without the independence property, we know  $S(A)$  (and  $S_{\Delta}(A)$ ) are relatively small (see [Sh 10], Keisler [Ke76], [Sh:a], III,§7,7.3,7.4,II,§4,4.9,4.10. Also for  $\lambda > |T|$ ,  $\{p \in S(A) : p \text{ does not split over some } B \subseteq A \text{ of cardinality } < \lambda\}$  is  $\lambda$ -dense (see [Sh:a],7.5 = [Sh:c],7.5,p.140).

Later this becomes interesting in the content of analyzing monadic logic (see Baldwin Shelah [BlSh 156] representation Baldwin [Bl]. By [Sh 197] if “no monadic expansion of  $T$  has the independence property” is a significant dividing line.

Lately, some model theorist have become interested in finitary versions called VC dimensions, see Laskowski [Lw92], Macintyre [Mcxx] (good bound for the case of expansion the real field) ? Mcxx ?

More relevant to the present work is the tree property, which is weaker than the strict order property (in [Sh:c],III,p.171).

In [Sh:93] we try to investigate theories without the tree property, so called simple. This can be looked at as a weakening of stable, so: simple  $\Leftrightarrow \kappa_{\text{cdt}}(T) < \infty \Leftrightarrow$  failure of the tree property  $\Leftrightarrow$  suitable local ranks  $< \infty$  are parallel of stable. We try to do the parallel to (parts of) Ch. II,III of [Sh:a], forking being generalized in some ways. But here instead showing the number of untrafilters of the Boolean Algebras of formulas  $\varphi(x, \bar{a})$  over  $A$  is small ( $\leq |A|^{|T|}$ ) we show that it can be decomposed to few subalgebras satisfying a strong chain condition. In this context we also succeed to get averages; but the Boolean algebras we get were derived from normal ones with a little twist. We did not start with generalizing the rest of [Sh:a] like supersimple (i.e.  $\kappa_{\text{cdt}}(T) = \aleph_0$ , equivalently suitable rank is  $< \infty$ ). The test problem in [Sh:93] was trying to characterize the class of pairs

$$SP(T) = \left\{ (\lambda, \kappa) : \text{every model of } T \text{ of cardinality } \lambda \text{ has a } \kappa\text{-saturated elementary extensions of cardinality } \lambda \right\}.$$

For simplicity we consider there only  $\lambda = \lambda^{|T|} > 2^{|T|+\kappa}$ ,  $\kappa > |T|$  and  $(\exists \mu)(\mu = \mu^{<\kappa} \leq \lambda \leq 2^\mu)$  (if this fails, see [Sh 576]). So by [Sh:93] for non-simple  $T$ , such  $(\lambda, \kappa)$  is in  $SP(T)$  iff  $\lambda = \lambda^{<\kappa}$ . If  $\mu = \mu^{<\mu} < \lambda = \lambda^{<\lambda}$ , after suitable forcing preserving  $\mu = \mu^{<\mu}$ , not collapsing cardinals and making  $2^\mu = \lambda$ , we have under suitable generalization of MA, so  $\kappa < \mu < \lambda < 2^\mu \Rightarrow (\lambda, \kappa) \in SP(T)$ .

It seems much better to use just the cardinal arithmetic assumptions (not the generalizations of MA). This call to investigate problem of  $\mathcal{P}^-(n)$ -amalgamation (see [Sh 87b], [Sh:c],XII,§5). For the case of  $n = 3$  this means that

- (\*)<sub>3</sub> if  $p_0(\bar{x}, \bar{y}), p_1(\bar{x}, \bar{z}), p_2(\bar{y}, \bar{z})$ , complete types over  $A$ , each saying the two sequences of variables are “independent” in suitable ways (like nonforking) then we extend the union of the three (preserving “independence”).

Now  $(*)_3$  can be proved. [Sh:93], Claim 7.8, p.201, (3.5, p.187). But the proof does not work for higher  $n$ , naturally counterexamples for the amalgamation should give counterexample to membership in SP. This was carried out by finding counterexamples in a wider framework: saturation inside  $P$  in [Sh 126]; but we could still hope that for the “true” one there is a positive one.

For long, I was occupied elsewhere and not look into it, but eventually Hrushovski becomes interested (and through him, others) and we try to explain the relevant research below. Also, it could be asked if simple unstable theories “occurs in nature”, “are important to algebraic applications”. The works cited below forms a positive answer (note that, quite natural, those examples concentrate on the lower part of hierarchy, like strongly minimal or finite Morley rank).

On the one hand, Hrushovski, continuing [Sh 126], prove that there are simple theories with bad behaviour for  $\mathcal{P}(n)$  so in the result above the cardinal arithmetic are not enough.

On the other hand, by Hrushovski Pillay [HrPi] in specific cases (finite ranks) relevant cases of  $(*)_n$  are proved, for  $n > 3$  under very specific conditions: for  $n = 3$  more general; but the relationship with [Sh:93], 7.8 of  $(*)$  was not clarified (in both cases the original rank does not work; the solution in [Sh:93] is to use *dnwd* (= “do not weakly divide”), Hrushovski changes the rank replacing “contradictory” by having small rank; this seems a reasonable approach only for supersimple theories and was carried only for ones with finite rank, and it gives more information in other respects.

In Hrushovski [Hr1] let  $\mathfrak{C}_0$  be the monster model for a strongly minimal theory with elimination of imaginaries,  $A \subseteq \mathfrak{C}$ ,  $A = \text{dcl } A$ , such that every  $p \in S^m(A)$  with multiplicity 1 is finitely satisfiable in  $A$ , now  $\text{Th}(\mathfrak{C}, A)$  is simple (of rank 1) and we can understand PAC in general content. Hrushovski [Hr2] does parallel thing for finite rank.

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We turn to the present work. First section deals with the existence of universal models. Note that existence of saturated models can be characterized nicely by stability (see [Sh:c]).

By Kojman Shelah [KjSh 409], the theory of linear order and more generally theories without the strict order property has universal models in “few” cardinals.

By [Sh 457] give a sufficient conditions for a consistency of “there is in  $\mu^{++}$  a model of  $T$  universal for models of  $T$  of cardinality  $\mu^+$ ”, which we use below.

The main aim is to show that all simple theories behave “better” in this respect than the theory of linear order. Specifically, it is consistent that  $\aleph_0 < \lambda = \lambda^{<\lambda}$ ,  $2^\lambda > \lambda^{++}$ , moreover, there is a club guessing  $\langle C_\delta : \delta < \lambda^+, \text{cf}(\delta) = \lambda \rangle$ , and every simple  $T$  of cardinality  $< \lambda$  has a model in  $\lambda^{++}$  universal for  $\lambda^+$ . For this we represent results of [Sh:93] and do the things needed specifically for the use of [Sh 457].

See 1.4A(2).

## §1 SIMPLE THEORIES HAVE MORE UNIVERSAL MODELS

We quote [Sh 457],5.1.

**1.1 Lemma.** *Suppose*

- (A)  *$T$  is first order, complete for simplicity with elimination of quantifiers (or just inductive theory with the amalgamation and disjoint embedding property).*
- (B)  *$K_{ap}$  is a simple  $\lambda$ -approximation system such that every  $M \in K_{ap}$  is a model of  $T$  hence every  $M_\Gamma$ , where for  $\Gamma \in K_{md}$  we let  $M_\Gamma = \cup\{M : M \in \Gamma\}$ .*
- (C) *Every model  $M$  of  $T$  of cardinality  $\lambda^+$  can be embedded into  $M_\Gamma$  for some  $\Gamma \in K_{md}$ .*

Then

- (a) *in 4.9 in  $V^P$ , there is a model of  $T$  of cardinality  $\lambda^{++}$  universal for models of  $T$  of cardinality  $\lambda^+$ .*
- (b) *So in  $V^P$ ,  $univ(\lambda^+, T) \leq \lambda^{++} < 2^\lambda$ .*

*Proof.* Straightforward.

*1.1 Fact.* 1) Assume  $M \prec N$ ,  $\bar{a} \in {}^\omega N$ , and  $\Delta$  a finite set of formulas possibly with parameters from  $M$ . Then there are a formula  $\psi(\bar{x}, \bar{b}) \in \text{tp}(\bar{a}, N)$  such that:

- (\*) for any  $\bar{a}' \in M$  realizing  $\psi(\bar{x}, \bar{b})$ , we can find a  $\Delta$ -2-indiscernible sequence  $\langle \bar{a}_i : i \leq \omega \rangle$  such that:  $\bar{a}_0 = \bar{a}'$ ,  $\bar{a}_{\omega+1} = \bar{a}$ ; hence we can find an indiscernible sequence  $\langle \bar{a}'_i : i < \omega \rangle$  (in  $\mathfrak{C}$ ) such that the  $\Delta$ -type of  $\bar{a}'_0 \hat{\ } \bar{a}'_1$  is the same as that of  $\bar{a}' \hat{\ } \bar{a}$ .

2) Assume  $2^{\theta+|T|} \leq \kappa$  and  $M \prec N$ , moreover

- ⊗ if  $A \subseteq M$ ,  $|A| \leq \kappa$ ,  $\bar{a} \in {}^\theta N$  then some  $\bar{a}' \in M$  realizes  $\text{tp}(\bar{a}, A, N)$ .

Then for any  $\bar{a} \in {}^\theta N$  and  $B \subseteq M$ ,  $|B| \leq \theta$ , there is  $A \subseteq M$ ,  $|A| \leq \kappa$ , such that for every  $\bar{a}' \in {}^\theta M$  realizing  $\text{tp}(\bar{a}, A, N)$  there is a sequence  $\langle \bar{a}_i : i \leq \kappa \rangle$  which is 2-indiscernible over  $B$ ,  $\bar{a}_0 = \bar{a}'$ ,  $\bar{a}_\kappa = \bar{a}$ , hence there is an indiscernible sequence  $\langle \bar{a}'_i : i < \omega \rangle$  such that  $\bar{a}'_0 \hat{\ } \bar{a}'_1$  realizes the same type as  $\bar{a}' \hat{\ } \bar{a}$  over  $B$ .

*Proof.* Obvious [notes on combination set theory].

1) Let  $\langle p_i : i < k \rangle$  list the possible  $\Delta$ -types of sequences of length  $\ell g(\bar{a}) + \ell(\bar{a})$ , so  $k < \omega$ . For each  $p_i$ ,  $\psi_i(\bar{x}, \bar{b}_i) \in \text{tp}(\bar{a}, M, N)$  such that, if possible for no  $\bar{a}' \in {}^{\ell g(\bar{a})} M$  realizing  $\psi_i(\bar{x}, \bar{b}_i)$  do we have:  $\bar{a}' \hat{\ } \bar{a}$  realizes  $p_i$ ; (if there..)

Now  $\psi(\bar{x}, \bar{b}) =: \bigwedge_{i < k} \psi_i(\bar{x}, \bar{b}_i)$  is as required.

2) Similar.

3)  $\langle p_i : i < k \rangle$  list the complete  $2\ell g(\bar{a})$ -types over  $B$ . Use  $p_i \subseteq \text{tp}(\bar{a}, M, N)$ ,  $|p_i| \leq \theta + |T|$  instead  $\psi_i(\bar{x}, \bar{b}_i)$ ,  $A = \bigcup_{i < k} \text{Dom } p_i$ .

**1.2 Theorem.** *If  $T$  is a complete simple (f.o.) theory,  $|T| < \lambda$  then  $T$  satisfies the assumption of 1.1 (hence its conclusions).*

*1.2A Remark.* 1) We can get results for a theory  $T$  of cardinality  $\leq \lambda$  under stronger assumptions on  $T$ .

2) Though not always necessary, in this section we'll assume  $T$  is simple.

3) Also this section is not written in a way focused on Theorem 1.2, but leisurely relook at simple theories.

*Proof.* Without loss of generality  $T$  has elimination of quantifiers.

We first represent (in 1.3 - 1.10) the needed definitions and facts on simple theories from [Sh:93] (adding notation and some facts), then say a little more and prove the theorem. So for a while we work in a fixed  $\bar{\kappa}$ -saturated model  $\mathfrak{C}$  of  $T$ ,  $\bar{\kappa}$  big enough. So  $M, N$  denotes elementary submodels of  $\mathfrak{C}$  of cardinality  $< \bar{\kappa}$ ,  $A, B, C, D$  denote subsets of  $\mathfrak{C}$  of cardinality  $< \bar{\kappa}$  and  $\bar{a}, \bar{b}, \bar{c}, \bar{d}$  denote sequences of elements of  $\mathfrak{C}$  of length  $< \bar{\kappa}$ , usually finite. Let  $\bar{a}/B = \text{tp}(\bar{a}, B) = \{\varphi(\bar{x}, \bar{b}) : \bar{b} \in {}^\omega B, \varphi \in L(T) \text{ is first order and } \mathfrak{C} \models \varphi[\bar{a}, \bar{b}]\}$ .

**1.3 Definition.** 1) We say that “ $p(\bar{x})$  does not weakly divide over  $(r, B)$ ” (in short  $p$  dnwd (that is does not weakly divide) over  $(r, B)$ ); we write over  $B$  when  $r = p \upharpoonright B$ , we write over  $r$  if  $B = \text{Dom}(r)$ , where  $r = r(\bar{x})$  is a type over  $B$  (and  $\bar{x}$  may be infinite) when: if  $\bar{b} \in B$  and  $\psi = \psi(\bar{x}^1, \dots, \bar{x}^n, \bar{y})$  a formula (where  $\ell g(\bar{x}^\ell) = \ell g(\bar{x}), \bar{x}^\ell$  with no repetition,  $\langle \bar{x}^\ell : \ell = 1, n \rangle \hat{\ } \langle \bar{y} \rangle$  pairwise disjoint) and (see definition 1.3(2) below)  $[r]^\psi$  is finitely satisfiable (in  $\mathfrak{C}$ ) then so is  $[r \cup p]^\psi$  (see Definition 1.3(2) below).

2) If  $\psi = \psi(\bar{x}^1, \dots, \bar{x}^n)$  (possibly with parameters),  $q = q(\bar{x})$  then

$$[q]^\psi = \{\psi\} \cup \bigcup_{\ell=1}^n q(\bar{x}^\ell).$$

3)  $p(\bar{x})$  divides over  $A$  if for some formula  $\psi(\bar{x}, \bar{a})$  we have  $p \vdash q(\bar{x}, \bar{a})$  and for some indiscernible sequence  $\langle \bar{a}_\ell : \ell < \omega \rangle$  over  $A, \bar{a} = \bar{a}_0$ , and  $\{\varphi(\bar{x}, \bar{a}_i) : i < \omega\}$  is  $(< \omega)$ -contradictory where a set  $p$  of formulas is  $n$ -contradictory if for any distinct  $\varphi_1, \dots, \varphi_n \in p, \{\varphi_1, \dots, \varphi_n\}$  is not realized (in  $\mathfrak{C}$ ), and  $(< \omega)$ -contradictory means:  $n$ -contradictory for some  $n$ . We write dnd for “does not divide”.

4) The type  $p$  fork over the set  $A$  if for some  $n < \omega$  and formulas  $\varphi_\ell(\bar{x}, \bar{a}_\ell)$  for  $\ell < n$  we have:  $p \vdash \bigvee_{\ell < n} \varphi_\ell(\bar{x}, \bar{a}_\ell)$  and for each  $\ell < n$  the formula  $\varphi_\ell(\bar{x}, \bar{a}_\ell)$  divides over  $A$ .

We use “dnf” as shortening for “does not fork”.

5) The type  $p$  is finitely satisfiable (finitely satisfiable) in  $A$  (or in  $\mathbf{I}$ ) if every finite subset  $p'$  of  $p$  is realized by some sequence from  $A$  (or a member of  $\mathbf{I}$ ).

6) If  $D$  is an ultrafilter on  $\text{Dom}(D) = \mathbf{I}$  (where all members of  $\mathbf{I}$  have the same length, say  $m$ ) then  $Av(B, D) = \{\varphi(\bar{x}, \bar{a}) : \{\bar{b} \in \text{Dom}(D) : \varphi(\bar{b}, \bar{a})\} \in D\}$ .

**1.4 Definition.** We say “ $\bar{a}/A$  (or  $\text{tp}(\bar{a}, A)$ ) weakly divides over  $B$ ” if  $B \subseteq A$  and  $\text{tp}(\bar{a}, A)$  weakly divides over  $(\text{tp}(\bar{a}, B), B)$ , (similarly for does not weakly divide).

