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# THE PCF THEOREM REVISITED
# SH506
# DEDICATED TO PAUL ERDÖS

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Abstract. The pcf theorem (of the possible cofinability theory) was proved for reduced products $\prod_{i<\kappa} \lambda_i/I$, where $\kappa < \min_{i<\kappa} \lambda_i$. Here we prove this theorem under weaker assumptions such as $\mathrm{wsat}(I) < \min_{i<\kappa} \lambda_i$, where $\mathrm{wsat}(I)$ is the minimal $\theta$ such that $\kappa$ cannot be delivered to $\theta$ sets $\notin I$ (or even slightly weaker condition). We also look at the existence of exact upper bounds relative to $<_I$ ($<_I$-eub) as well as cardinalities of reduced products and the cardinals $T_D(\lambda)$. Finally we apply this to the problem of the depth of ultraproducts (and reduced products) of Boolean algebras.

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## §0 Introduction

An aim of the pcf theory is to answer the question, what are the possible cofinalities (pcf) of the partial orders $\prod\limits_{i<\kappa} \lambda_i/I$, where $\mathrm{cf}(\lambda_i) = \lambda_{i>\kappa}$, for different ideals $I$ on $\kappa$. For a quick introduction to the pcf theory see [Sh 400a], and for a detailed exposition, see [Sh:g] and more history. In §1 and §2 we generalize the basic theorem of this theory by weakening the assumption $\kappa < \min_{i<\kappa} \lambda_i$ to the assumption that $I$ extends a fixed ideal $I^*$ with $\mathrm{wsat}(I^*) < \min_{i<\kappa} \lambda_i$, where $\mathrm{wsat}(I^*)$ is the minimal $\theta$ such that $\kappa$ cannot be divided to $\theta$ sets $\notin I^*$ (not just that the Boolean algebra $\mathscr{P}(\kappa)/I^*$ has no $\theta$ pairwise disjoint non zero elements). So §1, §2 follow closely [Sh:g, Ch.I=Sh345a], [Sh:g, II,3.1], [Sh:g, VIII,§1]. It is interesting to note that some of those proofs which look to be superceded when by [Sh 420, §1] we know that for regular $\theta < \lambda$, $\theta^+ < \lambda \Rightarrow \exists$ stationary $S \in I[\lambda]$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$, give rise to proofs here which seem neccessary. Note $\mathrm{wsat}(I^*) \leq |\mathrm{Dom}(I^*)|^+$ (and $\mathrm{reg}_*(I^*) \leq |\mathrm{Dom}(I^*)|^+$ so [Sh:g, I,§1,§2,II,§1,VII,2.1,2.2,2.6] are really a special case of the proofs here.

During the sixties the cardinalities of ultraproducts and reduced products were much investigated (see Chang and Keisler [\CK ]). For this the notion "regular filter" (and $(\lambda,\mu)$-regular filter) were introduced, as: if $\lambda_i \geq \aleph_0$, $D$ a regular ultrafilter (or filter) on $\kappa$ then $\prod\limits_{i<\kappa} \lambda_i/D = (\liminf_D \lambda_i)^\kappa$. We reconsider these problems in §3 (again continuing [Sh:g]). We also draw a conclusion on the depth of the reduced product of Boolen algebras partially answering a problem of Monk; and make it clear that the truth of the full expected result is translated to a problem on pcf. On those problems on Boolean algebras see Monk [M]. In this section we include known proofs for completeness (mainly 3.7).

Let us review the paper in more details. In 1.2, 1.4 we give basic definition of cofinality, true cofinality, $\mathrm{pcf}(\bar\lambda)$ and $J_{<\lambda}[\bar\lambda]$ where usually $\bar\lambda = \langle \lambda_i : i < \kappa \rangle$ is a sequence of regular cardinals, $I^*$ a fixed ideal on $\kappa$ such that we consider only ideals extending it (and filter disjoint to it). Let $\mathrm{wsat}(I^*)$ be the first $\theta$ such that we cannot partition $\kappa$ to $\theta$ $I^*$-positive set (so they are pairwise disjoint, not just disjoint modulo $I^*$). In 1.5, 1.8 we give the basic properties. In lemma 1.9 we phrase the basic property enabling us to do anything: (1.9 $(*)$): essentially if $\liminf_{I^*}(\bar\lambda) \geq \theta \geq \mathrm{wsat}(I^*)$ and $\prod \bar\lambda/I^*$ is $\theta^+$-directed then we prove that $\Pi\bar\lambda/J_{<\lambda}[\bar\lambda]$ is $\lambda$-directed. In 1.11, 1.13 we deduce more properties of $\langle J_{<\lambda}[\bar\lambda] : \lambda \in \mathrm{pcf}(\bar\lambda)\rangle$ and in 1.12 deal with $<_{J_{<\lambda}[\bar\lambda]}$-increasing sequence $\langle f_\alpha : \alpha < \lambda\rangle$ with no $<_{J_{<\lambda}[\bar\lambda]}$-bound in $\Pi\bar\lambda$. In 1.14 we prove $\mathrm{pcf}(\bar\lambda)$ has a last element. In 1.13 we deal with the connection between the true cofinality of $\prod\limits_{i<\kappa} \lambda_i/D^*$ and $\prod\limits_{i<\sigma} \mu_i/E$ when $\mu_i =: \ \mathrm{tcf}(\prod\limits_{i<\kappa} \lambda_i/D_i)$ and $D^*$ is the $E$-limit of the $D_i$'s.

In 2.1 we define normality of $\lambda$ for $\bar\lambda : J_{\leq\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + B_\lambda$ and we define semi-

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normality: $J_{\leq\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + \{B_\alpha : \alpha < \lambda\}$ where $B_\alpha/J_{<\lambda}[\bar\lambda]$ is increasing. We then (in 2.2) characterize semi-normality (there is a $<_{J_{<\lambda}[\bar\lambda]}$-increasing $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ cofinal in $\Pi\bar\lambda/D$ for every ultrafilter $D$ (disjoint to $I^*$ of course) such that $\mathrm{tcf}(\Pi\bar\lambda/D) = \lambda$) and when semi normality implies normality (if some such $\bar f$ has a $<_{J_{<\lambda}[\bar\lambda]} -$ eub).

We then deal with continuity system $\bar a$ and $<_{J_{<\lambda}[\bar\lambda]}$-increasing sequence obeying $\bar a$, in a way adapted to the basic assumption $(*)$ of 1.9.

Here as elsewhere if $\min(\bar\lambda) \geq \theta^+$ our life is easier than when we just assume $\limsup_{I^*}(\bar\lambda) \geq \theta$, $\Pi\bar\lambda/I^*$ is $\theta^+$-directed (where $\theta \geq \mathrm{wsat}(I^*)$, of course). In 2.3 we give the definitions, in 2.5 we quote existence theorem, show existence of obedient sequences (in 2.7), essential uniqueness (in 2.10) and better consequence to 1.12 (in the direction to normality). We define (2.12) generating sequence and draw a conclusion (2.13(1)). Now we get some desirable properties: in 2.11 we prove semi normality, in 2.13(2) we compute $\mathrm{cf}(\Pi\bar\lambda/I^*)$ as max $\mathrm{pcf}(\bar\lambda)$. Next we relook at the whole thing: define several variants of the pcf-th (Definition 2.16). Then (in 2.17) we show that e.g. if $\min(\bar\lambda) > \theta^+$, we get the strongest version (including normality using 2.9, i.e. obedience). Lastly, we try to map the implications between the various properties when we do not use the basic assumption 1.9 $(*)$ (in fact there are considerable dependence, see 2.18, 2.19).

In 3.1, 3.3 we present measures of regularity of filters, in 3.2 we present measures of hereditary cofinality of $\Pi\bar\lambda/D$: allowing to decrease $\bar\lambda$ and/or increase the filter. In 3.4 - 3.9 we try to estimate reduced products of cardinalities $\prod\limits_{i<\kappa}\lambda_i/D$ and in 3.10 we give a reasonable upper bound by hereditary cofinality ($\leq (\theta^\kappa/D + \mathrm{hcf}_{D,\theta}(\prod\limits_{i<\kappa}\lambda_I))^{<\theta}$ when $\theta \geq \mathrm{reg}_\otimes(D)$).

In 3.13 - 3.14 we return to existence of eub's and obedience (Saharon, new point over 2.9) and in 3.15 draw conclusion on "downward closure".

In 3.16 - 3.17 we estimate $T_D(\bar\lambda)$ and in 3.18 try to translate it more fully to pcf problem (countable cofinality is somewhat problematic (so we restrict ourselves to $T_D(\bar\lambda) > \mu = \mu^{\aleph_0}$). We also mention $\aleph_1$-complete filters; (3.19, 3.20) and see what can be done without relaying on pcf (3.23)).

Now we deal with depth: define it (3.21, see 3.22), give lower bound (3.25), compute it for ultraproducts of interval Boolean algebras of ordinals (3.27). Lastly we translate the problem "does $\lambda_i <$ Depth$^+(B_i)$ for $i < \kappa$ implies $\mu <$ Depth$^+(\prod\limits_{i<\kappa} B_i/D)$" at least when $\mu > 2^\kappa$ and $(\forall\alpha < \mu)[|\alpha|^{\aleph_0} < \mu]$, to a pcf problem (in 3.29).

In the last section we phrase a reason 1.9$(*)$ works (see 4.1), analyze the case we weaken to 1.9$(*)$ to $\lim\inf_{I^*}(\bar\lambda) \geq \theta \geq \mathrm{wsat}(I^*)$ proving the pseudo pcf-th (4.4).

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## §1 Basic pcf

1.1 Notation 1) $I, J$ denote ideals on a set $\mathrm{Dom}(I)$, $\mathrm{Dom}(J)$ resp., called its domain (possibly $\bigcup\limits_{A \in I} A \subset \mathrm{Dom}(I)$. If not said otherwise the domain is an infinite cardinal denoted by $\kappa$ and also the ideal is proper i.e. $\mathrm{Dom}(I) \notin I$. Similarly $D$ denotes a filter on a set $\mathrm{Dom}(D)$; we do not always distinguish strictly between an ideal on $\kappa$ and the dual filter on $\kappa$.
2) Let $\bar\lambda$ denote a sequence of the form $\langle \lambda_i : i < \kappa \rangle$. We say $\bar\lambda$ is regular if every $\lambda_i$ is regular, $\mathrm{Min}(\bar\lambda) = \mathrm{Min}\{\lambda_i : i < \kappa\}$ (of course also in $\bar\lambda$ we can replace $\kappa$ by another set), and let $\Pi\bar\lambda = \prod\limits_{i<\kappa} \lambda_i$; usually we are assuming $\bar\lambda$ is regular. Let $\bar A^*_\theta[\bar\lambda] = \langle A^*_\alpha : \alpha < \theta \rangle = \langle A^*_{\theta,\alpha}[\bar\lambda] : \alpha < \theta \rangle$ be defined by: $A^*_\alpha = \{i < \kappa : \lambda_i > \alpha\}$. But we can replace $\kappa$ by any set (in the definitions and claims). Let $I^*$ denote a fixed ideal on $\kappa$.
3) For $I$ a filter on $\kappa$ let $I^+ = \mathscr{P}(\kappa) \setminus I$ (similarly $D^+ = \{A \subseteq \kappa : \kappa \setminus A \notin D\}$), let

$$\lim \inf_I \bar\lambda = \min\{\mu : \{i < \kappa : \lambda_i \le \mu\} \in I^+\} \text{ and}$$

$$\lim \sup_I \bar\lambda = \mathrm{Min}\{\mu : \{i < \kappa : \lambda_i > \mu\} \in I\} \text{ and}$$

$$\mathrm{atom}_I \bar\lambda = \{\mu : \{i : \lambda_i = \mu\} \in I^+\}.$$

4) For a set $A$ of ordinals with no last element, $J^{\mathrm{bd}}_A = \{B \subseteq A : \sup(B) < \sup(A)\}$, i.e. the ideal of bounded subsets.
5) Generally, if $\mathrm{inv}(X) = \sup\{|y| :\vDash \varphi(X,y)\}$ then $\mathrm{inv}^+(X) = \sup\{|y|^+ :\vDash \varphi[X,y]\}$, and any $y$ such that $\vDash \varphi[X,y]$ is a wittness for $|y| \le \mathrm{inv}(X)$ (and $|y| < \mathrm{inv}^+(X)$), and it exemplifies this.
6) Let Ord be the class of ordinals.
7) Considering $\prod\limits_{i<\kappa} f(i)$, considering $\prod\limits_{i<\kappa} f(i)/I$ formally if $(\exists i) f(i) = 0$ then $\prod\limits_{i<\kappa} f(i) = \emptyset$; but we usually ignore this, particularly when $\{i : f(i) = 0\} \in I$.

**1.2 Definition.** 1) For a partial order[1] $P$:

(a) $P$ is $\lambda$-directed if: for every $A \subseteq P$, $|A| < \lambda$ there is $q \in P$ such that $\bigwedge\limits_{p \in A} p \le q$, and we say: $q$ is an upper bound of $A$;

[1] actually we do not require $p \le q \le p \Rightarrow p = q$ so we should say quasi order

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(b) $P$ has true cofinality $\lambda$ <u>if</u> there is a sequence $\bar{p} = \langle p_\alpha : \alpha < \lambda \rangle$ increasing and cofinal in $P$, i.e.: $\bigwedge\limits_{\alpha<\beta} p_\alpha < p_\beta$ and $\forall q \in P[\bigvee\limits_{\alpha<\lambda} q \le p_\alpha]$. We write $\text{tcf}(P) = \lambda$ for the minimal such $\lambda$, in fact it is unique and we say that $\bar{p}$ witness $\lambda = \text{tcf}(P)$. (Note: if $P$ is linearly ordered it always has a true cofinality but, e.g., $(\omega, <) \times (\omega_1, <)$ does not)

(c) $P$ is called endless if $\forall p \in P \exists q \in P[q > p]$ (so if $P$ is endless, in clauses (a), (b), (d) above we can replace $\le$ by $<$)

(d) $A \subseteq P$ is a cover (of $P$) <u>if</u>: $\forall p \in P \exists q \in A[p \le q]$; we also say "$A$ is cofinal in $P$"

(e) $\text{cf}(P) = \min\{|A| : A \subseteq P \text{ is a cover}\}$

(f) We say that, in $P$, $p$ is a lub (least upper bound) of $A \subseteq P$ <u>if</u>:
  - ($\alpha$) $p$ is an upper bound of $A$ (see (a))
  - ($\beta$) if $p'$ is an upper bound of $A$ then $p \le p'$

2) If $D$ is a filter on $S$, $\alpha_s$ (for $s \in S$) are ordinals, $f, g \in \prod\limits_{s \in S} \alpha_s$, then: $f/D < g/D$, $f <_D g$ and $f < g \bmod D$ all mean $\{s \in S : f(s) < g(s)\} \in D$. Also if $f$, $g$ are partial functions from $S$ to ordinals, $D$ a filter on $S$ <u>then</u> $f < g \bmod D$ means $\{i \in \text{Dom}(D) : i \notin \text{Dom}(f)$ <u>or</u> $f(i) < g(i)$ (so both are defined)$\}$ belongs to $D$. We write $X = A \bmod D$ if $\text{Dom}(D) \setminus [(X \setminus A) \cup (A \setminus X)]$ belongs to $D$. Similarly for $\le$, and we do not distinguish between a filter and the dual ideal in such notions. So if $J$ is an ideal on $\kappa$ and $f, g \in \Pi\bar{\lambda}$, then $f < g \bmod J$ iff $\{i < \kappa : \neg f(i) < g(i)\} \in J$. Similarly if we replace the $\alpha_s$'s by partial orders.
3) For $f$, $g : S \to$ Ordinals, $f < g$ means $\bigwedge\limits_{s \in S} f(s) < g(s)$; similarly $f \le g$. So $(\Pi\bar{\lambda}, \le)$ is a partial order, we denote it usually by $\Pi\bar{\lambda}$; similarly $\Pi f$ or $\prod\limits_{i<\kappa} f(i)$.
4) If $I$ is an ideal on $\kappa$, $F \subseteq {}^\kappa\text{Ord}$, we call $g \in {}^\kappa\text{Ord}$ an $\le_I$-eub (exact upper bound) of $F$ <u>if</u>:

- ($\alpha$) $g$ is an $\le_I$-upper bound of $F$ (in ${}^\kappa\text{Ord}$)
- ($\beta$) if $h \in {}^\kappa\text{Ord}, h <_I \text{Max}\{g, 1\}$ <u>then</u> for some $f \in F$, $h < \max\{f, 1\} \bmod I$
- ($\gamma$) if $A \subseteq \kappa, A \ne \emptyset \bmod I$ and $[f \in F \Rightarrow f \restriction A =_I 0_A$, i.e., $\{i \in A : f(i) \ne 0\} \in I]$ then $g \restriction A =_I 0_A$.

5a) We say the ideal $I$ (on $\kappa$) is $\theta$-weakly saturated if $\kappa$ cannot be divided to $\theta$ pairwise disjoint sets from $I^+$ (which is $\mathscr{P}(\kappa) \setminus I$).
5b) $\text{wsat}(I) = \text{Min}\{\theta : I \text{ is } \theta\text{-weakly saturated}\}$.

*1.3 Observation.* 1) Concerning 1.2(4), note: $g' = \text{Max}\{g, 1\}$ means $g'(i) = \text{Max}\{g(i), 1\}$ for each $i < \kappa$; if for every $f \in F, \{i < \kappa : f(i) = 0\} \in I$ we can replace $\text{Max}\{g, 1\}$, $\text{Max}\{f, 1\}$ by $g$, $f$ respectively in clause $(\beta)$ and omit clause $(\gamma)$.
2) The ideal $I$ on $\kappa$ is $\theta$-weakly saturated iff in the topological space of the ultrafilters on $\kappa$ the subspace $\{D : D$ an ultrafilter on $\kappa$ disjoint to $I\}$ has spread $< \theta$, or $\theta$ is a limit ordinal, it has spread $\theta$ but the spread is not obtained (hence $2^{\text{cf}(\theta)} \geq \theta$ but it is consistently singular, see [Sh 233], [JuSh 231]).

**1.4 Definition.** Below if $\Gamma$ is "the ultrafilters disjoint to $I$", we write $I$ instead of $\Gamma$. Recall that we can replace $\kappa$ by any set.
1) For a property $\Gamma$ of ultrafilters (if $\Gamma$ is the empty condition, we omit it):

$$\text{pcf}_\Gamma(\bar\lambda) = \text{pcf}(\bar\lambda, \Gamma) = \{\text{tcf}(\Pi\bar\lambda/D) : D \text{ is an ultrafilter on } \kappa \text{ satisfying } \Gamma\}$$

(so $\bar\lambda$ is a sequence of ordinals, usually of regular cardinals, note: as $D$ is an ultrafilter, $\Pi\bar\lambda/D$ is linearly ordered hence has true cofinality).
1A) More generally, for a property $\Gamma$ of ideals on $\kappa$ we let $\text{pcf}_\Gamma(\bar\lambda) = \{\text{tcf}(\Pi\bar\lambda/J) : J$ is an ideal on $\kappa$ satisfying $\Gamma$ such that $\Pi\bar\lambda/J$ has true cofinality$\}$; we call $\Gamma$ closed when if $I \in \Gamma$ and $A, B \in I^+$ are disjoint then $I + A \in \Gamma$ is a maximal ideal. Similarly below.
2) $J_{<\lambda}[\bar\lambda, \Gamma] = \{B \subseteq \kappa$: for no ultrafilter $D$ on $\kappa$ satisfying $\Gamma$ to which $B$ belongs, is $\text{tcf}(\Pi\bar\lambda/D) \geq \lambda\}$.
3) $J_{\leq\lambda}[\bar\lambda, \Gamma] = J_{<\lambda^+}[\bar\lambda, \Gamma]$.
4) $\text{pcf}_\Gamma(\bar\lambda, I) = \{\text{tcf}(\Pi\bar\lambda/D) : D$ is an ultrafilter on $\kappa$ disjoint to $I$ satisfying $\Gamma\}$.
5) If $B \in I^+$, $\text{pcf}_I(\bar\lambda \restriction B) = \text{pcf}_{I+(\kappa\setminus B)}(\bar\lambda)$ (so if $B \in I$ it is $\emptyset$), also $J_{<\lambda}(\bar\lambda \restriction B, I) \subseteq \mathscr{P}(B)$ is defined similarly.
6) If $I = I^*$ we may omit it, similarly in (2), (4).
7) If $\Gamma = \Gamma_{I^*} = \{D : D$ an ultrafilter on $\kappa$ disjoint to $I^*\}$ we may omit it.

*Remark.* We mostly use $\text{pcf}(\bar\lambda)$, $J_{<\lambda}[\bar\lambda]$. Below we list some of the obvious properties.

**1.5 Claim.** *0)* $(\Pi\bar\lambda, <_J)$ *and* $(\Pi\bar\lambda, \leq_J)$ *are endless (even when each* $\lambda_i$ *is just a limit ordinal).*
*1)* $\min(\text{pcf}_I(\bar\lambda)) \geq \liminf_I(\bar\lambda)$ *for* $\bar\lambda$ *regular.*
*2)*

*(i) If* $B_1 \subseteq B_2$ *are from* $I^+$ *then* $\text{pcf}_I(\bar\lambda \restriction B_1) \subseteq \text{pcf}_I(\bar\lambda \restriction B_2)$*;*

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(ii) *if* $I \subseteq J$ *then* $\mathrm{pcf}_J(\bar\lambda) \subseteq \mathrm{pcf}_I(\bar\lambda)$; *and*

(iii) *for* $B_1,\ B_2 \subseteq \kappa$ *we have* $\mathrm{pcf}_I(\bar\lambda \restriction (B_1 \cup B_2)) = \mathrm{pcf}_I(\bar\lambda \restriction B_1) \bigcup \mathrm{pcf}_I(\bar\lambda \restriction B_2)$. *Also*

(iv) $A \in J_{<\lambda}[\bar\lambda \restriction (B_1 \cup B_2)] \Leftrightarrow A \cap B_1 \in J_{<\lambda}[\bar\lambda \restriction B_1]\ \&\ A \cap B_2 \in J_{<\lambda}[\bar\lambda \restriction B_2]$

(v) *if* $A_1, A_2 \in I^+, A_1 \cap A_2 = \emptyset,\ A_1 \cup A_2 = \kappa$, *and* $\mathrm{tcf}(\Pi\bar\lambda \restriction A_\ell, <_I) = \lambda$ *for* $\ell = 1, 2$ *then* $\mathrm{tcf}(\Pi\bar\lambda, <_I) = \lambda$; *and if the sequence* $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ *witness both assumptions then it witness the conclusion.*

*3)*

(i) *if* $B_1 \subseteq B_2 \subseteq \kappa, B_1$ *finite and* $\bar\lambda$ *regular* *then*

$$\mathrm{pcf}_I(\bar\lambda \restriction B_2) \setminus \mathrm{Rang}(\bar\lambda \restriction B_1) \subseteq \mathrm{pcf}_I(\bar\lambda \restriction (B_2 \setminus B_1)) \subseteq \mathrm{pcf}_I(\bar\lambda \restriction B_2)$$

(ii) *if in addition* $i \in B_1 \Rightarrow \lambda_i < \mathrm{Min}(\mathrm{Rang}[\bar\lambda \restriction (B_2 \setminus B_1)])$, *then* $\mathrm{pcf}_I(\bar\lambda \restriction B_2) \setminus \mathrm{Rang}(\bar\lambda \restriction B_1) = \mathrm{pcf}_I(\bar\lambda \restriction (B_2 \setminus B_1))$.

*4) Let* $\bar\lambda$ *be regular (i.e., each* $\lambda_i$ *is regular);*

(i) *if* $\theta = \liminf_I \bar\lambda$ *then* $\Pi\bar\lambda/I$ *is* $\theta$*-directed*

(ii) *if* $\theta = \liminf_I \bar\lambda$ *is singular* *then* $\Pi\bar\lambda/I$ *is* $\theta^+$*-directed*

(iii) *if* $\theta = \liminf_I \bar\lambda$ *is a regular uncountable cardinal, for some club* $E$ *of* $\theta$,$\{i < \kappa : \lambda_i \in E$ *or* $\lambda_i = \theta\} \in I$ *then* $\Pi\bar\lambda/I$ *is* $\theta^+$*-directed. We can weaken the assumption to "*$I$ *is not lowly normal for* $(\theta, \bar\lambda)$*" (defined in 1.6 below, it is a weaker assumption)*

(iv) *If* $\{i : \lambda_i = \theta\} = \kappa$ *mod* $I$ *and* $I$ *is weakly normal then* $(\Pi\bar\lambda, <_I)$ *has true cofinality* $\theta$

(v) *If* $\Pi\bar\lambda/I$ *is* $\theta$*-directed then* $\mathrm{cf}(\Pi\bar\lambda/I) \geq \theta$ *and* $\min \mathrm{pcf}_I(\bar\lambda) \geq \theta$

(vi) $\mathrm{pcf}_I(\bar\lambda)$ *is non empty set of regular cardinals. [See part (7)].*

*5) Assume* $\bar\lambda$ *is regular and: if* $\theta' =: \limsup_I(\bar\lambda)$ *is regular then* $I$ *is not weakly normal for* $(\theta', \bar\lambda)$. *Then* $\mathrm{pcf}_I(\bar\lambda) \nsubseteq (\limsup_I(\bar\lambda))^+$; *in fact for some ideal* $J$ *extending* $I$, $\Pi\bar\lambda/J$ *is* $(\limsup_I(\bar\lambda))^+$*-directed.*

*6) If* $D$ *is a filter on a set* $S$ *and for* $s \in S$, $\alpha_s$ *is a limit ordinal* *then:*

(i) $\mathrm{cf}(\prod_{s\in S} \alpha_s, <_D) = \mathrm{cf}(\prod_{s\in S} \mathrm{cf}(\alpha_s), <_D) = \mathrm{cf}(\prod_{s\in S} (\alpha_s, <)/D)$, *and*

(ii) $\mathrm{tcf}(\prod_{s\in S} \alpha_s, <_D) = \mathrm{tcf}(\prod_{s\in S} (\mathrm{cf}(\alpha_s), <_D)) = \mathrm{tcf}(\prod_{s\in S} (\alpha_s, <)/D)$.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



*In particular, if one of them is well defined, <u>then</u> so are the others. This is true even if we replace $\alpha_s$ by linear orders or even partial orders with true confinality.*
*7) If $D$ is an ultrafilter on a set $S$, $\lambda_s$ a regular cardinal, <u>then</u> $\theta =: \mathrm{tcf}(\prod\limits_{s \in S} \lambda_s, <_D)$ is well defined and $\theta \in \mathrm{pcf}(\{\lambda_s : s \in S\})$.*
*8) If $D$ is a filter on a set $S$, for $s \in S$, $\lambda_s$ is a regular cardinal, $S^* = \{\lambda_s : s \in S\}$ and*

$$E =: \{B : B \subseteq S^* \text{ and } \{s : \lambda_s \in B\} \in D\}$$

*and $\lambda_s > |S|$ or at least $\lambda_s > |\{t : \lambda_t = \lambda_s\}|$ for any $s \in S$ <u>then</u>:*

- $(i)$ *$E$ is a filter on $S^*$, and if $D$ is an ultrafilter on $S$ <u>then</u> $E$ is an ultrafilter on $S^*$*
- $(ii)$ *$S^*$ is a set of regular cardinals and if $s \in S \Rightarrow \lambda_s > |S|$ <u>then</u> $(\forall \lambda \in S^*)\lambda > |S^*|$,*
- $(iii)$ *$F = \{f \in \prod\limits_{s \in S} \lambda_s : \lambda_s = \lambda_t \Rightarrow f(s) = f(t)\}$ is a cover of $\prod\limits_{s \in S} \lambda_s$,*
- $(iv)$ *$\mathrm{cf}(\prod\limits_{s \in S} \lambda_s/D) = \mathrm{cf}(\Pi S^*/E)$ and $\mathrm{tcf}\,(\prod\limits_{s \in S} \lambda_s/D) = \mathrm{tcf}(\Pi S^*/E)$.*

*9) Assume $I$ is an ideal on $\kappa$, $F \subseteq {}^\kappa Ord$ and $g \in {}^\kappa Ord$. If $g$ is a $\leq_I$-eub of $F$ <u>then</u> $g$ is a $\leq_I$-lub of $F$.*
*10) $\sup \mathrm{pcf}_I(\bar\lambda) \leq |\Pi\bar\lambda/I|$.*
*11) If $I$ is an ideal on $S$ and $(\prod\limits_{s \in S} \alpha_s, <_I)$ has true cofinality $\lambda$ as exemplified by $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ <u>then</u> the function $\langle \alpha_s : s \in S\rangle$ is a $<_I$-eub (hence $<_I$-lub) of $\bar f$.*
*12) The inverse of (11) holds: <u>if</u> $I$ is an ideal on $S$ and $f_\alpha \in {}^S\mathrm{Ord}$ for $\alpha < \lambda = \mathrm{cf}(\lambda)$, $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_I$-increasing with $<_I$-eub $f$ <u>then</u> $\mathrm{tcf}(\prod\limits_i f(i), <_I) = \mathrm{tcf}(\Pi \mathrm{cf}[f(i)], <_I) = \lambda$.*
*13) If $I \subseteq J$ are ideals on $\kappa$ <u>then</u>*

- $(a)$ $\mathrm{wsat}(I) \geq \mathrm{wsat}(J)$
- $(b)$ $\lim\inf_I(\bar\lambda) \leq \lim\inf_J(\bar\lambda)$
- $(c)$ *if $\lambda = \mathrm{tcf}(\prod\limits_{i<\kappa} \lambda_i, <_I)$ then $\lambda = \mathrm{tcf}(\prod\limits_{i<\kappa} \lambda_i, <_J)$.*

*14) If $f_1, f_2$ are $<_I$-lub of $F$ <u>then</u> $f_1 =_I f_2$.*

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



**1.6 Definition.** 1) Let "$I$ is not almost normal for $(\theta, \bar{\lambda})$" mean: for some $h \in \Pi\bar{\lambda}$, for no $j < \theta$ is $\{i < \kappa : \lambda_i \leq \theta \Rightarrow h(i) < j\} = \kappa \mod I$.
2) Let "$I$ is not lowly normal for $(\theta, \bar{\lambda})$" mean: for some $h \in \Pi\bar{\lambda}$, for no $\zeta < \theta$, is $\{i < \kappa : \lambda_i \leq \theta \Rightarrow h(i) < \zeta\} \in I^+$.

*Remark.* Note that weakly normals implies lowly normal.

*Proof.* They are all very easy, e.g.
0) We shall show $(\Pi\bar{\lambda}, <_J)$ is endless (assuming, of course, that $J$ is a proper ideal on $\kappa$). Let $f \in \Pi\bar{\lambda}$, then $g =: f + 1$ (defined $(f+1)(\gamma) = f(\gamma) + 1$) is in $\Pi\bar{\lambda}$ too as each $\lambda_\alpha$ being an infinite cardinal is a limit ordinal and $f < g \mod J$.
4) Clause *(iii)*:
First, assume that $I$ is not medium normal for $(\theta, \bar{\lambda})$, and let $h \in \Pi\bar{\lambda}$ witness this. Without loss of generality $\lambda_i > \theta \Rightarrow h(i) \geq \theta$. So assume that $f_\alpha \in \Pi\bar{\lambda}$ for $\alpha < \theta$. We now define a function $f$ with domain $\kappa$ by

$$f(i) = \cup\{f_\alpha(i) : \alpha < h(i)(\text{and } \alpha < \theta)\}.$$

Now first $i < \kappa \Rightarrow f(i) < \lambda_i$ because $\lambda_i$ is regular, $h(i) < \lambda_i$ and $\alpha < \theta \Rightarrow f_\alpha(i) < \lambda_i$. So $f \in \Pi\bar{\lambda}$.
Second, for any $\alpha < \theta$ we have

$$\{i < \kappa : \neg(f_\alpha(i) \leq f(i))\} \subseteq \{i < \kappa : \alpha \geq h(i)\}$$

and this set belongs to $I$ by the choice of $h$ above. So $\alpha < \theta \Rightarrow f_\alpha \leq_I f$. Together we are done. To finish we need

- ⊛ if there is a club $E$ of $\theta$ and $\{i : \lambda_i \in E \text{ or } \lambda_i = \theta\} \in I$ then $I$ is not medium normal for $(\theta, \bar{\lambda})$.
  [Why ⊛? Without loss of generality $\theta \leq \lambda_i$, we define a function $h$ with domain $\kappa, h(i) = \sup(E \cap \lambda_i)$ if $\lambda_i \notin E \cup \{\theta\}$ and $h(i) = 0$ if $\lambda_i \in E \cup \{\theta\}$. So $i < \kappa \Rightarrow h(i) < \lambda_i$ hence $h \in \Pi\bar{\lambda}$. Also for every $\alpha < \theta$ choose $\beta \in E, \beta > \alpha$ (e.g., $\text{Min}(E \backslash (\alpha + 1))$), the set $\{i < \lambda : h(i) < \alpha\}$ is included in $\{i < \kappa : \lambda_i = \theta\} \cup \{i < \kappa : \lambda_i \in E\} \cup \{i < \kappa : \lambda_i \leq \beta\}$.]

Now the first and second belong to $I$ by an assumption and the third as $\alpha < \theta = \liminf_I(\bar{\lambda})$, so we are done.
5) Let $\theta' =: \limsup_I(\bar{\lambda})$ and define

$$J =: \{A \subseteq \kappa : \text{ for some } \theta < \theta' \text{ the set } \{i < \kappa : \lambda_i > \theta \text{ and } i \in A\} \text{ belongs to } I\}.$$

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



Clearly $J$ is an ideal on $\kappa$ extending $I$ (and $\kappa \notin J$) and $\limsup_J(\bar{\lambda}) = \liminf_J(\bar{\lambda}) = \theta'$.

Case 1: $\theta'$ is $\aleph_0$.

We do not use the $J$ above. Now the desired conclusion fails then every ultrafilter on $\kappa$ disjoint to $I$ is $\aleph_1$-complete. Now if $\{i < \kappa : \lambda_i > \aleph_0\} \in J^+$ the construction is immediate so without loss of generality $i < \kappa \Rightarrow \lambda_i = \aleph_0$. But "not weakly normal for $(\theta, \bar{\lambda})$" then $j < \omega \Rightarrow A_j =: \{i < \kappa : h(i) < j\} \neq \kappa \bmod I$ but $\cup\{A_j : j < \omega\} = \kappa$. There is an ultrafilter $D$ on $\kappa$ disjoint to $J \cup \{A_j : j < \omega\}$ so $\langle A_j : j < \omega \rangle$ exemplifies $D$ is not $\aleph_1$-complete.

Case 2: $\theta'$ is singular.

By part (4), clause (ii), $\Pi\bar{\lambda}/I$ is $(\theta')^+$-directed and by part (4) clause (v) we get the desired conclusion.

Case 3: $\theta'$ is regular $> \aleph_0$.

Let $h^* \in \Pi\bar{\lambda}$ witness that "$I$ is not weakly normal for $(\theta', \bar{\lambda})$" and let

$$J^* = \{A \subseteq \kappa : \text{ for every } h \in \Pi\bar{\lambda}, \text{ for some } j < \theta' \text{ we have } \{i \in A : h(i) < j\} = A \bmod I\}.$$

Note that if $A \in J$ then for some $\theta < \theta'$ the set $A' =: \{i \in A : \lambda_i > \theta\} \in I$ hence for every $h \in \Pi\bar{\lambda}$, the choice $j =: \theta$ witness $A \in J^*$. So $J \subseteq J^*$. Also $J^* \subseteq \mathscr{P}(\kappa)$ by its definition. $J^*$ is closed under subsets (trivial) and under union [why? assume $A_0$, $A_1 \in J^*$, $A = A_0 \cup A_1$; for every $h \in \Pi\bar{\lambda}$, choose $j_0$, $j_1 < \theta'$ such that $A'_\ell =: \{i \in A_\ell : h(i) < j_\ell\} = A_\ell \bmod I$, so $j =: \max\{j_0, j_1\} < \theta'$ and $A' = \{i \in A : h(i) < j\} = A \bmod I$; so $A \in J^*$]. Also $\kappa \notin J^*$ [why? as $h^*$ witness that $I$ is not weakly normal for $(\theta', \bar{\lambda})$]. So together $J^*$ is an ideal on $\kappa$ extending $I$. Now $J^*$ is not medium normal for $(\theta', \bar{\lambda})$, as witnessed by $h^*$.
[Why? Let us check Definition 1.6(2), so let $\zeta < \theta'$. We should prove that $A_\zeta = \{i < \kappa : \lambda_i \leq \theta \Rightarrow h(i) < \zeta\} \notin J^+$; now $A^1_\zeta = \{i < \kappa : \lambda_i > \theta\} \in J \subseteq J^*$ and $A^2_\zeta = \{i < \kappa : h(i) < j\} \in J^*$ hence $A^1_\zeta \cup A^2_\zeta \in J$ but it includes $A_\zeta$, so we are done.]
Lastly, $\Pi\bar{\lambda}/J^*$ is $(\theta')^+$-directed (by part (4) clause (iii)), and so $\mathrm{pcf}_{J^*}(\bar{\lambda})$ is disjoint to $(\theta')^+$.
9) Let us prove $g$ is a $\leq_I$-lub of $F$ in $({}^\kappa\mathrm{Ord}, \leq_I)$. As we can deal separately with $I + A$, $I + (\kappa \setminus A)$ where $A =: \{i : g(i) = 0\}$, and the later case is trivial we can assume $A = \emptyset$. So assume $g$ is not a $\leq_I$-lub, so there is an upper bound $g'$ of $F$, but not $g \leq_I g'$. Define $g'' \in {}^\kappa\mathrm{Ord}$:

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



$$g''(i) = \begin{cases} 0 & \text{if } g(i) \leq g'(i) \\ g'(i) & \text{if } g'(i) < g(i). \end{cases}$$

Clearly $g'' <_I g$. So, as $g$ in an $\leq_I$-eub of $F$ for $I$, there is $f \in F$ such that $g'' <_I \max\{f, 1\}$, but $B =: \{i : g'(i) < g(i)\} \neq \emptyset \bmod(I)$ (as not $g \leq_I g'$) so $g' \restriction B = g'' \restriction B <_I \max\{f, 1\} \restriction B$. But we know that $f \leq_I g'$ (as $g'$ is an upper bound of $F$) hence $f \restriction B \leq_I g' \restriction B$, so by the previous sentence neccessarily $f \restriction B =_I 0_B$ hence $g' \restriction B =_I 0_B$; as $g'$ is a $\leq_I$-upper bound of $F$ we know $[f' \in F \Rightarrow f' \restriction B =_I 0_B]$, hence by $(\gamma)$ of Definition 1.2(4) we have $g \restriction B =_I 0_B$, a contradiction to $B \notin I$ (see above). $\square_{1.5}$

*1.7 Remark.* In 1.5 we can also have the straight monotonicity properties of

$$\mathrm{pcf}_I(\Pi\bar{\lambda}, \Gamma).$$

**1.8 Claim.** *1) $J_{<\lambda}[\bar{\lambda}]$ is an ideal (of $\mathscr{P}(\kappa)$, i.e., on $\kappa$, but the ideal may not be proper).*
*2) If $\lambda \leq \mu$, then $J_{<\lambda}[\bar{\lambda}] \subseteq J_{<\mu}[\bar{\lambda}]$.*
*3) If $\lambda$ is singular, $J_{<\lambda}[\bar{\lambda}] = J_{<\lambda^+}[\bar{\lambda}] = J_{\leq\lambda}[\bar{\lambda}]$.*
*4) If $\lambda \notin \mathrm{pcf}(\bar{\lambda})$, then $J_{<\lambda}[\bar{\lambda}] = J_{\leq\lambda}[\bar{\lambda}]$.*
*5) If $A \subseteq \kappa, A \notin J_{<\lambda}[\bar{\lambda}]$, and $f_\alpha \in \Pi\bar{\lambda} \restriction A$, $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_{J_{<\lambda}[\bar{\lambda}]}$-increasing cofinal in $(\Pi\bar{\lambda} \restriction A)/J_{<\lambda}[\bar{\lambda}]$ <u>then</u> $A \in J_{\leq\lambda}[\bar{\lambda}]$.*
*Also this holds when $A \subseteq \kappa, \langle f_\alpha : \alpha < \lambda\rangle$ is $<_J$-increasing cofinal in $(\Pi\bar{\lambda} \restriction A)/J$ for any ideal $J$ on $\kappa$ such that $I^* \subseteq J \subseteq J_{\leq\lambda}[\bar{\lambda}], A \notin J$.*
*6) The earlier parts hold for $J_{<\lambda}[\bar{\lambda}, \Gamma]$, too.*

*Proof.* Straight.

**1.9 Lemma.** *Assume*

(∗) *$\bar{\lambda}$ is regular and*
  - (α) $\mathrm{Min}(\bar{\lambda}) > \theta \geq \mathrm{wsat}(I^*)$ *(see 1.2(5)(b)) or at least*
  - (β) $\liminf_{I^*}(\bar{\lambda}) \geq \theta \geq \mathrm{wsat}(I^*)$, *and $\Pi\bar{\lambda}/I^*$ is $\theta^+$-directed.*[2]

[2] note, if $\mathrm{cf}(\theta) < \theta$ then "$\theta^+$-directed" follows from "$\theta$-directed" which follows from "lim $\inf_{I^*}(\bar{\lambda}) \geq \theta$", i.e. first part of clause (β). Note also that if clause (α) holds then $\Pi\bar{\lambda}/I^*$ is $\theta^+$-directed (even $(\Pi\bar{\lambda}, <)$ is $\theta^+$-directed), so clause (α) implies clause (β).

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



*If $\lambda$ is a cardinal $\geq \theta$, and $\kappa \notin J_{<\lambda}[\bar\lambda]$ then $(\Pi\bar\lambda, <_{J_{<\lambda}[\bar\lambda]})$ is $\lambda$-directed (remember: $J_{<\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda, I^*]$).*

*1.10 Remark.* Note that above $(\alpha) \Rightarrow (\beta)$ so in any case also $(\Pi\bar\lambda, \leq_{I^*})$ is $\theta^+$-directed.

*Proof.* Note: if $f \in \Pi\bar\lambda$ then $f < f+1 \in \Pi\bar\lambda$, (i.e., $(\Pi\bar\lambda, <_{J_\lambda[\bar\lambda]})$ is endless) where $f+1$ is defined by $(f+1)(i) = f(i)+1$). Let $F \subseteq \Pi\bar\lambda$, $|F| < \lambda$, and we shall prove that for some $g \in \Pi\bar\lambda$ we have $(\forall f \in F)(f \leq g \bmod J_{<\lambda}[\bar\lambda])$, this suffices. The proof is by induction on $|F|$. If $|F|$ is finite, this is trivial. Also if $|F| \leq \theta$, when $(\alpha)$ of $(*)$ holds it is easy: let $g \in \Pi\bar\lambda$ be $g(i) = \sup\{f(i) : f \in F\} < \lambda_i$; when $(\beta)$ of $(*)$ holds use second clause of $(\beta)$. So assume $|F| = \mu$, $\theta < \mu < \lambda$ so let $F = \{f^0_\alpha : \alpha < \mu\}$. By the induction hypothesis we can choose by induction on $\alpha < \mu$, $f^1_\alpha \in \Pi\bar\lambda$ such that:

(*a*) $f^0_\alpha \leq f^1_\alpha \bmod J_{<\lambda}[\bar\lambda]$
(*b*) for $\beta < \alpha$ we have $f^1_\beta \leq f^1_\alpha \bmod J_{<\lambda}[\bar\lambda]$.

If $\mu$ is singular, there is $C \subseteq \mu$ unbounded, $|C| = \mathrm{cf}(\mu) < \mu$, and by the induction hypothesis there is $g \in \Pi\bar\lambda$ such that for $\alpha \in C$, $f^1_\alpha \leq g \bmod J_{<\lambda}[\bar\lambda]$. Now $g$ is as required: $f^0_\alpha \leq f^1_\alpha \leq f^1_{\min(C\setminus\alpha)} \leq g \bmod J_{<\lambda}[\bar\lambda]$. So without loss of generality $\mu$ is regular. Let us define $A^*_\varepsilon =: \{i < \kappa : \lambda_i > |\varepsilon|\}$ for $\varepsilon < \theta$, so $\varepsilon < \zeta < \theta \Rightarrow A^*_\zeta \subseteq A^*_\varepsilon$ and $\varepsilon < \theta \Rightarrow A^*_\varepsilon = \kappa \bmod I^*$. Now we try to define by induction on $\varepsilon < \theta$, $g_\varepsilon$, $\alpha_\varepsilon = \alpha(\varepsilon) < \mu$ and $\langle B^\varepsilon_\alpha : \alpha < \mu\rangle$ such that:

(*i*) $g_\varepsilon \in \Pi\bar\lambda$
(*ii*) for $\varepsilon < \zeta$ we have $g_\varepsilon \restriction A^*_\zeta \leq g_\zeta \restriction A^*_\zeta$
(*iii*) for $\alpha < \mu$ let $B^\varepsilon_\alpha =: \{i < \kappa : f^1_\alpha(i) > g_\varepsilon(i)\}$
(*iv*) for each $\varepsilon < \theta$, for every $\alpha \in [\alpha_{\varepsilon+1}, \mu)$, $B^\varepsilon_\alpha \neq B^{\varepsilon+1}_\alpha \bmod J_{<\lambda}[\bar\lambda]$.

We cannot carry this definition: as letting $\alpha(*) = \sup\{\alpha_\varepsilon : \varepsilon < \theta\}$, then $\alpha(*) < \mu$ since $\mu = \mathrm{cf}(\mu) > \theta$. We know that $B^\varepsilon_{\alpha(*)} \cap A^*_{\varepsilon+1} \neq B^{\varepsilon+1}_{\alpha(*)} \cap A^*_{\varepsilon+1} \bmod J_{<\lambda}[\bar\lambda]$ for $\varepsilon < \theta$ (by (iv) and as $A^*_{\varepsilon+1} = \kappa \bmod I^*$ and $I^* \subseteq J_{<\lambda}[\bar\lambda]$) and $B^\varepsilon_{\alpha(*)} \subseteq \kappa$ (by (iii)) and $[\varepsilon < \zeta \Rightarrow B^\zeta_{\alpha(*)} \cap A^*_\zeta \subseteq B^\varepsilon_{\alpha(*)}]$ (by (ii)), together $\langle A^*_{\varepsilon+1} \cap (B^\varepsilon_{\alpha(*)} \setminus B^{\varepsilon+1}_{\alpha(*)}) : \varepsilon < \theta\rangle$ is a sequence of $\theta$ pairwise disjoint members of $(I^*)^+$, a contradiction[3]
Now for $\varepsilon = 0$ let $g_\varepsilon$ be $f^1_0$ and $\alpha_\varepsilon = 0$.

---

[3]i.e we have noted that for no $B_\varepsilon \subseteq \kappa$ $(\varepsilon < \theta)$ do we have: $B_\varepsilon \neq B_{\varepsilon+1} \bmod I^*$ and $\varepsilon < \zeta < \theta \Rightarrow B_\varepsilon \cap A_\zeta \subseteq B_\zeta$ where $A_\zeta = \kappa \bmod I^*$ (e.g., $A_\zeta = A^*_\zeta$) to the definition of $\theta = \mathrm{wsat}(I^*)$.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



For $\varepsilon$ limit let $g_\varepsilon(i) = \bigcup\limits_{\zeta<\varepsilon} g_\zeta(i)$ for $i \in A^*_\varepsilon$ and zero otherwise (note: $g_\varepsilon \in \Pi\bar\lambda$ as $\varepsilon < \theta$, $\lambda_i > \varepsilon$ for $i \in A^*_\varepsilon$ and $\bar\lambda$ is a sequence of regular cardinals) and let $\alpha_\varepsilon = 0$.
For $\varepsilon = \zeta + 1$, suppose that $g_\zeta$ hence $\langle B^\zeta_\alpha : \alpha < \mu\rangle$ are defined. If $B^\zeta_\alpha \in J_{<\lambda}[\bar\lambda]$ for unboundedly many $\alpha < \mu$ (hence for every $\alpha < \mu$) then $g_\zeta$ is an upper bound for $F$ mod $J_{<\lambda}[\bar\lambda]$ and the proof is complete. So assume this fails, then there is a minimal $\alpha(\varepsilon) < \mu$ such that $B^\zeta_{\alpha(\varepsilon)} \notin J_{<\lambda}[\bar\lambda]$. As $B^\zeta_{\alpha(\varepsilon)} \notin J_{<\lambda}[\bar\lambda]$, by Definition 1.4(2) for some ultrafilter $D$ on $\kappa$ disjoint to $J_{<\lambda}[\bar\lambda]$ we have $B^\zeta_{\alpha(\varepsilon)} \in D$ and $\mathrm{cf}(\Pi\bar\lambda/D) \geq \lambda$. But $\mu < \lambda$. Hence $\{f^1_\alpha/D : \alpha < \mu\}$ has an upper bound $h_\varepsilon/D$ where $h_\varepsilon \in \Pi\bar\lambda$. Let us define $g_\varepsilon \in \Pi\bar\lambda$:

$$g_\varepsilon(i) = \ \mathrm{Max}\{g_\zeta(i), h_\varepsilon(i)\}.$$

Now (i), (ii) hold trivially and $B^\varepsilon_\alpha$ is defined by (iii). Why does (iv) hold (for $\varepsilon$) with $\alpha_{\zeta+1} = \alpha_\varepsilon =: \alpha(\varepsilon)$? Suppose $\alpha(\varepsilon) \leq \alpha < \mu$. As $f^1_{\alpha(\varepsilon)} \leq f^1_\alpha \mod J_{<\lambda}[\bar\lambda]$ clearly $B^\zeta_{\alpha(\varepsilon)} \subseteq B^\zeta_\alpha \mod J_{<\lambda}[\bar\lambda]$. Moreover $J_{<\lambda}[\bar\lambda]$ is disjoint to $D$ (by its choice) so $B^\zeta_{\alpha(\varepsilon)} \in D$ implies $B^\zeta_\alpha \notin J_{<\lambda}[\bar\lambda]$.
On the other hand $B^\varepsilon_\alpha$ is $\{i < \kappa : f^1_\alpha(i) > g_\varepsilon(i)\}$ which is equal to $\{i < \kappa : f^1_\alpha(i) > g_\zeta(i), h_\varepsilon(i)\}$ which does not belong to $D$ ($h_\varepsilon$ was chosen such that $f^1_\alpha \leq h_\varepsilon \mod D$). We can conclude $B^\varepsilon_\alpha \notin D$, whereas $B^\zeta_\alpha \in D$; so they are distinct $\mod J_{<\lambda}[\bar\lambda]$ as required in clause (iv).
Now we have said that we cannot carry the definition for all $\varepsilon < \theta$, so we are stuck at some $\varepsilon$; by the above $\varepsilon$ is successor, say $\varepsilon = \zeta + 1$, and $g_\zeta$ is as required: an upper bound for $F$ modulo $J_{<\lambda}[\bar\lambda]$. $\square_{1.9}$

**1.11 Claim.** *If* $(*)$ *of 1.9, $D$ is an ultrafilter on $\kappa$ disjoint to $I^*$ and $\lambda = \mathrm{tcf}(\Pi\bar\lambda, <_D)$, <u>then</u> for some $B \in D$,$(\Pi\bar\lambda \restriction B, <_{J_{<\lambda}[\bar\lambda]})$ has true cofinality $\lambda$. (So $B \in J_{\leq\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda]$ by 1.8(5).)*

*Proof.* As $(\Pi\bar\lambda, \leq_{I^*})$ is $\theta^+$-directed (by 1.9) clearly $\lambda \geq \theta^+$. By the definition of $J_{<\lambda}[\bar\lambda]$ clearly $D \cap J_{<\lambda}[\bar\lambda] = \emptyset$.
Let $\langle f_\alpha/D : \alpha < \lambda\rangle$ be increasing unbounded in $\Pi\bar\lambda/D$ (so $f_\alpha \in \Pi\bar\lambda$). By 1.9 without loss of generality $(\forall\beta < \alpha)(f_\beta < f_\alpha \mod J_{<\lambda}[\bar\lambda])$.
Now 1.11 follows from 1.12 below: its hypothesis clearly holds. If $\bigwedge_{\alpha<\lambda} B_\alpha = \emptyset$ mod $D$, (see (A) of 1.12) then (see (D) of 1.12) $J \cap D = \emptyset$ hence (see (D) of 1.12) $g/D$ contradicts the choice of $\langle f_\alpha/D : \alpha < \lambda\rangle$. So for some $\alpha < \lambda$, $B_\alpha \in D$; by (C) of 1.12 we get the desired conclusion. $\square_{1.11}$

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



**1.12 Lemma.** *Suppose* $(*)$ *of 1.9,* $\mathrm{cf}(\lambda) > \theta$, $f_\alpha \in \Pi\bar\lambda$, $f_\alpha < f_\beta \mod J_{<\lambda}[\bar\lambda]$ *for* $\alpha < \beta < \lambda$, *and there is no* $g \in \Pi\bar\lambda$ *such that for every* $\alpha < \lambda$, $f_\alpha < g \mod J_{<\lambda}[\bar\lambda]$. *Then there are* $B_\alpha$ *(for* $\alpha < \lambda$*) such that:*

- $(A)$ $B_\alpha \subseteq \kappa$ *and for some* $\alpha(*) < \lambda : \alpha(*) \le \alpha < \lambda \Rightarrow B_\alpha \notin J_{<\lambda}[\bar\lambda]$
- $(B)$ $\alpha < \beta \Rightarrow B_\alpha \subseteq B_\beta \mod J_{<\lambda}[\bar\lambda]$ *(i.e.* $B_\alpha \setminus B_\beta \in J_{<\lambda}[\bar\lambda]$*)*
- $(C)$ *for each* $\alpha$, $\langle f_\beta \restriction B_\alpha : \beta < \lambda\rangle$ *is cofinal in* $(\Pi\bar\lambda \restriction B_\alpha, <_{J_{<\lambda}[\bar\lambda]})$ *(better restrict yourselves to* $\alpha \ge \alpha(*)$ *(see (A)) so that necessarily* $B_\alpha \notin J_{<\lambda}[\bar\lambda]$*);*
- $(D)$ *for some* $g \in \Pi\bar\lambda$, $\bigwedge_{\alpha<\lambda} f_\alpha \le g \mod J$ *where*[4] $J = J_{<\lambda}[\bar\lambda] + \{B_\alpha : \alpha < \lambda\}$; *in fact*
- $(D)^+$ *for some* $g \in \Pi\bar\lambda$ *for every* $\alpha < \lambda$, *we have* $f_\alpha \le g \mod (J_{<\lambda}[\bar\lambda] + B_\alpha)$, *in fact* $B_\alpha = \{i < \kappa : f_\alpha(i) > g(i)\}$
- $(E)$ *if* $g \le g' \in \Pi\bar\lambda$, *then for arbitrarily large* $\alpha < \lambda$:

$$\{i < \kappa : [g(i) \ge f_\alpha(i) \Leftrightarrow g'(i) \ge f_\alpha(i)]\} = \kappa \mod J_{<\lambda}[\bar\lambda]$$

*(hence for every large enough* $\alpha < \lambda$ *this holds)*

- $(F)$ *if* $\delta$ *is a limit ordinal* $< \lambda$, $f_\delta$ *is a* $\le_{J_{<\lambda}[\bar\lambda]}$*-lub of* $\{f_\alpha : \alpha < \delta\}$ *then* $B_\delta$ *is a lub of* $\{B_\alpha : \alpha < \delta\}$ *in* $\mathscr{P}(\kappa)/J_{<\lambda}[\bar\lambda]$.

*Proof.* Remember that for $\varepsilon < \theta$, $A^*_\varepsilon = \{i < \kappa : \lambda_i > |\varepsilon|\}$ so $A^*_\varepsilon = \kappa \mod I^*$ and $\varepsilon < \zeta \Rightarrow A^*_\zeta \subseteq A^*_\varepsilon$. We now define by induction on $\varepsilon < \theta$, $g_\varepsilon$, $\alpha(\varepsilon) < \lambda$, $\langle B^\varepsilon_\alpha : \alpha < \lambda\rangle$ such that:

- $(i)$ $g_\varepsilon \in \Pi\bar\lambda$
- $(ii)$ for $\zeta < \varepsilon$, $g_\zeta \restriction A^*_\varepsilon \le g_\varepsilon \restriction A^*_\varepsilon$
- $(iii)$ $B^\varepsilon_\alpha =: \{i \in \kappa : f_\alpha(i) > g_\varepsilon(i)\}$
- $(iv)$ if $\alpha(\varepsilon) \le \alpha < \lambda$ then $B^\varepsilon_\alpha \ne B^{\varepsilon+1}_\alpha \mod J_{<\lambda}[\bar\lambda]$.

For $\varepsilon = 0$ let $g_\varepsilon = f_0$, and $\alpha(\varepsilon) = 0$.
For $\varepsilon$ limit let $g_\varepsilon(i) = \bigcup_{\zeta<\varepsilon} g_\zeta(i)$ if $i \in A^*_\varepsilon$ and zero otherwise; now

$$[\zeta < \varepsilon \Rightarrow g_\zeta \restriction A^*_\varepsilon \le g_\varepsilon \restriction A^*_\varepsilon]$$

holds trivially and $g_\varepsilon \in \Pi\bar\lambda$ as each $\lambda_i$ is regular and $[i \in A^*_\varepsilon \Leftrightarrow \lambda_i > \varepsilon]$), and let $\alpha(\varepsilon) = 0$.

[4] Of course, if $B_\alpha = \kappa \mod J_{<\lambda}[\bar\lambda]$, this becomes trivial.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



For $\varepsilon = \zeta + 1$, if $\{\alpha < \lambda : B^\zeta_\alpha \in J_{<\lambda}[\bar\lambda]\}$ is unbounded in $\lambda$, then $g_\zeta$ is a bound for $\langle f_\alpha : \alpha < \lambda\rangle \mod J_{<\lambda}[\bar\lambda]$, contradicting an assumption. Clearly

$$\alpha < \beta < \lambda \ \Rightarrow B^\zeta_\alpha \subseteq B^\zeta_\beta \mod J_{<\lambda}[\bar\lambda]$$

hence $\{\alpha < \lambda : B^\zeta_\alpha \in J_{<\lambda}[\bar\lambda]\}$ is an initial segment of $\lambda$. So by the previous sentence there is $\alpha(\varepsilon) < \lambda$ such that for every $\alpha \in [\alpha(\varepsilon), \lambda)$, we have $B^\zeta_\alpha \notin J_{<\lambda}[\bar\lambda]$ (of course, we may increase $\alpha(\varepsilon)$ later). If $\langle B^\zeta_\alpha : \alpha < \lambda\rangle$ satisfies the desired conclusion, with $\alpha(\varepsilon)$ for $\alpha(*)$ in (A) and $g_\zeta$ for $g$ in (D), (D)$^+$ and (E), we are done. Now among the conditions in the conclusion of 1.12, clause (A) holds by the choice of $B^\zeta_\alpha$ and of $\alpha(\varepsilon)$, clause (B) holds by $B^\zeta_\alpha$'s definition as $\alpha < \beta \Rightarrow f_\alpha < f_\beta \mod J_{<\lambda}[\bar\lambda]$, (D)$^+$ holds with $g = g_\zeta$ by the choice of $B^\zeta_\alpha$ hence also clause (D) follows. Lastly if (E) fails, say for $g'$, then it can serve as $g_\varepsilon$. Now condition (F) follows immediately from (iii) (if (F) fails for $\delta$, then there is $B \subseteq B^\zeta_\delta$ such that $\bigwedge\limits_{\alpha<\delta} B^\zeta_\alpha \subseteq B \mod J_{<\lambda}[\bar\lambda]$ and $B^\zeta_\delta \setminus B \notin J_{<\lambda}[\bar\lambda]$; now the function $g^* =: (g_\zeta \restriction (\kappa \setminus B)) \cup (f_\delta \restriction B)$ contradicts "$f_\delta$ is a $\leq_{J_{<\lambda}[\bar\lambda]}$-lub of $\{f_\alpha : \alpha < \delta\}$", because: $g^* \in \Pi\bar\lambda$ (obvious), $\neg(f_\delta \leq g^* \mod J_{<\lambda}[\bar\lambda])$ [why? as $B^\zeta_\delta \setminus B \notin J_{<\lambda}[\bar\lambda]$ and $g^* \restriction (B^\zeta_\delta \setminus B) = g_\zeta \restriction (B^\zeta_\delta \setminus B) < f_\delta \restriction (B^\zeta_\delta \setminus B)$ by the choice of $B^\zeta_\delta$], and for $\alpha < \delta$ we have:

$$f_\alpha \restriction B \leq_{J_{<\lambda}[\bar\lambda]} f_\delta \restriction B = g^* \restriction B \text{ and}$$

$$f_\alpha \restriction (\kappa \setminus B) \leq_{J_{<\lambda}[\bar\lambda]} g_\zeta \restriction (\kappa \setminus B) = g^* \restriction (\kappa \setminus B)$$

(the $\leq_{J_{<\lambda}[\bar\lambda]}$ holds as $(\kappa \setminus B) \cap B^\zeta_\alpha \in J_{<\lambda}[\bar\lambda]$ and the definition of $B^\zeta_\alpha$). So only clause (C) (of 1.12) may fail, without loss of generality for $\alpha = \alpha(\varepsilon)$. I.e. $\langle f_\beta \restriction B^\zeta_{\alpha(\varepsilon)} : \beta < \lambda\rangle$ is not cofinal in $(\Pi\bar\lambda \restriction B^\zeta_{\alpha(\varepsilon)}, <_{J_{<\lambda}[\bar\lambda]})$. As this sequence of functions is increasing w.r.t. $<_{J_{<\lambda}[\bar\lambda]}$, there is $h_\varepsilon \in \Pi(\bar\lambda \restriction B^\zeta_{\alpha(\varepsilon)})$ such that for no $\beta < \lambda$ do we have $h_\varepsilon \leq f_\beta \restriction B^j_{\alpha(\varepsilon)} \mod J_{<\lambda}[\bar\lambda]$. Let $h'_\varepsilon = h_\varepsilon \cup 0_{(\kappa \setminus B^\zeta_{\alpha(\varepsilon)})}$ and $g_\varepsilon \in \Pi\bar\lambda$ be defined by $g_\varepsilon(i) = \operatorname{Max}\{g_\zeta(i), h'_\varepsilon(i)\}$. Now define $B^\varepsilon_\alpha$ by (iii) so (i), (ii), (iii) hold trivially, and we can check (iv).