*1.4A Remark.* 1) An equivalent formulation is “ $\bar{a}/A$  is an extension of  $\bar{a}/B$  with the same degree for most  $(\Delta, \aleph_0, k)$ ”; see 1.5(8) below.

2) On “divides”, “fork”, “weakly divide” see [Sh:93], Def.1.1,1.2, 2.7(2) respectively. On the first two also [Sh:a], but there the focus is on stable theories. On “finitely satisfiable” see [Sh:a], Ch.VII, §4. We present here most of their properties, ignoring mainly the connections with suitable degrees and indiscernibility and the derived Boolean algebras of formulas (satisfying chain conditions). For stable  $T$  the notions of Def. 1.3 collapse becoming equivalent (finitely satisfiable - only when the set is a model, see [Sh:a], Ch.III).

Basic properties are (mostly check directly, but 0A,6,8,9 are quoted).

**1.5 Claim.** 0) [Implications] If  $p$  divides over  $A$  then  $p$  forks over  $A$ .

0A) If  $p$  forks over  $A$  then  $p$  weakly divides over  $(p \upharpoonright A, \emptyset)$ , (by [Sh:93], 2.11(1), p.184, in its proof we have rely not only of [Sh:93], 2.10(2) + 2.9(3) but also on [Sh:93], 2.4(3)).

0B) If a type  $p$  is finitely satisfiable in  $A$  then  $p$  does not fork over  $A$ .

1) [Monotonicity] If  $B \subseteq A_1 \subseteq A_2 (\subseteq \mathfrak{C})$ ,  $\bar{a}/A_2$  does not widely divide over  $B$ .

1A) If  $p$  does not divide over  $A$ ,  $A \subseteq A_1$  and  $p_1 \subseteq p$  or at least  $p \vdash p_1$  then  $p_1$  does not divide over  $A_1$ .

1B) If  $p$  does not fork over  $A$ ,  $A \subseteq A_1$  and  $p_1 \subseteq p$  or at least  $p \vdash p_1$  then  $p_1$  does not fork over  $A_1$ .

1C) If  $p$  does not widely divide over  $(r, A)$ ,  $A \subseteq A_1$ ,  $r_1 \vdash r$  and  $p_1 \subseteq p$  or at least  $p \vdash p_1$  then  $p_1$  does not widely divide over  $(r_1, A_1)$ .

2) [Local character]  $\bar{a}/A$  does not widely divide over  $B$  iff for every finite subsequence  $\bar{a}'$  of  $\bar{a}$  and finite subset  $A'$  of  $A$ ,  $\bar{a}'/(A' \cup B)$  does not widely divide over  $B$ .

2A) The type  $p$  does not weakly divide over  $(r, B)$  iff every finite  $p' \subseteq p$  dnwd over  $(r, B)$ .

2B) The type  $p$  does not divide over  $A$  iff for every finite  $p' \subseteq p$  does not divide over  $A$ , iff some finite conjunction  $\varphi$  of members of  $p$  satisfies the requirement in Definition 1.3(1).

2C) The type  $p$  does not fork over  $A$  iff every finite  $p' \subseteq p$  does not fork over  $A$ .

3) [More monotonicity] Assume  $\text{Rang}(\bar{a}') = \text{Rang}(\bar{a}'')$ , then:  $\bar{a}'/A$  dnwd over  $B$  iff  $\bar{a}''/A$  dnwd over  $B$ .

3A) If  $B \subseteq A$ ,  $\text{Rang } \bar{a}'' \subseteq \text{acl}(B \cup \bar{a}')$  and  $\bar{a}' \setminus \bar{A}$  dnwd over  $B$  then  $\bar{a}''/A$  dnwd over  $B$ .

3B) Similarly to 3), 3A) for “does not divide” and for “does not fork” and for “dnwd over  $(r, B)$ ”.

4) [Transitivity] If  $A_0 \subseteq A_1 \subseteq A_2$  and  $\bar{a}/A_{\ell+1}$  dnwd over  $A_\ell$  for  $\ell = 0, 1$  then  $\bar{a}/A_2$  dnwd over  $A_0$ .

5) [Extendability] If  $B \subseteq A \subseteq A^+$ ,  $p$  an  $m$ -type over  $A$  and  $p$  does not fork over  $B$  then  $p$  has an extension  $q \in S^m(A^+)$  which does not fork over  $B$  (clear or see [Sh:93], 2.11(3)).

5A) If  $p$  is finitely satisfiable in  $A$  and  $(\text{Dom } p) \cup A \subseteq B$  then we can extend  $p$  to a complete type over  $B$  finitely satisfiable in  $A$ .

6) [Trivial nice behaviour]  $\bar{a}/A$  does not fork over  $A$  (by [Sh:93], 2.11(2)).

6A) For a set  $A$  and an  $m$ -type  $p$  we have:  $p$  does not widely divide over  $(p, A)$  (check).

6B) Every  $m$ -type over  $M$  is finitely satisfiable in  $M$ .

7) [Continuity] If  $p_i$  does not widely divide over  $(r_i, B_i)$  for  $i < \delta$  and

$i < j < \delta \Rightarrow p_i \subseteq p_j \ \& \ r_i \subseteq r_j \ \& \ B_i \subseteq B_j$  then  $\bigcup_{i < \delta} p_i$  does not widely divide

over  $(\bigcup_{i<\delta} r_i, \bigcup_{i<\delta} B_i)$ .

7A) If  $\langle A_i : i < \delta \rangle$  is increasing,  $\langle B_i : i < \delta \rangle$  is increasing and  $C/B_i$  is finite satisfiable in  $A_i$  for each  $i < \delta$  then  $C/\bigcup_{i<\delta} B_i$  is finite satisfiable in  $\bigcup_{i<\delta} A_i$ .

[Why? if  $p \subseteq C/\bigcup_{i<\delta} B_i$  is finite then for some  $j$  it is over  $B_j$  hence  $\subseteq C/B_j$  is satisfiable in  $A_j$  hence is satisfiable in  $\bigcup_{i<\delta} A_i$ ].

8) [Degree] Let  $\bar{x}_m = \langle x_\ell : \ell < m \rangle$ ,  $E_m$  be an ultrafilter on  $\Omega_m =: \{(\Delta, k) : \Delta = \Delta(\bar{x}_m) \subseteq L(T) \text{ finite, } k < \omega\}$  such that for every  $(\Delta_0, k_0) \in \Omega_m$  the following set belongs to  $E_m$ :

$$\{(\Delta, k) \in \Omega_m : \Delta \subseteq \Delta_0 \text{ and } k_0 < k\}.$$

If  $p(\bar{x})$  is a type over  $A$ ,  $lg(\bar{x}) = m$ , then for some complete type  $q(\bar{x})$  over  $A$  extending  $p$  for the  $E_m$ -majority of  $(\Delta(\bar{x}), k)$  we have  $D(q(\bar{x}), \Delta, \aleph_0, k) = D(p(\bar{x}), \Delta, \aleph_0, k)$  (by [Sh:93], 2.2(5), p.182; of course, we can use infinite  $\bar{x}$ ).

In such a case we say:  $q(\bar{x})$  is an  $E_m$ -nonforking extension of  $p(\bar{x})$  or  $q(\bar{x})$   $E_m$ -does not fork over  $p(\bar{x})$ . If  $p(\bar{x}) \in S^m(A)$  (so  $p = q \upharpoonright A$ ) we may replace "over  $p(\bar{x})$ " by "over  $A$ ".

9) [Additivity] If for every  $\alpha < \alpha^*$  the type  $tp(\bar{b}^\alpha, \bar{a} \cup A \cup \bigcup_{\beta < \alpha} \bar{b}^\beta)$  does not divide over

$A \cup \bigcup_{\beta < \alpha} \bar{b}^\beta$  then  $tp_*(\bigcup_{\beta < \alpha^*} \bar{b}^\beta, A \cup \bar{a})$  does not divide over  $A$  (by [Sh:93], 1.5, p.181).

10) [Finitely satisfiable is average] Let  $lg(\bar{x}) = m$  and  $p = p(\bar{x})$  a type. Then  $p$  is finitely satisfiable in  $\mathbf{I}$  iff for some ultrafilter  $D$  over  $\mathbf{I}$  we have  $p \subseteq Av(D, \text{Dom } p)$ .

11) If  $D$  is an ultrafilter on  $\mathbf{I}$ , then  $Av(D, A)$  belongs to  $S^m(A)$  and is a finitely satisfiable in  $\mathbf{I}$ .

**1.6 Claim.** 1) [small basis] If  $p \in S^\varepsilon(A)$  and  $B_0 \subseteq A$  then for some  $B$  we have:

- ( $\alpha$ )  $B_0 \subseteq B \subseteq A$
- ( $\beta$ )  $|B| \leq |\varepsilon| + |T| + |B_0|$
- ( $\gamma$ )  $p$  does not wind downward over  $(p \upharpoonright B, B)$ .

2) If  $\bar{a}/(A \cup \bar{b})$  dnwd over  $A$  and  $A \subseteq A^+ \subseteq \text{acl}[A \cup \{\bar{a}' \in \mathfrak{C} : \bar{a}' \text{ realizes } \bar{a}/A\}]$  then there is  $\bar{b}'$  (of the same length as  $\bar{b}$ ) such that:

- (a) if  $\bar{a}' \subseteq A^+$  and  $\bar{a}'/A = \bar{a}/A$  then  $\bar{b}' \hat{\wedge} \bar{a}'/A = \bar{b}' \hat{\wedge} \bar{a}/A$ .

3) [weak symmetry] If  $\bar{a}/(A \cup \bar{b})$  dnwd over  $A$  and then  $\bar{b}/(A \cup \bar{a})$  dnd over  $A$ .

4) Assume  $A \subseteq B \cap C$  (all  $\subseteq \mathfrak{C}$ ) and  $C/B$  is finitely satisfiable in  $A$  hence  $A = B \cap C$ . Then  $B/C$  dnwd over  $(B/A, A)$ .

*Proof.* 1) By [Sh:93], 3.3, p.186.

2) By [Sh:93], 2.13, p.185 we can get clause (a).

3) By [Sh:93], 2.14, p.185 it dnd.

4) Straightforward (e.g. use 1.5(10)).

□<sub>1.6</sub>

**1.7 Theorem.** *If  $M \prec N \prec \mathfrak{C}$ ,  $\|M\| = \mu$ ,  $\|N\| = \mu^+$ ,  $|T| < \kappa$ ,  $\mu = \mu^{<\kappa}$ , then there are  $M^+ \prec N^+$  such that  $N \prec N^+$ ,  $M \prec M^+$ ,  $\|M^+\| = \mu$ ,  $\|N^+\| = \mu^+$  and:*

- (\*)<sub>1</sub> *if  $B \subseteq A \subseteq N$ ,  $B \subseteq M$ ,  $|A| < \kappa$ ,  $m < \omega$ ,  $p \in S^m(A)$  and  $p$  dnwd over  $(p \upharpoonright B, \emptyset)$  then  $p$  is realized in  $M^+$ .*
- (\*)<sub>2</sub> *if  $B \subseteq A \subseteq N$ ,  $B \subseteq M$ ,  $|A| < \kappa$ ,  $C \subseteq \mathfrak{C}$ ,  $|C| < \kappa$  and  $A/(B \cup C)$  dnwd over  $B$  then there is  $C' \subseteq M^+$  realizing  $C/(B \cup A)$ .*

*Proof.* Clearly we can prove (\*)<sub>1</sub>, (\*)<sub>2</sub> separately. Now (\*)<sub>2</sub> is immediate from 1.6(2). As for (\*)<sub>1</sub>, this is proved in [Sh:93], §4 (read [Sh:93], 4.13, 4.14, 4.15, p.193 there, so we use [Sh:92], Theorem 3.1 which says that a Boolean algebra of cardinality  $\lambda^+$  satisfying the  $\kappa$ -c.c.,  $\lambda^{<\kappa} = \lambda$  is  $\lambda$ -centered, i.e. is the union of  $\leq \lambda$  ultrafilters, so if  $\kappa > 2^{|T|}$  we are done which is enough for our main theorem (1.2 when  $\lambda > |T|$ ). Actually repeating the proof of [Sh:93], Theorem 3.1 in the circumstances of [Sh:93], §4 show that  $\kappa > |T|$  is enough.  $\square_{1.7}$

**1.8 Definition.** 1)  $K_\lambda^0$  be

$$\left\{ \bar{M} : \bar{M} = \langle M_i : i < \lambda^+ \rangle \text{ is } \prec\text{-increasing continuous, each } M_i \text{ a model of } T \text{ of cardinality } \lambda \text{ and } |M_0| = \emptyset \text{ (we stipulate such a model } \prec M \text{ for every } M \models T) \right\}.$$

2)  $\leq^0$  is the following partial order on  $K_\lambda^0$ :  $\bar{M}^1 \leq^0 \bar{M}^2$  if for  $i < j < \lambda^+$  we have  $M_i^1 \prec M_i^2$ .

**1.9 Observation.** 1) If  $\langle \bar{M}^\alpha : \alpha < \delta \rangle$  is an  $\leq^0$ -increasing chain (in  $K_\lambda^0$ ) and  $\delta < \lambda^+$  then it has a lub  $\bar{M} : M_i = \bigcup_{\alpha < \delta} M_i^\alpha$ .

2) If  $M$  is a model of  $T$  of cardinality  $\lambda^+$ , then for some  $\bar{M} \in K_\lambda^0$ ,  $M = \bigcup_{i < \lambda^+} M_i$ .

3) If  $\bar{M}, \bar{N} \in K_\lambda^0$  and  $\bigcup_{\alpha < \lambda^+} M_\alpha \prec \bigcup_{\alpha < \lambda^+} N_\alpha$  then for some club  $E$  of  $\lambda^+$  for every  $\alpha \in E$ :  $M_\alpha \prec N_\alpha$  and  $N_\alpha / \bigcup_{\beta < \lambda^+} M_\beta$  dnwd over  $M_\beta$ .

*Proof.* 1) Immediate.

2) Use 1.5(2) + 1.6(1).  $\square_{1.9}$

Using 1.7 and 1.5(5A)  $\lambda^+ \times \lambda$  we get

**1.10 Observation.** Assume  $\lambda = \lambda^{<\kappa}$ . For every  $\bar{M} \in K_\lambda^0$  there is an  $\leq^0$ -increasing continuous sequence  $\langle \bar{N}_\zeta : \zeta \leq \lambda \rangle$ , in  $K_\lambda^0$ , (so  $\bar{N}_\zeta = \langle N_\alpha^\zeta : \alpha < \lambda^+ \rangle$ ),  $\bar{N}_0 = \bar{M}$



such that (letting  $N_\zeta = \bigcup_{\alpha < \lambda^+} N_\alpha^\zeta$ ) and [fixing  $\chi$ , letting  $\bar{a}_{\zeta, \alpha}$  an enumeration of  $|N_\alpha^\zeta|$  of length  $\lambda$ ] we can add: every type definable in  $(H(\chi), \in, <_\chi^*)$  from  $\langle \bar{N}_\varepsilon : \varepsilon \leq \zeta \rangle, \langle N_\alpha^{\zeta+1} : \alpha \leq \beta \rangle, \langle \bar{a}_{\varepsilon, \alpha} : \alpha < \lambda^+ \rangle : \varepsilon \leq \zeta, \langle \bar{a}_{\zeta, \alpha} : \alpha \leq \beta \rangle$  and  $\langle \alpha_{\zeta, \alpha, j} : j < i \rangle$  and finitely many ordinals  $< \lambda$  is realized in  $N_{\beta+1}^{\zeta+1}$ , hence:

- (\*)<sub>1</sub> if  $\alpha < \lambda^+, \zeta \leq \lambda, \text{cf}(\zeta) \in \{\lambda, 1\}, \text{cf}(\alpha) \in \{\lambda, 1\}, B \subseteq A \subseteq N_{\zeta-1}, B \subseteq N_\alpha^{\zeta-1}, |A| < \kappa, p \in S^m(A)$  and  $p$  dnwd over  $(p \upharpoonright B, \emptyset)$  then  $p$  is realized in  $N_\alpha^\zeta$ .  
(Note:  $\lambda - 1 = \lambda$ ).
- (\*)<sub>2</sub> if  $\alpha \leq \beta < \gamma < \lambda^+, \gamma$  is non-limit,  $B \subseteq A \subseteq A^+ \subseteq M_\gamma, |A^+| < \kappa, \bar{a} \in M_\gamma, \text{tp}(\bar{a}, A \cup M_\alpha)$  dnwd over  $B$  then either for some  $\bar{a}' \in N_\gamma$  we have:  $\text{tp}(\bar{a}, A^+ \cup M_\alpha) = \text{tp}(\bar{a}', A \cup M_\alpha)$  and  $\text{tp}(\bar{a}', A \cup N_\beta)$  dnwd over  $B$  or there is no such  $\bar{a}' \in \mathfrak{C}$ .
- (\*)<sub>3</sub> for  $\gamma$  non-limit,  $\zeta \leq \lambda, \text{cf}(\zeta) \in \{1, \lambda\}$  we have:  
 $N_\gamma^\zeta$  is  $\kappa$ -saturated (so when  $\kappa = \lambda$  it is saturated).