So we can define $g_\varepsilon$, $\alpha(\varepsilon)$ for $\varepsilon < \theta$, satisfying (i)–(iv). As in the proof of 1.9, this is impossible: because (remembering $\mathrm{cf}(\lambda) = \lambda > \theta$) letting $\alpha(*) =: \bigcup\limits_{\varepsilon<\theta} \alpha(\varepsilon) < \lambda$ we have: $\langle B^\varepsilon_{\alpha(*)} \cap A^*_\zeta : \varepsilon < \zeta\rangle$ is $\subseteq$-decreasing, for each $\zeta < \theta$, and $A^*_\varepsilon = \kappa \mod I^*$ and $B^{\varepsilon+1}_{\alpha(*)} \neq B^\varepsilon_{\alpha(*)} \mod J_{<\lambda}[\bar\lambda]$ so $\langle B^\varepsilon_{\alpha(*)} \cap A^*_{\varepsilon+1} \setminus B^{\varepsilon+1}_{\alpha(*)} : \varepsilon < \theta\rangle$ is a sequence of $\theta$ pairwise disjoint members of $(J_{<\lambda}[\bar\lambda])^+$ hence of $(I^*)^+$ which give the contradiction to $(*)$ of 1.9; so the lemma cannot fail. $\square_{1.12}$

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



**1.13 Lemma.** *Suppose* $(*)$ *of 1.9 and* $\theta < \lambda$.
*1) For every* $B \in J_{\leq\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda]$, *we have:*

$$(\Pi\bar\lambda \restriction B, <_{J_{<\lambda}[\bar\lambda]}); \text{ has true cofinality } \lambda; \text{ (hence } \lambda \text{ is regular).}$$

*2) If* $D$ *is an ultrafilter on* $\kappa$, *disjoint to* $I^*$, *<u>then</u>* $\mathrm{tcf}(\Pi\bar\lambda/D)$ *is* $\min\{\lambda : D \cap J_{\leq\lambda}[\bar\lambda] \neq \emptyset\}$.
*3)(i) For* $\lambda$ *a limit cardinal* $J_{<\lambda}[\bar\lambda] = \bigcup\limits_{\mu<\lambda} J_{<\mu}[\bar\lambda]$, *hence*
*(ii) For every* $\lambda$, $J_{<\lambda}[\bar\lambda] = \bigcup_{\mu<\lambda} J_{\leq\mu}[\bar\lambda]$.
*4)* $J_{\leq\lambda}[\bar\lambda] \neq J_{<\lambda}[\bar\lambda]$ *<u>iff</u>* $J_{\leq\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda] \neq \emptyset$ *<u>iff</u>* $\lambda \in \mathrm{pcf}(\bar\lambda)$.
*5)* $J_{\leq\lambda}[\bar\lambda]/J_{<\lambda}[\bar\lambda]$ *is* $\lambda$*-directed (i.e. if* $B_\gamma \in J_{\leq\lambda}[\bar\lambda]$ *for* $\gamma < \gamma^*$, $\gamma^* < \lambda$ *<u>then</u> for some* $B \in J_{\leq\lambda}[\bar\lambda]$ *we have* $B_\gamma \subseteq B \mod J_{<\lambda}[\bar\lambda]$ *for every* $\gamma < \gamma^*$.*)*

*Proof.* 1) Let

$$\begin{aligned} J = \{B \subseteq \kappa : & B \in J_{<\lambda}[\bar\lambda] \text{ or } B \in J_{\leq\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda] \text{ and} \\ & (\Pi\bar\lambda \restriction B, <_{J_{<\lambda}[\bar\lambda]}) \text{ has true cofinality } \lambda\}. \end{aligned}$$

By its definition clearly $J \subseteq J_{\leq\lambda}[\bar\lambda]$; it is quite easy to check it is an ideal (use 1.5(2)(v)). Assume $J \neq J_{\leq\lambda}[\bar\lambda]$ and we shall get a contradiction. Choose $B \in J_{\leq\lambda}[\bar\lambda] \setminus J$; as $J$ is an ideal, there is an ultrafilter $D$ on $\kappa$ such that: $D \cap J = \emptyset$ and $B \in D$. Now if $\mathrm{tcf}(\Pi\bar\lambda/D) \geq \lambda^+$, then $B \notin J_{\leq\lambda}[\bar\lambda]$ (by the definition of $J_{\leq\lambda}[\bar\lambda]$); contradiction.
On the other hand if $F \subseteq \Pi\bar\lambda$, $|F| < \lambda$ then there is $g \in \Pi\bar\lambda$ such that $(\forall f \in F)(f < g \mod J_{<\lambda}[\bar\lambda])$ (by 1.9), so $(\forall f \in F)[f < g \mod D]$ (as $J_{<\lambda}[\bar\lambda] \subseteq J$, $D \cap J = \emptyset$), and this implies $\mathrm{cf}(\Pi\bar\lambda/D) \geq \lambda$. By the last two sentences we know that $\mathrm{tcf}(\Pi\bar\lambda/D)$ is $\lambda$. Now by 1.11 for some $C \in D, (\Pi(\bar\lambda \restriction C), <_{J_{<\lambda}[\bar\lambda]})$ has true cofinality $\lambda$, of course $C \cap B \subseteq C$ and $C \cap B \in D$ hence $C \cap B \notin J_{<\lambda}[\bar\lambda]$. Clearly if $C' \subseteq C$, $C' \notin J_{<\lambda}[\bar\lambda]$ then also $(\Pi\bar\lambda \restriction C', <_{J_{<\lambda}[\bar\lambda]})$ has true cofinality $\lambda$, hence by the last sentence without loss of generality $C \subseteq B$; hence by 1.8(5) we know that $C \in J_{\leq\lambda}[\bar\lambda]$ hence by the definition of $J$ we have $C \in J$. But this contradicts the choice of $D$ as disjoint from $J$.
We have to conclude that $J = J_{\leq\lambda}[\bar\lambda]$ so we have proved 1.13(1).
2) Let $\lambda$ be minimal such that $D \cap J_{\leq\lambda}[\bar\lambda] \neq \emptyset$ (it exists as by 1.5(10) that is because $J_{<(\prod\bar\lambda)^+}[\bar\lambda] = \mathscr{P}(\kappa)$) and choose $B \in D \cap J_{\leq\lambda}[\bar\lambda]$. So $[\mu < \lambda \Rightarrow B \notin J_{\leq\mu}[\bar\lambda]]$ (by the choice of $\lambda$) hence by 1.13(3)(ii) below, we have $B \notin J_{<\lambda}[\bar\lambda]$. It similarly follows

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



that $D \cap J_{<\lambda}[\bar{\lambda}] = \emptyset$. Now $(\Pi\bar{\lambda} \restriction B, <_{J_{<\lambda}[\bar{\lambda}]})$ has true cofinality $\lambda$ by 1.11. As we know that $B \in D \cap J_{\leq\lambda}[\bar{\lambda}]$, and $J_{<\lambda}[\bar{\lambda}] \cap D = \emptyset$; clearly we have finished the proof.
3) Note that we should not use part (2)!

Clause (i):

Let $J =: \bigcup_{\mu<\lambda} J_{<\mu}[\bar{\lambda}]$. Now $J$ is an ideal by 1.8(2) and $(\Pi\bar{\lambda}, <_J)$ is $\lambda$-directed; i.e., if $\alpha^* < \lambda$ and $\{f_\alpha : \alpha < \alpha^*\} \subseteq \Pi\bar{\lambda}$, then there exists $f \in \Pi\bar{\lambda}$ such that

$$(\forall\alpha < \alpha^*)(f_\alpha < f \mod J).$$

[Why? If $\alpha^* < \theta^+$, as $(*)$ of 1.9 holds, this is obvious by 1.9. So without loss of generality $\alpha^* \geq \theta^+$ and $\alpha^* = \mathrm{cf}(\alpha^*)$; suppose not; $\lambda$ is a limit cardinal, hence there is $\mu^*$ such that $\alpha^* < \mu^* < \lambda$. Without loss of generality $|\alpha^*|^+ < \mu^*$. By 1.9, there is $f \in \Pi\bar{\lambda}$ such that $(\forall\alpha < \alpha^*)(f_\alpha < f \mod J_{<\mu^*}[\bar{\lambda}])$. Since $J_{<\mu^*}[\bar{\lambda}] \subseteq J$, it is immediate that

$$(\forall\alpha < \alpha^*)(f_\alpha < f \mod J).]$$

Clearly $\bigcup\limits_{\mu<\lambda} J_{<\mu}[\bar{\lambda}] \subseteq J_{<\lambda}[\bar{\lambda}]$ by 1.8(2). On the other hand, let us suppose that there is $B \in (J_{<\lambda}[\bar{\lambda}] \setminus \bigcup\limits_{\mu<\lambda} J_{<\mu}[\bar{\lambda}])$. Choose an ultrafilter $D$ on $\kappa$ such that $B \in D$ and $D \cap J = \emptyset$. Since $(\Pi\bar{\lambda}, <_J)$ is $\lambda$-directed and $D \cap J = \emptyset$, one has $\mathrm{tcf}(\Pi\bar{\lambda}/D) \geq \lambda$, but $B \in D \cap J_{<\lambda}[\bar{\lambda}]$, in contradiction to Definition 1.4(2).

Clause (ii):

If $\lambda$ limit — by part (i) and 1.8(2); if $\lambda$ successor — by 1.8(2) and Definition 1.4(3). Note that we hae not used part (2).
4) Easy.
5) Let $\langle f^\gamma_\alpha : \alpha < \lambda\rangle$ be $<_{J_{<\lambda}[\bar{\lambda}]+(\kappa\setminus B_\gamma)}$-increasing and cofinal in $\Pi\bar{\lambda} \mod J_{<\lambda}[\bar{\lambda}] + (\kappa\setminus\beta_\gamma)$ (for $\gamma < \gamma^*$). Let us choose by induction on $\alpha < \lambda$ a function $f_\alpha \in \Pi\bar{\lambda}$, as a $<_{J_{<\lambda}[\bar{\lambda}]}$-bound to $\{f_\beta : \beta < \alpha\} \cup \{f^\gamma_\alpha : \gamma < \gamma^*\}$, such $f_\alpha$ exists by 1.9 and apply 1.12 to $\langle f_\alpha : \alpha < \lambda\rangle$, getting $\langle B'_\alpha : \alpha < \lambda\rangle$, now $B'_\alpha$ for $\alpha$ large enough is as required. $\square_{1.13}$

*1.14 Conclusion.* If $(*)$ of 1.9, then $\mathrm{pcf}(\bar{\lambda})$ has a last element.

*Proof.* This is the minimal $\lambda$ such that $\kappa \in J_{\leq\lambda}[\bar{\lambda}]$. [$\lambda$ exists, since $\lambda^* =: |\Pi\bar{\lambda}| \in \{\lambda : \kappa \in J_{\leq\lambda}[\bar{\lambda}]\} \neq \emptyset$] and by 1.5(10). $\square_{1.14}$

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**1.15 Claim.** *Suppose* $(*)$ *of 1.9 holds. Assume for* $j < \sigma$, $D_j$ *is a filter on* $\kappa$ *extending* $\{\kappa \setminus A : A \in I^*\}$, $E$ *a filter on* $\sigma$ *and* $D^* = \{B \subseteq \kappa : \{j < \sigma : B \in D_j\} \in E\}$ *(a filter on* $\kappa$*). Let* $\mu_j =: \mathrm{tcf}(\Pi\bar{\lambda}, <_{D_j})$ *be well defined for* $j < \sigma$, *and assume further* $\mu_j > \sigma + \theta$.
*Let*

$$\lambda = \mathrm{tcf}(\Pi\bar{\lambda}, <_{D^*}), \mu = \mathrm{tcf}(\prod_{j<\sigma} \mu_j, <_E).$$

*Then* $\lambda = \mu$ *(in particular, if one is well defined, then so is the other).*

*Proof.* Without loss of generality $\sigma \geq \theta$. (Why? Otherwise we can add $\mu_j =: \mu_0$, $D_j =: D_0$ for $j \in \theta \setminus \sigma$, and replace $\sigma$ by $\theta$ and $E$ by $E' = \{A \subseteq \theta : A \cap \sigma \in E\}$). Let $\langle f^j_\alpha : \alpha < \mu_j \rangle$ be an $<_{D_j}$-increasing cofinal sequence in $(\Pi\bar{\lambda}, <_{D_j})$.
Now $\ell = 0, 1$, for each $f \in \Pi\bar{\lambda}$, define $G_\ell(f) \in \prod_{j<\sigma} \mu_j$ by $G_\ell(f)(j) = \min\{\alpha < \mu_j$: if $\ell = 1$ then $f \leq f^j_\alpha \mod D_j$ and if $\ell = 0$ then: not $f^j_\alpha \leq f \mod D_j\}$ (it is well defined for $f \in \Pi\bar{\lambda}$ by the choice of $\langle f^j_\alpha : \alpha < \mu_j \rangle$).
Note that for $f^1$, $f^2 \in \Pi\bar{\lambda}$ and $\ell < 2$ we have:

$$\begin{aligned}
f^1 \leq f^2 \mod D^* &\Leftrightarrow B(f^1, f^2) =: \{i < \kappa : f^1(i) \leq f^2(i)\} \in D^* \\
&\Leftrightarrow A(f^1, f^2) =: \{j < \sigma : B(f^1, f^2) \in D_j\} \in E \\
&\Leftrightarrow \text{ for some } A \in E, \text{ for every } i \in A \text{ we have } f^1 \leq_{D_i} f^2 \\
&\Rightarrow \text{ for some } A \in E \text{ for every } i \in A \text{ we have} \\
&\qquad G_\ell(f^1)(i) \leq G_\ell(f^2)(i) \\
&\Leftrightarrow G_\ell(f^1) \leq G_\ell(f^2) \mod E.
\end{aligned}$$

So

$\otimes_1$ $G_\ell$ is a mapping from $(\Pi\bar{\lambda}, \leq_{D^*})$ into $(\prod_{j<\sigma} \mu_j, \leq_E)$ preserving order.

Next we prove that

$\otimes_2$ for every $g \in \prod_{j<\sigma} \mu_j$ for some $f \in \Pi\bar{\lambda}$, we have $g \leq G_0(f) \mod E$.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



[Why? Note that $\min\{\mu_j : j < \sigma\} \geq \sigma^+ \geq \theta^+$ and $J_{\leq \theta}[\bar\lambda] \subseteq J_{\leq \sigma}[\bar\lambda]$. By 1.9 we know $(\Pi\bar\lambda, <_{J_{\leq\sigma}[\bar\lambda]})$ is $\sigma^+$-directed, hence for some $f \in \Pi\bar\lambda$:

$(*)_1$ for $j < \sigma$ we have $f^j_{g(j)} < f \mod J_{\leq\sigma}[\bar\lambda]$.

We here assumed $\sigma < \mu_j$, hence $J_{\leq\sigma}[\bar\lambda] \subseteq J_{<\mu_j}[\bar\lambda]$ (by 1.8(2)) but $J_{<\mu_j}[\bar\lambda]$ is disjoint to $D_j$ by the definition of $J_{<\mu_j}[\bar\lambda]$ (by 1.13(2) + 1.5(13)(c)) so together with $(*)_1$:

$(*)_2$ for $j < \sigma$, $f^j_{g(j)} < f \mod D_j$.

So for every $j < \sigma$ we have $g(j) < G_0(f)(j)$ hence clearly $g \leq G_0(f)$.]

$\otimes_3$ for $f \in \Pi\bar\lambda$ we have $G_0(f) \leq G_1(f)$.
[Why? Read the definitions].

$\otimes_4$ if $f_1, f_2 \in \Pi\bar\lambda$ and $G_1(f_1) <_E G_0(f_2)$ then $f_1 <_{D^*} f_2$.
[Why? As $G_1(f_1) <_E G_0(f_2)$ there is $B \in E$ such that: $j \in B \Rightarrow G_1(f_1)(j) < G_0(f_2)(j)$. For each $j \in \beta$ we have $f_1 \leq_{D_j} f^j_{G_1(f_1)(j)}$ by the definition of $G_1(f_1)$) and $f^j_{G_1(f_1)(j)} <_{D_j} f_2$ (as $G_1(f_1)(j) < G_0(f_2)(j)$ and the definition of $G_0(f_2)(j)$) so together $f_1 <_{D_j} f_2$. So $A(f_1, f_2) = \{i < \kappa : f_1(i) < f_2(i)\}$ satisfies: $A(f_1, f_2) \in D_j$ for every $j \in B$, hence $A(f_1, f_2) \in D^*$ (by the definition of $D^*$) hence $f_1 <_{D^*} f_2$ as required.]

Now first assume $\lambda = \operatorname{tcf}(\Pi\bar\lambda, <_{D^*})$ is well defined, so there is a sequence $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ of members of $\Pi\bar\lambda$, $<_{D^*}$-increasing and cofinal. So $\langle G_0(f_\alpha) : \alpha < \lambda\rangle$ is $\leq_E$-increasing in $\prod\limits_{j<\sigma} \mu_j$ (by $\otimes_1$), for every $g \in \prod\limits_{j<\sigma} \mu_j$ for some $f \in \Pi\bar\lambda$ we have $g \leq_E G_0(f)$ (why? by $\otimes_2$), but by the choice of $\bar f$ for some $\beta < \lambda$ we have $f <_{D^*} f_\beta$ hence by $\otimes_1$ we have $g \leq_E G_0(f) \leq_E G_0(f_\beta)$, so $\langle G_0(f_\alpha) : \alpha < \lambda\rangle$ is cofinal in $(\prod\limits_{j<\sigma} \mu_j, <_E)$. Also for every $\alpha < \lambda$, applying the previous sentence to $G_1(f_\alpha) + 1$ $(\in \prod\limits_{j<\sigma} \mu_j)$ we can find $\beta < \lambda$ such that $G_1(f_\alpha) + 1 \leq_E G_0(f_\beta)$, so $G_1(f_\alpha) <_E G_0(f_\beta)$, so for some club $C$ of $\lambda$, $\langle G_0(f_\alpha) : \alpha \in C\rangle$ is $<_E$-increasing cofinal in $(\prod\limits_{j<\sigma} \mu_j, <_E)$. So if $\lambda$ is well defined then $\mu = \operatorname{tcf}(\prod\limits_{j<\sigma} \mu_j, <_E)$ is well defined and equall to $\lambda$.

Lastly, assume that $\mu$ is well defined i.e. $\prod\limits_{j<\sigma} \mu_j/E$ has true cofinality $\mu$, let $\bar g = \langle g_\alpha : \alpha < \mu\rangle$ exemplifies it. Choose by induction on $\alpha < \mu$, a function $f_\alpha$ and ordinals $\beta_\alpha$, $\gamma_\alpha$ such that

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



(i) $f_\alpha \in \Pi\bar{\lambda}$

(ii) $g_{\beta_\alpha} <_E G_0(f_\alpha) \le_E G_1(f_\alpha) <_E g_{\gamma_\alpha}$ (so $\beta_\alpha < \gamma_\alpha$)

(iii) $\alpha_1 < \alpha_2 < \mu \Rightarrow \gamma_{\alpha_1} < \beta_{\alpha_2}$ (so $\beta_\alpha \ge \alpha$).

In stage $\alpha$, first choose $\beta_\alpha = \bigcup\{\gamma_{\alpha_1} + 1 : \alpha_1 < \alpha\}$, then choose $f_\alpha \in \Pi\bar{\lambda}$ such that $g_{\beta_\alpha} + 1 <_E G_0(f_\alpha)$ (possible by $\otimes_2$) then choose $\gamma_\alpha$ such that $G_1(f_\alpha) <_E g_{\gamma_\alpha}$. Now $G_0(f_\alpha) \le_E G_1(f_\alpha)$ by $\otimes_3$. By $\otimes_4$ we have $\alpha_1 < \alpha_2 \Rightarrow f_{\alpha_1} <_{D^*} f_{\alpha_2}$. Also if $f \in \prod \bar{\lambda}$ then $G_1(f) \in \prod\limits_{j<\sigma} \mu_j$ hence by the choice of $\bar{g}$, for some $\alpha < \mu$ we have $G_1(f) <_E g_\alpha$ but $\alpha \le \beta_\alpha$ so $G_1(f) <_E g_\alpha \le_E G_0(f_\alpha)$ hence by $\otimes_4$, $f <_{D^*} f_\alpha$. Altogether, $\langle f_\alpha : \alpha < \mu\rangle$ exemplifies that $(\Pi\bar{\lambda}, <_{D^*})$ has true cofinality $\mu$, so $\lambda$ is well defined and equal to $\mu$. $\square_{1.15}$

*1.16 Conclusion.* If $(*)$ of 1.9 holds, and $\sigma$, $\bar{\mu} = \langle \mu_j : j < \sigma\rangle, \langle D_j : j < \sigma\rangle$ are as in 1.15 and $\sigma + \theta < \min(\bar{\mu})$, and $J$ is an ideal on $\sigma$ and $I$ an ideal on $\kappa$ such that $I^* \subseteq I \subseteq \{A \subseteq \kappa$: for some $B \in J$ for every $j \in \sigma \setminus B$ we have $A \notin D_j\}$, $A \in I \Rightarrow \bigwedge\limits_{j<\sigma} (\kappa \setminus A) \in D_j$ (e.g. $I = I^*$) then $\mathrm{pcf}_J(\{\mu_j : j < \sigma\}) \subseteq \mathrm{pcf}_I(\bar{\lambda})$.

*Proof.* Assume $\lambda \in \mathrm{pcf}_J(\{\mu_j : j < \sigma\})$. Let $E$ be an ultrafilter on $\sigma$ disjoint to $J$ such that $\lambda = \mathrm{tcf}(\prod\limits_{j<\sigma} \mu_j/E)$ then we can define an ultrafilter $D^*$ on $\kappa$ as in 1.15, so clearly $D^*$ is disjoint to $I$ and $\lambda = \mathrm{tcf}(\Pi\bar{\lambda}/I)$ hence $\lambda \in \mathrm{pcf}_I(\bar{\lambda})$ as required. $\square_{1.16}$

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



## §2 Normality of $\lambda \in \mathrm{pcf}(\bar\lambda)$ for $\bar\lambda$

Having found those ideals $J_{<\lambda}[\bar\lambda]$, we would like to know more. As $J_{<\lambda}[\bar\lambda]$ is increasing continuous in $\lambda$, the question is how $J_{<}[\bar\lambda]$, $J_{<\lambda^+}[\bar\lambda]$ are related.

The simplest relation is $J_{<\lambda^+}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + B$ for some $B \subseteq \kappa$, and then we call $\lambda$ normal (for $\bar\lambda$) and denote $B = B_\lambda[\bar\lambda]$ though it is unique only modulo $J_{<\lambda}[\bar\lambda]$. We give a sufficient condition for exsitence of such $B$, using this in 2.11; giving the necessary definition in 2.3 and needed information in 2.5, 2.7, 2.9; lastly 2.10 is the essential uniqueness of cofinal sequences in appropriate $\Pi\bar\lambda/I$.

**2.1 Definition.** 1) We say $\lambda \in \mathrm{pcf}(\bar\lambda)$ is normal (for $\bar\lambda$) if for some $B \subseteq \kappa$, $J_{\le\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + B$.
2) We say $\lambda \in \mathrm{pcf}(\bar\lambda)$ is semi-normal (for $\bar\lambda$) if there are $B_\alpha$ for $\alpha < \lambda$ such that:

$(i)$ $\alpha < \beta \Rightarrow B_\alpha \subseteq B_\beta \mod J_{<\lambda}[\bar\lambda]$ and
$(ii)$ $J_{\le\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + \{B_\alpha : \alpha < \lambda\}$.

3) We say $\bar\lambda$ is normal if every $\lambda \in \mathrm{pcf}(\bar\lambda)$ is normal for $\bar\lambda$. Similarly for semi normal.
4) In (1), (2), (3) instead $\bar\lambda$ we can say $(\bar\lambda, I)$ or $\Pi\bar\lambda/I$ or $(\Pi\bar\lambda, <_I)$ if we replace $I^*$ by $I$ (an ideal on $\mathrm{Dom}(\bar\lambda)$).

*2.2 Fact.* Suppose $(*)$ of 1.9 and $\lambda \in \mathrm{pcf}(\bar\lambda)$.
Now:
1) $\lambda$ is semi-normal for $\bar\lambda$ <u>iff</u> for some $F = \{f_\alpha : \alpha < \lambda\} \subseteq \Pi\bar\lambda$ we have: $[\alpha < \beta \Rightarrow f_\alpha < f_\beta \mod J_{<\lambda}[\bar\lambda]]$ and for every ultrafilter $D$ over $\kappa$ disjoint to $J_{<\lambda}[\bar\lambda]$, $F$ is unbounded in $(\Pi\bar\lambda, <_D)$ whenever $\mathrm{tcf}(\Pi\bar\lambda, <_D) = \lambda$.
2) In 2.1(2), without loss of generality, we may assume that
either: $B_\alpha = B_0 \mod J_{<\lambda}[\bar\lambda]$ (so $\lambda$ is normal)
or: $B_\alpha \neq B_\beta \mod J_{\le\lambda}[\bar\lambda]$ for $\alpha < \beta < \lambda$.
3) Assume $\lambda$ is semi normal for $\bar\lambda$. Then $\lambda$ is normal for $\bar\lambda$ <u>iff</u> for some $F$ as in part (1) (of 2.2) $F$ has a $<_{J_{<\lambda}[\bar\lambda]}$-exact upper bound $g \in \prod_{i<\kappa}(\lambda_i + 1)$ and then $B =: \{i < \kappa : g(i) = \lambda_i\}$ generates $J_{\le\lambda}[\bar\lambda]$ over $J_{<\lambda}[\bar\lambda]$.
4) If $\lambda$ is semi normal for $\bar\lambda$ <u>then</u> for some $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$, $\bar B = \langle B_\alpha : \alpha < \lambda\rangle$ we have: $\bar B$ is increasing modulo $J_{<\lambda}[\bar\lambda]$, $J_{\le\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + \{B_\alpha : \alpha < \lambda\}$, and $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_{J_{<\lambda}[\bar\lambda]}$-increasing and $\bar f$, $\bar B$ as in 1.12.

*Proof.* 1) For the direction $\Rightarrow$, given $\langle B_\alpha : \alpha < \lambda\rangle$ as in Definition 2.1(2), for each $\alpha < \lambda$, by 1.13(1) we have $(\Pi\bar\lambda \restriction B_\alpha, <_{J_{<\lambda}[\bar\lambda]})$ has true cofinality $\lambda$, and let it be

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



exemplified by $\langle f^\alpha_\beta : \beta < \lambda\rangle$. By 1.9 we can choose by induction on $\gamma < \lambda$ a function $f_\gamma \in \Pi\bar\lambda$ such that: $\beta, \gamma \le \alpha \Rightarrow f^\alpha_\beta \le_{J_{<\lambda}[\bar\lambda]} f_\gamma$ and $\beta < \gamma \Rightarrow f_\beta <_{J_{<\lambda}[\bar\lambda]} f_\gamma$.

Now $F =: \{f_\alpha : \alpha < \lambda\}$ is as required. [Why? First, obviously $\alpha < \beta \Rightarrow f_\alpha < f_\beta$ mod $J_{<\lambda}[\bar\lambda]$. Second, if $D$ is an ultrafilter on $\kappa$ disjoint to $I^*$ and $(\prod\bar\lambda, <_D)$ has true cofinality $\lambda$, then by 1.11 for some $B \in J_{\le\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda]$ we have $B \in D$, so by the choice of $\langle B_\alpha : \alpha < \lambda\rangle$ for some $\alpha < \lambda$, $B \subseteq B_\alpha \mod J_{<\lambda}[\bar\lambda]$ hence $B_\alpha \in D$. As $f^\alpha_\beta \le_{J_{<\lambda}[\bar\lambda]} f_\beta$ for $\beta \in [\alpha, \lambda)$ clearly $F$ is cofinal in $(\Pi\bar\lambda, <_D)$.]

The other direction, $\Leftarrow$ follows from 1.12 applied to $F = \{f_\alpha : \alpha < \lambda\}$. [Why? By 1.12 there is a sequence $\langle B_\alpha : \alpha < \lambda\rangle$ as there, in particular $B_\alpha \in J_{\le\lambda}[\bar\lambda]$ increasing modulo $J_{<\lambda}[\bar\lambda]$ so $J =: J_{<\lambda}[\bar\lambda] + \{B_\alpha : \alpha < \lambda\} \subseteq J_{\le\lambda}[\bar\lambda]$.

If equality does not hold then for some ultrafilter $D$ over $\kappa$, $D \cap J = \emptyset$ but $D \cap J_{\le\lambda}[\bar\lambda] \ne \emptyset$ so by clause (D) of 1.12, $F$ is bounded in $\Pi\lambda/D$ whereas by 1.13(1),(2), $\mathrm{tcf}(\Pi\bar\lambda, <_D) = \lambda$ contradicting the assumption on $F$.]

2) Because we can replace $\langle B_\alpha : \alpha < \lambda\rangle$ by $\langle B_{\alpha_i} : i < \lambda\rangle$ whenever $\langle \alpha_i : i < \lambda\rangle$ is non decreasing, non eventually constant.

3) If $\lambda$ is normal for $\bar\lambda$, let $B \subseteq \kappa$ be such that $J_{\le\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + B$. By 1.13(1) we know that $(\prod(\bar\lambda \restriction B), <_{J_{<\lambda}[\bar\lambda]})$ has true cofinality $\lambda$, so let it be exemplified by $\langle f^0_\alpha : \alpha < \lambda\rangle$. Let $f_\alpha = f^0_\alpha \cup 0_{(\kappa\setminus B)}$ for $\alpha < \lambda$. Now $\langle f_\alpha : \alpha < \lambda\rangle$ is as required by 1.5(11).

Now suppose $\langle f_\alpha : \alpha < \lambda\rangle$ is as in part (1) of 2.2 and $g$ is a $<_{J_{<\lambda}[\bar\lambda]}$-eub of $F$, $g \in \prod\limits_{i<\kappa}(\lambda_i + 1)$ and $B = \{i : g(i) = \lambda_i\}$. Let $D$ be an ultrafilter on $\kappa$ disjoint to $J_{<\lambda}[\bar\lambda]$. If $B \in D$ then for every $f \in \prod\bar\lambda$, let $f' = (f \restriction B) \cup 0_{(\kappa\setminus B)}$, now necessarily $f' < \max\{g, 1\}$ (as $[i \in B \Rightarrow f'(i) < \lambda_i = g(i)]$ and $[i \in \kappa \setminus B \Rightarrow f'(i) = 0 \le g(i)]$), hence (see Definition 1.4(4)) for some $\alpha < \lambda$ we have $f' < \max\{f_\alpha, 1\} \mod J_{<\lambda}[\bar\lambda]$ hence for some $\alpha < \lambda$, $f' \le f_\alpha \mod J_{<\lambda}[\bar\lambda]$ hence $f \le f' \le f_\alpha \mod D$; also $\alpha < \beta \Rightarrow f_\alpha < f_\beta \mod D$, hence together $\langle f_\alpha : \alpha < \lambda\rangle$ exemplifies $\mathrm{tcf}(\Pi\bar\lambda, <_D) = \lambda$. If $B \notin D$ then $\kappa \setminus B \in D$ so $g' = g \restriction (\kappa \setminus B) \cup 0_B = g \mod D$ and $\alpha < \lambda \Rightarrow f_\alpha <_D f_{\alpha+1} \le_D g =_D g'$, so $g' \in \Pi\bar\lambda$ exemplifies $F$ is bounded in $(\Pi\bar\lambda, <_D)$ so as $F$ is as in 2.2(1), $\mathrm{tcf}(\Pi\bar\lambda, <_D) = \lambda$ is impossible. As $D$ is disjoint to $J_{<\lambda}[\bar\lambda]$, necessarily $\mathrm{tcf}(\Pi\bar\lambda, <_D) > \lambda$. The last two arguments together give, by 1.13(2) that $J_{\le\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + B$ as required in the definition of normality.

4) Should be clear. $\square_{2.2}$

We shall give some sufficient conditions for normality.

*Remark.* In the following definitions we slightly deviate from [Sh:g, Ch.I] = [Sh 345a]. The ones here are perheps somewhat artificial but enable us to deal also with case $(\beta)$ of 1.9$(*)$. I.e. in Definition 2.3 below we concentrate on the first $\theta$ elements of an $a_\alpha$ and for "obey" we also have $\bar A^* = \langle A_\alpha : \alpha < \theta\rangle$ and we want to cover also the case $\theta$ is singular.

**2.3 Definition.** Let there be given regular $\lambda$, $\theta < \mu < \lambda$,$\mu$ possibly an ordinal, $S \subseteq \lambda$, $\sup(S) = \lambda$ and for simplicity $S$ is a set of limit ordinals or at least have no two successive members.
1) We call $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ a special continuity condition for $(S, \mu, \theta)$ (or is an $(S, \mu, \theta)$-continuity condition) <u>if</u>: $S$ is an unbounded subset of $\lambda$, $a_\alpha \subseteq \alpha$, $\mathrm{otp}(a_\alpha) < \mu$, and $[\beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta]$ and, for every club $E$ of $\lambda$, for some[5] $\delta \in S$ we have $\theta = \mathrm{otp}\{\alpha \in a_\delta : \mathrm{otp}(a_\alpha) < \theta$ and for no $\beta \in a_\delta \cap \alpha$ is $(\beta, \alpha) \cap E = \emptyset\}$. We say $\bar{a}$ is continuous in $S^*$ if $\alpha \in S^* \Rightarrow \alpha = \sup(a_\alpha)$.
2) Assume $f_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \lambda$ and $\bar{A}^* = \langle A^*_\alpha : \alpha < \theta \rangle$ is a decreasing sequence of subsetes of $\kappa$ such that $\kappa \backslash A^*_\alpha \in I^*$. We say $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ obeys $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ for $\bar{A}^*$ if:

- $(i)$ for $\beta \in a_\alpha$, if $\varepsilon =: \mathrm{otp}(a_\alpha) < \theta$ then we have $f_\beta \restriction A^*_\varepsilon \leq f_\alpha \restriction A^*_\varepsilon$ (note: $\bar{A}^*$ determine $\theta$).

2A) Let $\kappa$, $\bar{\lambda}$, $I^*$ be as usual. We say $\bar{f}$ obeys $\bar{a}$ for $\bar{A}^*$ continuously on $S^*$ if: $\bar{a}$ is continuous in $S^*$ and $\bar{f}$ obeys $\bar{a}$ for $\bar{A}^*$ and in addition $S^* \subseteq S$ and for $\alpha \in S^*$ (a limit ordinal) we have $f_\alpha = f_{a_\alpha}$ from (2B) below, i.e., for every $i < \kappa$ we have $f_\alpha(i) = \sup\{f_\beta(i) : \beta \in a_\alpha\}$ when $|a_\alpha| < \lambda_i$.
2B) For given $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$, $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ where $f_\alpha \in \Pi\bar{\lambda}$ and $a \subseteq \lambda$, and $\theta$ let $f_a \in \Pi\bar{\lambda}$ be defined by: $f_a(i)$ is 0 if $|a| \geq \lambda_i$ and $\cup\{f_\alpha(i) : \alpha \in a\}$ if $|a| < \lambda_i$.
3) Let $(S, \theta)$ stands for $(S, \theta{+}1, \theta)$; $(\lambda, \mu, \theta)$ stands for "$(S, \mu, \theta)$ for some unbounded subset $S$ of $\lambda$" and $(\lambda, \theta)$ stands for $(\lambda, \theta + 1, \theta)$.
If each $A^*_\alpha$ is $\kappa$ then we may omit "for $\bar{A}^*$" (but $\theta$ should be fixed or said).
4) We add to "continuity condition" (in part (1)) the adjective "weak" ["$\theta$-weak"] if "$\beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta$" is replaced by "$\alpha \in S$ & $\beta \in a_\alpha \Rightarrow (\exists \gamma < \alpha)[a_\alpha \cap \beta \subseteq a_\gamma$ & $\gamma < \min(a_\alpha \setminus (\beta + 1))$ & $[|a_\alpha \cap \beta| < \theta \Rightarrow |a_\gamma \cap \beta| < \theta]]$" [and we demand that $\gamma$ exists only if $\mathrm{otp}(a_\alpha \cap \beta) < \theta$]. (Of course a continuity condition is a weak continuity condition which is a $\theta$-weak continuity condition).

*2.4 Remark.* There are some obvious monotonicity implications, we state below only 2.5(3).

*2.5 Fact.* 1) Let $\theta_r = \begin{cases} \theta & \mathrm{cf}(\theta) = \theta \\ \theta^+ & \mathrm{cf}(\theta) < \theta \end{cases}$ and assume $\lambda = \mathrm{cf}(\lambda) > \theta_r^+$. <u>Then</u> for some stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta_r\}$, there is a continuity condition $\bar{a}$ for $(S, \theta_r)$; moreover, it is continuous in $S$ and $\delta \in S \Rightarrow \mathrm{otp}(a_\delta) = \theta_r$; so for every

[5]Note: if $\mathrm{otp}(a_\delta) = \theta$ and $\delta = \sup(a_\delta)$ (holds if $\delta \in S$, $\mu = \theta + 1$ and $\bar{a}$ continuous in $S$ (see below)) <u>then</u> $\delta \in E$.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



club $E$ of $\lambda$ for some $\delta \in S$, $\forall \alpha, \beta[\alpha < \beta \ \& \ \alpha \in a_\delta \ \& \ \beta \in a_\delta \to (\alpha, \beta) \cap E \neq \emptyset\}]$.
2) Assume $\lambda = \theta^{++}$, then for some stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\theta)\}$ there is a continuity condition for $(S, \theta + 1, \theta)$.
3) If $\bar{a}$ is a $(\lambda, \mu, \theta_1)$-continuity condition and $\theta_1 \geq \theta$ <u>then</u> there is a $(\lambda, \theta + 1, \theta)$-continuity condition.

*Proof.* 1) By [Sh 420, §1].
2) By [Sh 351, 4.4](2) and[6].
3) Check. $\square_{2.5}$

*2.6 Remark.* Of course also if $\lambda = \theta^+$ the conclusion of 2.5(2) may well hold. We suspect but do not know that the negation is consistent with ZFC.

*2.7 Fact.* Suppose $(*)$ of 1.9, $f_\alpha \in \Pi\bar{\lambda}$ for $\alpha < \lambda$, $\lambda = \mathrm{cf}(\lambda) \leq \theta$ (of course $\kappa = \mathrm{dom}(\bar{\lambda})$) and $\bar{A}^* = \bar{A}^*[\bar{\lambda}]$ is as in the proof of 1.9 (i.e., $A^*_\alpha = \{i < \kappa : \lambda_i > \alpha\}$). <u>Then</u>
1) Assume $\bar{a}$ is a $\theta$-weak continuity condition for $(S, \theta)$, $\lambda = \sup(S)$, <u>then</u> we can find $\bar{f}' = \langle f'_\alpha : \alpha < \lambda \rangle$ such that:

- (i) $f'_\alpha \in \Pi\bar{\lambda}$,
- (ii) for $\alpha < \lambda$ we have $f_\alpha \leq f'_\alpha$
- (iii) for $\alpha < \beta < \lambda$ we have $f'_\alpha <_{J_{<\lambda}[\bar{\lambda}]} f'_\beta$
- (iv) $\bar{f}'$ obeys $\bar{a}$ for $\bar{A}^*$.

2) If in addition $\min(\bar{\lambda}) > \mu$, $S^* \subseteq S$ are stationary subsets of $\lambda$ and $\bar{a}$ is a continuity condition for $(S, \mu, \theta)$ <u>then</u> we can find $\bar{f}' = \langle f'_\alpha : \alpha < \lambda \rangle$ such that:

- (i) $f'_\alpha \in \Pi\bar{\lambda}$
- (ii) for $\alpha \in \lambda \backslash S^*$ we have $f_\alpha \leq f'_\alpha$ and $\alpha = \beta + 1 \in \lambda \backslash S^* \ \& \ \beta \in S^* \Rightarrow f_\beta \leq f'_\alpha$
- (iii) for $\alpha < \beta < \lambda$ we have $f'_\alpha <_{J_{<\lambda}[\bar{\lambda}]} f'_\beta$
- (iv) $\bar{f}'$ obeys $\bar{a}$ for $\bar{A}^*$ continuously on $S^*$.

3) Suppose $\langle f'_\alpha : \alpha < \lambda \rangle$ obeys $\bar{a}$ continuously on $S^*$ and satisfies 2.7(2)(ii) (and 2.7(2)'s assumption holds). If $g_\alpha \in \Pi\bar{\lambda}$ and $\langle g_\alpha : \alpha < \lambda \rangle$ obeys $\bar{a}$ continuously on $S^*$ and $[\alpha \in \lambda \setminus S^* \Rightarrow g_\alpha \leq f_\alpha]$ <u>then</u> $\bigwedge\limits_\alpha g_\alpha \leq f'_\alpha$.

[6] the definition of $B^\alpha_i$ in the proof of [Sh:g, III,2.14](2) should be changed as in [Sh 351, 4.4](2), [Sh:g, III,2.14](2),clause(c),p.135-7

4) If $\zeta < \theta$, for $\varepsilon < \zeta$ we have $\bar{f}^\varepsilon = \langle f^\varepsilon_\alpha : \alpha < \lambda \rangle$, where $f^\varepsilon_\alpha \in \Pi\bar{\lambda}$, then in 2.7(1) (and 2.7(2)) we can find $f'$ as there for all $\bar{f}^\varepsilon$ simultaneously. Only in clause (ii) we replace $f_\alpha \le f'_\alpha$ by $f_\alpha \restriction A^*_\zeta \le f'_\alpha \restriction A^*_\zeta$ (and $f_\beta \le f'_\alpha$ by $f_\beta \restriction A^*_\zeta \le f'_\alpha \restriction A^*_\zeta$.

*Proof.* Easy (using 1.9 of course).

**2.8 Claim.** *In 2.7 we can replace "$(*)$ from 1.9" by "$\Pi\bar{\lambda}/J_{<\lambda}[\bar{\lambda}]$ is $\lambda$-directed".*

**2.9 Claim.** *Assume $(*)$ of 1.9 and let $\bar{A}^*$ be as there.*
*1) In 1.12, if $\langle f_\alpha : \alpha < \lambda \rangle$ obeys some $(S,\theta)$-continuity condition or just a $\theta$-weak one for $\bar{A}^*$ (where $S \subseteq \lambda$ is unbounded) then we can deduce also:*
*(G) the sequence $\langle B_\alpha / J_{<\lambda}[\bar{\lambda}] : \alpha < \lambda \rangle$ is eventually constant.*
*2) If $\theta^+ < \lambda$ then $J_{\le\lambda}[\bar{\lambda}]/J_{<\lambda}[\bar{\lambda}]$ is $\lambda^+$-directed (hence if $\lambda$ is semi normal for $\bar{\lambda}$ then it is normal to $\bar{\lambda}$).*

*Proof.* 1) Assume not, so for some club $E$ of $\lambda$ we have

$$(*) \quad \alpha < \delta < \lambda \ \& \ \delta \in E \Rightarrow B_\alpha \ne B_\delta \mod J_{<\lambda}[\bar{\lambda}].$$

As $\bar{a}$ is a $\theta$-weak $(S,\theta)$-continuity condition, there is $\delta \in S$ such that $b =: \{\alpha \in a_\delta : \mathrm{otp}(a_\delta \cap \alpha) < \theta$ and for no $\beta \in a_\delta \cap \alpha$ is $(\beta,\alpha) \cap E = \emptyset\}$ has order type $\theta$. Let $\{\alpha_\varepsilon : \varepsilon < \theta\}$ list $b$ (increasing with $\varepsilon$). So for every $\varepsilon < \theta$ there is $\gamma_\varepsilon \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap E$, and let $\beta_\varepsilon < \alpha_{\varepsilon+1}$ be such that $a_\delta \cap \alpha_\varepsilon \subseteq a_{\beta_\varepsilon}$ and $\mathrm{otp}(a_{\beta_\varepsilon} \cap \alpha_\varepsilon) < \theta$; by shrinking and renaming without loss of generality $\beta_\varepsilon < \gamma_\varepsilon$ and $\alpha_\varepsilon \in a_{\beta_\varepsilon}$. Let $\xi(\varepsilon) =: \mathrm{otp}(a_{\beta_\varepsilon} \cap \alpha_\varepsilon)$.

Lastly, let $B^0_\varepsilon =: \{i < \kappa : f_{\alpha_\varepsilon}(i) < f_{\beta_\varepsilon}(i) < f_{\gamma_\varepsilon}(i) < f_{\alpha_{\varepsilon+1}}(i)\}$, clearly it is $= \kappa \mod I^*$ and let (remember $(*)$ above) $B^*_\varepsilon =: A^*_{\xi(\varepsilon)+1} \cap (B_{\gamma_\varepsilon} \setminus B_{\beta_\varepsilon}) \cap B^0_\varepsilon$, now $B_{\alpha_\varepsilon} \subseteq B_{\beta_\varepsilon} \subseteq B_{\gamma_\varepsilon} \mod J_{<\lambda}[\bar{\lambda}]$ by clause (B) of 1.12, and $B_{\gamma_\varepsilon} \ne B_{\beta_\varepsilon}$ by $(*)$ above hence $B_{\gamma_\varepsilon} \setminus B_{\beta_\varepsilon} \ne \emptyset \mod J_{<\lambda}[\bar{\lambda}]$. Now $B^0_\varepsilon$, $A^*_{\xi(\varepsilon)+1} = \kappa \mod I^*$ by the previous sentence and by 1.9$(*)$ which we are assuming respectively and $I^* \subseteq J_{<\lambda}[\bar{\lambda}]$ by the later's definition; so we have gotten $B^*_\varepsilon \ne \emptyset \mod J_{<\lambda}[\bar{\lambda}]$. But for $\varepsilon < \zeta < \theta$ we have $B^*_\varepsilon \cap B^*_\zeta = \emptyset$, for suppose $i \in B^*_\varepsilon \cap B^*_\zeta$, so $i \in A^*_{\xi(\varepsilon)+1}$ and also $f_{\gamma_\varepsilon}(i) < f_{\alpha_{\varepsilon+1}}(i) \le f_{\beta_\zeta}(i)$ (as $i \in B^0_\varepsilon$ and as $\alpha_{\varepsilon+1} \in a_{\beta_\zeta}$ & $i \in A^*_{\xi(\zeta)+1}$ respectively); now $i \in B^*_\varepsilon$ hence $i \in B_{\gamma_\varepsilon}$ i.e., (where $g$ is from 1.12 clause (D)$^+$) $f_{\gamma_\varepsilon}(i) > g(i)$ hence (by the above) $f_{\beta_\zeta}(i) > g(i)$ hence $i \in B_{\beta_\zeta}$ hence $i \notin B^*_\zeta$, contradiction. So $\langle B^*_\varepsilon : \varepsilon < \theta \rangle$ is a sequence of $\theta$ pairwise disjoint members of $(J_{<\lambda}[\bar{\lambda}])^+$, contradiction.
2) The proof is similar to the proof of 1.13(4), using 2.9(1) instead 1.12 (and $\bar{a}$ from 2.5(1) if $\lambda > \theta^+_r$ or 2.5(2) if $\lambda = \theta^{++}$). $\square_{2.9}$

We note also (but shall not use):

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



**2.10 Claim.** *Suppose* $(*)$ *of 1.9 and*

(a) $f_\alpha \in \Pi\bar\lambda$ *for* $\alpha < \lambda$, $\lambda \in \mathrm{pcf}(\bar\lambda)$ *and* $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ *is* $<_{J_{<\lambda}[\bar\lambda]}$*-increasing*

(b) $\bar f$ *obeys* $\bar a$ *continuously on* $S^*$, *where* $\bar a$ *is a continuity condition for* $(S, \theta)$ *and* $\lambda = \sup(S)$ *(hence* $\lambda > \theta$ *by the last phrase of 2.3(1))*

(c) $J$ *is an ideal on* $\kappa$ *extending* $J_{<\lambda}[\bar\lambda]$, *and* $\langle f_\alpha/J : \alpha < \lambda\rangle$ *is cofinal in* $(\Pi\bar\lambda, <_J)$ *(e.g.,* $J = J_{<\lambda}[\bar\lambda] + (\kappa \setminus B)$, $B \in J_{\leq\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda]$*).*

(d) $\langle f'_\alpha : \alpha < \lambda\rangle$ *satisfies (a), (b) above*

(e) $f_\alpha \leq f'_\alpha$ *for* $\alpha \in \lambda \setminus S^*$ *(alternatively:* $\langle f'_\alpha : \alpha < \lambda\rangle$ *satisfies (c))*

(f) *if* $\delta \in S^*$ *<u>then</u>* $J$ *is* $\mathrm{cf}(\delta)$*-indecomposable (i.e., if* $\langle A_\varepsilon : \varepsilon < \mathrm{cf}(\delta)\rangle$ *is a* $\subseteq$*-increasing sequence of members, of* $J$ *then* $\bigcup\limits_{\varepsilon < \mathrm{cf}(\delta)} A_\varepsilon \in J$*).*

*<u>Then</u>:*

(A) *the set*

$$\{\delta < \lambda : \text{ if } \delta \in S^* \text{ and } \mathrm{otp}(a_\delta) = \theta \text{ then } f'_\delta = f_\delta \mod J\}$$

*contains a club of* $\lambda$

(B) *the set*

$$\{\delta < \lambda : \text{if } \alpha \in S \text{ and } \delta = \sup(\delta \cap a_\alpha) \text{ and } \mathrm{otp}(\alpha \cap a_\delta) = \theta \\ \text{then } f'_{\alpha \cap a_\delta} = f_{\alpha \cap a_\delta} \mod J\}$$

*contains a club of* $\lambda$.

*Proof.* We concentrate on proving (A).

Suppose $\delta \in S^*$, and $f_\delta \neq f'_\delta \mod J$. Let

$$A_{1,\delta} = \{i < \kappa : f_\delta(i) < f'_\delta(i)\}$$

$$A_{2,\delta} = \{i < \kappa : f_\delta(i) > f'_\delta(i)\}.$$

So $A_{1,\delta} \cup A_{2,\delta} \in J^+$, suppose first $A_{1,\delta} \in J^+$. By Definition 2.3(2A), for every $i \in A_{1,\delta}$ for every large enough $\alpha \in a_\delta$, $f_\delta(i) < f'_\alpha(i)$, say for $\alpha \in a_\delta \setminus \beta_i$. As $J$ is $\mathrm{cf}(\delta)$-indecomposable for some $\beta < \alpha$ we have $\{i < \kappa : \beta_i < \beta\} \in J^+$ so $f_\delta \restriction A_{1,\delta} < f'_\beta \restriction A_{1,\delta}$ (and $\beta < \delta$). Now by clause (c), $E =: \{\delta < \lambda$: for every $\beta < \delta$ we have $f'_\beta < f_\delta \mod J\}$ is a club of $\lambda$, and so we have proved

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



$$\delta \in E \Rightarrow A_{1,\delta} \in J.$$

If $\bigwedge_{\alpha<\lambda} f_\alpha \le f'_\alpha$ (first possibility in clause (e)) also $A_{2,\delta} \in J$ hence for no $\delta \in S^* \cap E$ do we have $f_\delta \ne f'_\delta \mod J$. If the second possibility of clause (e) holds, we can interchange $\bar f$, $\bar f'$ hence $[\delta \in E \Rightarrow A_{2,\delta} \in J]$ and we are done. $\square_{2.10}$

We now return to investigating the $J_{<\lambda}[\bar\lambda]$, first without using continuity conditions.

**2.11 Lemma.** *Suppose* $(*)$ *of 1.9 and* $\lambda = \mathrm{cf}(\lambda) \in \mathrm{pcf}(\bar\lambda)$. *Then* $\lambda$ *is semi normal for* $\bar\lambda$.

*Proof.* We assume $\lambda$ is not semi normal for $\bar\lambda$ and eventually get a contradiction. Note that by our assumption $(\Pi\bar\lambda, <_I)$ is $\theta^+$-directed hence $\lambda \ge \min \mathrm{pcf}_I(\bar\lambda) \ge \theta^+$ (by 1.5(4)(v)) hence let us define by induction on $\xi \le \theta$, $\bar f^\xi = \langle f^\xi_\alpha : \alpha < \lambda\rangle$, $B_\xi$ and $D_\xi$ such that:

- $(I)(i)$ $f^\xi_\alpha \in \Pi\bar\lambda$
- $(ii)$ $\alpha < \beta < \lambda \Rightarrow f^\xi_\alpha \le f^\xi_\beta \mod J_{<\lambda}[\bar\lambda]$
- $(iii)$ $\alpha < \lambda$ & $\xi < \theta \Rightarrow f^\xi_\alpha \le f^\theta_\alpha \mod J_{<\lambda}[\bar\lambda]$
- $(iv)$ for $\zeta < \xi < \theta$ and $\alpha < \lambda : f^\zeta_\alpha \restriction A^*_\xi \le f^\xi_\alpha \restriction A^*_\xi$
- $(II)(i)$ $D_\xi$ is an ultrafilter on $\kappa$ such that: $\mathrm{cf}(\Pi\bar\lambda/D_\xi) = \lambda$
- $(ii)$ $\langle f^\xi_\alpha/D_\xi : \alpha < \lambda\rangle$ is not cofinal in $\Pi\bar\lambda/D_\xi$
- $(iii)$ $\langle f^{\xi+1}_\alpha/D_\xi : \alpha < \lambda\rangle$ is increasing and cofinal in $\Pi\bar\lambda/D_\xi$; moreover
- $(iii)^+$ $B_\xi \in D_\xi$ and $\langle f^{\xi+1}_\alpha : \alpha < \lambda\rangle$ is increasing and cofinal in $\Pi\bar\lambda/(J_{<\lambda}[\bar\lambda] + (\kappa \setminus B_\xi))$
- $(iv)$ $f^{\xi+1}_0/D_\xi$ is above $\{f^\xi_\alpha/D_\xi : \alpha < \lambda\}$.

<u>For $\xi = 0$</u>: No problem. [Use 1.13(1)+(4)].

<u>For $\xi$ limit $< \theta$</u>: Let $g^\xi_\alpha \in \Pi\bar\lambda$ be defined by $g^\xi_\alpha(i) = \sup\{f^\zeta_\alpha(i) : \zeta < \xi\}$ for $i \in A^*_\xi$ and $f^\xi_\alpha(i) = 0$ else, (remember that $\kappa \setminus A^*_\xi \in I^*$). Then choose by induction on $\alpha < \lambda$, $f^\xi_\alpha \in \Pi\bar\lambda$ such that $g^\xi_\alpha \le f^\xi_\alpha$ and $\beta < \alpha \Rightarrow f_\beta < f_\alpha \mod J_{<\lambda}[\bar\lambda]$. This is possible by 1.9 and clearly the requirements (I)(i),(ii),(iv) are satisfied.
Use 2.2(1) to find an appropriate $D_\xi$ (i.e.. satisfying II(i)+(ii)). Now $\langle f^\xi_\alpha : \alpha < \lambda\rangle$ and $D_\xi$ are as required.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



<u>For $\xi = \theta$</u>: Choose $f^\theta_\alpha$ by induction of $\alpha$ satisfying I(i), (ii), (iii) (possible by 1.9).

<u>For $\xi = \zeta + 1$</u>: Use 1.11 to choose $B_\zeta \in D_\zeta \cap J_{\le\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda]$. Let $\langle g^\xi_\alpha : \alpha < \lambda\rangle$ be cofinal in $(\Pi\bar\lambda, <_{D_\xi})$ and even in $(\Pi\bar\lambda, <_{J_<[\bar\lambda]+(\kappa\setminus B_\xi)})$ and without loss of generality $\bigwedge_{\alpha<\lambda} f^\zeta_\alpha/D_\zeta < g^\xi_0/D_\zeta$ and $\bigwedge\limits_{\alpha<\lambda} f^\zeta_\alpha \restriction A^*_\xi \le g^\xi_\alpha \restriction A^*_\xi$. We get $\langle f^\xi_\alpha : \alpha < \lambda\rangle$ increasing and cofinal $\mod (J_{<\lambda}[\bar\lambda] + (\kappa \setminus B_\xi))$ such that $g^\xi_\alpha \le f^\xi_\alpha$ by 1.9 from $\langle g^\xi_\alpha : \alpha < \lambda\rangle$. Then get $D_\xi$ as in the case "$\xi$ limit".

So we have defined the $f^\xi_\alpha$'s and $D_\xi$'s. Now for each $\xi < \theta$ we apply (II) (iii)$^+$ for $\langle f^{\xi+1}_\alpha : \alpha < \lambda\rangle$, $\langle f^\theta_\alpha : \alpha < \lambda\rangle$. We get a club $C_\xi$ of $\lambda$ such that:

$$(*)\qquad \alpha < \beta \in C_\xi \Rightarrow f^\theta_\alpha \restriction B_\xi < f^{\xi+1}_\beta \restriction B_\xi \mod J_{<\lambda}[\bar\lambda]$$

So $C =: \bigcap\limits_{\xi<\theta} C_\xi$ is a club of $\lambda$. By 2.2(1) applied to $\langle f^\theta_\alpha : \alpha < \lambda\rangle$ (and the assumption "$\lambda$ is not semi-normal for $\bar\lambda$") there is $g \in \Pi\bar\lambda$ such that

$$(*)_1\qquad \neg g \le f^\theta_\alpha \mod J_{<\lambda}[\bar\lambda] \text{ for } \alpha < \lambda$$

by 1.9 without loss of generality

$$(*)_2\qquad f^\xi_0 < g \mod J_{<\lambda}[\bar\lambda] \text{ for } \xi < \theta$$

For each $\xi < \theta$, by II (iii), (iii)$^+$ for some $\alpha_\xi < \lambda$ we have

$$(*)_3\qquad \xi < \theta \Rightarrow g \restriction B_\xi < f^{\xi+1}_{\alpha_\xi} \restriction B_\xi \mod J_{<\lambda}[\bar\lambda]$$

Let $\alpha(*) = \sup_{\xi<\theta} \alpha_\xi$, so $\alpha(*) < \lambda$ and so

$$(*)_4\qquad \xi < \theta \Rightarrow g \restriction B_\xi < f^{\xi+1}_{\alpha(*)} \restriction B_\xi \mod J_{<\lambda}[\bar\lambda]$$

For $\zeta < \theta$, let $B^*_\zeta = \{i \in A^*_\zeta : g(i) < f^\zeta_{\alpha(*)}(i)\}$. By $(*)_4$, clearly $\beta^*_{\xi+1} \subseteq \beta_\xi \mod J_{<\lambda}[\bar\lambda]$, but $\beta_\xi \in D_\xi$ by $(II)(iii)^+$ hence $B^*_{\xi+1} \in D_\xi$; by (II)(iv)+$(*)_2$ we know $B^*_\xi \notin D_\xi$, hence $B^*_\xi \ne B^*_{\xi+1} \mod D_\xi$ hence $B^*_\xi \ne B^*_{\xi+1} \mod J_{<\lambda}[\bar\lambda]$.