**1.11 Definition.** Let  $A, B, C$  be given ( $\subseteq \mathfrak{C}$ ).

- 0)  $A \leq_B^{-1} C$  means that for every  $\bar{b} \subseteq B, \bar{b}/(A \cup C)$  dnwd over  $(\frac{\bar{b}}{A}, A)$ .
- 1)  $A \leq_B^0 C$  means that for every  $\bar{c} \subseteq C, \bar{c}/(A \cup B)$  dnwd over  $(\frac{\bar{c}}{A}, \emptyset)$ .
- 2)  $A \leq_B^1 C$  means there is an increasing continuous sequence  $\langle A - \alpha : \alpha \leq \beta \rangle$  such that:  $A = A_0, A \cup C = A_\beta$  and

$$\alpha \text{ an even ordinal } < \beta \Rightarrow A_\alpha \leq_B^0 A_{\alpha+1}$$

$$\alpha \text{ an odd ordinal } < \beta \Rightarrow A_\alpha \leq_B^{-1} A_{\alpha+1}.$$

- 3)  $A \leq_B^2 C$  means that for some  $C', C \subseteq C'$  and  $A \leq_B^1 C'$ .
- 4)  $A \leq_B^3 C$  means that for some increasing continuous sequence  $\langle A_\alpha : \alpha \leq \beta \rangle$  we have  $A = A_0, A \cup C = A_\beta$  and  $A_\alpha \leq_B^2 A_{\alpha+1}$ .

**1.12 Claim.** 0)  $A \leq_B^e A$  (for  $e = -1, 0, 1, 2, 3$ ).

1)  $A \leq_B^e C$  iff  $A \leq_{A \cup B}^e A \cup C$  (for  $e = -1, 0, 1, 2, 3$ ).

[Why? For  $e = -1$  by 1.5(3A) for  $e = 0$  by the definition of  $\leq_B^0$ ; then continued].

2) If  $A \subseteq B_1 \subseteq B \cup A$  and  $A \leq_B^e C$  then  $A \leq_{B_1}^e C$  (for  $e = -1, 0, 1, 2, 3$ ).

[Why? By part (1) and: for  $e = -1$  trivially, for  $e = 0$  by 1.5(1C) and for  $e = 1, 2, 3$  use earlier cases].

3) For  $e = 1, 3$  we have:  $\leq_B^e$  is a partial order.

[Why? Read their definition].

4) If  $e = 1, 3$  and  $\langle A_\alpha : \alpha \leq \beta \rangle$  is increasing continuous and  $A_\alpha \leq_B^e A_{\alpha+1}$  for  $\alpha < \beta$  then  $A_0 \leq_B^e A_\beta$ .

[Why? Check].

5) For  $(e^1, e^2) \in \{(-1, 1), (0, 1), (1, 2), (2, 3)\}$ , we have:  $A \leq_B^{e^1} C$  implies  $A \leq_B^{e^2} C$ .

[Why? Read].

6) If for every  $\bar{b} \subseteq B, \bar{b}/(A \cup C)$  is finitely satisfiable in  $A$  then  $A \leq_B^0 C$ .

[Why? By 1.6(4) (and Definition 1.11(1))].

7) If  $A \leq_B^2 C$  and  $C' \subseteq C$  then  $A \leq_B^2 C'$ .

[Why? Read Definition 1.11(2)].

8)  $A \leq_B^0 C$  iff  $A \leq_C^{-1} B$ .

[Why? Read the definitions].

**1.13 Claim.** Let  $M \prec N$  and  $M \subseteq A$ . Then the following are equivalent:

(a)  $M \leq_N^3 A$

(b) there are  $M_0 \prec M_1 \prec M_2$  such that:

(i)  $M = M_0$

(ii) the type  $tp_*(N, M_1)$  is finitely satisfiable in  $M_0$  and the type  $tp_*(M_2, M_1 \cup N)$  is finitely satisfiable in  $M_1$

(iii) for some elementary map  $f, f(A) \subseteq M_2$  and  $f \upharpoonright N = \text{identity}$

(c) like (b) with  $\|M_2\| \leq |T| + |A|$

(d)  $M \leq_N^2 A$ .

13A Remark. 1) Clause (ii) of (b) implies  $M_0 \leq_n^0 M_1 \leq_N^{-1} M_2$ .

2) An equivalent formulation of (b) is

(b)' for some  $M_0, M_1, M_2, f$  we have  $M = M_0 \leq_{f(N)} M_1 \leq_{f(N)} M_2, f \upharpoonright M_0 = \text{id}_{M_0}, f(A) \subseteq M_2$ .

3) Another formulation is

(b)'' like (b)' but  $f = \text{id}_{A \cup N}$ .

*Proof.* (c)  $\Rightarrow$  (b) Trivial.

(b)  $\Rightarrow$  (c).

By the Lowenkeim Skolem argument.

(b)  $\Rightarrow$  (d)

By 1.12(6) clearly  $M_0 \leq_N^0 M_1$  and similarly  $M_1 \leq_{M_2}^0 N$ , hence by 1.2(8) we have  $M_1 \leq_N^{-1} M_2$ . Hence by 1.12(5),  $M_e \leq_N^1 M_{e+1}$  (for  $e = 0, 1$ ), so by 1.12(3)  $M_0 \leq_N^1 M_2$  hence by Definition 1.11(3) (and clause (iii) of 1.13(b)),  $M = M_0 \leq_N^2 A$  as required.

(d)  $\Rightarrow$  (a) Trivial (by 1.12(5)).

So the only (and main) part left is:

(a)  $\Rightarrow$  (b) We know  $M \leq_N^3 A$ , by 1.12(1) without loss of generality  $M \subseteq A$ , hence there is an increasing continuous sequence  $\langle A_\varepsilon : \varepsilon \leq \zeta \rangle$  such that:  $A_0 = M, A_\zeta = A$  and  $A_\varepsilon \leq_N^2 A_{\varepsilon+1}$ . By the Definition of  $\leq_n^2, \leq_N^1$  there is an increasing continuous sequence  $\langle B_{\varepsilon+i} : i \leq i_\varepsilon \rangle$  such that  $B_{\varepsilon,0} = A_\varepsilon, A_{\varepsilon+1} \subseteq B_{\varepsilon, i_\varepsilon}$  and  $B_{\varepsilon, i} \leq_N^{\ell(\varepsilon)} B_{\varepsilon, i+1}$  (where for  $i < i_\varepsilon$  we have  $\ell(\varepsilon) \in \{-1, 0\}$  and  $\varepsilon = \ell(\varepsilon) \bmod 2$ ). Let  $\theta = 2^{|T|} + |N| + \sum_{\varepsilon < \zeta} (|i_\varepsilon| + |B_{\varepsilon, i_\varepsilon}|)^+$  and choose regular  $\mu = \mu^\theta$ .

We choose by induction on  $\alpha < \mu^+$ ,  $M_\alpha, N_\alpha$  such that: medskip

- (i)  $M_\alpha \prec \mathfrak{C}$  is increasing continuous
- (ii)  $\|M_\alpha\| = \mu$ ,  $M \subseteq M_0$ ,
- (iii)  $f_\alpha$  is an elementary mapping,  $\text{Dom}(f_\alpha) = N$ ,  $\text{Rang}(f_\alpha) = N_\alpha$  and  $f_\alpha \upharpoonright M = \text{id}_M$
- (iv)  $\text{tp}_*(N_\alpha, M_\alpha)$  is finitely satisfiable in  $M$
- (v)  $N_\alpha \subseteq M_{\alpha+1}$
- (vi)  $M_{\alpha+1}$  is  $\theta^+$ -saturated.

There is no problem to carry the definition. (First choose  $M_\alpha$ : if  $\alpha = 0$  to satisfy (i) + (ii), if  $\alpha$  is a limit ordinal, as  $\bigcup_{\beta < \alpha} M_\beta$ , and if  $\alpha = \beta + 1$  to satisfy (i) + (ii) + (v).

Second choose  $f_\alpha, N_\alpha$  satisfying (iii) + (iv) which exists by 1.5(10) + (11)).

By using 1.7,  $\lambda^+$  times we can find  $\bar{M}^+ = \langle M_\alpha^+ : \alpha < \lambda^+ \rangle$  such that:

- (A)  $\bar{M}^+$  is an increasing continuous sequence of elementary submodels of  $\mathfrak{C}$
  - (B)  $\|M_\alpha^+\| \leq \mu$ ,  $M_\alpha \prec M_\alpha^+$
  - (C)<sub>1</sub> if  $\alpha < \beta < \mu^+$ ,  $B_1 \subseteq M_\alpha$  and  $\text{cf}(\alpha) = \mu$ ,  $B_1 \subseteq B_2 \subseteq M_\beta$ ,  $|B_2| < \theta$ ,  $C \subseteq \mathfrak{C}$  and  $C/B_2$  dnwd over  $(C/B_2, \emptyset)$  (equivalently, for every finite  $\bar{c} \subseteq C$ ,  $\bar{c}/B_2$  dnwd over  $(\bar{c}/B_1, \emptyset)$ ) then  $C/B_2$  is realized in  $M_\alpha$
  - (C)<sub>2</sub> similarly, but we replace the dnwd assumption by “ $B_2/(C_1 \cup C)$  dnwd over  $(B_2/B_1, B_1)$ ”.
- [Note: we use  $(*)_1$  from 1.7 for (C)<sub>1</sub> and  $(*)_2$  from 1.7 for (C)<sub>2</sub>].

Now let  $M = \bigcup_{\alpha < \mu} M_\alpha$ ,  $M^+ = \bigcup_{\alpha < \mu^+} M_\alpha^+$ ; and let

$E = \{\delta < \mu^+ : \delta \text{ a limit ordinal and } (M_\delta^+, M_\delta) \prec (M^+, M)\}$ . Clearly  $E$  is a club of  $\mu^+$  and

(\*)  $\delta \in E \Rightarrow \text{tp}(M_\delta^+, M)$  is finitely satisfiable in  $M_\delta$ .

Choose  $\delta \in E$  of cofinality  $\mu$ . Now we choose  $g_\varepsilon$  by induction on  $\varepsilon \leq \zeta$  such that:

- ( $\alpha$ )  $g_\varepsilon$  an elementary mapping
- ( $\beta$ )  $\text{Dom}(g_\varepsilon) = N \cup A_\varepsilon$
- ( $\gamma$ )  $g_\varepsilon$  is increasing continuous in  $\varepsilon$
- ( $\delta$ )  $g_\varepsilon \upharpoonright N = f_\delta$
- ( $\varepsilon$ )  $\text{Rang}(g_\varepsilon \upharpoonright A_\varepsilon) \subseteq M_\delta^+$ .

If we succeed, then we get the desired conclusion (i.e. prove clause (b)).

[Why? First note that in clause (b) we can omit  $f \upharpoonright N = \text{id}$  by  $f \upharpoonright M = \text{the identity}$  if in clause (ii) we use  $f(N)$ ; we call this (b)'. Now (b)' holds with  $M, M_\delta, M_\delta^+, g_\zeta$  here standing to  $M_0, M_1, M_2, f$  there). So it is enough to carry the induction on  $\varepsilon$ . For  $\varepsilon = 0$  let  $g_\varepsilon = f_\delta$ , and for  $\varepsilon$  a limit ordinal let  $g_\varepsilon = \bigcup_{\xi < \varepsilon} g_\xi$ ; lastly for  $\varepsilon$  a successor ordinal say  $\varepsilon = \xi + 1$ , we choose  $g_{\varepsilon, i}$  by induction on  $i \leq i_\varepsilon$  such that:

- ( $\alpha$ )'  $g_{\varepsilon, i}$  an elementary mapping
- ( $\beta$ )'  $\text{Dom}(g_{\varepsilon, i}) = N \cup B_{\varepsilon, i}$
- ( $\gamma$ )'  $g_{\varepsilon, i}$  is increasing continuous in  $i$
- ( $\delta$ )'  $g_{\varepsilon, 0} = g_\varepsilon$
- ( $\varepsilon$ )'  $\text{Rang}(g_{\varepsilon, i}) \subseteq M_\delta^+$ .

If we succeed then  $g_{\varepsilon, i_\varepsilon} \upharpoonright A_{\varepsilon+1}$  is as required. So it is enough to carry the induction on  $i$ . For  $i = 0$  let  $g_{\varepsilon, i} = g_\varepsilon$ , for  $i$  limit let  $g_{\varepsilon, i} = \bigcup_{j < i} g_{\varepsilon, j}$  and for  $i$  a successor ordinal say  $j + 1$ , use clause  $(C)_1$  in the choice of  $M_\alpha^+$  if  $j$  even, remembering Definition 1.11(1) and use clause  $(C)_2$  in the choice of  $M_\alpha^+$  if  $j$  is odd remembering Definition 1.11(0).  $\square_{1.13}$

**1.13A Claim.** *If  $M \prec N, M \leq_N^2 A_\ell$  for  $\ell = 1, 2$  then there are  $M^+, f_1, f_2$  such that:  $M \prec M^+, M \leq_N^2 M^+$  and for  $\ell = 1, 2, f_\ell$  is an elementary mapping,  $\text{Dom}(f_\ell) = N \cup A_\ell, f_\ell \upharpoonright N = \text{id}_N, f_\ell(A_\ell) \subseteq M^+$ .*

*Proof.* Same proof as 1.13 (just shorter).

**1.14 Definition.** 1) Let  $K_0^{pr} = \{(M, N) : M \prec N < \mathfrak{C}\}$  and  $(M_1, N_1) \leq^* (M_2, N_2)$  iff  $((M_e, N_e) \in K_0^{pr}$  for  $e = 1, 2$  and  $M_1 \prec M_2, N_1 \prec N_2$  and  $M_1 \leq_{N_1}^2 M_2$  (equivalently,  $M_1 \leq_{N_1}^3 M_2$  (by 1.13))).  
2) We define  $(M_1, N_1) \leq_{fs} (M_2, N_2)$  similarly replacing “ $M_1 \leq_{N_1}^2 M_2$ ” by “ $N_1/M_2$  is finitely satisfiable in  $M_1$ ”.

**1.15 Claim.** 1)  $\leq^*$  is a partial order on  $K_0^{pr}$ .  
2) If  $\langle (M_\alpha, N_\alpha) : \alpha \leq \beta \rangle$  is increasing continuous and  $(M_\alpha, N_\alpha) \leq^* (M_{\alpha+1}, N_{\alpha+1})$  for  $\alpha < \beta$  then  $(M_0, N_0) \leq^* (M_\beta, N_\beta)$ .  
3) If  $M \prec N$  and  $M \leq_N^2 A$  then for some  $(M_1, N_1)$  we have:  $A \subseteq M_2$  and  $(M, N) \leq^* (M_1, N_1) \in K_0^{pr}$ .

*Proof.* 1) If  $(M_0, N_0) \leq^* (M_1, N_1) \leq^* (M_2, N_2)$  then

- (i)  $M_0 \subseteq M_1 \subseteq M_2$  and  $N_0 \subseteq N_1 \subseteq N_2$
- (ii)  $M_0 \leq_{N_0}^2 M_1$  and
- (iii)  $M_1 \leq_{N_1}^2 M_2$
- (iv)  $M_\ell \subseteq N_\ell$ .