On the other hand by (I)(iv) for each $\zeta < \theta$ we have $\langle B^*_\xi \cap A^*_\zeta : \xi \le \zeta\rangle$ is $\subseteq$-increasing and (as $A^*_\zeta = \kappa \mod J_{<\lambda}[\bar\lambda]$ for each $\zeta < \theta$) we have $\langle B^*_\xi/I^* : \xi < \theta\rangle$ is $\subseteq$-increasing, and by the previous sentence $B^*_\xi \ne B^*_{\xi+1} \mod J_{<\lambda}[\bar\lambda]$ hence $\langle B^*_\xi/I^* : \xi < \theta\rangle$ is strictly $\subseteq$-increasing. Together clearly $\langle B^*_{\xi+1} \cap A^*_{\xi+1} \setminus B^*_\xi : \xi < \theta\rangle$ is a sequence of $\theta$ pairwise disjoint members of $(J_{<\lambda}[\bar\lambda])^+$, hence of $(I^*)^+$; contradiction to $\theta \ge \mathrm{wsat}(I^*)$. $\square_{2.11}$

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



**2.12 Definition.** 1) We say $\bar{B} = \langle B_\lambda : \lambda \in \mathfrak{c}\rangle$ is a <u>generating sequence</u> for $\bar{\lambda}$ <u>if</u>:

(i) $B_\lambda \subseteq \kappa$ and $\mathfrak{c} \subseteq \mathrm{pcf}(\bar{\lambda})$
(ii) $J_{\leq\lambda}[\bar{\lambda}] = J_{<\lambda}[\bar{\lambda}] + B_\lambda$ for each $\lambda \in \mathfrak{c}$.

2) We call $\bar{B} = \langle B_\lambda : \lambda \in \mathfrak{c}\rangle$ <u>smooth</u> if:

$$i \in B_\lambda \;\;\&\;\; \lambda_i \in \mathfrak{c} \Rightarrow B_{\lambda_i} \subseteq B_\lambda.$$

3) We call $\bar{B} = \langle B_\lambda : \lambda \in \mathrm{Rang}(\bar{\lambda})\rangle$ <u>closed</u> if for each $\lambda$

$$B_\lambda \supseteq \{i < \kappa : \lambda_i \in \mathrm{pcf}(\bar{\lambda} \restriction B_\lambda)\}.$$

4) We call $\bar{B} = \langle B_\lambda : \lambda \in \mathfrak{c}\rangle$ <u>full</u> when $\mathfrak{c} = \mathrm{pcf}(\bar{\lambda})$.

*2.13 Fact.* Assume $(*)$ of 1.9.
1) Suppose $\mathfrak{c} \subseteq \mathrm{pcf}(\bar{\lambda})$, $\bar{B} = \langle B_\lambda : \lambda \in \mathfrak{c}\rangle$ is a generating sequence for $\bar{\lambda}$, and $B \subseteq \kappa$. If $\mathrm{pcf}(\bar{\lambda} \restriction B) \subseteq \mathfrak{c}$ <u>then</u> for some finite $\mathfrak{d} \subseteq \mathfrak{c}$, $B \subseteq \bigcup\limits_{\mu\in\mathfrak{d}} B_\mu \mod I^*$.

*2.14 Remark.* For another proof of 2.13(2) see 2.17(2) + 2.17(4) and for another use of the proof of 2.13(2) see 2.19(1).

*Proof.* 1) If not, then $I = I^* + \{B \cap \bigcup\limits_{\mu\in\mathfrak{d}} B_\mu : \mathfrak{d} \subseteq \mathfrak{c}, \mathfrak{d}$ finite$\}$ is a family of subsets of $\kappa$, closed under union, $B \notin I$, hence there is an ultrafilter $D$ on $\kappa$ disjoint from I to which $B$ belongs. Let $\mu =: \mathrm{cf}(\prod\limits_{i<\kappa} \lambda_i/D)$; necessarily $\mu \in \mathrm{pcf}(\bar{\lambda} \restriction B)$, hence by the last assumption of 2.13(1) we have $\mu \in \mathfrak{c}$. By 1.13(2) we know $B_\mu \in D$ hence $B \cap B_\mu \in D$, contradicting the choice of $D$.

**2.15 Claim.** *1)* $\mathrm{cf}\,(\Pi\bar{\lambda}/I^*) =$ *max pcf*$(\bar{\lambda})$.
*2) The case* $\theta = \aleph_0$ *is trivial (as wsat*$(I^*) \leq \aleph_0$ *implies* $\mathscr{P}(\kappa)/I^*$ *is a Boolean algebra satisfying the* $\aleph_0$*-c.c. (as here we can substract) hence this Boolean algebra is finite hence also pcf*$(\bar{\lambda})$ *is finite) so we assume* $\theta > \aleph_0$*.*
*For* $B \in (I^*)^+$ *let* $\lambda(B) =$ *max* $\mathrm{pcf}_{I^*\restriction B}(\bar{\lambda} \restriction B)$*.*

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



*We prove by induction on $\lambda$ that for every $B \in (I^*)^+$, $\mathrm{cf}(\Pi\bar{\lambda}, <_{I^*+(\kappa\setminus B)}) = \lambda(B)$ when $\lambda(B) \le \lambda$; this will suffice (use $B = \kappa$ and $\lambda = |\prod_{i<\kappa} \lambda_i|^+$). Given $B$ let $\lambda = \lambda(B)$, by renaming without loss of generality $B = \kappa$. By 1.14, pcf$(\Pi\bar{\lambda})$ has a last element, necessarily it is $\lambda =: \lambda(B)$. Let $\langle f_\alpha : \alpha < \lambda\rangle$ be $<_{J_{<\lambda}[\bar{\lambda}]}$ increasing cofinal in $\Pi\bar{\lambda}/J_{<\lambda}[\bar{\lambda}]$, it clearly exemplifies max pcf$(\bar{\lambda}) \le$ cf$(\Pi\bar{\lambda}/I^*)$. Let us prove the other inequality. For $A \in J_{<\lambda}[\bar{\lambda}] \setminus I^*$ choose $F_A \subseteq \Pi\bar{\lambda}$ which is cofinal in $\Pi\bar{\lambda}/(I^* + (\kappa\setminus A))$, $|F_A| = \lambda(A) < \lambda$ (exists by the induction hypothesis). Let $\chi$ be a large enough regular, and we now choose by induction on $\varepsilon < \theta$, $N_\varepsilon$, $g_\varepsilon$ such that:*

(A)(i) $N_\varepsilon \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ *sn*
  (ii) $\|N_\varepsilon\| = \lambda$
  (iii) $\langle N_\varepsilon : \xi \le \varepsilon\rangle \in N_{\varepsilon+1}$
  (iv) $\langle N_\varepsilon : \varepsilon < \theta\rangle$ *is increasing continuous*
  (v) $\{\varepsilon : \varepsilon \le \lambda + 1\} \subseteq N_0$, $\{\bar{\lambda}, I^*\} \in N_0$, $\langle f_\alpha : \alpha < \lambda\rangle \in N_0$ *and the function* $A \mapsto F_A$ *belongs to* $N_0$

(B)(i) $g_\varepsilon \in \Pi\bar{\lambda}$ *and* $g_\varepsilon \in N_{\varepsilon+1}$
  (ii) *for no* $f \in N_\varepsilon \cap \Pi\bar{\lambda}$ *do we have* $g_\varepsilon <_{I^*} f$
  (iii) $\zeta < \varepsilon$ & $\lambda_i > |\varepsilon| \Rightarrow g_\zeta(i) < g_\varepsilon(i)$.

*There is no problem to define $N_\varepsilon$, and if we cannot choose $g_\varepsilon$ this means that $N_\varepsilon \cap \Pi\bar{\lambda}$ exemplifies cf$(\Pi\bar{\lambda}, <) \le \lambda$ as required. So assume $\langle N_\varepsilon, g_\varepsilon : \varepsilon < \theta\rangle$ is defined. For each $\varepsilon < \theta$ for some $\alpha(\varepsilon) < \lambda$, $g_\varepsilon < f_{\alpha(\varepsilon)} \mod J_{<\lambda}[\bar{\lambda}]$ hence $\alpha(\varepsilon) \le \alpha < \lambda \Rightarrow g_\varepsilon <_{J_{<\lambda}[\bar{\lambda}]} f_\alpha$. As $\lambda =$ cf$(\lambda) > \theta$, we can choose $\alpha < \lambda$ such that $\alpha > \bigcup_{\varepsilon<\theta} \alpha(\varepsilon)$. Let $B_\varepsilon = \{i < \kappa : g_\varepsilon(i) \ge f_\alpha(i)\}$; so for each $\xi < \theta$ we have $\langle B_\varepsilon \cap A^*_\xi : \varepsilon < \xi\rangle$ is increasing with $\varepsilon$, (by clause (B)(iii)), hence as usual as $\theta \ge$ wsat$(I^*)$ (and $\theta > \aleph_0$) we can find $\varepsilon(*) < \theta$ such that $\bigwedge_n B_{\varepsilon(*)+n} = B_{\varepsilon(*)} \mod I^*$ [why do we not demand $\varepsilon \in (\varepsilon(*), \theta) \Rightarrow B_\varepsilon = B_{\varepsilon(*)} \mod I^*$? as $\theta$ may be singular]. Now as $g_{\varepsilon(*)} \in N_{\varepsilon(*)+1}$ and $f_\alpha \in N_0 \prec N_{\varepsilon(*)+1}$ clearly, by its definition, $B_{\varepsilon(*)} \in N_{\varepsilon(*)+1}$ hence $F_{B_{\varepsilon(*)}} \in N_{\varepsilon(*)+1}$. Now:*

$$\begin{aligned} g_{\varepsilon(*)+1} \restriction (\kappa \setminus B_{\varepsilon(*)}) =_{I^*} g_{\varepsilon(*)+1} \restriction (\kappa \setminus B_{\varepsilon(*)+1}) &< f_\alpha \restriction (\kappa \setminus B_{\varepsilon(*)+1}) \\ &=_{I^*} f_\alpha \restriction (\kappa \setminus B_{\varepsilon(*)}). \end{aligned}$$

*[Why first equality and last equality? As $B_{\varepsilon(*)+1} = B_{\varepsilon(*)} \mod I^*$, why the $<$ in the middle? By the definition of $B_{\varepsilon(*)+1}$].*

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



*But* $g_{\varepsilon(*)+1} \restriction B_{\varepsilon(*)} \in \prod_{i \in B_{\varepsilon(*)}} \lambda_i$, *and* $B_{\varepsilon(*)} \in J_{<\lambda}[\bar\lambda]$ *as* $g_\varepsilon < f_{\alpha(\varepsilon)} \le f_\alpha$ mod $J_{<\lambda}[\bar\lambda]$ *so for some* $f \in F_{B_{\varepsilon(*)}} \subseteq \Pi\bar\lambda$ *we have* $g_{\varepsilon(*)+1} \restriction B_{\varepsilon(*)} < f \restriction B_{\varepsilon(*)}$ mod $I^*$. *By the last two sentences*

$$(*) \qquad g_{\varepsilon(*)+1} < \max\{f, f_\alpha\} \mod I^*$$

*Now* $f_\alpha \in N_{\varepsilon(*)+1}$ *and* $f \in N_{\varepsilon(*)+1}$ *(as* $f \in F_{B_{\varepsilon(*)}}$, $|F_{B_{\varepsilon(*)}}| \le \lambda$, $\lambda + 1 \subseteq N_{\varepsilon(*)+1}$ *the function* $B \mapsto F_B$ *belongs to* $N_0 \prec N_{\varepsilon(*)+1}$ *and* $B_{\varepsilon(*)} \in N_{\varepsilon(*)+1}$ *as* $\{g_{\varepsilon(*)}, f_\alpha\} \in N_{\varepsilon(*)+1}$*) so together*

$$(**) \qquad max\{f, f_\alpha\} \in N_{\varepsilon(*)+1};$$

*But* $(*)$, $(**)$ *together contradict the choice of* $g_{\varepsilon(*)+1}$ *(i.e., clause (B)(ii)).* $\square_{\mathbf{2.15}}$

$\rightarrow$ *scite{2.10A} ambiguous*

**2.16 Definition.** 1) We say that $I^*$ satisfies the pcf-th for (the regular) $(\bar\lambda, \theta)$ <u>if</u>:

- $(a)$ $\Pi\bar\lambda/I^*$ is $\theta$-directed and
- $(b)$ for every $\lambda \in \text{pcf}_{I^*}(\bar\lambda)$, $(\Pi\bar\lambda, <_{J_{<\lambda}[\bar\lambda]})$ is $\lambda$-directed and
- $(c)$ we can find $\langle B_\lambda : \lambda \in \text{pcf}_{I^*}(\bar\lambda)\rangle$, such that:
  - $\circledast_{\bar B}$ $(\alpha)$ $B_\lambda \subseteq \kappa$,
  - $(\beta)$ $J_{<\lambda}[\bar\lambda, I^*] = I^* + \{B_\mu : \mu \in \lambda \cap \text{pcf}_{I^*}(\bar\lambda)\}$,
  - $(\gamma)$ $B_\lambda \notin J_{<\lambda}[\bar\lambda, I^*]$ and
  - $(\delta)$ $\Pi(\bar\lambda \restriction B_\lambda)/J_{<\lambda}[\bar\lambda, I^*]$ has true cofinality $\lambda$ (so $B_\lambda \in J_{\le\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda]$ and $J_{\le\lambda}[\bar\lambda] = J_{<\lambda}[\bar\lambda] + B_\lambda$).

1A) We say that $I^*$ satisfies the weak pcf-th for $(\bar\lambda, \theta)$ <u>if</u>:

- $(a)$ $(\Pi\bar\lambda, <_{I^*})$ is $\theta$-directed
- $(b)$ $(\Pi\bar\lambda, <_{J_{<\lambda}[\bar\lambda]})$ is $\lambda$-directed for each $\lambda \in \text{pcf}_{I^*}(\bar\lambda)$
- $(c)$ there are $B_{\lambda,\alpha} \subseteq \kappa$ for $\alpha < \lambda \in \text{pcf}_{I^*}(\bar\lambda)$ such that
  - $(\alpha)$ $\alpha < \beta < \mu \in \text{pcf}_{I^*}(\bar\lambda) \Rightarrow B_{\mu,\alpha} \subseteq B_{\mu,\beta} \mod J_{<\mu}[\bar\lambda, I^*]$
  - $(\beta)$ $J_{<\lambda}[\bar\lambda] = I^* + \{B_{\mu,\alpha} : \alpha < \mu < \lambda, \mu \in \text{pcf}_{I^*}(\bar\lambda)\}$ and
  - $(\gamma)$ $(\Pi\bar\lambda, <_{J_{<\lambda}[\bar\lambda]})$ is $\lambda$-directed and
  - $(\delta)$ $(\Pi(\bar\lambda \restriction B_{\mu,\alpha}), <_{J_{<\lambda}[\bar\lambda]})$ has true cofinality $\lambda$.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



1B) We say that $I^*$ satisfies the weaker pcf-th for $(\bar\lambda, \theta)$ if:

(a) $(\Pi\bar\lambda, <_{I^*})$ is $\theta$-directed

(b) each $(\Pi\bar\lambda, <_{J_{<\lambda}[\bar\lambda]})$ is $\lambda$-directed

(c) for any ultrafilter $D$ on $\kappa$ disjoint to $J_{<\theta}[\bar\lambda]$ letting $\lambda = \operatorname{tcf}(\Pi\bar\lambda, <_D)$ we have: $\lambda \geq \theta$ and for some $B \in D \cap J_{\leq\lambda}[\bar\lambda] \setminus J_{<\lambda}[\bar\lambda]$, the partial order $(\Pi(\bar\lambda \restriction B), <_{J_{<\lambda}[\bar\lambda]})$ has true cofinality $\lambda$.

1C) We say that $I^*$ satisfies the weakest pcf-th for $(\bar\lambda, \theta)$ if:

(a) $(\Pi\bar\lambda, <_{I^*})$ is $\theta$-directed and

(b) $(\Pi\bar\lambda, <_{J_{<\lambda}[\bar\lambda]})$ is $\lambda$-directed for any $\lambda \geq \theta$.

1D) Above we write $\bar\lambda$ instead $(\bar\lambda, \theta)$ when we mean

$$\theta = \max\{\theta : (\Pi\bar\lambda, <_{I^*}) \text{ is } \theta\text{-directed}\}.$$

2) We say that $I^*$ satisfies the pcf-th for $\theta$ if for any regular $\bar\lambda$ such that lim $\inf_{I^*}(\bar\lambda) \geq \theta$, we have: $I^*$ satisfies the pcf-th every for $\bar\lambda$. We say that $I^*$ satisfies the pcf-th above $\mu$ if it satisfies the pcf-th for $\bar\lambda$ with lim $\inf_{I^*}(\bar\lambda) > \mu$. Similarly (in both cases) for the weak pcf-th and the weaker pcf-th.
3) Given $I^*$, $\theta$ let

$$J_\theta^{\mathrm{pcf}} = \{A \subseteq \kappa : A \in I^* \text{ or } A \notin I^* \text{ and } I^* + (\kappa \setminus A) \text{ satisfies the pcf-theorem for } \theta\}.$$

$$J_\theta^{\mathrm{wsat}} =: \{A \subseteq \kappa : \mathrm{wsat}(I^* \restriction A) \leq \theta \text{ or } A \in I^*\}$$

similarly $J_\theta^{\mathrm{wpcf}}$; we may write $J_\theta^x[I^*]$.
4) We say that $I^*$ satisfies the pseudo pcf-th for $\bar\lambda$ if for every ideal $I$ on $\kappa$ extending $I^*$, for some $A \in I^+$ we have $(\Pi(\bar\lambda \restriction A), <_I)$ has a true cofinality.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



**2.17 Claim.** *1) If $(*)$ of 1.9 then $I^*$ satisfies the weak pcf-th for $(\bar\lambda, \theta^+)$.*
*2) If $(*)$ of 1.9 holds, and $\Pi\bar\lambda/I^*$ is $\theta^{++}$-directed (e.g., $\theta^+ < \min\bar\lambda$) or just there is a continuity condition for $(\theta^+,\theta)$) then $I^*$ satisfies the pcf-th for $(\bar\lambda, \theta^+)$.*
*3) If $I^*$ satisfy the pcf-th for $(\bar\lambda,\theta)$ then $I^*$ satisfy the weak pcf-th for $(\bar\lambda,\theta)$ which implies that $I^*$ satisfies the weaker pcf-th for $(\bar\lambda,\theta)$, which implies that $I^*$ satisfies the weakest pcf-th for $(\bar\lambda,\theta)$.*

*Proof.* 1) Let appropriate $\bar\lambda$ be given. By 1.9, 1.13 most demands holds, but we are left with normality. By 2.11, if $\lambda \in \text{pcf}(\bar\lambda)$, then $\bar\lambda$ is semi normal for $\lambda$. This finishing the proof of (1).
2) Let $\lambda \in \text{pcf}(\bar\lambda)$ and let $\bar f$, $\bar B$ be as in 2.2(4). By 2.5(1)+(2) there is $\bar a$, a $(\lambda,\theta)$-continuity condition; by 2.7(1) without loss of generality $\bar f$ obeys $\bar a$, by 2.9(1) the relevant $B_\alpha/I^*$ are eventually constant which suffices by 2.2(2).
3) Should be clear. $\square_{2.17}$

**2.18 Claim.** *Assume $(\Pi\bar\lambda, <_{I^*})$ is given (but possibly $(*)$ of 1.9 fails).*
*1) If $I^*$, $\bar\lambda$ satisfies (the conclusions of) 1.11, then $I^*$, $\bar\lambda$ satisfy (the conclusion of) 1.13(1), 1.13(2), 1.13(3), 1.13(4), 1.14.*
*1A) If $I^*$ satisfies the weaker pcf-th for $\bar\lambda$ then they satisfy the conclusion of 1.11 (and 1.9).*
*2) If $I^*$, $\bar\lambda$ satisfies (the conclusion of) 1.9 then $I^*$, $\bar\lambda$ satisfies (the conclusion of) 1.15.*
*2A) If $I^*$ satisfies the weakest pcf-th for $\bar\lambda$ then $I^*$, $\bar\lambda$ satisfy the conclusion of 1.9.*
*3) If $I^*$, $\bar\lambda$ satisfies 1.9, 1.11 then $I^*$, $\bar\lambda$ satisfies 2.2(1) (for 2.2(2) - no assumptions).*

*4) If $I^*$, $\bar\lambda$ satisfies 1.13(1), 1.13(2) then $I^*$, $\bar\lambda$ satisfies 2.2(3).*
*5) If $I^*$, $\bar\lambda$ satisfies 1.13(2) then $I^*$, $\bar\lambda$ satisfies 2.13(1).*
*6) If $I^*$ $\bar\lambda$ satisfy 1.13(1) + 1.13(3)(i) then $I^*$, $\bar\lambda$ satisfies 1.13(2).*
*7) If $I^*$, $\bar\lambda$ satisfies 1.13(1) + 1.13(2) and is semi normal then 2.13(2) holds, i.e.,*

$$\text{cf}(\Pi\bar\lambda, <_{I^*}) \le \sup \text{pcf}_{I^*}(\lambda).$$

*Proof.* 1) We prove by parts.

*Proof of 1.13(2).* Let $\lambda = \text{tcf}(\Pi\bar\lambda/D)$; by the definition of pcf, $D \cap J_{<\lambda}[\bar\lambda] = \emptyset$. Also by 1.11 for some $B \in D$ we have $\lambda = \text{tcf}(\Pi(\bar\lambda \restriction B), <_{J_{<\lambda}[\bar\lambda]})$, so by the previous sentence $B \notin J_{<\lambda}[\bar\lambda]$, and by 1.8(5) we have $B \in J_{\le\lambda}[\bar\lambda]$, together we finish.

*Proof of 1.13(1).* Repeat the proof of 1.13(1) replacing the use of 1.9 by 1.13(2).

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



*Proof of 1.8(3)(i).* Let $J =: \bigcup\limits_{\mu<\lambda} J_{<\mu}[\bar\lambda]$, so $J \subseteq J_{<\lambda}[\bar\lambda]$ is an ideal because $\langle J_{<\mu}[\bar\lambda] : \mu < \lambda\rangle$ is $\subseteq$-increasing (by 1.8(2)), if equality fail choose $B \in J_{<\lambda}[\bar\lambda] \setminus J$ and choose $D$ an ultrafilter on $\kappa$ disjoint to $J$ to which $B$ belongs. Now if $\mu = \text{cf}(\mu) < \lambda$ then $\mu^+ < \lambda$ (as $\lambda$ is a limit cardinal) and $\mu = \text{cf}(\mu)\ \&\ \mu^+ < \lambda \Rightarrow D \cap J_{\leq\mu}[\bar\lambda] = D \cap J_{<\mu^+}[\bar\lambda] = \emptyset$ hence by 1.13(2) we have $\mu \neq \text{cf}(\Pi\bar\lambda/D)$. Also if $\mu = \text{cf}(\mu) \geq \lambda$ then $D \cap J_{<\mu}[\bar\lambda] \subseteq D \cap J_{<\lambda}[\bar\lambda] = \emptyset$ hence by 1.13(2) we have $\mu \neq \text{cf}(\Pi\bar\lambda/D)$. Together contradiction by 1.5(7).

*Proof of 1.13(3)(ii).* Follows.

*Proof of 1.13(4).* Follows.

*Proof of 1.14.* As in 1.14.
1) Check.
2) Read the proof of 1.15.
2A) Check.
3) The direction $\Rightarrow$ is proved directly as in the proof of 2.2(1) (where the use of 1.13(1) is justified by 2.18(1)).

So let us deal with the direction $\Leftarrow$. So assume $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$ is a sequence of members of $\prod\bar\lambda$ which is $<_{J_{<\lambda}[\bar\lambda]}$-increasing such that for every ultrafilter $D$ on $\kappa$ disjoint to $J_{<\lambda}[\bar\lambda]$ we have: $\lambda = \text{tcf}(\Pi\bar\lambda, <_D)$ iff $\bar f$ is unbounded (equivalently cofinal) in $(\Pi\bar\lambda, <_D)$. By (the conclusion of) 1.9 without loss of generality $\bar f$ is $<_{J_{<\lambda}[\bar\lambda]}$-increasing, and let

$$J =: \{A \subseteq \kappa : A \in J_{<\lambda}[\bar\lambda] \text{ or } \bar f \text{ is cofinal in } (\Pi\bar\lambda, <_{J_{<\lambda}[\bar\lambda]+(\kappa\setminus A)}\}.$$

Clearly $J$ is an ideal on $\kappa$ (by 1.5(2)(v)), and $J_{<\lambda}[\bar\lambda] \subseteq J \subseteq J_{\leq\lambda}[\bar\lambda]$. If $J \neq J_{<\lambda}[\bar\lambda]$ choose $A \in J_{\leq\lambda}[\bar\lambda] \setminus J$ and an ultrafilter $D$ on $\kappa$ disjoint to $J$ to which $A$ belongs.

By (the conclusion of) 1.11, there is $A \in J \cap D$; contradiction, so actually $J = J_{\leq\lambda}[\bar\lambda]$. By 1.9 there is $g \in \Pi\bar\lambda$ such that $f_\alpha < g \mod J_{\leq\lambda}[\bar\lambda]$ for each $\alpha < \lambda$, and let $B_\alpha =: \{i < \kappa : g(i) \leq f_\alpha(i)\}$. Hence $B_\alpha \in J_{\leq\lambda}[\bar\lambda]$ (by the previous sentence) and $\langle B_\alpha/J_{<\lambda}[\bar\lambda] : \alpha < \lambda\rangle$ is $\subseteq$-increasing (as $\langle f_\alpha : \alpha < \lambda\rangle$ is $<_{J_{<\lambda}[\bar\lambda]}$-increasing). Lastly if $B \in J_{\leq\lambda}[\bar\lambda]$, but $B \setminus B_\alpha \notin J_{<\lambda}[\bar\lambda]$ for each $\alpha < \lambda$, let $D$ be an ultrafilter on $\kappa$ disjoint to $J_{<\lambda}[\bar\lambda] + \{B_\alpha : \alpha < \lambda\}$ but to which $B$ belongs, so $\text{tcf}(\Pi\bar\lambda, g_D) = \lambda$ (by 1.13(3) which holds by 2.17(1)) but $\{f_\alpha/D : \alpha < \lambda\}$ is bounded by $g/D$ (as $f_\alpha/D \leq g/D$ by the definition of $B_\alpha$), contradiction. So the sequence $\langle B_\alpha : \alpha < \lambda\rangle$ is as required.
4) – 6) Left to the reader.

7) For $\lambda \in \text{ pcf}(\bar{\lambda})$ let $\langle B_i^\lambda : i < \lambda \rangle$ be such that $J_{\leq \lambda}[\bar{\lambda}] = J_{<\lambda}[\bar{\lambda}] + \{B_i^\lambda : i < \lambda\}$ (exists by semi-normality; we use only this equality). Let $\langle f_\alpha^{\lambda,i} : \alpha < \lambda \rangle$ be cofinal in $(\Pi(\bar{\lambda} \restriction B_i^\lambda), <_{J_{<\lambda}[\bar{\lambda}]})$, it exists by 1.13(1). Let $F$ be the closure of $\{f_\alpha^{\lambda,i} : \alpha < \lambda, i < \lambda, \lambda \in \text{ pcf}(\bar{\lambda})\}$, under the operation $\max\{g, h\}$. Clearly $|F| \leq$ sup pcf$(\bar{\lambda})$, so it suffice to prove that $F$ is a cover of $(\prod \bar{\lambda}, <_{I^*})$. Let $g \in \Pi\bar{\lambda}$, if $(\exists f \in F)(g \leq f)$ we are done, if not

$$I = \{A \cup \{i < \kappa : f(i) > g(i)\} : f \in F, A \in I^*\}$$

is $\aleph_0$-directed, $\kappa \notin I$, so there is an ultrafilter $D$ on $\kappa$ disjoint to $I$, (so $f \in F \Rightarrow g <_D f$) and let $\lambda = \text{ tcf}(\Pi\bar{\lambda}/D)$, so by 1.13(2) we have $D \cap J_{\leq \lambda}[\bar{\lambda}] \setminus J_{<\lambda}[\bar{\lambda}] \neq \emptyset$, hence for some $i < \lambda$, $B_i^\lambda \in D$, and we get contradiction to the choice of the $\{f_\alpha^{\lambda,\alpha} : \alpha < \lambda\}$ $(\subseteq F)$. $\square_{2.18}$

**2.19 Claim.** *If $I^*$ satisfies pseudo* pcf-*th* *then*

(1) $\text{cf}(\Pi\bar{\lambda}, <_{I^*}) = \sup \text{pcf}_{I^*}(\bar{\lambda})$
(2) *We can find $\langle (J_\zeta, \theta_\zeta) : \zeta < \zeta^* \rangle$, $\zeta^*$ a successor ordinal such that $J_0 = I^*$, $J_{\zeta+1} = \{A \subseteq \kappa :$ if $A \notin J_\zeta$ then* $\text{tcf}(\Pi(\bar{\lambda} \restriction A), <_{J_\zeta}) = \theta_\zeta$ *and for no $A \in (J_\zeta)^+$ does $(\Pi(\bar{\lambda} \restriction A), <_{J_\zeta})$ has true cofinality which is $< \theta_\zeta\}$*
(3) *If $I^*$ satisfies the weaker* pcf-*th for $\bar{\lambda}$* *then* *$I^*$ satisfies the pseudo* pcf-*th for $\bar{\lambda}$.*

*Proof.* 1) Similar to the proof of 2.13(2).
2) Check (we can also present those ideals in other ways).
3) Check. $\square_{2.19}$

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



## §3 Reduced products of cardinals

We characterize here the cardinalities $\prod\limits_{i<\kappa} \lambda_i/D$ and $T_D(\langle \lambda_i : i < \kappa\rangle)$ using pcf's and the amount of regularity of $D$ (in 3.1 - 3.4). Later we give sufficient conditions for the existence of $<_D$-lub or $<_D$-eub. Remember the old result of Kanamori [Kn] and Ketonen [Kt]: for $D$ an ultrafilter the sequence $\langle \alpha/D : \alpha < \kappa\rangle$ (i.e., the constant functions) has a $<_D$-lub if $\mathrm{reg}(D) < \kappa$; and see [Sh:g, III,3.3] (for filters). Then we turn to depth of ultraproducts of Boolean algebras.

The questions we would like to answer are (restricting ourselves to "$\lambda_i \ge 2^\kappa$" or "$\lambda_i \ge 2^{2^\kappa}$" and $D$ an ultrafilter on $\kappa$ will be good enough).

Question A: What can be $\mathrm{Car}_D =: \{\prod\limits_{i<\kappa} \lambda_i/D : \lambda_i$ a cardinal for $i < \kappa\}$, i.e., characterize it by properties of $D$; (or at least $\mathrm{Card}_D \setminus 2^\kappa$) (for $D$ a filter also $\{T_D(\prod\limits_{i<\kappa} \lambda_i) : \lambda_i$ a cardinal for $i < \kappa$ is natural).

Question B: What can be $\mathrm{DEPTH}^+_D = \{\mathrm{Depth}^+(\prod\limits_{i<\kappa} \lambda_i/D) : \lambda_i$ a regular cardinal$\}$ (at least $\mathrm{DEPTH}^+_D \setminus 2^\kappa$, see Definition 3.21).

If $D$ is an $\aleph_1$-complete ultrafilter, the answer is clear. For $D$ a regular ultrafilter on $\kappa$, $\lambda_i \ge \aleph_0$ the answer to question A is known ([\CK ]) in fact it was the reason for defining "regularity of filters" (for $\lambda_i < \aleph_0$ see [Sh 7], [Sh:a, VI,§3,Th. 3.12,pp.357-370] better [Sh:c, VI,§3] and Koppleberg [Ko].) For $D$ a regular ultrafilter on $\kappa$, the answer to the question is essentially completed in 3.25(1), the remaining problem can be answered by pp (see [Sh:g]) except the restriction $(\forall \alpha < \lambda)(|\alpha|^{\aleph_0} < \lambda)$, which can be removed if the cov = pp problem is completed (see [Sh:g, AG]). So the problem is for the other ultrafilters $D$, on which we give a reasonable amount on information translating to a pcf problem, sometimes depending on the pcf theorem.

**3.1 Definition.** 1) For a filter $D$ let $\mathrm{reg}(D) = \mathrm{Min}\{\theta : D$ is not $\theta$-regular$\}$ (see below).
2) A filter $D$ is $\theta$-regular if there are $A_\varepsilon \in D$ for $\varepsilon < \theta$ such that the intersection of any infinitely many $A_\varepsilon$-s' is empty.
3) For a filter $D$ let

$$\begin{aligned}\mathrm{reg}_*(D) = \mathrm{Min}\{\theta :\ &\text{there are no } A_\varepsilon \in D^+ \text{ for } \varepsilon < \theta \text{ such that}\\ &\text{no } i < \kappa \text{ belongs to infinitely many } A_\varepsilon\text{'s}\}\end{aligned}$$

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



and

$$\begin{aligned}\text{reg}_{\otimes}(D) =: \{\theta : &\text{ there are no } A_\varepsilon \in D^+ \text{ for } \varepsilon < \theta \text{ such that :}\\ &\varepsilon < \zeta \Rightarrow A_\zeta \subseteq A_\varepsilon \mod D \text{ and no } i < \kappa\\ &\text{belongs to infinitely many } A_\varepsilon\text{'s}\}.\end{aligned}$$

4) $\text{reg}^\sigma(D) = \min\{\theta : D$ is not $(\theta, \sigma)$-regular$\}$ where "$D$ is $(\theta, \sigma)$-regular" means that there are $A_\varepsilon \in D$ for $\alpha < \theta$ such that the intersection of any $\sigma$ of them is empty.

Lastly, $\text{reg}^\sigma_*(D)$, $\text{reg}^\sigma_\otimes(D)$ are defined similarly using $A_\varepsilon \in D^+$. Of course, $\text{reg}(I)$, etc., means $\text{reg}(D)$ where $D$ is the dual filter.

**3.2 Definition.** 1) Let

$$\begin{aligned}\text{htcf}_{D,\mu}(\Pi\gamma_i) = \sup\{\text{tcf}(\prod_{i<\kappa}\lambda_i/D) :&\mu \le \lambda_i = \text{cf}\lambda_i \le \gamma_i \text{ for } i < \kappa \text{ and}\\ &\text{tcf}(\Pi\lambda_i/D) \text{ is well defined}\}\end{aligned}$$

and

$$\text{hcf}_{D,\mu}(\prod_{i<\kappa}\gamma_i) = \sup\{\text{cf}(\Pi\lambda_i/D) : \mu \le \lambda_i = \text{cf}\lambda_i \le \gamma_i\};$$

if $\mu = \aleph_0$ we may omit it.
2) For $E$ a family of filters on $\kappa$ let $\text{hcf}_{E,\mu}(\prod\limits_{i<\kappa}\alpha_i)$ be

$$\begin{aligned}\sup\{\text{tcf}(\prod_{i<\kappa}\lambda_i/D) :&D \in E \text{ and } \mu \le \lambda_i = \text{cf}\lambda_i \le \alpha_i \text{ for } i < \kappa \text{ and}\\ &\text{tcf}(\prod_{i<\kappa}\lambda_i/D) \text{ is well defined}\}.\end{aligned}$$

Similarly for $\text{hcf}_{E,\mu}$ (using cf instead of tcf).
3) $\text{hcf}^*_{D,\mu}(\prod\limits_{i<\kappa}\alpha_i)$ is $\text{hcf}_{E,\mu}(\prod\limits_{i<\kappa}\alpha_i)$ for $E = \{D' : D'$ a filter on $\kappa$ extending $D\}$. Similarly for $\text{htcf}^*_{D,\mu}$.
4) When we write $I$, e.g., in $\text{hcf}_{I,\mu}$ we mean $\text{hcf}_{D,\mu}$ where $D$ is the dual filter.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



**3.3 Claim.** *1)* $\mathrm{reg}(D)$ *is always regular.*
*2) If* $\theta < \mathrm{reg}_*(D)$ *<u>then</u> some filter extending* $D$ *is* $\theta$*-regular.*
*3)* $\mathrm{wsat}(D) \le \mathrm{reg}_*(D)$.
*4)* $\mathrm{reg}(D) \le \mathrm{reg}_\otimes(D) \le \mathrm{reg}_*(D)$.
*5)* $\mathrm{reg}_*(D) = \min\{\theta$*: no ultrafilter* $D_1$ *on* $\kappa$ *extending* $D$ *is* $\theta$*-regular*$\}$.
*6) If* $D \subseteq E$ *are filters on* $\kappa$ *then:*

(a) $\mathrm{reg}(D) \le \mathrm{reg}(E)$
(b) $\mathrm{reg}_*(D) \ge \mathrm{reg}_*(E)$.

*Proof.* Should be clear. E.g.
2) Let $\langle u_\varepsilon : \varepsilon < \theta\rangle$ list the finite subsets of $\theta$, and let $\{A_\varepsilon : \varepsilon < \theta\} \subseteq D^+$ exemplify "$\theta <$ $\mathrm{reg}_*(D)$". Now let $D^* =: \{A \subseteq \kappa$: for some finite $u \subseteq \theta$, for every $\varepsilon < \theta$ we have: $u \subseteq u_\varepsilon \Rightarrow A_\varepsilon \subseteq A \mod D\}$, and let $A^*_\varepsilon = \bigcup\{A_\zeta : \varepsilon \in u_\zeta\}$. Now $D^*$ is a filter on $\kappa$ extending $D$ and for $\varepsilon < \theta$ we have $A^*_\varepsilon \in D$.

Finally, the intersection of $A^*_{\varepsilon_0} \cap A^*_{\varepsilon_1} \cap \ldots$ for distinct $\varepsilon_n < \theta$ is empty, because for any memeber $j$ of it we can find $\zeta_n < \theta$ such that $j \in A_{\zeta_n}$ and $\varepsilon_n \in u_{\zeta_n}$. Now <u>if</u> $\{\zeta_n : n < \omega\}$ is infinite then there is no such $j$ by the choice of $\langle A_\varepsilon : \varepsilon < \theta\rangle$, and <u>if</u> $\{\zeta_n : n < \omega\}$ is finite then without loss of generality $\bigwedge\limits_{n<\omega} \zeta_n = \zeta_0$ contradicting "$u_{\zeta_0}$ is finite" as $\bigwedge\limits_{n<\omega} \varepsilon_n \in u_{\zeta_n}$. $\square_{3.3}$

*3.4 Observation.* $|\prod\limits_{i<\kappa} \lambda_i/I| \ge |\aleph_0^\kappa/I|$ holds when $\bigwedge\limits_{i<\kappa} \lambda_i \ge \aleph_0$.

*3.5 Observation.* 1) $|\prod\limits_{i<\kappa} \lambda_i/I| \ge$ $\mathrm{htcf}^*_I(\prod\limits_{i<\kappa} \lambda_i)$.
2) If $I^*$ satisfies the pcf-th for $\bar\lambda$ or even the weaker pcf-th or even the pseudo pcf-th for $\bar\lambda$ (see Definition 2.16) <u>then</u>: $\mathrm{cf}(\Pi\bar\lambda/I^*) =$ $\max \mathrm{pcf}_{I^*}(\bar\lambda)$.
3) If $I^*$ satisfies the pcf-th for $\mu$ for and $\min(\bar\lambda) \ge \mu$ <u>then</u>

$$\mathrm{hcf}_{D,\mu}(\Pi\bar\lambda) = \ \mathrm{hcf}^*_{D,\mu}(\Pi\bar\lambda) = \ \mathrm{htcf}^*_{D,\mu}(\Pi\bar\lambda)$$

whenever $D$ is disjoint to $I^*$.
4) $\mathrm{hcf}_{E,\mu}(\prod\limits_{i<\kappa} \lambda_i) =$ $\mathrm{hcf}^*_{E,\mu}(\prod\limits_{i<\kappa} \lambda_i)$.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



5) $\prod\limits_{i<\kappa} \lambda_i/I \geq \ \mathrm{hcf}_{I,\mu}(\prod\limits_{i<\kappa} \lambda_i) = \ \mathrm{hcf}^*_{I,\mu}(\prod\limits_{i<\kappa} \lambda_i) \geq \ \mathrm{htcf}^*_{I,\mu}(\prod\limits_{i<\kappa} \lambda_i)$ and $\mathrm{hcf}_{I,\mu}(\prod\limits_{i<\kappa} \lambda_i) \geq \mathrm{htcf}_{I,\mu}(\prod\limits_{i<\kappa} \lambda_i)$.

*3.6 Remark.* In 3.5(3) concerning $\mathrm{htcf}_{D,\mu}$ see 3.13.

*Proof.* 1) By the definition of $\mathrm{htcf}^*_I$ it suffices to show $|\prod\limits_{i<\kappa} \lambda_i/I| \geq \ \mathrm{tcf}(\Pi\lambda'_i/I')$, when $I'$ is an ideal on $\kappa$ extending $I$, $\lambda'_i = \ \mathrm{cf}\lambda'_i \leq \lambda_i$ for $i < \kappa$ and $\mathrm{tcf}(\prod\limits_{i<\kappa} \lambda'_i/I')$ is well defined. Now $|\prod\limits_{i<\kappa} \lambda_i/I| \geq |\prod\limits_{i<\kappa} \lambda'_i/I| \geq |\prod\limits_{i<\kappa} \lambda'_i/I'| \geq \ \mathrm{cf}(\Pi\lambda'_i/I')$, so we have finished.
2) By 2.18(1) and 1.14 and 2.19.
3) Left to the reader (see Definition 2.16(2)).
4), 5) Check. $\square_{3.5}$

**3.7 Claim.** *If $\lambda = |\prod\limits_{i<\kappa} \lambda_i/I|$ (and $\lambda_i \geq \aleph_0$ and, of course, $I$ an ideal on $\kappa$) and $\theta < \mathrm{reg}(I)$ <u>then</u> $\lambda = \lambda^\theta$.*

*Proof.* For each $i < \kappa$, let $\langle \eta^i_\alpha : \alpha < \lambda_i \rangle$ list the finite sequences from $\lambda_i$. Let $M_i = (\lambda_i, F_i, G_i)$ where $F_i(\alpha) = \lg(\eta^i_\alpha)$, $G_i(\alpha,\beta)$ is $\eta^i_\alpha(\beta)$ if $\beta < \ell g(\eta^i_\alpha)$ $(= F_i(\alpha))$, and $F(\alpha,\beta) = 0$ otherwise; let $M = \prod\limits_{i<\kappa} M_i/I$ so $\|M\| = |\Pi\lambda_i/I|$ and let $M = (\Pi\lambda_i/I, F, G)$. Let $\langle A_i : i < \theta \rangle$ exemplifies $I$ is $\theta$-regular. Now

- $(*)_1$ We can find $f \in {}^\kappa\omega$ and $f_\varepsilon \in \prod\limits_{i<\kappa} f(i)$ for $\varepsilon < \theta$ such that: $\varepsilon < \zeta < \theta \Rightarrow f_\varepsilon <_I f_\zeta$ [just for $i < \kappa$ let $w_i = \{\varepsilon < \theta : i \in A_\varepsilon\}$, it is finite and let $f(i) = |w_i|$ and $f_\varepsilon(i) = |\varepsilon \cap w_i| < f(i)$, and note $\varepsilon < \zeta \& i \in A_\varepsilon \cap A_\zeta \Rightarrow f_\varepsilon(i) < f_\zeta(i)$].
- $(*)_2$ For every sequence $\bar g = \langle g_\varepsilon : \varepsilon < \theta \rangle$ of members of $\prod\limits_{i<\kappa} \lambda_i$, there is $h \in \prod\limits_{i<\kappa} \lambda_i$ such that $\varepsilon < \theta \Rightarrow M \models F(h/I, f_\varepsilon/I) = g_\varepsilon/I$.
  [Why? Let, in the notation of $(*)_1$, $h(i)$ be such that $\eta^i_{h(i)} = \langle g_\varepsilon(i) : \varepsilon \in w_i \rangle$ (in the natural order).]

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



So in $M$, every $\theta$-sequence of members is coded by at least one member so $\|M\|^\theta = \|M\|$, but $\|M\| = |\prod\limits_{i<\kappa} \lambda_i/I|$ hence we have proved 3.7. $\square_{3.7}$

*3.8 Fact.* 1) For $D$ a filter on $\kappa$, $\langle A_1, A_2\rangle$ a partition of $\kappa$ and (non zero) cardinals $\lambda_i$ for $i < \kappa$ we have

$$|\prod_{i<\kappa} \lambda_i/D| = |\prod_{i<\kappa} \lambda_i/(D + A_1)| \times |\prod_{i<\kappa} \lambda_i/(D + A_2)|$$

(note: $|\prod\limits_{i<\kappa} \lambda_i/\mathscr{P}(\kappa)| = 1$).

2) $D^{[\mu]} =: \{A \subseteq \kappa : |\prod\limits_{i<\kappa} \lambda_i/(D+(\kappa \setminus A))| < \mu\}$ is a filter on $\kappa$ ($\mu$ an infinite cardinal of course) and if $\aleph_0 \le \mu \le \prod\limits_{i<\kappa} \lambda_i/D$ then $D^{[\mu]}$ is a proper filter.

3) If $\lambda \le |\prod\limits_{i<\kappa} \lambda_i/I|$, ($\lambda_i$ infinite, of course, $I$ an ideal on $\kappa$) and $A \in I^+ \Rightarrow |\prod\limits_{i\in A} \lambda_i/I| \ge \lambda$ and $\sigma <$ $\text{reg}_\otimes(I)$ <u>then</u> $|\Pi\lambda_i/I| \ge \lambda^\sigma$.

*Proof.* Check (part (3) is like 3.7).

**3.9 Claim.** *If $D \subseteq E$ are filters on $\kappa$ <u>then</u>*

$$|\prod_{i<\kappa} \lambda_i/D| \le |\prod_{i<\kappa} \lambda_i/E| + \sup_{A\in E\setminus D} |\prod_{i<\kappa} \lambda_i/(D + (\kappa \setminus A))| + (2^\kappa/D) + \aleph_0.$$

*We can replace $2^\kappa/D$ by $|\mathscr{P}|$ if $\mathscr{P}$ is a maximal subset of $E$ such that $A \ne B \in \mathscr{P} \Rightarrow (A \setminus B) \cup (B \setminus A) \ne \emptyset \mod D$.*

*Proof.* Think.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



**3.10 Lemma.** $|\prod\limits_{i<\kappa}\lambda_i/D| \le (\theta^\kappa/D + \mathrm{hcf}_{D,\theta}(\prod\limits_{i<\kappa}\lambda_i))^{<\theta}$ *(see Definition 3.2(1)) provided that:*

$$(*) \qquad \theta \ge \mathrm{reg}_\otimes(D)$$

*3.11 Remark.* 1) If $\theta = \theta_1^+$, we can replace $\theta^\kappa/D$ by $\theta_1^\kappa/D$. In general we can replace $\theta^\kappa/D$ by $\sup\{\prod\limits_{i<\kappa} f(i)/D : f \in \theta^\kappa\}$.
2) If $D$ satisfies the pcf-th above $\theta$ (see 2.16(1A), 2.17(2)) then by 3.5(3) we can use htcf* (sometime even htcf, see 3.13). But by 3.8(1) we can ignore the $\lambda_i \le \theta$, and when $i < 2 \Rightarrow \lambda_i > \theta$ we know that 1.9(*)($\alpha$) holds by 3.3(3).

*Proof.* Let $\lambda = \theta^\kappa/D +\ \mathrm{hcf}_{D,\theta}(\prod\limits_{i<\kappa}\lambda_i)$. Let for $\zeta < \theta$, $\mu_\zeta =: \lambda^{\|\zeta\|}$, i.e., $\mu_\zeta =: (\theta^\kappa/D +\ \mathrm{hcf}_{D,\theta}\prod\limits_{i<\kappa}\lambda_i)^{|\zeta|}$, clearly $\mu_\zeta = \mu_\zeta^{|\zeta|}$. Let $\chi = \beth_8(\sup_{i<\kappa}\lambda_i)^+$ and $N_\zeta \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ be such that $\|N_\zeta\| = \mu_\zeta$, $N^{\le|\zeta|} \subseteq N_\zeta$, $\lambda + 1 \subseteq N_\zeta$ and $\{D, \langle\lambda_i : i < \kappa\rangle\} \in N_\zeta$ and $[\varepsilon < \zeta \Rightarrow N_\varepsilon \prec N_\zeta]$. Let $N = \cup\{N_\zeta : \zeta < \theta\}$. Let $g^* \in \prod\limits_{i<\kappa}\lambda_i$ and we shall find $f \in N$ such that $g^* = f \mod D$, this will suffice. We shall choose by induction on $\zeta < \theta$, $f^e_\zeta (e < 3)$ and $\bar{A}^\zeta$ such that:

- $(a)$ $f^e_\zeta \in \prod\limits_{i<\kappa}(\lambda_i + 1)$
- $(b)$ $f^1_\zeta \in N_\zeta$ and $f^2_\zeta \in N_\zeta$
- $(c)$ $\bar{A}^\zeta = \langle A^\zeta_i : i < \kappa\rangle \in N_\zeta$
- $(d)$ $\lambda_i \in A^\zeta_i \subseteq \lambda_i + 1$, $|A^\zeta_i| \le |\zeta| + 1$, and $\langle A^\zeta_i : \zeta < \theta\rangle$ is increasing continuous (in $\zeta$)
- $(e)$ $f^0_\zeta(i) =\ \mathrm{Min}(A^\zeta_i \setminus g^*(i))$; note: it is well defined as $g^*(i) < \lambda_i \in A^\zeta_i$
- $(f)$ $f^1_\zeta = f^0_\zeta \mod D$
- $(g)$ $g^* < f^2_\zeta < f^1_\zeta \mod (D + \{i < \kappa : g^*(i) \ne f^1_\zeta(i)\})$
- $(h)$ if $g^*(i) \ne f^1_\zeta(i)$ then $f^2_\zeta(i) \in A^{\zeta+1}_i$.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



So assume everything is defined for every $\varepsilon < \zeta$. If $\zeta = 0$, let $A^\zeta_i = \{\lambda_i\}$, if $\zeta$ limit $A^\zeta_i = \bigcup_{\varepsilon<\zeta} A^\varepsilon_i$, for $\zeta = \varepsilon + 1$, $A^\zeta_i$ will be defined in stage $\varepsilon$. So arriving to $\zeta$, $\bar{A}^\zeta$ is well defined and it belongs to $N_\zeta$: for $\zeta = 0$ check, for $\zeta = \varepsilon+1$, done in stage $\varepsilon$, for $\zeta$ limit it belongs to $N_\zeta$ as we have $N^{\le|\zeta|}_\zeta \subseteq N_\zeta$ and $\xi < \zeta \Rightarrow N_\xi \prec N_\zeta$. Now use clause (e) to define $f^0_\zeta/D$. As $\langle A^\zeta_i : i < \kappa\rangle \in N_\zeta$, $|A^\zeta_i| < \theta$ and $\theta^\kappa/D \le \lambda < \lambda + 1 \subseteq N_\zeta$, clearly $|\prod\limits_{i<\kappa} |A^\zeta_i|/D| \le \lambda$ hence $\{f/D : f \in \prod\limits_{i<\kappa} A^\zeta_i\} \subseteq N_\zeta$ hence $f^0_\zeta/D \in N_\zeta$ hence there is $f^1_\zeta \in N_\zeta$ such that $f^1_\zeta \in f^0_\zeta/D$ i.e. clause (f) holds. As $g^* \le f^0_\zeta$ clearly $g^* \le f^1_\zeta \mod D$, let $y^\zeta_0 =: \{i < \kappa : g^*(i) \ge f^1_\zeta(i)\}$, $y^\zeta_1 =: \{i < \kappa : i \notin y^\zeta_0$ and $\mathrm{cf}(f^1_\zeta(i)) < \theta\}$ and $y^\zeta_2 =: \kappa \setminus y^\zeta_0 \setminus y^\zeta_1$. So $\langle y^\zeta_e : e < 3\rangle$ is a partition of $\kappa$ and $g^* < f^1_\zeta \mod (D + y^\zeta_e)$ for $e = 1, 2$.

Let $y^\zeta_4 = \{i < \kappa : \mathrm{cf}(f^1_\zeta(i)) \ge \theta\}$ so $f^1_\zeta \in N_\zeta$, and $\theta \in N_\zeta$ hence $y^\zeta_4 \in N_\zeta$, so $(\prod\limits_{i<\kappa} f^1_\zeta(i), <_{D+y^\zeta_4}) \in N_\zeta$. Now

$$\mathrm{cf}(\prod_{i<\kappa} f^1_\zeta(i), <_{D+y^\zeta_4}) \le \ \mathrm{hcf}_{D+y^\zeta_4,\theta}(\prod_{i<\kappa} \lambda_i) \le \ \mathrm{hcf}_{D,\theta}(\prod_{i<\kappa} \lambda_i) \subseteq \lambda < \lambda + 1 \subseteq N_\zeta$$

hence there is $F \in N_\zeta$, $|F| \le \lambda$, $F \subseteq \prod\limits_{i\in y^\zeta_4} f^1_\zeta(i)$ such that:

$$(\forall g)[g \in \prod_{i\in y^\zeta_4} f^1_\zeta(i) \Rightarrow (\exists f \in F)(g < f \mod (D + y^\zeta_4)))].$$

As $\lambda + 1 \subseteq N$ necessarily $F \subseteq N_\zeta$. Apply the property of $F$ to $(g \restriction y^\zeta_2) \cup 0_{(\kappa\setminus y^\zeta_2)}$ and get $f^\zeta_4 \in F \subseteq N$ such that $g^* < f^\zeta_4 \mod (D + y^\zeta_2)$. Now use similarly $\prod\limits_{i<\kappa} \mathrm{cf}(f^1_\zeta(i))/(D + y^\zeta_1) \le |\theta^\kappa/D| \le \lambda$; by the proof of 3.8(1) there is a function $f^2_\zeta \in N_\zeta \cap \prod\limits_{i<\kappa} f^1_\zeta(i)$ such that $g^* \restriction (y^\zeta_1 + y^\zeta_2) < f^2_\zeta \mod D$. Let $A^{\zeta+1}_i$ be: $A^\zeta_i$ if $i \in y^\zeta_0$ and $A^\zeta_i \cup \{f^2_\zeta(i)\}$ if $i \in y^\zeta_1 \cup y^\zeta_2$.

It is easy to check clauses (g), (h). So we have carried the definition.
Let

$$X_\zeta =: \{i < \kappa : f^0_{\zeta+1}(i) < f^0_\zeta(i)\}.$$

Note that by the choice of $f^1_\zeta$, $f^1_{\zeta+1}$ we know $X_\zeta = y^\zeta_1 \cup y^\zeta_2 \mod D$, if this last set is not $D$-positive then $g^* \ge f^1_\zeta \mod D$, hence $g^*/D = f^1_\zeta/D \in N_\zeta$, contradiction, so

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



$y_1^\zeta \cup y_2^\zeta \neq \emptyset \mod D$ hence $X_\zeta \in D^+$. Also $\langle y_1^\zeta \cup y_2^\zeta : \zeta < \theta\rangle$ is $\subseteq$-decreasing hence $\langle X_\zeta / D : \zeta < \theta\rangle$ is $\subseteq$-decreasing.

Also if $i \in X_{\zeta_1} \cap X_{\zeta_2}$ and $\zeta_1 < \zeta_2$ then $f^0_{\zeta_2}(i) \leq f^0_{\zeta_1+1}(i) < f^0_{\zeta_1}(i)$ (first inequality: as $A_i^{\zeta_1+1} \subseteq A_i^{\zeta_2}$ and clause (e) above, second inequality by the definition of $X_{\zeta_1}$), hence for each ordinal $i$ the set $\{\zeta < \theta : i \in X_\zeta\}$ is finite. So $\theta <\ \mathrm{reg}_\otimes(D)$, contradiction to the assumption $(*)$. $\square_{3.10}$

Note we can conclude

**3.12 Claim.**

$$\prod_{i<\kappa} \lambda_i / D = \sup\{(\prod_{i<\kappa} f(i))^{<\mathrm{reg}_\otimes(D_1)} + \mathrm{hcf}_{D_1}(\prod_{i<\kappa}\lambda_i)^{<\mathrm{reg}_\otimes(D_1)} : D_1 \textit{ is a filter on } \kappa \textit{ extending } D \textit{ such that } A \in D_1^+ \Rightarrow \prod_{i<\kappa}\lambda_i/(D_1 + A) = \prod_{i<\kappa}\lambda_i/D_1 \textit{ and } f \in \theta^\kappa, f(i) \leq \lambda_i\}$$

*Proof.* The inequality $\geq$ should be clear by 3.8(3). For the other direction let $\mu$ be the right side cardinality and let $D_1 = \{\kappa \setminus A : \text{if } A \in D^+ \text{ then } \prod_{i<\kappa}\lambda_i/D \leq \mu\}$, so we know by 3.8(2) that $D_1$ is a filter on $\kappa$ extending $D$. Now $\mu \geq {\aleph_0}^\kappa/D$ (by the term $(\prod_i f(i)/D_1)^{<\mathrm{reg}_\otimes(D_1)})$ so by 3.9 we have $\prod_{i<\kappa}\lambda_i/D_1 > \mu$. By 3.10 (see 3.11(1)) we get a contradiction. $\square_{3.12}$

Next we deal with existence of $<_D$-eub.

**3.13 Claim.** *1) Assume $D$ a filter on $\kappa$, $g^*_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \delta$, $\bar g^* = \langle g^*_\alpha : \alpha < \delta\rangle$ is $\leq_D$-increasing, and*

$$(*) \qquad \mathrm{cf}(\delta) \geq \theta \geq \mathrm{reg}_*(D).$$

*Then at least one of the following holds:*

- *(A) $\langle g^*_\alpha : \alpha < \delta\rangle$ has a $<_D$-eub $g \in {}^\kappa\mathrm{Ord}$; moreover, $\theta \leq \liminf_D \langle \mathrm{cf}[g(i)] : i < \kappa\rangle$*
- *(B) $\mathrm{cf}(\delta) = \mathrm{reg}_*(D)$*

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



(*C*) *for some club $C$ of $\delta$ and some $\theta_1 < \theta$ and $\gamma_i < \theta_1^+$ and $w_i \subseteq \mathrm{Ord}$ of order type $\gamma_i$ for $i < \kappa$, there are $f_\alpha \in \prod\limits_{i<\kappa} w_i$ (for $\alpha \in C$) such that $f_\alpha(i) = \min(w_i \setminus g^*_\alpha(i))$ and $\alpha \in C \ \& \ \beta \in C \ \& \ \alpha < \beta \Rightarrow f_\alpha \le_D f_\beta \ \& \ \neg f_\alpha =_D f_\beta \ \& \ \neg f_\alpha \le_D g^*_\beta \ \& \ g^*_\alpha \le f_\alpha$.*

*2) In (C) above if for simplicity $D$ is an ultrafilter we can find $w_i \subseteq \mathrm{Ord}$, $\mathrm{otp}(w_i) = \gamma_i$, $\langle \alpha_\xi : \xi < \mathrm{cf}(\delta)\rangle$ increasing continuous with limit $\delta$, and $h_\varepsilon \in \prod\limits_{i<\kappa} w_i$ such that $f_{\alpha_\varepsilon} <_D h_\varepsilon <_D f_{\alpha_{\varepsilon+1}}$, moreover, $\bigwedge\limits_{i<\kappa} \gamma_i < \omega$.*

*Proof.* 1) Let $\sigma = \mathrm{reg}_*(D)$. We try to choose by induction on $\zeta < \sigma$, $g_\zeta$, $f_{\alpha,\zeta}$ (for $\alpha < \delta$), $\bar A^\zeta$, $\alpha_\zeta$ such that:

(*a*) $\bar A^\zeta = \langle A^\zeta_i : i < \kappa\rangle$

(*b*) $A^\zeta_i = \{f_{\alpha_\varepsilon,\varepsilon}(i), g_\zeta(i) : \varepsilon < \zeta\} \cup \{[\sup_{\alpha<\delta} g^*_\alpha(i)] + 1\}$

(*c*) $f_{\alpha,\zeta}(i) = \mathrm{Min}(A^\zeta_i \setminus g^*_\alpha(i))$ (and $f_{\alpha,\zeta} \in {}^\kappa\mathrm{Ord}$, of course)

(*d*) $\alpha_\zeta$ is the first $\alpha$, $\bigcup\limits_{\varepsilon<\zeta} \alpha_\varepsilon < \alpha < \delta$ such that $[\beta \in [\alpha,\delta) \Rightarrow f_{\beta,\zeta} = f_{\alpha,\zeta} \mod D]$ if there is one

(*e*) $g_\zeta \le f_{\alpha_\zeta,\zeta}$; moreover, $g_\zeta < \max\{f_{\alpha_\zeta,\zeta}, 1_\kappa\}$ but for no $\alpha < \delta$ do we have $g_\zeta < \max\{g^*_\alpha, 1\} \mod D$

Let $\zeta^*$ be the first for which they are not defined (so $\zeta^* \le \sigma$).
Note

$$(*) \quad \varepsilon < \xi < \zeta^* \ \& \ \alpha_\xi \le \alpha < \delta \Rightarrow f_{\alpha_\varepsilon,\varepsilon} =_D f_{\alpha,\varepsilon} \ \& \ f_{\alpha,\xi} \le f_{\alpha,\varepsilon} \ \& \ f_{\alpha,\xi} \ne_D f_{\alpha,\varepsilon}.$$

[Why last phrase? applying clause (e) above, second phrase with $\alpha$, $\varepsilon$ here standing for $\alpha$, $\zeta$ there we get $A_0 =: \{i < \kappa : \max\{g^*_\alpha(i), 1\} \le g_\varepsilon(i)\} \in D^+$ and applying clause (e) above first phrase with $\varepsilon$ here standing for $\zeta$ there we get $A_1 = \{i < \kappa : g_\varepsilon(i) < f_{\alpha,\varepsilon}(i)$ or $g_\varepsilon(i) = 0 = f_{\alpha,\varepsilon}(i)\} \in D$, hence $A_0 \cap A_1 \in D^+$, and $g_\varepsilon(i) > 0$ for $i \in A_0 \cap A_1$ (even for $i \in A_0$). Also by clause (c) above $g^*_\alpha(i) \le g_\varepsilon(i) \Rightarrow f_{\alpha,\xi}(i) \le g_\varepsilon(i)$. Now by the last two sentences $i \in A_0 \cap A_1 \Rightarrow g^*_\alpha(i) \le g_\varepsilon(i) < f_{\alpha,\varepsilon}(i) \Rightarrow f_{\alpha,\xi}(i) \le g_\varepsilon(i) < f_{\alpha,\varepsilon}(i)$, together $f_{\alpha,\xi} \ne_D f_{\alpha,\varepsilon}$ as required.]