By 1.12(2) + (1) and clause (iii) above

- (v)  $M_1 \leq_{N_0}^2 M_2$ ,

by 1.12(5) we have (by (ii) and (v) respectively)

- (ii)'  $M_0 \leq_{N_0}^3 M_1$
- (v)'  $M_1 \leq_{N_0}^3 M_2$

hence by 1.13

- (vii)  $M_0 \leq_{N_0}^2 M_2$

hence  $(M_0, N_0) \leq^* (M_2, N_2)$  holds by (i), (iv) and (vii).

2) Similarly using 1.12(4) + (1.13).

3) Use 1.13 (see 1.13a(3)) so there are  $M_0 \prec M_1 \prec M_2$  such that  $M = M_0, N/M_1$  fs in  $M_0, M_2/(M_1 \cup N)$  fs in  $M_1$  and  $A \subseteq M_2$ . So by 1.x  $M_0 \leq_N^{-1} M_1, M_1 \leq_N^0 M_1$  hence (see 1.12(x)  $M_0 \leq_N^3 M_0 \leq_N^3 M_1$  hence (see 1.12(4))  $M_0 \leq_N^3 M_2$  hence for any  $N^*, M_2 \cup N \subseteq N^* \prec \mathfrak{C}$  we have  $(M, n) \leq^* (M_2, N^*) \in K_0^{pr}$ .  $\square_{1.15}$

**1.16 Definition.**  $K_2^{pr} = \{(M, N) : \text{the pair } (M, N) \in K_0^{pr} \text{ and if } (M, N) \leq^* (M', N') \in K_0^{pr} \text{ then } M'/N \text{ is fs in } M\}$ .

**1.17 Claim.** *If  $(M, N) \in K_0^{pr}$  then for some  $(M', N')$  we have:*

- (a)  $(M, N) \leq (M', N') \in K_0^{pr}$
- (b)  $\|N'\| \leq \|N\| + |T|$
- (c)  $(M', N') \in K_1^{pr}$  i.e.  
if  $(M', N') \leq^* (M'', N'')$  then  $M''/N'$  is fs in  $M'$ .

*Proof.* Let  $\mu = \|N\| + |T|$ , assume the conclusion fails. We now choose by induction on  $\alpha < \mu^+$ ,  $(M_\alpha, N_\alpha)$  such that:

- (i)  $(M_0, N_0) = (M, N)$
- (ii)  $(M_\alpha, N_\alpha) \in K_0^{pr}$ ,  $\|N_\alpha\| \leq \mu$
- (iii)  $\beta < \alpha \Rightarrow (M_\beta, N_\beta) \leq^* (M_\alpha, N_\alpha)$
- (iv) for limit  $\delta$  we have  $(M_\delta, N_\delta) = \left( \bigcup_{\alpha < \delta} M_\alpha, \bigcup_{\alpha < \delta} N_\alpha \right)$
- (v)  $M_{\alpha+1}/N_\alpha$  is not fs in  $M_\alpha$ .

For  $\alpha = 0$  see (i) for  $\alpha$  limit see (iv) and 1.15(2) if  $\alpha = \beta + 1$  find  $(M_\alpha, N_\alpha)$  satisfying  $(M_\beta, N_\beta) \leq^* (M_\alpha, N_\alpha) \in K_1^{pr}$  and satisfying (v). By Lowenheim Skolem argument without loss of generality  $\|N_\alpha\| \leq \mu$  and by 1.15(1) also clause (iii) holds. For a club of  $\delta < \mu^+$  we get contradiction to clause (v).  $\square_{1.17}$

**1.18 Fact.** 1) If  $(M, N) \in K_1^{pr}$  and  $(M', N') \in K_0^{pr}$  and  $(M, N) \leq^* (M', N')$  then  $(M, M') \leq^* (N, N')$ .  
2) If  $(M, N) \in K_1^{pr}$  and  $M \leq_N^2 A$  then  $A/N$  is fs in  $M$ .

*Proof.* 1) By 1.17 we know  $M'/N$  is fs in  $M$  hence by 1.13, (b)  $\Rightarrow$  (d) we know  $M \leq_{M'}^2 N$  which give the desired conclusion.

2) By 1.13A.

**1.19 Claim.** 1) *If  $(M_\alpha, N_\alpha) \in K_1^{pr}$  for  $\alpha < \delta$  and  $\langle (M_\alpha, N_\alpha) : \alpha < \delta \rangle$  is  $<^*$ -increasing then for  $\alpha < \delta$*

$$(M_\alpha, N_\alpha) \leq^* \left( \bigcup_{i < \delta} M_i, \bigcup_{i < \delta} N_i \right) \in K_1^{pr}.$$

2) *If  $(M_\alpha, N_\alpha) \in K_1^{pr}$  and  $\langle (M_\alpha, N_\alpha) : \alpha \leq \delta \rangle$  is  $<^*$ -increasing then*  

$$\left( \bigcup_{\alpha < \delta} M_\alpha, \bigcup_{\alpha < \delta} N_\alpha \right) \in K_1^{pr} \text{ and } \left( \bigcup_{\alpha < \delta} M_\alpha, \bigcup_{\alpha < \delta} N_\alpha \right) \leq^* (M_\delta, N_\delta).$$

*Proof.* We prove both together by induction on  $\delta$ .

0) By the induction hypothesis without loss of generality  $\langle (M_\alpha, N_\alpha) : \alpha < \delta \rangle$  is

increasing continuous.

1) Clearly  $(M_\alpha, N_\alpha) \leq^* \left( \bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i \right) \in K_0^{pr}$  (see 1.17(2)). Suppose

$\left( \bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i \right) \leq^* (M, N)$ . So by 1.17(1), for  $\alpha < \delta$ ,  $(M_\alpha, N_\alpha) \leq^* (M, N)$ , but  $(M_\alpha, N_\alpha) \in K_1^{pr}$  hence  $M/N_\alpha$  is fs in  $M_\alpha$ . But this implies  $M/\bigcup_{\alpha<\delta} N_\alpha$  is fs in

$\bigcup_{\alpha<\delta} M_\alpha$  by 1.5(7A).

2) As we are proving by induction on  $\delta$ ; without loss of generality  $\langle (M_\alpha, N_\alpha) : \alpha < \delta \rangle$

is  $\leq^*$ -increasing continuous, so by part (1),  $(M_\alpha, N_\alpha) \leq^* \left( \bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i \right) \in K_1^{pr}$

for  $\alpha < \delta$ . Now for  $\alpha < \delta$ ,  $(M_\alpha, N_\alpha) \leq^* (M_\delta, N_\delta)$  and  $(M_\alpha, N_\alpha) \in K_1^{pr}$  clearly  $M_\delta/N_\alpha$  is fs in  $M_\alpha$  hence by 1.5(7A),  $M_\delta/\bigcup_{\alpha<\delta} N_i$  is fs in  $\bigcup_{i<\delta} M_i$ , hence by 1.13

$\bigcup_{i<\delta} M_i \leq^3 \bigcup_{\alpha<\delta} N_i \bigcup_{i<\delta} N_i$  hence  $\left( \bigcup_{i<\delta} M_i, \bigcup_{i<\delta} N_i \right) \leq^* (M_{\delta+1}, N_{\delta+1})$ . □<sub>1.19</sub>

Now we want to apply 1.1. Toward this (for  $\lambda$  as there) we define:

**1.20 Definition.** 1)  $K_{ap}^0 = K_{ap}^0 = K_{ap}^0[T] = K_{ap}^0[T, \lambda]$  is the set of models  $M$  of  $T$  with universe  $\subseteq \lambda^+$  and cardinality  $< \lambda$  such that:  $M \cap \lambda \neq \emptyset$  and  $0 < \alpha < \lambda^+$  implies  $M \upharpoonright (\lambda \times \alpha) \prec M$ . For such  $M$  let  $\text{Dom}(M) = \{\alpha < \lambda^+ : [\lambda \times \alpha, \lambda \times \alpha + \lambda) \cap M \neq \emptyset\}$ . We now define  $<_{K_{ap}^0}$  by:  $M \leq_{K_{ap}^0} N$  if (both are in  $K_{ap}^0$  and  $M \prec N$  and): for every  $\alpha \in (0, \lambda^+)$ ,  $M \upharpoonright (\lambda \times \alpha) <_{M \upharpoonright (\lambda \times \alpha + \lambda)}^2 N \upharpoonright (\lambda \times \alpha)$ .

*1.21 Observation.* So  $M \leq_{K_{ap}^0} N$  iff both are in  $K_{ap}^0$ ,  $M < N$  and for  $\alpha \in (0, \lambda^+)$  we have  $(M \upharpoonright (\lambda \times \alpha), M \upharpoonright (\lambda \times \alpha + \lambda)) \leq^* (N \upharpoonright (\lambda \times \alpha), N \upharpoonright (\lambda \times \alpha + \lambda))$ .

**1.22 Claim.** 1)  $\leq_{K_{ap}^0}$  is a partial order on  $K_{ap}^0$ .

[Why? By 1.15(1)].

2) If  $\langle M_i : i < \delta \rangle$  is  $\leq_{K_{ap}^0}$ -increasing,  $\sum_{i<\delta} \|M_i\| < \lambda$  then  $M_i \leq_{K_{ap}^0} \bigcup_{j<\delta} M_j \in K_{ap}^0$ .

[Why? By 1.15(2)].

**1.23 Claim.** Let  $\bar{N}^\zeta, N_\zeta$  (for  $\zeta \leq \lambda$ ) be as in 1.10. Let  $E \subseteq \lambda^+$  be a thin enough club of  $\lambda^+$ ,  $\{\varepsilon(\alpha) : \alpha < \lambda^+\}$  enumerate  $\{0\} \cup E$ ,  $H$  a 1-to-1 map from  $N_\lambda$  onto  $\lambda^+$  mapping  $N_{\varepsilon(\alpha)}^\lambda$  onto  $\lambda \times \alpha$ . Let  $N_\alpha^* = H(N_{\varepsilon(\alpha)}^\lambda)$ ,  $N^* = \bigcup_{\alpha < \lambda} N_\alpha^*$ .

1) If  $M \in K_{ap}^0$  then there is a lawful  $f$  (see [Sh 457], 4.1) which is an elementary embedding of  $M$  into  $N^*$  such that for  $\alpha \in \text{Dom}(M)$ ,  $f(M \upharpoonright (\lambda \times \alpha)) <_{f(M \upharpoonright (\lambda \times \alpha + \lambda))}^2 N^* \upharpoonright (\lambda \times \alpha)$ .

*Proof.* Straightforward.

[Saharon: put old proof of 1.27 from AP here?]

But we want more, not only universality but also homogeneity.

**1.24 Definition.**  $K_{ap}^1 = K_{ap}^1[T, \lambda]$  is the set of  $M \in K_{ap}^0$  such that for every  $\alpha \in (0, \lambda^+)$ , if  $\neg(M \subseteq \lambda \times \alpha)$  then  $(M \upharpoonright (\lambda \times \alpha), M) \in K_1^{pr}$ .

Let  $\leq_{K_{ap}^1}$  be  $\leq_{K_{ap}^0} \upharpoonright K_{ap}^1$ .

**1.25 Claim.** 1)  $\leq_{K_{ap}^1}$  is a partial order on  $K_{ap}^1$ .

[Why? By 1.22 and Definition 1.2].

2) If  $\langle M_i : i < \delta \rangle$  is  $\leq_{K_{ap}^1}$ -increasing,  $\sum_{i < \delta} \|M_i\| < \lambda$  then  $M_i \leq_{K_{ap}^1} \bigcup_{j < \delta} M_j \in K_{ap}^1$ .

[Why? By 1.23(1) and 1.19(1)].

**1.26 Claim.** Let  $\bar{N}^\zeta, N_\zeta$  (for  $\zeta \leq \lambda$ ) be as in 1.10. Let  $E \subseteq \lambda^+$  be a thin enough club of  $\lambda^+$ ,  $\{\varepsilon(\alpha) : \alpha < \lambda^+\}$  enumerate  $\{0\} \cup E$ ,  $H$  a 1-to-1 map from  $N_\lambda$  onto  $\lambda^+$  mapping  $N_{\varepsilon(\alpha)}^\lambda$  onto  $\lambda \times \alpha$ . Let  $N_\alpha^* = H(N_{\varepsilon(\alpha)}^\lambda)$ ,  $N^* = \bigcup_{\alpha < \lambda} N_\alpha^*$ .

1) If  $M \in K_{ap}^1$  then there is a lawful  $f$  (see [Sh 457], 4.1) which is an elementary embedding of  $M$  into  $N^*$  such that for  $\alpha \in \text{Dom}(M)$ ,  $f(M \upharpoonright (\lambda \times \alpha)) <_{f(M \upharpoonright (\lambda \times \alpha + \lambda))}^2 N^* \upharpoonright (\lambda \times \alpha)$ .

2) If  $M_0 \leq_{K_{ap}^1} M_1$  and  $(M_0, f_0)$  is as in part (1) then we can find  $f_1, f_0 \subseteq f_1$  such that  $(M_1, f_1)$  is as in part (1). Moreover, if  $f_0 \cup (f_1 \upharpoonright (M_1 \upharpoonright (M_1 \upharpoonright (\lambda \times \alpha))))$  has been determined we can continue.

**1.27 Amalgamation Claim.** Assume  $M_0 \leq_{K_{ap}^0} M_\ell$  for  $\ell = 1, 2$  and (for simplicity)  $|M_1| \cap |M_2| = |M_0|$ . Then there is  $M \in K_{ap}^0$  such that  $M_1 \leq_{K_{ap}^0} M$  and  $M_2 \leq_{K_{ap}^0} M$ .

*Proof.* Follows from 1.26(1) + (2) (Q: domain?)

**1.28 Claim.**  $(K_{ap}, \leq^*)$  is a  $\lambda$ -system (see [Sh 457], §4).

*Proof.* Check.

**1.29 Claim.**  $(K_{ap}, \leq^*)$  is simple (see [Sh 457], §4).

*Proof.* Included in the proof of amalgamation (see last clause of 1.26(2)).

**1.30 Claim.** If  $M$  is a model of  $T$  of cardinality  $\lambda^+$  then for some  $\Gamma \in K_\lambda^{md}$ ,  $M$  can be elementarily embedded into  $M_\Gamma$ .

*Proof.* Use 1.10 with  $M = \bigcup_{\alpha < \lambda} M_\alpha$  so we get  $N^*, N_\alpha^*(\alpha < \lambda)$  as in 1.18. Check.

1.31 Proof of 1.2. Use the above claims.

## §2 ON THE STRONG ORDER PROPERTIES AND FINITARY VERSIONS

*2.0 Discussion.* By [Sh 457], for some non-simple (first order complete) the answer to the following is yes:

- $\bigoplus_T$  if  $\lambda = \lambda^{<\lambda} > |T|$ ,  $2^\lambda = \lambda^+$ , is there a ( $\lambda$ -complete),  $\lambda^+$ -c.c. forcing notion  $Q$ ,  $\Vdash_Q$  “ $\text{univ}(\lambda^+, T) \leq \lambda^{++} < 2^\lambda$ ”?
- $\bigoplus'_T$  and by [?] if  $\lambda = \lambda^{<\lambda} > |T|$ ,  $2^\lambda = \lambda^+$  is there a  $\lambda$ -complete  $\lambda^+$ -c.c. forcing notion  $Q$ ,  $\Vdash_Q$  “ $\text{univ}(\lambda^+, T) = 1$ ,  $\lambda^+ < 2^\lambda$ ”?

We know that for theories  $T$  with the strict order property the answer is no (by [KjSh 409], or see [Sh 457], §3). We would like to characterize the answer by a natural property of  $T$  (hence show that the answer to all reasonable variants is the same, e.g. does not depend on  $\lambda$ ,  $\bigoplus_T \equiv \bigoplus'_T$ , etc.) So the results we mention above give a lower bound (simple theories  $+T_{qef} + T_{trf}$ ) and an upper bound (failure of the strict order property) to the family of  $T$ 's with a positive answer. However, we can lower the upper bound. We suggest below a strictly weaker property. From another point of view, a major theme of citeSh:a, [Sh:c] was to find natural dividing lines for the family of first order theories (so the main ones there were stable, superstable and also NTOP, deepness NOTOP). Now [Sh:93] suggests another one: simplicity. Note that the negation of simple, the tree property has been touched upon in [Sh:a] but there were conclusions only for one side. [Sh:93] establishes this dividing line by having consequences for both the property and its negation and having “semantical characterization” for  $T$  simple: when  $|T| \leq \kappa < \lambda = \lambda^{<\lambda} < \mu = \mu^\kappa$  we can force by a  $\lambda^+$ -c.c.  $\lambda$ -complete forcing notion  $Q$  that  $2^\lambda > \mu$  and every model of  $T$  of cardinality  $\mu$  can be extended to a  $\kappa^+$ -saturated one, and the tree property implies a strong negation. Of course, both the inner theory and such “outside”, “semantical” characterization are much weaker than those for stable theories.