<u>Case A</u>: $\zeta^* = \sigma$ and $\bigcup\limits_{\zeta<\sigma} \alpha_\zeta < \delta$.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



Let $\alpha(*) = \bigcup_{\zeta<\sigma} \alpha_\zeta$, for $\zeta < \sigma$ let $y_\zeta = \{i < \kappa : f_{\alpha(*),\zeta}(i) \neq f_{\alpha(*),\zeta+1}(i)\} \neq \emptyset$ mod $D$. Now for $i < \kappa$, $\langle f_{\alpha(*),\zeta}(i) : \zeta < \sigma\rangle$ is non increasing so $i$ belongs to finitely many $y_\zeta$'s only, so $\langle y_\zeta : \zeta < \sigma\rangle$ contradict $\sigma \geq \mathrm{reg}_*(D)$.

Case B: $\zeta^* = \sigma$ and $\bigcup\limits_{\zeta<\sigma} \alpha_\zeta = \delta$.

So possibility (B) of Claim 3.13 holds.

Case C: $\zeta^* < \sigma$.

Still $A_i^{\zeta^*}$ $(i < \kappa)$, $f_{\alpha,\zeta^*}$ $(\alpha < \delta)$ are well defined.

Subcase C1: $\alpha_{\zeta^*}$ cannot be defined.

Then possibility $C$ of 3.13 holds (use $w_i =: A_i^{\zeta^*}$, $f_\beta = f_{\alpha_{\zeta^*}+\beta,\zeta^*}$).

Subcase C2: $\alpha_{\zeta^*}$ can be defined.

Then $f_{\alpha_{\zeta^*},\zeta^*}$ is a $<_D$-eub of $\langle g^*_\alpha : \alpha < \delta\rangle$ as otherwise there is $g_{\zeta^*}$ as required in clause (e). Now $f_{\alpha^*_\zeta,\zeta^*}$ is almost as required in possibility (A) of Claim 3.13 only the second phrase is missing. If for no $\theta_1 < \theta$, $\{i < \kappa : \mathrm{cf}[f_{\alpha_{\zeta^*},\zeta^*}(i)] \leq \theta_1\} \in D^+$, then possibility (A) holds.

So assume $\theta_1 < \theta$ and $B =: \{i < \kappa : \aleph_0 \leq \mathrm{cf}[f_{\alpha_{\zeta^*},\zeta^*}(i)] \leq \theta_1\}$ belongs to $D^+$, we shall try to prove that possibility (C) holds, thus finishing. Now we choose $w_i$ for $i < \kappa$: for $i \in \kappa$ we let $w_i^0 =: \{f_{\alpha_{\zeta^*},\zeta^*}(i), [\sup\limits_{\alpha<\delta} g^*_\alpha(i)] + 1\}$, for $i \in B$ let $w_i^1$ be an unbounded subset of $f_{\alpha_{\zeta^*},\zeta^*}(i)$ of order type $\mathrm{cf}[f_{\alpha_{\zeta^*},\zeta^*}(i)]$ and for $i \in \kappa \setminus B$ let $w_i^1 = \emptyset$, lastly let $w_i = w_i^0 \cup w_i^1$, so $|w_i| \leq \theta_1$ as required in possibility (C). Define $f_\alpha \in {}^\kappa\mathrm{Ord}$ by $f_\alpha(i) = \min(w_i \setminus g^*_\alpha(i))$ (by the choice of $w_i^0$ it is well defined). So $\langle f_\alpha : \alpha < \delta\rangle$ is $\leq_D$-increasing; if for some $\alpha^* < \delta$, for every $\alpha \in [\alpha^*, \delta)$ we have $f_\alpha/D = f_{\alpha^*}/D$, we could define $g_{\zeta^*} \in {}^\kappa\mathrm{Ord}$ by:

$g_{\zeta^*} \restriction B = f_{\alpha^*}$ (which is $< f_{\alpha_{\zeta^*},\zeta^*}$),

$g_{\zeta^*} \restriction (\kappa \setminus B) = 0_{\kappa\setminus B}$.

Now $g_{\zeta^*}$ is as required in clause (e) so we get contradiction to the choice of $\zeta^*$. So there is no $\alpha^* < \delta$ as above so for some club $C$ of $\delta$ we have $\alpha < \beta \in C \Rightarrow f_\alpha \neq_D f_\beta$, so we have actually proved possibility (C).

2) Easy (for $\bigwedge\limits_i \gamma_i < \omega$, without loss of generality $\theta = \mathrm{reg}_*(D)$ but $\mathrm{reg}_*(D) = \mathrm{reg}(D)$ so $\theta_1 < \mathrm{reg}(D)$). $\square_{3.13}$

**3.14 Claim.** *1) In 3.13(1), if $\lambda = \delta = \mathrm{cf}(\lambda)$, $\bar g^*$ obeys $\bar a$ ($\bar a$ as in 2.1), $\bar a$ a $\theta$-weak $(S,\theta)$-continually condition, $S \subseteq \lambda$ unbounded, then clause (C) of 3.13 implies:*
*(C)$'$ there are $\theta_1 < \mathrm{reg}_*(D)$ and $A_\varepsilon \in D^+$ for $\varepsilon < \theta$ such that the intersection of*

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



*any* $\theta_1^+$ *of the sets* $A_\varepsilon$ *is empty (equivalently* $i < \kappa \Rightarrow (\exists^{\leq \theta_1} \varepsilon)[i \in A_\varepsilon]$ *(reminds* $(\sigma, \theta_1^+)$*-regularity of ultrafilters).*
*2) We can in 3.13(1) weaken the assumption* $(*)$ *to* $(*)'$ *below if in the conclusion we weaken clause (A) to (A)' where*

$(*)'$ $\mathrm{cf}(\delta) \geq \theta \geq \mathrm{reg}(D)$

(A)' *there is a* $\leq_D$*-upper bound* $f$ *of* $\{g^*_\alpha : \alpha < \delta\}$ *such that no* $f' <_D f$ *(of course* $f' \in {}^\kappa\mathrm{Ord}$*) is a* $\leq_D$*-upper bound of* $\{g^*_\alpha : \alpha < \delta\}$ *and* $\theta \leq$ *lim inf*$_D \langle \mathrm{cf}[f(i)] : i < \kappa \rangle$.

*3) If* $g^*_\alpha \in {}^\kappa\mathrm{Ord}$, $\langle g^*_\alpha : \alpha < \delta \rangle$ *is* $<_D$*-increasing and* $f \in {}^\kappa\mathrm{Ord}$ *satisfies* $(A)'$ *above and*

$(*)''$ $\mathrm{cf}(\delta) \geq \mathrm{wsat}(D)$ *and for some* $A \in D$ *for every* $i < \kappa$, $\mathrm{cf}(f(i)) \geq \mathrm{wsat}(D)$

*then for some* $B \in D^+$ *we have* $\prod\limits_{i<\kappa} \mathrm{cf}[f(i)]/(D + B)$ *has true cofinality* $\mathrm{cf}(\delta)$.

*Remark.* Compare with 2.9.

*Proof.* 1) By the choice of $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ as $C$ (in clause (c) of 3.14(1)) is a club of $\lambda$, we can find $\beta < \lambda$ such that letting $\langle \alpha_\varepsilon : \varepsilon < \theta \rangle$ list $\{\alpha \in a_\beta : \mathrm{otp}(\alpha \cap a_\beta) < \theta\}$ (or just a subset of it) we have $(\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap C \neq \emptyset$.

Let $\gamma_\varepsilon \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap C$, and $\xi_\varepsilon \in (\alpha_\varepsilon, \alpha_{\varepsilon+1})$ be such that $\{\alpha_\zeta : \zeta \leq \varepsilon\} \subseteq a_{\xi_\varepsilon}$, and as we can use $\langle \alpha_{2\varepsilon} : \varepsilon < \theta \rangle$, without loss of generality $\xi_\varepsilon < \gamma_\varepsilon$. For $\zeta < \theta$ let $B_\zeta = \{i < \kappa : f_{\alpha_\zeta}(i) < f_{\beta_\zeta}(i) < f_{\gamma_\zeta}(i) < f_{\alpha_{\zeta+1}}(i)$ and $\sup\{f_{\alpha_\xi}(i) + 1 : \xi < \zeta\} < \sup\{f_{\alpha_\xi}(i) + 1 : \xi < \zeta + 1\}$.
2) In the proof of 3.13 we replace clause (e) by

$(e')$ $g_\zeta \leq f_{\alpha_\zeta, \zeta}$ and for $\alpha < \delta$ we have $f_\alpha \leq g_\zeta \mod D$.

3) By 1.13(1). $\square_{3.14}$

**3.15 Claim.** *1) Assume* $\lambda = \mathrm{tcf}(\prod \bar{\lambda}/D)$ *and* $\mu = \mathrm{cf}(\mu) < \lambda$ *then there is* $\bar{\lambda}' <_D \bar{\lambda}$, $\bar{\lambda}'$ *a sequence of regular cardinals and* $\mu = \mathrm{tcf}(\prod \bar{\lambda}'/D)$ *provided that*

$(*)$ $\mu > \mathrm{reg}_*(D), \min(\bar{\lambda}) > \mathrm{reg}_*^{\sigma^+}(D)$ *whenever* $\sigma < \mathrm{reg}_*(D)$

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



*2) Let $I^*$ be the ideal dual to $D$, and assume $(*)$ above. If $(*)(\alpha)$ of 1.9 holds and $\mu$ is semi-normal (for $(\bar\lambda, I^*)$) then it is normal.*

*Proof.*
Case 1 $\mu <$ lim $\inf_D(\bar\lambda)$.
We let

$$\lambda' = \begin{cases} \mu & \text{if } \mu < \lambda_i \\ 1 & \text{if } \mu \geq \lambda_i \end{cases}$$

and we are done.

Case 2: lim $\inf_D(\bar\lambda) \geq \theta \geq \operatorname{reg}_*(D)$, $\mu > \theta$, and $(\forall \sigma < \operatorname{reg}_*(D))[\operatorname{reg}^\sigma_*(D) < \theta]$.
Let $\theta =: \operatorname{reg}_*(D)$. There is an unbounded $S \subseteq \mu$ and an $(S,\theta)$-continuity system $\bar a$ (see 2.5). As $\Pi\bar\lambda/D$ has true cofinality $\lambda$, $\lambda > \mu$ clearly there are $g^*_\alpha \in \Pi\bar\lambda$ for $\alpha < \mu$ such that $\bar g^* = \langle g^*_\alpha : \alpha < \mu\rangle$ obeys $\bar a$ (exists as $\theta \leq$ lim $\inf_D(\bar\lambda)$).
Now if in claim 3.13(1) for $\bar g^*$ possibility (A) holds, we are done. By 3.14(1) we get that for some $\sigma < \operatorname{reg}_*(D), \operatorname{reg}^\sigma_*(I) \geq \mu$, contradiction.

Case 3: lim $\inf_D(\bar\lambda) \geq \theta$ $\operatorname{reg}_*(D)$, $\mu \geq \theta$, and $(\forall \sigma < \operatorname{reg}_*(D))[\operatorname{reg}^\sigma_*(D) < \theta]$.
Like the proof of [Sh:g, Ch.II,1.5B] using the silly square.

* * *

We turn to other measures of $\Pi\bar\lambda/D$.

**3.16 Definition.**

(a) $T^0_D(\bar\lambda) = \sup\{|F| : F \subseteq \Pi\bar\lambda \text{ and } f_1 \neq f_2 \in F \Rightarrow f_1 \neq_D f_2\}$
(b)

$$T^1_D(\bar\lambda) = \operatorname{Min}\{|F| : \begin{array}{ll} (i) & F \subseteq \Pi\bar\lambda \\ (ii) & f_1 \neq f_2 \in F \Rightarrow f_1 \neq_D f_2 \\ (iii) & F \text{ maximal under } (i)+(ii)\} \end{array}$$

(c) $T^2_D(\bar\lambda) = \operatorname{Min}\{|F| : F \subseteq \Pi\bar\lambda$ and for every $f_1 \in \Pi\bar\lambda$, for some $f_2 \in F$ we have $\neg f_1 \neq_D f_2\}$
(d) If $T^0_D(\bar\lambda) = T^1_D(\bar\lambda) = T^2_D(\bar\lambda)$ then let $T_D(\bar\lambda) = T^l_D(\bar\lambda)$ for $l < 3$
(e) for $f \in {}^\kappa\mathrm{Ord}$ and $\ell < 3$ let $T^l_D(f)$ means $T^l_D(\langle f(\alpha) : \alpha < \kappa\rangle)$.

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



**3.17 Theorem.** *0) If $D_0 \subseteq D_1$ are filters on $\kappa$ then $T^\ell_{D_0}(\bar\lambda) \le T^\ell_{D_1}(\bar\lambda)$ for $\ell = 0,2$. Also if $\kappa = A_0 \cup A_1, A_0 \in D^+$, and $A_1 \in D^+$ then $T^\ell_D(\bar\lambda) = \min\{T^\ell_{D+A_0}(\bar\lambda), T^\ell_{D+A_1}(\bar\lambda)\}$ for $\ell = 0,2$.*
*1) $\mathrm{htcf}_D(\Pi\bar\lambda) \le T^2_D(\bar\lambda) \le T^1_D(\bar\lambda) \le T^0_D(\bar\lambda)$.*
*2) If $T^0_D(\bar\lambda) > |\mathscr{P}(\kappa)/D|$ or just $T^0_D(\lambda) > \mu$, and $\mathscr{P}(\kappa)/D$ satisfies the $\mu^+$-c.c. then $T^0_D(\bar\lambda) = T^1_D(\bar\lambda) = T^2_D(\bar\lambda)$ so the supremum in 3.16(a) is obtained (so, e.g., $T^0_D(\bar\lambda) > 2^\kappa$ suffice).*
*3) $T^0_D(\bar\lambda)^{<\mathrm{reg}(D)} = T^0_D(\bar\lambda)$ (each $\lambda_i$ infinite of course).*
*4) $[\mathrm{htcf}_D \prod_{i<\kappa} f(i)] \le T^2_D(f) \le [\mathrm{htcf}_D \prod_{i<\kappa} f(i)]^{<\mathrm{reg}(D)} + \mathrm{wsat}(D)^\kappa/D$.*
*5) If $D$ is an ultrafilter $|\Pi\bar\lambda/D| = T^e_D(\bar\lambda)$ for $e \le 2$.*
*6) In (4), if $\bigwedge_{i<\kappa} f(i) \ge 2^\kappa$ (or just $(\mathrm{wsat}(D)+2)^\kappa/D \le \min_{i<\kappa} f(i)$), the second and third terms are equal.*
*7) If the $\sup$ in the definition of $T^0_D(\bar\lambda)$ is not obtained then it has cofinality $\ge \mathrm{reg}(D)$ and even is regular.*

*Proof.* 0) Check.
1) First assume $\mu =: T^2_D(\bar\lambda) < \mathrm{htcf}_D(\Pi\bar\lambda)$; then we can find $\mu^* = \mathrm{cf}(\mu^*) \in (\mu, \mathrm{htcf}_D(\prod\bar\lambda)]$ and $\bar\mu = \langle \mu_i : i < \kappa\rangle$, a sequence of regular cardinals, $\bigwedge_{i<\kappa} \mu_i \le \lambda_i$ such that $\mu^* = \mathrm{tcf}(\Pi\bar\mu/D)$ and let $\langle f_\alpha : \alpha < \mu^*\rangle$ exemplify this. Now let $F$ exemplify $\mu = T^2_D(\bar\lambda)$, for each $g \in F$ let

$$g' \in \prod_{i<\kappa} \mu_i \text{ be } : g'(i) = \begin{cases} g(i) & \text{if } g(i) < \mu_i \\ 0 & \text{otherwise.}\end{cases}$$

So there is $\alpha(g) < \mu^*$ such that $g' <_D f_{\alpha(g)}$. Let $\alpha^* = \sup\{\alpha(g) : g \in F\}$, now $\alpha^* < \mu^*$ (as $\mu^* = \mathrm{cf}(\mu^*) > \mu = |F|$). So $g \in F \Rightarrow g \ne_D f_{\alpha^*}$, contradiction. So really $T^2_D(\bar\lambda) \le \mathrm{htcf}_D(\Pi\bar\lambda)$ as required.

If $F$ exemplifies the value of $T^1_D(\bar\lambda)$, it also exemplifies $T^2_D(\bar\lambda) \le |F|$ hence $T^2_D(\bar\lambda) \le T^1_D(\bar\lambda)$.

Lastly if $F$ exemplifies the value of $T^1_D(f)$ it also exemplifies $T^0_D(\bar\lambda) \ge |F|$, so $T^1_D(\bar\lambda) \le T^0_D(\bar\lambda)$.
2) Let $\mu$ be $|\mathscr{P}(\kappa)/D|$ or at least $\mu$ is such that the Boolean algebra $\mathscr{P}(\kappa)/D$ satisfies the $\mu^+$-c.c. Assume that the desired conclusion fails so $T^2_D(\bar\lambda) < T^0_D(\bar\lambda)$, so there is $F_0 \subseteq \Pi\bar\lambda$, such that $[f_1 \ne f_2 \in F_0 \Rightarrow f_1 \ne_D f_2]$, and $|F_0| > T^2_D(\bar\lambda) + \mu$ (by the definition of $T^0_D(\bar\lambda)$). Also there is $F_2 \subseteq \Pi\bar\lambda$ exemplifying the value of $T^2_D(\bar\lambda)$. For every $f \in F_0$ there is $g_f \in F_2$ such that $\neg f \ne_D g_f$ (by the choice of $F_2$). As $|F_0| > T^2_D(\bar\lambda) + \mu$ for some $g \in F_2$, $F^* =: \{f \in F_0 : g_f = g\}$ has cardinality

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



$> T^2_D(f) + \mu$. Now for each $f \in F^*$ let $A_f = \{i < \kappa : f(i) = g(i)\}$ clearly $A_f \in D^+$. Now $f \mapsto A_f/D$ is a function from $F^*$ into $\mathscr{P}(\kappa)/D$, hence, if $\mu = |\mathscr{P}(\kappa)/D|$, it is not one to one (by cardinality consideration) so for some $f' \neq f''$ from $F^*$ (hence form $F_0$) we have $A_{f'}/D = A_{f''}/D$; but so

$$\{i < \kappa : f'(i) = f''(i)\} \supseteq \{i < \kappa : f'(i) = g(i)\} \cap \{i < \kappa : f''(i) = g(i)\} = A_{f'}/D$$

hence is $\neq \emptyset \mod D$, so $\neg f' \neq_D f''$, contradition the choice of $F_0$. If $\mu \neq |\mathscr{P}(\kappa)/D|$ (as $F^* \subseteq F_0$ by the choice of $F_0$) we have:

$$f_1 \neq f_2 \in F^* \Rightarrow A_{f_1} \cap A_{f_2} = \emptyset \mod D$$

so $\{A_f : f \in F^*\}$ contradicts "the Boolean algebra $\mathscr{P}(\kappa)/D$ satisfies the $\mu^+$-c.c.".
3) Assume that $\theta <$ reg$(D)$ and[7] $\mu \leq^+ T^0_D(\bar\lambda)$. As $\mu \leq^+ T^0_D(\bar\lambda)$ we can find $f_\alpha \in \Pi\bar\lambda$ for $\alpha < \mu$ such that $[\alpha < \beta \Rightarrow f_\alpha \neq_D f_\beta]$. Also (as $\theta <$ reg$(D)$) we can find $\{A_\varepsilon : \varepsilon < \theta\} \subseteq D$ such that for every $i < \kappa$ the set $w_i =: \{\varepsilon < \theta : i \in A_\varepsilon\}$ is finite. Now for every function $h : \theta \to \mu$ we define $g_h$, a function with domain $\kappa$:

$$g_h(i) = \{(\varepsilon, f_{h(\varepsilon)}(i)) : \varepsilon \in w_i\}.$$

So $|\{g_h(i) : h \in {}^\theta\mu\}| \leq (\lambda_i)^{|w_i|} = \lambda_i$, and if $h_1 \neq h_2$ are from ${}^\theta\mu$ then for some $\varepsilon < \theta$, $h_1(\varepsilon) \neq h_2(\varepsilon)$ so $B_{h_1,h_2} = \{i : f_{h_1(\varepsilon)}(i) \neq f_{h_2(\varepsilon)}(i)\} \in D$ that is $B_{h_1,h_2} \cap A_\varepsilon \in D$ so

- $\otimes_1$ if $i \in B_{h_1,h_2} \cap A_\varepsilon$ then $\varepsilon \in w_i$, so $g_{h_1}(i) \neq g_{h_2}(i)$
- $\otimes_2$ $B_{h_1,h_2} \cap A_\varepsilon \in D$.

So $\langle g_h : h \in {}^\theta\mu\rangle$ exemplifies $T^0_D(\bar\lambda) \geq \mu^\theta$. If the supremum in the definition of $T^0_D(\bar\lambda)$ is obtained we are done. If not then $T^0_D(\bar\lambda)$ is a limit cardinal, and by the proof above:

$$[\mu < T^0_D(\bar\lambda) \quad \& \quad \theta < \text{reg}(D) \quad \Rightarrow \quad \mu^\theta < T^0_D(\bar\lambda)].$$

So if $T^0_D(\bar\lambda)$ has cofinality $\geq$ reg$(D)$ we are done; otherwise let it be $\sum_{\varepsilon<\theta}\mu_\varepsilon$ with $\mu_\varepsilon < T^0_D(\bar\lambda)$ and $\theta <$ reg$(D)$. Note that by the previous sentence $T^0_D(\bar\lambda)^\theta = T^0_D(\bar\lambda)^{<\text{reg}(D)} = \prod_{\varepsilon<\theta}\mu_\varepsilon$, and let $\{f^\varepsilon_\alpha : \alpha < \mu_\varepsilon\} \subseteq \Pi\bar\lambda$ be such that $[\alpha < \beta \Rightarrow f^\varepsilon_\alpha \neq_D f^\varepsilon_D]$ and repeat the previous proof with $f^\varepsilon_{h(\varepsilon)}$ replacing $f_{h(\varepsilon)}$.

[7] $\leq^+$ means the left side is a supremum, right bigger than the left or equal but the supremum is obtained

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



4) For the first inequality assume it fails so $\mu =: T^2_D(f) < \text{ htcf}_D(\prod\limits_{i<\kappa} f(i))$ hence for some $g \in \prod\limits_{i<f(i)} (f(i)+1)$, $\text{tcf}(\prod\limits_{i<\kappa} g(i), <_D)$ is $\lambda$ with $\lambda = \text{ cf}(\lambda) > \mu$. Let $\langle f_\alpha : \alpha < \lambda\rangle$ exemplifies this. Let $F$ be as in the definition of $T^2_D(f)$, now for each $h \in F$, there is $\alpha(h) < \lambda$ such that

$$\{i < \kappa : \text{ if } h(i) < g(i) \text{ then } h(i) < f_{\alpha(g)}(i)\} \in D.$$

Let $\alpha^* = \sup\{\alpha(h) + 1 : h \in F\}$, now $f_{\alpha^*} \in \prod\limits_{i<\kappa} f(i)$ and $h \in F \Rightarrow h \neq_D f_{\alpha^*}$ contradicting the choice of $F$.

For the second inequality repeat the proof of 3.10 except that here we prove $F =: \bigcup\limits_{\zeta<\theta}(N_\zeta \cap \prod\limits_{i<\kappa} f(i))$ exemplifies $T^2_D(f) \leq \lambda$; we replace clause (g) in the proof by

$(g)'$ $g^* < f^2_{\zeta+1} < f^1_\zeta \mod D$

the construction is for $\zeta < \text{ reg}(D)$ and if we find satisfy $\neg f^1_\zeta \neq_D g^*$ we are done.
5) Straightforward.
6) Note that all those cardinals are $\geq 2^\kappa$ and $2^\kappa \geq \text{wsat}(D)^\kappa/D$. Now write successively inequalities from (2), (4), (1) and (3):

$$T^0_D(f) = T^2_D(f) \leq [\text{htcf}_D \prod_{i<\kappa} f(i)]^{<\text{reg}(D)} \leq [T^0_D(f)]^{<\text{reg}(D)} = T^0_D(f).$$

7) See proof of part (3). Moreover, if $\mu = \sum\limits_{\varepsilon<\tau} \mu_\varepsilon$, $\tau < T^0_D(\bar\lambda)$, $\mu_\varepsilon < T^0_D(\bar\lambda)$ as exemplified by $\{f_\varepsilon : \varepsilon < \tau\}$, $\{f^\varepsilon_\alpha : \alpha < \mu_\varepsilon\}$ respectively. Let $g_\alpha$ be: if $\sum\limits_{\varepsilon<\zeta} \mu_\varepsilon < \alpha < \sum\limits_{\varepsilon\leq\zeta} \mu_\varepsilon$ then $g_\alpha(i) = (f_\varepsilon(i), f^\varepsilon_\alpha(i))$. So $\{g_\alpha : \alpha < \mu\}$ show: if $T^0_D(\bar\lambda)$ is singular then the supremum is obtained. $\square_{3.17}$

**3.18 Claim.** *Assume $D$ is a filter on $\kappa$, $f \in {}^\kappa\text{Ord}$, $\mu^{\aleph_0} = \mu$ and $2^\kappa < \mu$, $T_D(f)$, (see Definition 3.16(d) and Theorem 3.17(2)). If $\mu < T_D(f)$ then for some sequence $\bar\lambda \leq f$ of regulars, $\mu^+ = \text{tcf}(\Pi\bar\lambda/D)$, or at least*

(∗) *there are* $\langle\langle\lambda_{i,n} : n < n_i\rangle : i < \kappa\rangle$, $\lambda_{i,n} = \mathrm{cf}(\lambda_{i,n}) < f(i)$ *and a filter* $D^*$ *on* $\bigcup_{i<\kappa}\{i\} \times n_i$ *such that:* $\mu^+ = \mathrm{tcf}(\prod_{(i,n)} \lambda_{i,n}/D^*)$ *and* $D = \{A \subseteq \kappa : \bigcup_{i\in A}\{i\} \times n_i \in D^*\}$.

*Also the inverse is true.*

*Remark 3.15A.* 1) It is not clear whether the first possibility may fail. We have explained earlier the doubtful role of $\mu^{\aleph_0} = \mu$.
2) We can replace $\mu^+$ by any regular $\mu$ such that $\bigwedge_{\alpha<\mu} |\alpha|^{\aleph_0} < \mu$ and then we use 3.17(4) to get $\mu \leq^+ T_D(f)$.
(3) The assumption $2^\kappa < \mu$ can be omitted.

*Proof.* The inverse should be clear (as in the proof of 3.7, by 3.17(3)). Without loss of generality $f(i) > 2^\kappa$ for $i < \kappa$, and trivially $\mathrm{wsat}(D))^\kappa/D \leq 2^\kappa$, so by 3.17(4)

$$T_D(f) \leq [\mathrm{htcf}_D(\prod_{i<\kappa} f(i)]^{<\mathrm{reg}(D)}.$$

If $\mu <$ $\mathrm{htcf}_D(\prod_{i<\kappa} f(i))$ we are done (by 3.15(1)), so assume $\mathrm{htcf}_D(\prod_{i<\kappa} f(i)) \leq \mu$, but we have assumed $\mu < T_D(f)$ so we can conclude $\mu^{<\mathrm{reg}(D)} \geq \mu^+$. Let $\chi \leq \mu$ be minimal such that $\bigvee_{\theta< \mathrm{reg}(D)} \chi^\theta \geq \mu$, and let $\theta =:$ $\mathrm{cf}(\chi)$ so, as $\mu > 2^\kappa$ we know $\chi^{\mathrm{cf}\chi} = \chi^{<\mathrm{reg}(D)} = \mu^{<\mathrm{reg}(D)} \geq \mu^+, \chi > 2^\kappa, \bigwedge_{\alpha<\chi} |\alpha|^{<\mathrm{reg}(D)} < \chi$. By the assumption $\mu = \mu^{\aleph_0}$ we know $\theta > \aleph_0$ (of course $\theta$ is regular). By [Sh:g, Ch.VIII,1.6](2),IX,3.5 and [Sh 513, 6.12] there is a strictly increasing sequence $\langle\mu_\varepsilon : \varepsilon < \theta\rangle$ of regular cardinals with limit $\chi$ such that $\mu^+ =$ $\mathrm{tcf}(\prod_{\varepsilon<\theta} \mu_\varepsilon J^{bd}_\theta)$.

As clearly $\chi \leq$ $\mathrm{htcf}_D(\prod_{i<\kappa} f(i))$, we can find for each $\varepsilon < \theta$, a sequence $\bar\lambda^\varepsilon = \langle\lambda^\varepsilon_i : i < \kappa\rangle$ such that $\lambda^\varepsilon_i =$ $\mathrm{cf}(\lambda^\varepsilon_i) \leq f(i)$, and $\mathrm{tcf}(\prod_{i<\kappa} \lambda^\varepsilon_i/D) = \mu_\varepsilon$, also without loss of generality $\lambda^\varepsilon_i > 2^\kappa$. Let $\langle A_\varepsilon : \varepsilon < \theta\rangle$ exemplify $\theta <$ $\mathrm{reg}(D)$ and $n_i = |\{\varepsilon < \theta : i \in A_\varepsilon\}|$ and $\{\lambda_{i,n} : n < \omega\}$ enumerate $\{\lambda^\varepsilon_i : \varepsilon$ satisfies $i \in A_\varepsilon\}$, so we have gotten (∗). $\square_{3.18}$

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3.19 Conclusion Suppose $D$ is an $\aleph_1$-complete filter on $\kappa$. If $\lambda_i \geq 2^\kappa$ for $i < \kappa$ and $\sup_{A \in D^+} T_{D+A}(\bar{\lambda}) > \mu^{\aleph_0}$ then for some $\lambda'_i = \mathrm{cf}(\lambda'_i) \leq \lambda_i$ we have

$$\sup_{A \in D^+} \mathrm{tcf}_{D+A}(\prod_{i<\kappa} \lambda'_i) > \mu.$$

3.20 Conclusion Let $D$ be an $\aleph_1$-complete filter on $\kappa$. If for $i < \kappa, B_i$ is a Boolean algebra and $\lambda_i < \mathrm{Depth}^+(B_i)$ (see below) and

$$2^\kappa < \mu^{\aleph_0} < \sup_{A \in D^+} T_{D+A}(\bar{\lambda})$$

then $\mu^+ < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$.

*Proof.* Use 3.28 below and 3.19 above.

**3.21 Definition.** For a partial order $P$ (e.g., a Boolean algebra) let $\mathrm{Depth}^+(P) = \mathrm{Min}\{\lambda$:we cannot find $a_\alpha \in P$ for $\alpha < \lambda$ such that $\alpha < \beta \Rightarrow a_\alpha <_P a_\beta\}$.

3.22 Discussion 1) We conjecture that in 3.19 (and 3.20) the assumption "$D$ is $\aleph_1$-complete" can be omitted.
2) Note that our results are for $\mu = \mu^{\aleph_0}$ only; to remove this we need to improve the theorem on pp = cov (i.e., to prove $\mathrm{cf}(\lambda) = \aleph_0 < \lambda \Rightarrow \mathrm{pp}(\lambda) = \mathrm{cov}(\lambda, \lambda, \aleph_1, 2)$ (or $\sup\{\mathrm{pp}(\mu) : \mathrm{cf}(\mu) = \aleph_0 < \mu < \lambda\} = \mathrm{cf}(S_{\leq \aleph_0}(\lambda), \subseteq)$ (see [Sh:g], [Sh 430, §1]), which seems to me a very serious open problem (see [Sh:g, Analytic guide,14]).
3) In 3.20, if we can find $f_\alpha \in \prod_{i<\kappa} \lambda_i$ for $\alpha < \lambda : [\alpha < \beta < \lambda \Rightarrow f_\alpha \leq f_\beta \mod D]$ and $\neg f_\alpha =_D f_{\alpha+1}$ then $\lambda < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$. But this does not help for $\lambda$ regular $> 2^\kappa$.
4) We can approach 3.18 differently, by 3.23 - 3.26 below.

**3.23 Claim.** *If $2^{2^\kappa} \leq \mu < T_D(\bar{\lambda})$, (or at least $2^{|D|+\kappa} \leq \mu < T_D(\bar{\lambda})$) and $\mu^{<\theta} = \mu$, then for some $\theta$-complete filter $E \subseteq D$ we have $T_E(\bar{\lambda}) > \mu$.*

*Proof.* Without loss of generality $\theta$ is regular (as $\mu^{<\theta} = \mu$ & $\mathrm{cf}(\theta) < \theta \Rightarrow \mu^{<\theta^+} = \mu$). Let $\{f_\alpha : \alpha < \mu^+\} \subseteq \Pi\bar{\lambda}$, be such that $[\alpha < \beta \Rightarrow f_\alpha \neq_D f_\beta]$. We choose by induction on $\zeta$, $\alpha_\zeta < \mu^+$ as follows: $\alpha_\zeta$ is the minimal ordinal $\alpha < \mu^+$ such that $E_{\zeta,\alpha} \subseteq D$ where $E_{\zeta,\alpha}$ = the $\theta$-complete filter generated by

$$\{\{i < \kappa : f_{\alpha_\varepsilon}(i) \neq f_\alpha(i)\} : \varepsilon < \zeta\}$$

(note: each generator of $E_{\zeta,\alpha}$ is in $D$ but not necessarily $E_{\zeta,\alpha} \subseteq D$!).

Let $\alpha_\zeta$ be well defined if $\zeta < \zeta^*$, clearly $\varepsilon < \zeta \Rightarrow \alpha_\varepsilon < \alpha_\zeta$. Now if $\zeta^* < \mu^+$, then clearly $\alpha^* = \bigcup\limits_{\zeta<\zeta^*} \alpha_\zeta < \mu^+$ and for every $\alpha \in (\alpha^*, \mu^+)$, $E_{\zeta^*,\alpha} \nsubseteq D$, so for every such $\alpha$ there are $A_\alpha \in D^+$ and $a_\alpha \in [\zeta^*]^{<\theta}$ such that $A_\alpha = \bigcup_{\varepsilon\in a_\alpha}\{i < \kappa : f_{\alpha_\varepsilon}(i) = f_\alpha(i)\}$. But for every $A \in D^+$, $a \in [\zeta^*]^{<\theta}$ we have

$$\{\alpha : \alpha \in (\alpha^*, \mu^+), A_\alpha = A, a_\alpha = a\} \subseteq \{\alpha : f_\alpha \restriction A \in \prod_{i<\kappa}\{f_{\alpha_\varepsilon}(i) : \varepsilon \in a_\alpha\}\},$$

hence has cardinality $\leq \theta^\kappa \leq 2^\kappa < \mu$. Also $|[\zeta^*]^{<\theta}| \leq \mu^{<\theta} < \mu^+, \|D^+\| \leq 2^\kappa < \mu^\kappa$ so we get easy contradiction.

So $\zeta^* = \mu^+$, but the number of possible $E$'s is $\leq 2^{2^\kappa}$, hence for some $E$ we have $|\{\varepsilon < \mu^+ : E_{\varepsilon,\alpha_\varepsilon} = E\}| = \mu^+$. Necessarily $E \subseteq D$ and $E$ is $\theta$-complete, and $\{f_{\alpha_\varepsilon} : \varepsilon < \mu^+$, and $E_{\alpha_\varepsilon} = E\}$ exemplifies $T_E(\bar\lambda) > \mu$, so $E$ is as required. $\square_{3.23}$

<u>3.24 Fact</u> 1) In 3.23 we can replace $\mu^+$ by $\mu^*$ if $2^{2^\kappa} <$ cf$(\mu^*) \leq \mu^* \leq T^0_D(\bar\lambda)$ and $\bigwedge\limits_{\alpha<\mu^*} |\alpha|^{<\theta} < \mu^*$.

2) We can, in 3.23, [and 3.24(1)] replace "$T_D(\bar\lambda) > \mu$" by "$\Pi\bar\lambda/D$ has an increasing sequence of lengths $> \mu[\geq \mu]$", we can deduce this also otherwise.

**3.25 Claim.** *1) If $2^\kappa < |\Pi\bar\lambda/D|, D$ an ultrafilter on $\kappa$, $\mu =$ cf$(\mu) \leq |\Pi\bar\lambda/D|$, $\bigwedge\limits_{i<\kappa} |i|^{\aleph_0} < \mu$, and $D$ is regular <u>then</u> $\mu <$ Depth$^+(\prod\limits_{i<\kappa} \lambda_i/D)$.*

*2) Similarly for $D$ just a filter.*

*Proof.* Without loss of generality $\lambda = \lim_D\bar\lambda = \sup(\bar\lambda)$, so $|\Pi\bar\lambda/D| = \lambda^\kappa$ (by [\CK ]). If $\mu \leq \lambda$ we are done; otherwise let $\chi =$ Min$\{\chi : \chi^\kappa = \lambda^\kappa\}$, so $\chi^{\text{cf}(\chi)} = \lambda^\kappa$, cf$(\chi) \leq \kappa$ but $\lambda < \mu \leq \lambda^\kappa$ hence $\lambda^{\aleph_0} < \mu$ hence cf$(\chi) > \aleph_0$, also by $\chi's$ minimality $\bigwedge\limits_{i<\chi} |i|^{\text{cf}(\chi)} \leq |i|^\kappa < \chi$, and remember $\chi < \mu =$ cf$(\mu) \leq \chi^{\text{cf}\chi}$ so by [Sh:g, VIII,1.6](2) there is $\langle\mu_\varepsilon : \varepsilon <$ cf$(\chi)\rangle$ strictly increasing sequence of regular cardinals

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



with limit $\chi$, $\prod\limits_{\varepsilon < \mathrm{cf}(\chi)} \mu_\varepsilon / J^{\mathrm{bd}}_{\mathrm{cf}(\chi)}$ has true confinality $\mu$. Let $\chi_\varepsilon = \sup\{\mu_\zeta : \zeta < \varepsilon\} + 2^\kappa$, let $i : \kappa \to \mathrm{cf}(\chi)$ be $i(i) = \sup\{\varepsilon + 1 : \lambda_i \geq \chi_\varepsilon\}$. If there is a function $h \in \prod\limits_{i<\kappa} \mathfrak{i}(i)$ such that $\bigwedge\limits_{j<\mathrm{cf}(\chi)} \{i < \kappa : h(i) < j\} = \emptyset \mod D$ then $\prod\limits_{i<\kappa} \mu_{h(i)}/D$ has true cofinality $\mu$ as required; if not $(D, \mathfrak{i})$ is weakly normal (i.e. there is no such $h$ - see [Sh 420]). But for $D$ regular, $D$ is $\mathrm{cf}(\chi)$-regular, some $\langle A_\varepsilon : \varepsilon < \mathrm{cf}(\kappa)\rangle$ exemplifies it and $h(i) = \max\{\varepsilon : \varepsilon < \mathfrak{i}(i)$ and $i \in A_\varepsilon\}$ (maximum over a finite set) is as required.

$\square_{3.25}$

<u>3.26 Discussion</u> 1) In 3.23 (or 3.24) ' we can apply [Sh 410, §6] so $\mu = \mathrm{tcf}(\Pi \bigcup\limits_{i<\mu} \mathfrak{a}_i / D^*$, where $D = \{A \subseteq \kappa : \bigcup\limits_{i \in A} \mathfrak{a}_i \in D^*\}$ and each $\mathfrak{a}_i$ is finite.

In 3.18 we have gotten this also for $\mu \in (2^\kappa, 2^{2^\kappa})$.

**3.27 Claim.** *If $D$ is a filter on $\kappa$, $B_i$ is the interval Boolean algebra on the ordinal $\alpha_i$, and $|\prod\limits_{i<\kappa} \alpha_i / D| > 2^\kappa$ <u>then</u> for regular $\mu$ we have $\mu < \mathrm{Depth}^+(\prod\limits_{i<\kappa} B_i/D)$ iff for some $\mu_i \leq \alpha_i$ (for $i < \kappa$) and $A \in D^+$, the true cofinality of $\prod\limits_{i<\kappa} \mu_i/(D + A))$ is well defined and equal to $\mu$.*

*Proof.* The $\Rightarrow$ (i.e., only if direction) is clear. For the $\Leftarrow$ direction assume $\mu$ is regular $< \mathrm{Depth}^+(\prod\limits_{i<\kappa} B_i/D)$ so there are $f_\alpha \in \prod\limits_{i<\kappa} B_i$ such that $\prod\limits_{i<\kappa} B_i/D \models f_\alpha/D < f_\beta/D$ for $\alpha < \beta$.

Without loss of generality $\mu > 2^\kappa$. Let $f_\alpha(i) = \bigcup\limits_{\ell < n(\alpha,i)} [j_{\alpha,i,2\ell}, j_{\alpha,i.2\ell+1})$ where $j_{\alpha,i,\ell} < j_{\alpha,i,\ell+1} < \alpha_i$ for $\ell < 2n(\alpha,i)$. As $\mu = \mathrm{cf}(\mu) > 2^\kappa$ without loss of generality $n_{\alpha,i} = n_i$. By [Sh 430, 6.6D] (see more [Sh 513, 6.1]) we can find $A \subseteq A^* =: \{(i,\ell) : i < \kappa, \ell < 2n_\alpha\}$ and $\langle \gamma^*_{i,\ell} : i < \kappa, \ell < 2n_i\rangle$ such that $(i,\ell) \in A \Rightarrow \gamma^*_{i,\ell}$ is a limit ordinal and

$(*)$ for every $f \in \prod\limits_{(i,\ell)\in A} \gamma^*_{i,\ell}$ and $\alpha < \mu$ there is $\beta \in (\alpha, \mu)$ such that

$$(i,\ell) \in A^* \setminus A \Rightarrow j_{\alpha,i,\ell} = \gamma^*_{i,\ell}$$
$$(i,\ell) \in A \Rightarrow f(i,\ell) < j_{\alpha,i,\ell} < \gamma^*_{i,\ell}$$
$$(i,\ell) \in A \Rightarrow \mathrm{cf}(\gamma^*_{i,\ell}) > 2^\kappa.$$

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



Let $\ell(i) = \max\{\ell < 2n(i) : (i,\ell) \in A\}$ and let $B = \{i : \ell(i)$ well defined$\}$. Clearly $B \in D^+$ (otherwise we can find $\alpha < \beta < \mu$ such that $f_\alpha/D = f_\beta/D$, contradiction). For $(i,\ell) \in A$ define $\beta^*_{i,\ell}$ by $\beta^*_{i,\ell} = \sup\{\gamma^*_{j,m}+1 : (j,m) \in A^*$ and $\gamma^*_{j,m} < \gamma^*_{i,\ell}\}$. Now $\beta^*_{i,\ell} < \gamma^*_{i,\ell}$ as $\mathrm{cf}(\gamma^*_{i,\ell}) > 2^\kappa$. Let

$$Y = \{\alpha < \mu : \text{if}(i,\ell) \in A^* \setminus A \text{ then } j_{\alpha,i,\ell} = \gamma^*_{i,\ell} \\ \text{and if } (i,\ell) \in A \text{ then } \beta^*_{i,\ell} < j_{\alpha,\ell,i} < \gamma^*_{\ell,i}\}.$$

Let $B_1 = \{i \in B : \ell(i) \text{ is odd}\}$. Clearly $B_1 \subseteq B$ and $B \setminus B_1 = \emptyset \mod D$ (otherwise as in $(*)_1$, $(*)_2$ below get contradiction) hence $B_1 \in D^+$. Now

$(*)_1$ for $\alpha < \beta$ from $Y$ we have

$$\langle j_{\alpha,i,\ell(i)} : i \in B_1\rangle \le \langle j_{\beta,i,\ell(i)} : i \in B_1\rangle \mod (D \restriction B_1)$$

[Why? as $f_\alpha/D$ was non decreasing in $\prod\limits_{i<\kappa} B_i/D$]

$(*)_2$ for every $\alpha \in Y$ for some $\beta$, $\alpha < \beta \in Y$ we have

$$\langle j_{\alpha,i,\ell(i)} : i \in B_1\rangle < \langle j_{\beta,i,\ell(i)} : i \in B_1\rangle \mod (D \restriction B_1)$$

[Why? by $(*)$ above.]

Together for some unbounded $Z \subseteq Y$, $\big\langle\langle j_{\alpha,\ell,\ell(i)} : i \in B_1\rangle/(D \restriction B_1) : \alpha \in Z\big\rangle$ is $<_{D\restriction B_1}$-increasing, so it has a $<_{(D\restriction B_1)}$-eub (as $\mu > 2^\kappa$), say $\langle j^*_i : i \in B_1\rangle$ hence $\prod\limits_{i\in B_1} j^*_i/(D \restriction B_1)$ has true cofinality $\mu$, and clearly $j^*_i \le \gamma^*_{i,\ell(i)} \le \alpha_i$, so we have finished. $\square_{3.27}$

**3.28 Claim.** *If $D$ is a filter on $\kappa$, $B_i$ a Boolean algebra, $\lambda_i < \mathrm{Depth}^+(B_i)$ then*

(a) $\mathrm{Depth}(\prod\limits_{i<\kappa} B_i/D) \ge \sup\limits_{A\in D^+} \mathrm{tcf}(\prod\limits_{i<\kappa} \lambda_i/(D+A))$ *(i.e., on the cases tcf is well defined)*

(b) $\mathrm{Depth}^+(\prod\limits_{i<\kappa} B_i/D)$ *is* $\ge \mathrm{Depth}^+(\mathscr{P}(\kappa)/D)$ *and is at least*

$$\sup\{[\mathrm{tcf}(\prod_{i<\kappa} \lambda'_i/(D+A))]^+ : \lambda'_i < \mathrm{Depth}^+(B_i), A \in D^+\}.$$

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



*Proof.* Check.

**3.29 Claim.** *Let $D$ be a filter on $\kappa$, $\langle \lambda_i : i < \kappa \rangle$ a sequence of cardinals and $2^\kappa < \mu = \mathrm{cf}(\mu)$. Then $(\alpha) \Leftrightarrow (\beta) \Rightarrow (\gamma) \Rightarrow (\delta)$, and if $(\forall \sigma < \mu)(\sigma^{\aleph_0} < \mu)$ we also have $(\gamma) \Leftrightarrow (\delta)$ where*

- $(\alpha)$ *if $B_i$ is a Boolean algebra, $\lambda_i < \mathrm{Depth}^+(B_i)$ <u>then</u> $\mu < \mathrm{Depth}^+(\prod\limits_{i<\kappa} B_i/D)$*
- $(\beta)$ *there are $\mu_i = \mathrm{cf}(\mu_i) \leq \lambda_i$ for $i < \kappa$ and $A \in D^+$ such that $\mu = \mathrm{tcf}(\Pi \mu_i/(D+A))$*
- $(\gamma)$ *there are $\langle \langle \lambda_{i,n} : n < n_i \rangle : i < \kappa \rangle$, $\lambda_{i,n} = \mathrm{cf}(\lambda_{i,n}) < \lambda_i$ and a filter $D^*$ on $\bigcup\limits_{i<\kappa} \{i\} \times n_i$ such that:*

$$\mu = \mathrm{tcf}(\prod_{(i,n)} \lambda_{i,n}/D^*) \text{ and } D = \{A \subseteq \kappa : \text{ the set } \bigcup_{i \in A} \{i\} \times n_i \text{ belongs to } D^*\}$$

- $(\delta)$ *for some $A \in D^+$, $\mu \leq T_{D+A}(\langle \lambda_i : i < \kappa \rangle)$.*

*Remark.* So the question whether $(\alpha) \Leftrightarrow (\delta)$ assuming $(\forall \sigma < \mu)(\sigma^{\aleph_0} < \mu)$ is equivalent to $(\beta) \leftrightarrow (\gamma)$ which is a "pure" pcf problem.

*Proof.* Note $(\gamma) \Rightarrow (\delta)$ is easy (as in 3.18, i.e., as in the proof of 3.7, only easier). Now $(\beta) \Rightarrow (\gamma)$ is trivial and $(\beta) \Rightarrow (\alpha)$ by 3.28. Next $(\alpha) \Rightarrow (\beta)$ holds as we can use $(\alpha)$ for $B_i =$: the interval Boolean algebra of the order $\lambda_i$ and use 3.27. Lastly assume $(\forall \sigma < \mu)(\sigma^{\aleph_0} < \mu)$, now $(\gamma) \Leftrightarrow (\delta)$ by 3.18. $\square_{3.29}$

<u>Discussion</u>: We would like to have (letting $B_i$ denote Boolean algebra)

$$\mathrm{Depth}^{(+)}(\prod_{i<\kappa} B_i/D) \geq \prod_{i<\kappa} \mathrm{Depth}^{(+)}(B_i)/D$$

if $D$ is just filter we should use $T_D$ and so by the problem of attainment (serious by Magidor Shelah [MgSh 433]), we ask

⊗ for $D$ an ultrafilter on $\kappa$, does $\lambda_i < \mathrm{Depth}^+(B_i)$ for $i < \kappa$ implies

$$\prod_{i<\kappa} \lambda_i/D < \mathrm{Depth}^+(\prod_{i<\kappa} B_i/D)$$

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



at least when $\lambda_i > 2^\kappa$;

$\otimes'$ for $D$ a filter on $\kappa$, does $\lambda_i <$ Depth$^+(B_i)$ for $i < \kappa$ implies, assuming $\lambda_i > 2^\kappa$ for simplicity,

$$T_D(\langle \lambda_i : i < \kappa \rangle) < \text{Depth}^+(\prod_{i<\kappa} B_i/D).$$

As explained in 3.29 this is a pcf problem.

However changing the invariant (closing under homomorphisms, see [M]) we get a nice result; this will be presented in [Sh 580].

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



## §4 Remarks on the conditions for the pcf analysis

We consider a generalization whose interest is not so clear.

**4.1 Claim.** *Suppose $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ is a sequence of regular cardinals, and $\theta$ is a cardinal and $I^*$ is an ideal on $\kappa$; and $H$ is a function with domain $\kappa$. We consider the following statements:*

$(**)_H$ *lim inf*$_{I^*}(\bar{\lambda}) \geq \theta \geq$ *wsat*$(I^*)$ *and $H$ is a function from $\kappa$ to $\mathscr{P}(\theta)$ such that:*

- $(a)$ *for every $\varepsilon < \theta$ we have $\{i < \kappa : \varepsilon \in H(i)\} = \kappa \mod I^*$*
- $(b)$ *for $i < \kappa$ we have $otp(H(i)) \leq \lambda_i$ or at least $\{i < \kappa : |H(i)| \geq \lambda_i\} \in I^*$*

$(**)^+$ *similarly but*

- $(b)^+$ *for $i < \kappa$ we have $otp(H(i)) < \lambda_i$.*

*1) In 1.9 we can replace the assumption $(*)$ by $(**)_H$ above.*
*2) Also in 1.11, 1.12, 1.13, 1.14, 1.15,* **?** *we can replace 1.9$(*)$ by $(**)_H$.*
$\rightarrow$ *scite{1.11} undefined*
*3) Suppose in Definition 2.3(2) we say $\bar{f}$ obeys $\bar{a}$ for $H$ (instead of for $\bar{A}^*$) if*

$(i)$ *for $\beta \in a_\alpha$ such that $\varepsilon =:$ $otp(a_\alpha) < \theta$ we have*

$$otp(a_\beta), otp(a_\alpha) \in H(i) \Rightarrow f_\beta(i) \leq f_\alpha(i)$$

*and in 2.3(2A), $f_\alpha(i) = \sup\{f_\beta(i) : \beta \in a_\alpha$ and $otp(a_\beta)$, $otp(a_\alpha) \in H(i)\}$.*

*Then we can replace 1.9$(*)$ by $(**)_H$ in 2.7, 2.8,* **?***; and replace 1.9$(*)$ by $(**)^+_H$ in*
$\rightarrow$ *scite{2,6} undefined*
*2.10 (with the natural changes).*

*Proof.* 1) Like the proof of 1.9, but defining the $g_\varepsilon$'s by induction on $\varepsilon$ we change requirement (ii) to

$(ii)'$ if $\zeta < \varepsilon$, and $i \in H(\zeta) \cap H(\varepsilon)$ then $g_\zeta(i) < g_\varepsilon(i)$.

We can not succeded as

$$\langle (B^\varepsilon_{\alpha(*)} \setminus B^{\varepsilon+1}_{\alpha(*)}) \cap \{i < \kappa : \varepsilon, \varepsilon + 1 \in H(i)\} : \varepsilon < \theta \rangle$$

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



is a sequence of $\theta$ pairwise disjoint member of $(I^*)^+$.

In the induction, for $\varepsilon$ limit let $g_\varepsilon(i) < \cup\{g_\zeta(i) : \zeta \in H(i) \text{ and } \varepsilon \in H(i)\}$ (so this is a union at most $\mathrm{otp}(H(i) \cap \varepsilon)$ but only when $\varepsilon \in H(i)$ hence is $\langle\ \mathrm{otp}(H(i)) \leq \lambda_i)$.
2) The proof of 1.11 is the same, in the proof of 1.12 we again replace (ii) by (ii)$'$. Also the proof of the rest is the same.
3) Left to the reader. $\square_{4.1}$

We want to see how much weakening $(*)$ of 1.9 to "$\lim\inf_{I^*}(\bar\lambda) \geq \theta \geq \mathrm{wsat}(I^*)$ suffices. If $\theta$ singular or $\lim\inf_{I^*}(\bar\lambda) > \theta$ or just $(\Pi\bar\lambda, <_{I^*})$ is $\theta^+$-directed then case $(\beta)$ of 1.9 applies. This explains $(*)$ of 4.2 below.

**4.2 Claim.** *Suppose $\bar\lambda = \langle\lambda_i : i < \kappa\rangle$, $\lambda_i = \mathrm{cf}(\lambda_i)$, $I^*$ an ideal on $\kappa$, and*

$$(*) \qquad \lim\inf_I(\bar\lambda) = \theta \geq \mathrm{wsat}(I^*), \quad \theta \text{ regular.}$$

*<u>Then</u> we can define a sequence $\bar J = \langle J_\zeta : \zeta < \zeta(*)\rangle$ and an ordinal $\zeta(*) \leq \theta^+$ such that*

- $(a)$ *$\bar J$ is an increasing continuous sequence of ideals on $\kappa$*
- $(b)$ *$J_0 = I^*$, $J_{\zeta+1} =: \{A : A \subseteq \kappa$ and: $A \in J_\zeta$ or we can find $h : A \to \theta$ such that $\lambda_i > h(i)$ and $\varepsilon < \theta \Rightarrow \{i : h(i) < \varepsilon\} \in J_\zeta\}$*
- $(c)$ *for $\zeta < \zeta(*)$ and $A \in J_{\zeta+1} \setminus J_\zeta$, the pair $(\Pi\bar\lambda, J_\zeta + (\kappa \setminus A))$ (equivalently $\Pi\bar\lambda \restriction A, J_\zeta \restriction A)$) satisfies condition 1.9$(*)$ (case $(\beta)$) hence its consequences, (in particular it satisfies the weak pcf-th for $\theta$)*
- $(d)$ *if $\kappa \notin \cup_{\zeta<\zeta(*)} J_\zeta$ then $(\Pi\bar\lambda, \cup_{\zeta<\zeta(*)} J_\zeta)$ has true cofinality $\theta$.*

*Proof.* Straight. (We define $J_\zeta$ for $\zeta \leq \theta^+$ by clause (b) for $\zeta = 0$, $\zeta$ successor and as $\bigcup\limits_{\varepsilon<\zeta} J_\varepsilon$ for $\zeta$ limit. Clause $(c)$ holds by claim 4.4 below. It should be clear that $J_{\theta^++1} = J_{\theta^+}$, and let $\zeta(*) = \min\{\zeta : J_{\zeta+1} = \bigcup\limits_{\varepsilon<\zeta} J_\varepsilon\}$ so we are left with checking clause (d). If $A \in J^+_{\zeta(*)}$, $h \in \prod\limits_{i\in A} \lambda_i$, choose by induction on $\zeta < \theta$, $\varepsilon(\zeta) < \theta$ increasing with $\zeta$ such that $\{i < \kappa : h(i) \in (\varepsilon(\zeta), \varepsilon(\zeta+1)) \in J^+_{\zeta(*)}$. If we succeed we contradict $\theta \geq \mathrm{wsat}(I^*)$ as $\theta$ is regular. So for some $\zeta < \theta$, $\varepsilon(\zeta)$ is well defined but not $\varepsilon(\zeta+1)$. As $J_{\zeta(*)} = J_{\zeta(*)+1}$, clearly $\{i < \kappa : h(i) \leq \varepsilon(\zeta)\} = \kappa \mod J_{\zeta(*)}$. So $g_\varepsilon(i) = \begin{cases} \varepsilon & \text{if } \varepsilon < \lambda_i \\ 0 & \text{if } \varepsilon \geq \lambda_i \end{cases}$ exemplifies $\mathrm{tcf}(\Pi\bar\lambda/J_{\zeta(*)}) = 0$. $\square_{4.2}$

Now:

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**4.3 Conclusion.** *Under the assumptions of 4.2, $I^*$ satisfies the pseudo pcf-th (see Definition 2.16(4)), hence $cf(\Pi\bar\lambda, <_{I^*}) = \sup\ pcf_{I^*}(\bar\lambda)$ (see 2.19).*

**4.4 Claim.** *Under the assumption of 4.2, if $J$ is an ideal on $\kappa$ extending $I^*$ the following conditions are equivalent*

(a) *for some $h \in \Pi\bar\lambda$, for every $\varepsilon < \theta$ we have $\{i \in A : h(i) < \varepsilon\} \in J$*

(b) *$(\Pi\bar\lambda, <_{J+(\kappa\setminus A)})$ is $\theta^+$-directed.*

*Proof.* (a) $\Rightarrow$ (b)

Let $f_\zeta \in \Pi\bar\lambda$ for $\zeta < \theta$, we define $f^* \in \Pi\bar\lambda$ by

$$f^*(i) = \sup\{f_\zeta(i) + 1 : \zeta < h(i)\}.$$

Now $f^*(i) < \lambda_i$ as $h(i) < \lambda_i = \text{cf}(\lambda_i)$ and $f_\zeta \restriction A <_J f^* \restriction A$ as $\{i \in A : h(i) < \zeta\} \in J$.

(b) $\Rightarrow$ (a):

Let $f_\zeta$ be the following function with domain $\kappa$:

$$f_\zeta(i) = \begin{cases} \zeta & \text{if } \zeta < \lambda_i \\ 0 & \text{if } \zeta \geq \lambda_i \end{cases}$$

As $\lim \inf_{I^*} \geq \theta$, clearly $\varepsilon < \zeta \Rightarrow f_\varepsilon <_{I^*} f_\zeta$ and of course $f_\zeta \in \Pi\bar\lambda$. By our assumption (b) there is $h \in \Pi\bar\lambda$ such that $\zeta < \theta \Rightarrow f_\zeta \restriction A < h \restriction A \mod J$. Clearly $h$ is as required. $\square_{4.4}$

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



regular cardinals and cardinal invariants of Boolean Algebra,

Paper Sh:506, version 2008-02-10_10. See https://shelah.logic.at/papers/506/ for possible updates.



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