The strict order property has no such results only several consequences. We suggest below weaker properties (first the strong order property then the  $n$ -version of it for  $n < \omega$ ) which has similar consequences and so may be the right dividing line (for some questions). Remember (this is in equivalent formulations).

**2.1 Definition.**  $T$  has the strict order property if some formula  $\varphi(\bar{x}, \bar{y})$  (with  $lg\bar{x} = lg\bar{y}$ ) define in some model  $M$  of  $T$ , a partial order with infinite chains.

**2.2 Definition.** 1) A first order complete  $T$  has the strong order property if some sequence  $\bar{\varphi} = \langle \varphi_n(\bar{x}^n; \bar{y}^n) : n < \omega \rangle$  of formulas exemplifies it which means that for every  $\lambda$ :

(\*) $_{\bar{\varphi}}^\lambda(a)$   $lg\bar{x}^n = lg\bar{y}^n$  are finite,  $\bar{x}^n$  an initial segment of  $\bar{x}^{n+1}$

$\bar{y}^n$  an initial segment of  $\bar{y}^{n+1}$

- (b)  $T, \varphi_{n+1}(\bar{x}^{n+1}, \bar{y}^{n+1}) \vdash \varphi_n(\bar{x}^n, \bar{y}^n)$
- (c) for  $m \leq n$ ,  $\neg(\exists \bar{x}^{n,0} \dots \bar{x}^{n,m-1})[\bigwedge \{\varphi_n(\bar{x}^{n,k}, \bar{x}^{n,\ell}) : k = \ell + 1 \pmod m\}]$  belongs to  $T$
- (d) there is a model  $M$  of  $T$  and  $\bar{a}_\alpha^n \in M$  (of length  $\bar{y}^n$ , for  $n < \omega$ ,  $\alpha < \lambda$ ) such that  $\bar{a}_\alpha^n = \bar{a}_\alpha^{n+1} \upharpoonright lg\bar{y}^n$  and  $M \models \varphi_n[\bar{a}_\beta^n, \bar{a}_\alpha^n]$  for  $n < \omega$  and  $\alpha < \beta < \lambda$ .



- 2) The finitary strong order property is defined similarly but  $\bar{x}^n = \bar{x}, \bar{y}^n = \bar{y}$ .  
 3) We use the shorthand SOP, FSOP and for the negation NSOP, NFSOP (similarly later for NSOPn).

**2.3 Claim.** 1) *The strict order property implies finitary strong order property which implies the strong order property.*  
 2) *There is a first order complete  $T$ , which has the strong order property (even the finitary one) but not the strict order property.*  
 3) *Also some first order complete  $T$  has the strong order property but not the finitary strong order property, i.e. no  $\langle \varphi_n(\bar{x}, \bar{y}) : n < \omega \rangle$  exemplifies it (i.e. with  $lg \bar{x}_n$  constant).*

*Proof.* 1) Immediate.

2) For  $\ell \leq n < \omega$  let  $<_{n,\ell}$  be a two-place relation. Let  $<_n = <_{n,0}$ . Let  $T_0$  say:

- (a)  $x <_{n,m-1} y \Rightarrow x <_{n,m} y$
- (b)  $x <_{n,n} y$
- (c)  $\neg(x <_{n,n-1} x)$
- (d) if  $\ell + k + 1 = m \leq n$  then  $x <_{n,\ell} y$  &  $y <_{n,k} z \Rightarrow x <_{n,m} z$ .

We shall now prove that  $T_0$  has the amalgamation property; it also has the point embedding property (as the latter is easier we leave its checking to the reader).

Now suppose  $M_i \models T_0, M_0 \subseteq M_i$  for  $i = 0, 1, 2$  and  $M_1 \cap M_2 = M_0$ . We define a model  $M$ : its universe is  $M_1 \cup M_2$  and

$$\begin{aligned} <_{n,m}^M = \left\{ (a, b) \in M \times M : \text{if } m < n \text{ then for some } i \in \{1, 2\} \text{ we have :} \right. \\ & \quad (a, b) \in <_{n,m}^{M_i} \text{ or } a \in M_i \setminus M_0, b \in M_{3-i} \setminus M_0 \\ & \quad \text{and for some } c \in M_0 \text{ and } \ell, k \text{ we have :} \\ & \quad \left. m = \ell + k + 1, (a, c) \in <_{n,\ell}^{M_i}, (c, b) \in <_{n,k}^{M_{3-i}} \right\}. \end{aligned}$$

Now clearly  $M$  extends  $M_1$  and  $M_2$ : trivially  $<_{n,m}^M \upharpoonright M_i = <_{n,m}^{M_i}$ . Is  $M$  a model of  $T_0$ ? Let us check.

Clause (a) holds: For  $x, y \in M_i$  as  $M_i \subseteq M$ ; for  $i = 1, 2$  and  $x \in M_i \setminus M_0, y \in M_{3-i} \setminus M_0$ , without loss of generality  $m < n$ ; let  $c \in M$  witness  $(a, b) \in <_{n,m-1}^M$  i.e. for some  $\ell, k$  we have  $\ell + k + 1 = m - 1, (a, c) \in <_{n,\ell}^{M_i}$  and  $(c, b) \in <_{n,k}^{3-i}$ . Now by clause (a) applied to  $M_i, (a, c) \in <_{n,\ell+1}^{M_i}$  now apply the definition to get  $(a, b) \in <_{n,(\ell+1)+k+1}^M = <_{n,m}^M$ .

Clause (b) holds: Check as defining  $<_{n,m}^M$  we say: “if  $m < n$  then ...” so if  $n = m$  there is no requirement.

Clause (c): As  $M_i \subseteq M$  and  $M_i \models T_0$ .

Clause (d): Check by cases, i.e. for some  $i \in \{1, 2\}$  one of the following cases hold.

- (1)  $\{x, y, z\} \subseteq M_i$ :  
use “ $M_i$  is a model of  $T_0$  and  $M_i$  a submodel of  $M$ ”.
- (2)  $\{x, y\} \subseteq M_i, \{y, z\} \subseteq M_{3-i}$ :  
use the definition of  $<_{n,m}^M$ .
- (3)  $y \in M_i \setminus M_0, \{x, z\} \subseteq M_{3-i} \setminus M_0$ .

As  $x <_{n,\ell} y$  there are  $\ell_1, \ell_2$  and  $x_1 \in M_0$  such that  $x <_{n,\ell_1} x_1, x_1 \leq_{n,\ell_2} y$  and  $\ell_1 + \ell_2 + 1 = \ell$ .

As  $y <_{n,k} z$  there are  $k_1, k_2$  and  $z_1 \in M_0$  such that  $y \leq_{n,k_1}^{M_i} z_1, z_1 \leq_{n,k_2}^{M_{3-i}} z, k_1 + k_2 + 1 = k$ . In  $M_i$  we have  $x_1 <_{n,\ell_2}^{M_i} y <_{n,k_1}^{M_i} z_1$  hence  $x_1 \leq_{n,\ell_2+k_1+1}^{M_i} z_1$  and as  $\{x_1, z_1\} \subseteq M_0 \subseteq M_i$  clearly  $x_1 <_{n,\ell_2+k_1+1}^{M_0} z_1$ . Now in  $M_{3-i}$  we have  $x <_{n,\ell_1}^{M_{3-i}} x_1 <_{n,\ell_2+k_1+1}^{M_{3-i}} z_1$  hence  $x \leq_{n,\ell_1+\ell_2+k_1+2}^{M_{3-i}} z_1$  so  $x <_{n,\ell_1+\ell_2+k_1+2}^{M_{3-i}} z_1 <_{n,k_2}^{M_{3-i}} z$  hence  $x <_{n,\ell_1+\ell_2+k_1+k_2+3}^{M_{3-i}} z$  but  $\ell_1 + \ell_2 + k_1 + k_2 + 3 = \ell + k + 1 = m$  so  $x <_{n,m}^{M_{3-i}} z$  as required.

- (4)  $y \in M_i \setminus M_0, x \in M_{3-i} \setminus M_0, z \in M_0$ .  
Similar to case (3) but with no  $x_1$ .
- (5)  $y \in M_i \setminus M_0, x \in M_0, z \in M_{3-i} \setminus M_i$ .  
Similar to case (3) but with no  $z_1$ .

Let  $T$  be the model completion of  $T^0$ ; easy to check that it exists and has elimination of quantifiers. Let  $\varphi_n(x, y) = \bigwedge_{\ell \leq n} x <_{n,\ell} y$  (remember  $x <_{n,\ell} y$  means  $x <_{n,\delta} y$ ) now  $\langle \varphi_n : n < \omega \rangle$  exemplifies that  $T$  has the (finitary) strong order property. On the other hand we shall show that for every  $n(*) < \omega$  the theory  $T_{n(*)} =: T \upharpoonright \{<_{n,\ell} : \ell \leq n \leq n(*)\}$  does not have the strict order property (as  $T = \bigcup_{n < \omega} T_n$ , this clearly implies that  $T$  does not have the strict order property).

First note that also  $n(*)$  has elimination of quantifiers and then check directly.

(3) Let  $T^0$  say:

- (a)  $P_n$  (for  $n < \omega$ ) are pairwise disjoint ( $P_n$  unary predicates)
- (b)  $F_n$  a partial one place function from  $P_{n+1}$  into  $P_n$
- (c)  $<_{n,\ell}$  are two-place relations on  $P_n$  for  $\ell \leq n < \omega$ ; and let  $<_n = <_{n,0}$ 
  - ( $\alpha$ )  $x <_{n,m-1} y \rightarrow x <_{n,m} y$
  - ( $\beta$ )  $P_n(x) \ \& \ P_n(y) \rightarrow x <_{n,n} y$
  - ( $\gamma$ )  $\neg(x <_{n,n-1} x)$
  - ( $\delta$ ) if  $\ell + k + 1 = m \leq n$  then:  $x <_{n,\ell} y \ \& \ y <_{n,k} z \rightarrow x <_{n,m} z$

(d)  $x <_{n+1,\ell} y \rightarrow F_n(x) <_{n,\ell} F_n(y)$ .

Again  $T$  will be the model completion of  $T^0$  and it has elimination of quantifiers and we shall use  $\bar{x}_n = \langle x_i : i < n \rangle, \bar{y}_n = \langle y_i : i < n \rangle$  and  $\varphi_n(\bar{x}_n, \bar{y}_n) = \bigwedge_{i < n} F_i(x_{i+1}) = x_i \ \& \ \bigwedge_{i < n} F_i(y_{i+1}) = y_i \ \& \ \bigwedge_{i < n} x_i <_i y_i$ . □<sub>2.3</sub>

**2.4 Claim.** 1) *The following are equivalent (for  $\lambda \geq |T|$ ):*

- (A)  *$T$  has the strong order property*
- (B) $_\lambda$  *there is a  $\lambda^+$ -saturated model  $M$  of  $T$ , a  $L_{\infty, \lambda^+}$ -formula  $\varphi = \varphi(\bar{x}, \bar{y}), \varepsilon = \ell g \bar{x} = \ell g \bar{y} \leq \lambda$ , possible with  $\leq \lambda$  parameters, such that in  $M, \varphi$  defines a partial linear order with a chain of length  $\geq \beth_2(\lambda)^+$ .*

2) *The following are equivalent ( $\lambda \geq |T|$ ):*

- (A)'  *$T$  has the finitary strong order property*
- (B)' $_\lambda$  *like (B) $_\lambda$  but  $\varepsilon < \omega$ .*

*Proof.* 1)  $(A) \Rightarrow (B)_\lambda$  Straight: for a given  $\bar{\varphi} = \langle \varphi_n(\bar{x}_n, \bar{y}_n) : n < \omega \rangle$ , let  $\bar{x}, \bar{y}$  be the limit of  $\bar{x}_n, \bar{y}_n$  respectively and write  $\psi^*(\bar{x}, \bar{y}) = \bigvee_m (\exists \bar{z}_0, \dots, \bar{z}_m)[\bar{x} = \bar{z}_0 \ \& \ \bar{y} = \bar{z}_m \ \& \ \bigwedge_{\ell < m} \varphi_\omega(\bar{z}_\ell, \bar{z}_{\ell+1})]$  where  $\varphi_\omega(\bar{x}, \bar{y}) =: \bigwedge_n \varphi_n(\bar{x}, \bar{y})$ .

$(B)_\lambda \Rightarrow (A)$  Let  $\bar{a}_\alpha \in {}^\varepsilon M$  for  $\alpha < \beth_2(\lambda)^+$  form a chain. Without loss of generality the order  $\varphi$  defines is strict (i.e.  $\vdash \varphi(\bar{x}, \bar{x})$ ) and no parameters (just add them to the  $\bar{a}_\alpha$ 's). By Erdos Rado theorem without loss of generality for some type  $q = q(\bar{x}, \bar{y})$  for all  $\alpha < \beta < \omega$  the sequence  $\bar{a}_\alpha \hat{\ } \bar{a}_\beta$  realizes  $q$ .

For every  $n, \bigcup \{q(\bar{x}_\ell, \bar{x}_k) : k = \ell + 1 \bmod n \text{ and } k, \ell < n\}$  cannot be realized in  $M$  (as if  $\bar{b}_0 \hat{\ } \dots \hat{\ } \bar{b}_{n-1}$  realizes if we get a contradiction to “ $\varphi(\bar{x}, \bar{y})$  defines a strict partial order”). By saturation there is  $\varphi_n^0(\bar{x}, \bar{y}) \in q(\bar{x}, \bar{y})$  such that  $\{\varphi_n^0(\bar{x}_\ell, \bar{x}_k) : k = \ell + 1 \bmod n \text{ and } k, \ell < n\}$  is not realized in  $M$ . The rest should be clear.

2) Left to the reader. □<sub>2.4</sub>

**2.5 Definition.** 1)  $T$  has the  $n$ -stronger order property ( $SOP_n$ ) if there is a formula  $\varphi(\bar{x}, \bar{y})$  having this property for  $T$  which means:  $\ell g \bar{x} = \ell g \bar{y}$  (allowing parameters changes nothing) and there is a model  $M$  of  $T$  and  $\bar{a}_k \in {}^{\ell g \bar{x}} M$  for  $k < \omega$  such that:

- (a)  $M \models \varphi[\bar{a}_k, \bar{a}_m]$  for  $k < m < \omega$
- (b)  $M \models \neg \exists \bar{x}_0 \dots \bar{x}_{n-1} (\bigwedge \{\varphi(\bar{x}_\ell, \bar{x}_k) : \ell, k < n \text{ and } k = \ell + 1 \bmod n\})$ .

2) “ $T, \varphi(\bar{x}, \bar{y})$  have the  $SOP_{\leq n}$ ” is defined similarly except that in (b) we replace  $n$  by each  $m \leq n$ .

**2.6 Claim.**  $SOP \Rightarrow SOP_{n+1}, SOP_{n+1} \Rightarrow SOP_n, SOP_{\leq n+1} \Rightarrow SOP_{\leq n}$  and  $SOP_n \Leftrightarrow SOP_{\leq n}$  for any given  $T$ , (we did not say “for any  $\varphi$ ”).

*Proof.* The first clause is immediate. The second clause is straight too: let  $\varphi(\bar{x}, \bar{y}), M, \langle \bar{a}_m : m < \omega \rangle$  exemplify  $SOP_{n+1}$  and without loss of generality the sequence  $\langle \bar{a}_m : m < \omega \rangle$  is an indiscernible sequence. Does  $M \models (\exists \bar{x}_0, \dots, \bar{x}_{n-1})[\bar{x}_0 = \bar{a}_1 \ \& \ \bar{x}_{n-1} = \bar{a}_0 \ \& \ \bigwedge \{\varphi(\bar{x}_\ell, \bar{x}_k) : \ell, k < n \text{ and } k = \ell + 1 \bmod n\}]$ ? If the answer is yes we can replace  $\bar{a}_1$  by  $\bar{a}_2$  (by indiscernability), let  $\bar{c}_0, \dots, \bar{c}_{n-1}$  be as required above on  $\bar{x}_0, \dots, \bar{x}_{n-1}$  and  $\bar{b}_0 =: \bar{a}_1, \bar{b}_1 =: \bar{a}_2 (= \bar{c}_0), \bar{b}_2 =: \bar{c}_1, \dots, \bar{b}_{n-1} =: \bar{c}_{n-2}, \bar{b}_n =: \bar{c}_{n-1} = a_0$ ; now they satisfy the requirement mentioned in (b) of 2.5(1) on

$\bar{x}_0, \dots, x_n$  (for  $SOP_{n+1}$ ), contradicting clause (b) of 2.5(1). So assume “no” and now  $\varphi'(\bar{x}, \bar{y})$  have  $SOP_n$  for  $T$  where:  $\varphi'(\bar{x}, \bar{y}) =: \varphi(\bar{x}, \bar{y}) \ \& \ \neg(\exists \bar{x}_0, \dots, x_{n-1})[\bar{x}_0 = \bar{x} \ \& \ \bar{x}_1 = \bar{y} \ \& \ \bigwedge \{\varphi(\bar{x}_\ell, \bar{x}_k) : \ell, k < n \text{ and } k = \ell \bmod n\}]$ .

As for  $SOP_n \Leftrightarrow SOP_{\leq n}$ , the implications  $\rightarrow$  is really included in the proof above, (i.e. by it, if  $\langle \bar{a}_\ell : \ell < \omega \rangle, \varphi_n$  exemplifies  $SOP_n$ , for some  $\varphi_{n-1}$  we have  $\langle \bar{a}_\ell : \ell < \omega \rangle, \varphi_{n-1}$  exemplifies  $SOP_{n-1}$  (with  $n, n-1$  here corresponding to  $n+1, n$  there), and we can define  $\varphi_{n-2}, \dots$  similarly; now  $\langle \bar{a}_\ell : \ell < \omega \rangle, \bigwedge_{i \leq n} \varphi_i$  exemplifies  $SOP_{\leq n}$ . The implication  $\Leftarrow$  is trivial. Now the third clause  $SOP_{\leq n+1} \Rightarrow SOP_{\leq n}$  is trivial (read the definition).  $\square_{2.6}$

**2.7 Claim.** *Let  $T$  be complete. If  $T$  has  $SOP_3$  then  $T$  has the tree property (i.e. is not simple).*

*Proof.* Let  $\kappa = \text{cf}(\kappa) > |T|$  and  $\lambda > \kappa$  be a strong limit singular cardinal of cofinality  $\kappa$ . Let  $J = {}^\kappa \lambda, I = \{\eta \in {}^\kappa \lambda : \eta(i) = 0 \text{ for every } i < \kappa \text{ large enough}\}$ . Let  $\varphi(\bar{x}, \bar{y})$  exemplify the  $SOP_3$ . By the definition we can find a model  $M$  of  $T$  and  $\bar{a}_\eta \in M$  (for  $\eta \in J$ ) such that:

$$(*) \ \eta <_{\ell x} \nu \text{ in } I \Rightarrow M \models \varphi[\bar{a}_\eta, \bar{a}_\nu].$$

Without loss of generality  $\|M\| \geq \lambda$ ,  $M$  is  $\kappa^+$ -saturated. So for every  $\eta \in {}^\kappa(\lambda \setminus \{0\}) \setminus I$  we can find  $\bar{a}_\eta \in M$  such that it realizes  $p_\eta = \{\varphi(a_{(\eta \upharpoonright i) \frown 0_{[i, \kappa]}}), \bar{x}\} \ \& \ \varphi(\bar{x}, a_{(\eta \upharpoonright i) \frown (\eta(i)+1) \frown 0_{[i, \kappa]}}) : i < \kappa\}$ . But if  $\eta_1 <_{\ell x} \eta_2 \in {}^\kappa(\lambda \setminus \{0\})$  then we can find  $\nu, \rho \in I$  such that:  $\eta_1 <_{\ell x} \nu <_{\ell x} \rho <_{\ell x} \eta_2$  and  $\varphi(\bar{x}, \bar{a}_\nu) \in p_{\eta_1}, \varphi(\bar{a}_\rho, \bar{x}) \in p_{\eta_2}$  and by (\*) we have  $M \models \varphi[\bar{a}_\nu, \bar{a}_\rho]$ , so  $p_{\eta_1} \cup p_{\eta_2}$  is contradictory (by clause (b) of 2.5(1) for “ $\varphi$  have the  $SOP_3$ ”). So  $\langle p_\eta : \eta \in {}^\kappa(\lambda \setminus \{0\}) \rangle$  are pairwise contradictory,  $|p_\eta| = \kappa$ , and  $\lambda^\kappa > \lambda = \lambda^{< \kappa} > 2^{|T|}$  and  $\bigcup \{\text{Dom } p_\eta : \eta \in {}^\kappa(\lambda \setminus \{0\})\}$  has cardinality  $\leq \lambda$  and  $\kappa > |T|$ .

By [Sh:a],III,7.7 = [Sh:c],III,7.7,p.141 this implies that  $T$  has the tree property.  $\square_{2.7}$

**2.8 Claim.** 1) *The theory  $T_n =: T \upharpoonright \{\langle n, \ell : \ell \leq n \rangle$  from 2.3(2) has  $SOP_n$  but not  $SOP_{n+1}$ .*

2)  *$T_{\text{trf}}^{\text{mc}}$ , the model completion of the theory of triangle free graphs has  $SOP_3$  but not  $SOP_4$ .*

3) *For  $n \geq 3$  the model completion  $T_n^{\text{mc}} = T_{\text{dcf}(n)}^{\text{mc}}$  of the theory  $T_n = T_{\text{dcf}(n)}$  of graphs (= directed graphs, no loops or multiple edge for simplicity) with no directed circle of length  $\leq n$  has  $SOP_n$  but not  $SOP_{n+1}$ .*

4) *For odd  $n \geq 3$ , the model completion  $T_n^{\text{mc}} = T_{\text{ocf}(n)}^{\text{mc}}$  of the theory  $T_n = T_{\text{ocf}(n)}$  of graphs with no odd circle of length  $\leq n$ , has  $SOP_n$  but not  $SOP_{n+1}$ .*

5) *For  $n \geq 3$ , the model completion  $T_n^{\text{mc}} = T_{\text{cf}(n)}^{\text{mc}}$  of the theory  $T_n = T_{\text{cf}(n)}$  of graphs with no circles of length  $\leq n$ , has  $SOP_3$  but not  $SOP_4$ .*

6) *The theory  $T_{\text{qcf}}$  (see [Sh 457] does not have  $SOP_3$  (but is not simple).*

2.8A Remark. 1) Note that  $\text{univ}(\lambda, T_{\text{cf}(n)}^{\text{mc}}) = \text{univ}(\lambda, T_{\text{cf}(n)})$ .

2) For those theories,  $D(T^{\text{mc}})$  is an uncountable; they have no universal model in  $\lambda < 2^{\aleph_0}$ .

*Proof.* 1) Proved really in 2.3.

2) This is included in part (5).

3), 4), 5) We discuss the existence of model completion later; note that the meaning of  $T_n$  depends on the part we are proving.

Let  $xRy$  mean  $(x, y)$  is an edge; when we say  $(x, y)$  is an edge, for graphs we mean  $\{x, y\}$  is an edge. Let  $\bar{y} = \langle y_\ell : \ell < n \rangle$ ,  $\varphi(\bar{x}, \bar{y}) = \bigwedge_{\ell < n-1} x_\ell R y_{\ell+1} \ \& \ x_{n-1} R y_0$ . First

we note there  $T_n \vdash \neg(\exists \bar{x}_0, \dots, \bar{x}_{n-1}) \bigwedge \{\varphi(\bar{x}_i, \bar{x}_k) : \ell, k < n, k = \ell + 1 \text{ mod } n\}$ , otherwise there are  $M \models T_n$  and  $\bar{a}_\ell = \langle a_{\ell,0}, \dots, a_{\ell,n-1} \rangle \in {}^n M$  as forbidden but then  $a_{0,0}, a_{1,1}, \dots, a_{n-1,n-1}$  is a circle, so in all cases this is impossible.

For parts 3), 4) let  $M$  be the following model of  $T_n$ ; elements  $a_i^\ell$  ( $i < \omega, \ell < n$ ),  $R = \{(a_i^\ell, a_j^{\ell+1}) : i < j < \omega, \ell < n-1\} \cup \{(a_i^{n-1}, a_j^0) : i < j < \omega\}$  (but for graphs we put all such pairs and the inverted pair as  $R$  should be symmetric and irreflexive relation). Clearly for  $i < j < \omega$ ,  $M \models \varphi[\bar{a}_i, \bar{a}_j]$  where  $\bar{a}_i = \langle a_i^0, \dots, a_i^{n-1} \rangle$ . Lastly  $M \models T$ : for part (3) as  $R$  is not symmetric the absence of any circle should be clear,  $M \models a_{i(1)}^{\ell(1)} R a_{i(2)}^{\ell(2)} \Rightarrow i(1) < i(2)$ ; for part (4) there are circles but even or long and  $M \models a_{i(1)}^{\ell(1)} R a_{i(2)}^{\ell(2)} \Rightarrow \ell(1) = \ell(2) + 1 \text{ mod } n$ , so  $T_{\text{dcf}(n)}, T_{\text{ocf}(n)}$  (and  $T_{\text{ocf}(n)}^{\text{mc}}$ ) has even or long  $M \models a_{i(1)}^{\ell(1)} R a_{i(2)}^{\ell(2)} \Rightarrow \ell(1) = \ell(2) + 1 \text{ mod } n$ . So  $T_{\text{dcf}(n)}, T_{\text{ocf}(n)}$  (and  $T_{\text{dcf}(n)}^{\text{mc}}$  and  $T_{\text{ocf}(n)}^{\text{mc}}$ ) has  $\text{SOF}_n$ .

Let  $n = 3$ . Now  $T_{\text{cf}(3)} = T_{\text{ocf}(3)}$  so we can ignore part (5). Also  $T_{\text{dcf}(n)}, T_{\text{ocf}(n)}$  has the amalgamation property and joining embedding property. Thus, it is enough to show that  $T_{\text{dcf}(n)}^{\text{mc}}, T_{\text{ocf}(n)}^{\text{mc}}$  fails the  $\text{SOP}_4$ . As  $T^{\text{md}}$  has elimination of quantifiers the reader can check directly that  $T_n$  does not have  $\text{SOP}_4$ .

Let  $n > 3$ . Though  $T_n^{\text{mc}}$  does not have elimination of quantifiers, every formula is equivalent to a Boolean combination of formulas of the form:  $x = y, xRy$  for  $m < n$ ,  $\varphi_m(x, y) =: (\exists x_0, \dots, x_m)[x = x_0 \ \& \ y = x_m \ \& \ \bigwedge_{\ell < m} x_\ell R x_{\ell+1}]$  (i.e. the

distance from  $x$  to  $y$  is  $\leq m$ , directed from  $x$  to  $y$  in the case of di-graphs). For part (5) of 2.8, we should add for  $\ell < m < n/2, \ell > 0$  a partial function  $F_{m,\ell}$  defined by:  $F_{m,\ell}(x, y) = z$  iff there are  $t_0, \dots, t_m$  with no repetition such that  $x = t_0, y = t_m, z = t_\ell$  and  $\bigwedge_{\ell < m} t_\ell R t_{\ell+1}$  and lastly  $\psi_{m,\ell}(x, y) =: (\exists z)[F_{m,\ell}(x, z) = z]$ . Let  $T_n^2$

be the set of obvious (universal) axioms for those relations. Then easily  $T_n^2$  has amalgamation and has model completion,  $T_n^1$  which has elimination of quantifiers (but the closure of a finite set under those functions may be infinite). Moreover, assume  $M \models T_n^2, \langle \bar{a}_m : m < \omega \rangle$  is an indiscernible sequence in  $M$ ,  $\bar{a}_m = \langle a_\ell^m : \ell < k \rangle$ , with  $k < \omega$ . Then there is  $w \subseteq k$  such that  $[a_\ell^m = a_\ell^{m+1} \Leftrightarrow \ell \in w]$  and without loss of generality  $[\ell_1 < \ell_2 \Rightarrow a_{\ell_1}^m \neq a_{\ell_2}^m]$ . Let for  $u \subseteq \omega, M_u$  be the submodel of  $\mu$  generated by  $\bigcup_{m \in u} \bar{a}_m$  for parts (3), (4),  $M_u = \bigcup \{M_v : v \subseteq u \text{ and } |v| \leq 1\}$  so

things are simple. By the indiscernibility (increasing the  $\bar{a}_m$ 's e.g. taking  $\omega$  blocks) without loss of generality

(\*)  $M_u \cap M_\nu = M_{u \cap \nu}$  and the universe of  $M_{\{m\}}$  is the range of  $\bar{a}_m$ .

Let  $m = n$  for parts (3), (4) of 2.8,  $m = 3$  for part (5). For part (5) note: the distance between  $a_{\ell_0}^0, a_{\ell_1}^1$  is  $> \frac{n}{4}$ .

[Why? If not there is a path  $C^{i,j}$  of length  $\leq \frac{n}{4}$  for  $a_{\ell_0}^i$  to  $a_{\ell_1}^j$ , now  $C^{0,3} \cup C^{1,2} \cup C^{1,4} \cup C^{0,4}$  is a circle of length  $\leq n$ , may cross itself but still there is a too small circle].

We can now define models  $N_{\{\ell\}}$  (for  $\ell < n + 1$ ),  $N_{\{\ell, \ell+1\}}$  ( $\ell < n$ ) and  $N_{\{n, 0\}}$  and isomorphisms  $h_\ell, g_\ell$  ( $\ell < n + 1$ ) such that:

- (a) for  $\ell < n + 1$   $h_\ell$  an isomorphism from  $M_{\{\ell\}}$  onto  $N_{\{\ell\}}$
- (b) for  $\ell < n$ ,  $g_\ell$  an isomorphism  $M_{\{\ell, \ell+1\}}$  onto  $N_{\{\ell, \ell+1\}}$  extending  $h_\ell, h_{\ell+1}$ .
- (c) for  $\ell = n$ ,  $g_\ell$  an isomorphism from  $M_{\{n, n+1\}}$  onto  $N_{\{n, 0\}}$  extending  $h_n$  and  $h_0 \circ f$  where  $f$  is the isomorphism from  $M_{\{n+1\}}$  onto  $M_{\{0\}}$  taking  $\bar{a}_{n+1}$  onto  $\bar{a}_0$ .
- (d)  $N_\emptyset =: g_\ell(M_\emptyset)$  does not depend on  $\ell$ .
- (e)  $N_u \cap N_\nu = N_{u \cap \nu}$  if  $u, \nu$  are among  $\emptyset, \{\ell\}, \{m, m+1\}, \{n, 0\}$  ( $\ell < n+1, m < n$ ).

Now,

⊗ There is a model of  $T_n^2$  extending all  $N_{\{\ell, \ell+1\}}, N_{\{n, 0\}}$  ( $\ell < n$ ).

This is enough for showing that  $T_n^1$  lacks the  $\text{SOP}_{m+1}$ .

Lastly the reader can check that  $T_{\text{cf}(n)}^{\text{mc}}$  has  $\text{SOP}_3$  [choose  $k \in (\frac{n}{3}, n)$ ,  $\bar{a}_\ell = \langle a_{\ell, 0} \rangle, \ell_1 < \ell_2 \Rightarrow \varphi_k(a_{\ell, 0}, a_{\ell_2, 0})$ ]. □<sub>2.8</sub>

**2.9 Theorem.** *Let  $T$  be first order complete,  $\lambda \geq |T|$  and  $T$  has the  $\text{SOP}_3$ . Then:*

- 1)  $T$  is maximal in the Keisler order  $\triangleleft_\lambda$ , i.e. for a regular filter  $D$  on  $\lambda$  and some (= every) model  $M \models T$  we have  $M^\lambda/D$  is  $\lambda^+$ -saturated iff  $D$  is a good ultrafilter.
- 2) Moreover, in 2.10  $T$  is  $\triangleleft^\ell$ -maximal, (see Definition 2.10 below).

*We delay the proof.*

*Remark.* The order  $\triangleleft$  was introduced and investigated by Keisler [Ke76]; further investigated in [Sh 42], [Sh:a], CH.VI, new version [Sh:c], CH.VI. The following is a generalization.

**2.10 Definition.** 1) For models  $M_0, M_1$  we say  $M_0, \triangleleft_\lambda^* M_1$  if the following holds: for some model  $\mathfrak{B}_0$  in which  $M_0, M_1$  are interpreted (so  $M_i = M_i^{\mathfrak{B}_0}$ ), for every elementary extension  $\mathfrak{B}$  of  $\mathfrak{B}_0$ , which is  $(\aleph_0 + |\tau(M_0)| + |\tau(M_1)|)^+$ -saturated we have: [ $M_1^{\mathfrak{B}}$  is  $\lambda^+$ -saturated  $\Rightarrow M_0^{\mathfrak{B}}$  is  $\lambda^+$ -saturated].

2)  $M_0 \triangleleft^* M_1$  if for every  $\lambda \geq \aleph_0 + |\tau(M_0)| + |\tau(M_1)|$  we have  $M_0 \triangleleft_\lambda^* M_1$ .

3) Using the superscript  $\ell$  instead of  $*$  means in the saturation we use only  $\varphi$ -types for some  $\varphi = \varphi(\bar{x}, \bar{y})$  (so any  $\varphi$  is O.K., but for each type  $\varphi$  is constant) and omit the saturation demand on  $\mathfrak{B}$ .

4) For complete theories  $T_1, T_2$  we say  $T_1 \triangleleft_\lambda^* T_2$  **iff** for every model  $M_1$  of  $T_1$  for some model  $M_2$  of  $T_2$ ,  $M_1 \triangleleft_\lambda^* M_2$ . Similarly for  $T_1 \triangleleft_\lambda^\ell T_2, T_1 \triangleleft_{(\lambda)}^\ell T_2$ .

*2.11 Observation.* 1) In 2.11(1) we can just use  $\mathfrak{B}_0$  of the form

$(H(\chi), \in, <_\chi^*, M_0, M_1)$  with  $\chi$  strong limit.

2)  $\triangleleft_\lambda^*$  is a partial order, also  $\triangleleft_\lambda^\ell, \triangleleft^\ell$  are partial orders;  $M \triangleleft_\lambda^* M$ , and if  $M_0$  is interpretable in  $M_1$  then  $M_0 \triangleleft_\lambda^* M_1$ .

2A) For models of countable vocabulary, similar statements hold for  $\triangleleft^*$  (without the countability if  $|\tau(M_1)| > |\tau(M_0)| + |\tau(M_2)| + \aleph_0$ , we can get a silly situation).<sup>1</sup>

3) If  $\lambda \geq \aleph_0 + |\tau(M_0)| + |\tau(M_1)|$  then:  $M_0 \triangleleft_\lambda^* M_1$  iff for every finite  $\tau \subseteq \tau(M_0)$ ,  $M_0 \upharpoonright \tau \triangleleft_\lambda^* M_1$ .

4)  $M_1 \otimes_\lambda^\ell M_2 \Rightarrow M_1 \triangleleft_\lambda^* M_2$ .

5) Parallel results hold for theories.

6) Any (complete first order) theory of any infinite linear order is  $\triangleleft^\ell$ -maximal hence  $\triangleleft_\lambda^*$ -maximal for every  $\lambda \geq |T| + \aleph_0$ .

7) All countable stable theories without the f.c.p. (e.g.  $T = Th(\omega_1 =)$ ) are  $\triangleleft^\ell$ -equivalent.

8) All countable stable theories with the f.c.p. are equivalent

(e.g.  $T_{\text{eq}} = Th(\bigcup_n (\{n\} \times n), E)$  where  $E$  is equally of first coordinates).

9) If  $T_1$  is countable unstable, then  $T = \triangleleft_\lambda^\ell T_{\text{eq}} \triangleleft_\lambda^\ell T$  moreover  $\lambda > \aleph_0 \Rightarrow T_{\text{eq}} \triangleleft_\lambda^* T, \lambda \geq 2^{\aleph_0} \Rightarrow T \triangleleft_\lambda^* T_{\text{eq}}$ .

*Proof.* 1) - 4) Obvious.

5) The proof of [Sh:a], VI,2.6 = [Sh:c], VI,2.6,p.337 gives this, too.

6), 7), 8) As there [references].

□<sub>2.11</sub>

*2.12 Proof of Theorem 2.9(1),(2).* Without loss of generality  $\tau(T)$  is finite. Remember: if  $T'$  has infinite linear orders as models then it is  $\triangleleft^\ell$ -maximal. Let  $J$  be a dense linear order, such that:

(a)  $J$  has a closed interval which is  $I$

(b) for any regular  $\mu_1, \mu_2 \leq |J|$ ,  $J$  has an interval isomorphic to  $[(\{1\} \times \mu_1) \cup (\{2\} \times \mu_2)]$  ordered by  $i_1, \alpha_1 <_J (i_2, \alpha_2) \Leftrightarrow (i_1 = 1 \ \& \ i_2 = 2) \vee (i_1 = 1, i_2 \ \& \ \alpha_1 < \alpha_2) \vee (i_1 = 2 = i_2 \ \& \ \alpha_1 > \alpha_2)$ .

Let  $\varphi(x, y)$  exemplify the SOP<sub>3</sub>. Let  $M$  be a model of  $T$  and  $F : I \rightarrow {}^{\ell g \bar{x}} M$  be such that  $I \models \eta < \nu \Rightarrow M \models \varphi[F(\eta), F(\nu)]$  and for every  $c \in M^{\mathfrak{B}}$  we can find a finite  $I' \subseteq I$  such that: if  $[t_1, t_2 \in (I \setminus I')] \ \& \ \bigwedge_{s \in I'} [s <_I t \equiv s <_I t_2]$

then  $M^{\mathfrak{B}} \models \varphi[F(t_1), c] \equiv \varphi[F(t_2), c]$  and  $M^{\mathfrak{B}} \models \varphi[c, F(t_1)] \equiv \varphi[c, F(t_2)]$ . Let  $\mathfrak{B}_0 = (H(\chi), \in, <_\chi^*, J, I, F, M)$ , and  $\mathfrak{B}$  be a model,  $\mathbf{j}$  an elementary embedding of  $\mathfrak{B}_0$  into  $\mathfrak{B}$  such that  $M^* = M^{\mathfrak{B}} \upharpoonright L(T)$  is locally  $\lambda^+$ -saturated but  $I^{\mathfrak{B}} = \mathbf{j}(I)$  is not  $\lambda^+$ -saturated (for 2.9:  $\mathfrak{B}^* = \mathfrak{B}^\lambda/D$ ).

<sup>1</sup>so to overcome this, we may in Definition 2.10(2) replace “every  $\lambda > \dots$ ” by “every large enough  $\lambda$ ”

As  $\mathbf{j}(I)$  is not  $\lambda^+$ -saturated, we can find  $\lambda_0, \lambda_1 \leq \lambda$  and  $\alpha_i^\ell \in \mathbf{j}(I)$  (for  $i < \lambda_\ell$ ,  $\ell < 2$ ) such that:

- ( $\alpha$ )  $I^{\mathfrak{B}} \models a_i^0 < a_j^0$  for  $i < j < \lambda_0$
- ( $\beta$ )  $I^{\mathfrak{B}} \models a_i^1 > a_j^1$  for  $i < j < \lambda_1$
- ( $\gamma$ )  $I^{\mathfrak{B}} \models a_i^0 < a_j^1$  for  $i < \lambda_0, j < \lambda_1$
- ( $\delta$ )  $I^{\mathfrak{B}} \models \neg(\exists x)[\bigwedge_{\substack{i < \lambda_0 \\ j < \lambda_1}} a_i^0 < x < a_j^1]$ .

Clearly  $\{\varphi(\bar{a}_i^0, \bar{x}), \varphi(\bar{x}, \bar{a}_j^1) : i < \lambda_0, j < \lambda_1\}$  is finitely satisfiable in  $\mathbf{j}(M)$ . Now as  $\mathbf{j}(M)$  is locally  $\lambda^+$ -saturated there is  $\bar{a} \in {}^{\ell g \bar{x}}(M^*)$  such that  $M \models \varphi(\bar{a}_i^0, \bar{a})$  &  $\varphi(\bar{a}, \bar{a}_j^1)$  for  $i < \lambda_0, j < \lambda_1$ . In  $\mathfrak{B}$  we can define:

$$I_-^{\mathfrak{B}}[\bar{a}] =: \left\{ \eta \in I^{\mathfrak{B}} : \text{there is } \nu \in I^{\mathfrak{B}} \text{ such that : } I^{\mathfrak{B}} \models \text{“}\eta \leq \nu\text{” and } \mathbf{j}(M) \models \varphi(\bar{a}_\nu, \bar{a}) \right\}$$

$$I_+^{\mathfrak{B}}[\bar{a}] =: \left\{ \eta \in I^{\mathfrak{B}} : \text{there is } \nu \in I^{\mathfrak{B}} \text{ such that : } I^{\mathfrak{B}} \models \text{“}\nu \leq \eta\text{”} \right. \\ \left. \text{and } \mathbf{j}(M) \models \varphi[\bar{a}, \bar{a}_\nu] \right\}.$$

Clearly

- (a)  $I_-^{\mathfrak{B}}[\bar{a}]$  is an initial segment of  $\mathbf{j}(I)$  which belongs to  $\mathfrak{B}$ .
- (b)  $I_+^{\mathfrak{B}}[t]$  is an end segment of  $\mathbf{j}(I)$  which belongs to  $\mathfrak{B}$ .
- (c) For every  $i < \lambda_0$ 
  - (\*) $_0$   $a_i^0 \in I_-^{\mathfrak{B}}[\bar{a}]$
- (d) for every  $j < \lambda_1$ 
  - (\*) $_1$   $a_j^1 \notin I_-^{\mathfrak{B}}[\bar{a}]$
- (e) By the choice of  $\varphi$ 
  - (\*) $_3$   $I_-^{\mathfrak{B}}[\bar{a}] \cap I_+^{\mathfrak{B}}[t] = \emptyset$ .

If for some  $c \in \mathfrak{B}$ ,  $\mathfrak{B} \models \text{“}c \in \mathbf{j}(J)\text{”}$  and  $(\forall x \in I_-[t]) < t \leq_I c$  and  $(\forall x \in I_+[t])[c \leq_I x]$  we are done. So  $\mathfrak{B}$  thinks  $I_-^{\mathfrak{B}}[t], I_+^{\mathfrak{B}}[t]$  is a Dedekind cut, so let  $\mathfrak{B} \models \text{“}cf(I_-^{\mathfrak{B}}[t], <_I) = t_-, cf(I_+^{\mathfrak{B}}[t], >_I) = t_+$  and the (outside) cofinalities of  $t_-, t_+$  are  $\mu_1, \mu_2$  respectively. If  $\mu_1, \mu_2$  are infinite, we use clause (b) of the choice of  $J$  (and the choice of  $\mu$ ). We are left with the case where  $\mu_1 = 1 < \mu_2$  (the other case is the same). Use what  $\mathfrak{B}$  “thinks” is a  $(t_1, t_2)$  Dedekind cut of  $J$  to show  $\mu_2 \geq \mu^+$  a contradiction.  $\square_{2.9}$



**2.13 Theorem.** 1) *The theorems on non-existence of a universal model in  $\lambda$  for linear order from [KjSh 409], [Sh 457], §3 hold for any theory with  $SOP_4$ .*  
 2) *We can use embedding (not necessarily elementary) if  $\varphi(\bar{x}, \bar{y})$  is quantifier free or even existential.*

*Proof.* We concentrate on the case  $\lambda$  is regular and part (1). We will concentrate on the new part relative to [KjSh 409]. Let  $\varphi(\bar{x}, \bar{y})$  exemplify  $SOP_{\leq 4}$  (exists by 2.6(1)) in a complete first order theory  $T$ . Without loss of generality  $lg\bar{x} = lg\bar{y} = 1$  and  $T \vdash \neg\varphi(\bar{x}, \bar{x})$ .

Let  $M$  be a model of  $T$  with universe  $\lambda$ ,  $I$  a linear order,  $a_s \in M$ ,  $M \models \varphi[a_s, a_t]$  for  $s <_I t$  (from  $I$ ).

We do not have a real Dedekind cut (as  $\varphi(x, y)$  is not transitive), but we use replacements. Now for every  $b \in M$ , let  $I^-[b] = \{t : M \models \varphi[a_t, b]\}$  and  $I^+[b] = \{t : M \models \varphi[b, a_t]\}$ . As  $\varphi$  exemplifies also  $SOP_{\leq 3}$  clearly the following is satisfied:

- (\*)  $s \in I^-[b] \ \& \ t \in I^+[t] \Rightarrow s < t$   
 (if  $t < s$  from a counterexample,  $b, t, s$  gives a contradiction).

Note:  $I^-[a_s] = \{t : t <_I s\}$ ,  $I^+[a_s] = \{t : s <_I t\}$ .

Let  $P = \{a_s : s \in I\}$ ,  $<^* = \{(a_s, a_t) : s <_I t\}$  and

$J^-[t, a] = \{s \in I : s <_I t, a_s \in \alpha\}$ ,  $J^+[t, \alpha] = \{s \in I : t <_I s, a_s \in \alpha\}$

(remember:  $|M|$ , the universe of  $M$ , is  $\lambda$ ,  $\alpha = \{\beta : \beta < \alpha\}$ ).

Hence  $C = \{\delta < \lambda : (M, P, <^*, <) \upharpoonright \delta \prec M^+ =: (M, P, <^*, <)\}$  is a club of  $\lambda$ .

Clearly:

- (\*\*) let  $\delta \in C$ ,  $b \in M \cap \delta$ ;
- (i) if  $(I^-[b], <_I)$  has cofinality  $< \lambda$ , then  $I^-[b] \cap M_\delta$  is  $<_I$ -cofinal in it
  - (ii) if  $(I^+[b], >_I)$  has cofinality  $< \lambda$ , then  $I^+[b] \cap M_\delta$  is  $(>_I)$ -cofinal in it
  - (iii) if there is  $t$  such that  $I^-[b] \leq_I t \leq_I I^+[b]$  then there is such  $t \in M \cap \delta$ .

Now suppose that  $\delta_1 < \delta_2$  are in  $C$ ,  $t(*) \in I$ ,  $a_{t(*)} \in P \setminus \delta_2$ .

Case 1: For some  $s(*) \in M \cap \delta_2$  we have  $(\forall s \in J^-[t(*), \delta_1])(s <_I s(*) <_I t(*))$ . Let  $b =: a_{s(*)}$ . Hence for every  $c \in M \cap \delta_1$ : if  $\varphi(c, a_{t(*)})$  then for every  $t', t''$  satisfying  $t'' <_I t(*) <_I t'$ ,  $t' \in I$ ,  $a_{t'} \in \delta_1$ ,  $t'' \in I$ ,  $a_{t''} \in \delta_1$  we have  $M^+ \models (\exists x)[x \in P \ \& \ \varphi(c, x) \ \& \ a_{t'} <^* x <^* a_{t''}]$ . Clearly (or see the middle of the proof of case 2 below) necessarily for arbitrary  $<_I$ -large  $t \in J^-[t(*), \delta_1]$  we have  $\varphi[c, a_t]$  but for any such  $t$ ,  $\varphi[a_t, a_{s(*)}]$  i.e.  $\varphi[a_t, b]$  hence

- (\*)<sub>1</sub>  $(\forall c \in M \cap \delta_1)[\varphi(c, a_{t(*)}) \rightarrow (\exists y \in M \cap \delta_1)[\varphi(c, y) \ \& \ \varphi(y, b)]]$ .

Of course,

- (\*)<sub>2</sub>  $b \in M \cap \delta_2$   
 (\*)<sub>3</sub>  $\varphi[b, a_{t(*)}]$ .

Note: those three properties speak on  $M, \delta_1, \delta_2, a_{t(*)}, b$  but not on  $I, <^*, P, <$ .

Case 2: For no  $s(*) \in I, a_{s(*)} \in \delta_2$  do we have  $(\forall s \in J^-[t(*), \delta_1])[s <_I s(*) <_I t(*)]$  we assume:

(A)  $\{a_s : s \in I, a_s \in \delta_2, s <_I t(*)\}$  is not definable in  $M^+ \upharpoonright \delta_2$ .

We shall now show that for no  $b \in M \cap \delta_2$  do we have  $(*)_1 + (*)_2 + (*)_3$ , so assume  $b$  is like that and we shall get a contradiction.

Without loss of generality (A) holds. By  $(*)_3$  we have  $\varphi[b, a_{t(*)}]$  hence for arbitrarily  $<_I$ -large  $t \in J^-[t(*), \delta_2]$  we have  $\varphi[b, a_t]$ ; choose such  $t_0$ .

[Why? Otherwise  $I^+[b] \cap J^-[t(*), \delta_2]$  is bounded say by some  $t^*$ , so  $\theta(x, b, a_{t^*}) =: x \in P \ \& \ (\exists y)[y \in P \ \& \ y \leq^* x \ \& \ \varphi(b, y) \ \& \ t^* <^* x]$  define in  $M^+$  a set which is an end segment of  $(P, <^*)$ , include  $t^*$  (check) but so  $s \in \delta_2, s <^* t^*$ . So in  $M^+ \upharpoonright \delta_2$  it defines the set  $\{a_s : s \in J^+[t^*), \delta_2]\}$  hence  $\neg\theta(x, b, a_{t^*})$  define in  $M^+ \upharpoonright \delta_2$  the set  $\{a_s : s \in J^-[t^*), \delta_2]\}$ , hence by the assumption of the case,  $M^+ \upharpoonright \delta_2$  satisfies:

$$(\forall z)[z \in P \ \& \ \neg\theta(z, b, a_{t^*}) \rightarrow (\exists y < \delta_1)(y \in P \ \& \ z \leq^* y \ \& \ \neg\theta(z, b, a_{t^*}))]$$

contradicting (A) above.

So by the assumption of the case (i.e. that  $t_0 < \delta_1$  cannot serve as  $s(*)$  and  $t_0 <_I t(*)$ ) for some  $t_1 \in J^-[t(*), \delta_1]$  we have  $t_0 <_I t_1$  and clearly  $t_1 <_I t(*)$  hence  $\varphi[a_{t_1}, a_{t(*)}]$ . So by  $(*)_1$  applied with  $a_{t_1}$  standing for  $c$  for some  $y \in M \cap \delta_1$  we have  $\varphi[a_{t_1}, y] \ \& \ \varphi[y, b]$ . Now  $b, a_{t_0}, a_{t_1}, y$  contradicts “ $\varphi(x, y)$  exemplifies SOP<sub>4</sub>”.

Hence we get together

$\bigoplus$  if  $\delta_1 < \delta_2 \in C, a_{t(*)} \in P \setminus \delta_2$  (so  $t(*) \in I$ ) then the following conditions are equivalent:

( $\alpha$ ) for some  $s(*) \in \delta_2$ , (so necessarily  $s(*) \neq t(*)$ ) we have  $(\forall s \in I)[a_s \in \delta_1 \Rightarrow s <_I s(*) \equiv s <_I t(*)]$  we have:

(A)  $\{a_s : s \in I, a_s \in \delta_2, s <_I t(*)\}$  is definable in  $M^+ \upharpoonright \delta_2$   
(with parameters)

(B)  $\{a_s : s \in I, a_s \in \delta_2, t(*) <_I s\}$  is definable in  $M^+ \upharpoonright \delta_2$   
(with parameters)

( $\beta$ ) for some  $b \in \delta_2$  the conditions  $(*)_1, (*)_2, (*)_3$  above holds for  $\varphi(x, y)$  or for  $\varphi^-(x, y)$  where  $\varphi^-(x, y) = \varphi(y, x)$ .

*Proof.* If clause ( $\alpha$ ) holds and  $s(*) <_I t(*)$  holds use Case 1 above. If clause ( $\alpha$ ) holds and  $s(*) <_I t(*)$  fails, then  $t(*) <_I s(*)$  inverts the order of  $I$ , use  $\varphi^-$  and now apply Case 1 above. So assume  $\neg(\alpha)$ . We first want to apply Case 2 to prove there is no  $b$  satisfying  $(*)_1, (*)_2, (*)_3$ . For this we need clause (A) there. We claim it holds.

[Why? Assume  $\bar{d} \in (M^2 \upharpoonright \delta_2), \psi$  a first order formula (in the vocabulary of  $M^+$ ), such that for every  $e \in M^+ \upharpoonright \delta_2$  we have:  $M^+ \upharpoonright \delta_2 \models \psi[e, \bar{d}]$  iff  $e \in \{a_s : s \in I, a_s \in \delta_2, s <_I t(*)\}$ . So  $M^+ \models (\exists z)[P(z) \ \& \ (\forall y)(y < \delta_1 \ \& \ P(y) \Rightarrow y <^* z \equiv \psi[y, \bar{d}])]$

as  $z \mapsto a_{t(*)}$  satisfies it, but  $M^+ \upharpoonright \delta_2 \prec M^+$  hence there is  $z^* \in \delta_2$  satisfying this. So  $z^* \in P$  hence for some  $s(*)$ ,  $z^* = a_{s(*)}$ , so  $s(*)$  contradicts the assumptions  $\neg(\alpha)$ . So we have proved the failure of the first possibility from clause  $(\beta)$ . The second is proved similarly inverting the order of  $I$ , using  $\varphi^-$  (noting that this transformation preserves the statement (A) from Case 2].

**2.14 Definition.** Let  $M$  be a model with universe  $\lambda$  and  $\varphi(x, y)$  a formula exemplifying  $\text{SOP}_4$  (possibly with parameters) let  $\varphi^+(x, y) =: \varphi(x, y), \varphi^-(x, y) =: \varphi(y, x)$ . Assume  $\bar{C} = \langle C_\delta : \delta \in S \rangle$  is a club system,  $S \subseteq \lambda$  stationary, guessing club<sup>2</sup> (i.e. for every club  $E$  of  $\lambda$  for stationarily many  $\delta < \lambda, \delta \in S, C_\delta \subseteq E$ )

(a) for  $x \in |M|$  and  $\delta \in S$  let

$$\text{inv}_\varphi(x, C_\delta, M) = \left\{ \alpha \in \text{nacc } C_\delta : \text{letting } \delta_2 = \alpha, \delta_1 = \sup(C_\delta \cap \alpha) \text{ (well defined),} \right.$$

for some  $b$  conditions  $(*)_1, (*)_2, (*)_3$  of Case 1

$$\left. \text{holds for } \varphi^+ \text{ or for } \varphi^- \right\}$$

(b)  $\text{Inv}_\varphi(C_\delta, M) = \{\text{inv}(x, C_\delta, M) : x \in M\}$   
 $\text{INV}_\varphi(M, \bar{C}) = \langle \text{Inv}_\varphi(C_\delta, M) : \delta \in S \rangle$   
 $\text{INV}_\varphi(M < \bar{C}) = \text{INV}_\varphi(M, \bar{C}) / \text{id}^\alpha(\bar{C})$  where:

**2.15 Definition.**

$\text{id}^\alpha(\bar{C}) = \{S' \subseteq \lambda : \text{for some club } E \text{ of } \lambda \text{ the set of } \delta \in S' \cap S \text{ for which } C_\delta \subseteq E \text{ is not stationary}\}$ .

*2.16 Observation.* If  $M' \cong M''$  are models of  $T$  and both have universe  $\lambda$  in  $M$  then  $\text{INV}_\varphi(M', \bar{C}) = \text{INV}_\varphi(M'', \bar{C})$  so  $\text{INV}_\varphi(M, \bar{C})$  can be defined for any model of cardinality  $\lambda$ .

*Proof.* Let  $f$  be from  $M'$  onto  $M''$ , so  $f$  is a permutation of  $\lambda$ . So  $E_0 = \{\delta < \lambda : \delta \text{ a limit ordinal, } f \text{ maps } \delta \text{ onto } \delta\}$ . Assume  $C_\delta \subseteq E$ , then for  $x \in M' \setminus \delta, \delta \in S$  we have  $\text{inv}(x, C_\delta, M') = \text{inv}_\varphi(f(x), C_\delta, M'')$ . [Why? Read  $(*)_1, (*)_2, (*)_3$ ]. Hence  $\text{Inv}_\varphi(C_\delta, M') \in \text{Inv}_\varphi(C_\delta, M'')$ . By the definition of  $\text{id}^\alpha(\bar{C})$  we are done. □<sub>2.16</sub>

*2.17 Observation.* If  $\bar{C} = \langle C_\delta : \delta \in S \rangle, S \subseteq \lambda$  stationary,  $C_\delta \subseteq \delta = \sup(C_\delta), C_\delta$  closed,  $I$  a linear order with the set of elements being  $\lambda$  we let:

- (a) for  $x \in \lambda, \delta \in S, \text{inv}(x, C_\delta, I) = \{\alpha \in \text{nacc}(C_\delta) : \text{there are } y, z \in \alpha \text{ such that } y <_I x <_I z \text{ such that } (\forall s)[s < \sup(C_\delta \cap \alpha) \Rightarrow s <_I y \vee z <_I s]\}$
- (b)  $\text{Inv}(C_\delta, I) = \{\text{inv}(x, C_\delta, I) : x \in M\}$
- (c)  $\text{INV}(I, \bar{C}) = \langle \text{Inv}(C_\delta, I) : \delta \in S \rangle$
- (d)  $\text{INV}(I, \bar{C}) = \text{INV}(I, \bar{c}) / \text{id}^\alpha(\bar{C})$

<sup>2</sup>otherwise dull

*2.18 Observation.*  $\text{INV}(I, \bar{C}) = \text{INV}(I', \bar{C})$  if  $I \cong I'$ , so actually it is well defined for any linear order with cardinality  $\lambda$ .

*2.19 Observation.* If  $M$  is a model with universe  $\lambda$  and  $\varphi, \langle a_s : s \in I \rangle$  as above and,  $\emptyset \notin \text{id}^\alpha(\bar{C})$  then  $\text{INV}(I, \bar{C}) \leq \text{INV}_\varphi(M, \bar{C})$  i.e. for some club  $E$  of  $\lambda$ ,  $\delta \in S$  &  $C_\delta \subseteq E \Rightarrow \text{Inv}(C_\delta, I) \subseteq \text{Inv}_\varphi(C_\delta, M)$ .

*Proof.* By  $\oplus$  above.

*Conclusion of the proof of 2.13.* As in [KjSh 409]. □<sub>2.13</sub>

**2.20 Claim.** *For a complete  $T$ , the following are equivalent:*

- (a)  $T$  does not have  $\text{SOP}_3$
- (b) if in  $\mathfrak{C}$ ,  $\langle \bar{a}_i : i < \alpha \rangle$  is an indiscernible sequence,  $\alpha$  infinite and  $\{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}$  contradictory and for each  $j$  for some  $\bar{b}_j$  we have  $i \leq j \Rightarrow \models \varphi[\bar{a}_i, \bar{b}_j]$  and  $i > j \Rightarrow \models \psi[\bar{b}_j, \bar{a}_i]$  then for  $i < j$  we have  $(\exists \bar{x})(\varphi(\bar{x}, \bar{a}_j) \ \& \ \psi(\bar{x}, \bar{a}_i))$
- (c) in clause (b) we replace the conclusion: for every finite disjoint  $u, v \subseteq \omega$

we have  $(\exists \bar{x}) \left( \bigwedge_{i \in u} \varphi(\bar{x}, \bar{a}_i) \ \& \ \bigwedge_{j \in v} \psi(\bar{x}, \bar{a}_j) \right)$ .

*Proof.* (c)  $\Rightarrow$  (b): Trivial.

$\neg(c) \Rightarrow \neg(b)$ : Choose counterexample with  $|u \cup v|$  minimal, assume  $\alpha > \omega + |u \cup v|$ .

$\neg(a) \Rightarrow \neg(b)$ : Straight by the Definition of  $\text{SOP}_3$ , etc.

$\neg(b) \Rightarrow \neg(a)$ : Without loss of generality  $\langle \bar{a}_i \hat{\ } \bar{b}_i : i < \alpha \rangle$  is an indiscernible sequence. Now we cannot find  $\bar{c}_0, \bar{c}_1, \bar{c}_2$  such that  $\bar{c}_0 \hat{\ } \bar{c}_1, \bar{c}_1 \hat{\ } \bar{c}_2, \bar{c}_2 \hat{\ } \bar{c}_0$  realizes the same type as  $(\bar{a}_0 \hat{\ } \bar{b}_0) \hat{\ } (\bar{a}_1 \hat{\ } \bar{b}_1)$ , so  $\text{SOP}_3$  is exemplified. □<sub>2.20</sub>

## REFERENCES

- [Bl] John T. Baldwin. Definable second order quantifiers. In J. Barwise and S. Feferman, editors, *Model Theoretic Logics*, Perspectives in Mathematical Logic, chapter XII, pages 445–477. Springer-Verlag, New York Berlin Heidelberg Tokyo, 1985.
- [BlSh 156] John T. Baldwin and Saharon Shelah. Second-order quantifiers and the complexity of theories. *Notre Dame Journal of Formal Logic*, **26**:229–303, 1985. Proceedings of the 1980/1 Jerusalem Model Theory year.
- [ChHr] Gregory Cherlin and Ehud Hrushovski. Quasi-finite  $\aleph_0$ -categorical. *preprint*. title keeps changing.
- [Hr2] Ehud Hrushovski. Paralel to finite rank. *notes*. ???
- [Hr1] Ehud Hrushovski. PAC structures. *preprint*, 1990.
- [HrPi2] Ehud Hrushovski and Anand Pillay. .
- [HrPi1] Ehud Hrushovski and Anand Pillay. Groups of finite rank. *preprint*.
- [HrPi] Ehud Hrushovski and Anand Pillay. Groups definable in local fields and pseudo-finite fields. *Israel Journal of Mathematics*, **85**:203–262, 1994.
- [Ke76] H. Jerome Keisler. Six classes of theories. *Journal of Australian Mathematical Society*, **21**:257–275, 1976.
- [KjSh 409] Menachem Kojman and Saharon Shelah. Non-existence of Universal Orders in Many Cardinals. *Journal of Symbolic Logic*, **57**:875–891, 1992.
- [Lw92] Michael C. Laskowski. Vapnik-Chervonenkis classes of definable sets. *Journal of the London Mathematical Society*, **45**:377–384, 1992.
- [Sh 10] Saharon Shelah. Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory. *Annals of Mathematical Logic*, **3**:271–362, 1971.
- [Sh 42] Saharon Shelah. The monadic theory of order. *Annals of Mathematics*, **102**:379–419, 1975.
- [Sh:a] Saharon Shelah. *Classification theory and the number of nonisomorphic models*, volume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, xvi+544 pp, \$62.25, 1978.
- [Sh:92] Saharon Shelah. Remarks on Boolean algebras. *Algebra Universalis*, **11**:77–89, 1980.
- [Sh:93] Saharon Shelah. Simple unstable theories. *Annals of Mathematical Logic*, **19**:177–203, 1980.
- [Sh 126] Saharon Shelah. On saturation for a predicate. *Notre Dame Journal of Formal Logic*, **22**:239–248, 1981.
- [Sh 87b] Saharon Shelah. Classification theory for nonelementary classes, I. The number of uncountable models of  $\psi \in L_{\omega_1, \omega}$ . Part B. *Israel Journal of Mathematics*, **46**:241–273, 1983.

- [Sh 197] Saharon Shelah. Monadic logic: Hanf numbers. In *Around classification theory of models*, volume 1182 of *Lecture Notes in Mathematics*, pages 203–223. Springer, Berlin, 1986.
- [Sh:c] Saharon Shelah. *Classification theory and the number of nonisomorphic models*, volume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, xxxiv+705 pp, 1990.
- [Sh 457] Saharon Shelah. The Universality Spectrum: Consistency for more classes. In *Combinatorics, Paul Erdős is Eighty*, volume 1, pages 403–420. Bolyai Society Mathematical Studies, 1993. Proceedings of the Meeting in honour of P.Erdős, Keszthely, Hungary 7.1993; A corrected version available as <ftp://ftp.math.ufl.edu/pub/settheory/shelah/457.tex>.
- [Sh 576] Saharon Shelah. Categoricity of an abstract elementary class in two successive cardinals. *Israel Journal of Mathematics*, **126**:29–128, 2001.