IF THERE IS AN EXACTLY λ-FREE ABELIAN GROUP THEN THERE IS AN EXACTLY λ-SEPARABLE ONE IN λ

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Abstract. We give a solution stated in the title to problem 3 of part 1 of the problems listed in the book of Eklof and Mekler [EM],(p.453). There, in pp. 241-242, this is discussed and proved in some cases. The existence of strongly λ-free ones was proved earlier by the criteria in [Sh:161] in [MkSh:251]. We can apply a similar proof to a large class of other varieties in particular to the variety of (non-commutative) groups.

I thank Alice Leonhardt for the beautiful typing
Typed December 12/93 - Done Fall '93;§2 done 28 (+1) Feb.94
last revision 3/13/95

§0 Introduction

Convention. In §0 and §1, "group" here means "abelian group", and "free" means in this variety.

We assume there is a λ-free, non-free (abelian) group of cardinality λ. We shall prove that there is a λ-separable non-free abelian group of cardinality λ, apriori a stronger statement. We rely on the characterization of λ as in the hypothesis from [Sh 161]: the existence of $S, \langle < s^\ell_\eta : \ell < n >: \eta \in S_f \rangle$, $\langle \ell(k) : k \rangle$ as there (see appendix; i.e. §3 here). Mekler Shelah [MkSh 251] dealt with a similar weaker problem in a parallel way: if there is a λ-free not free abelian group of cardinality λ then there is a strongly λ-free one. In Eklof Mekler [EM], the present problem was raised, discussed and sufficient conditions were given, depending on the form of S, see [EM], p.242-242, the problem in [EM],p.453. The direct sufficient condition is that for every $S' \subseteq S_f$ of cardinality λ there is a well ordering $<^*$ such that for each $\eta \in S_f$, $\bigcup\limits_{\ell<n} s^\ell_\eta$ is almost disjoint to $\cup\{\bigcup\limits_{\ell<n} s^\ell_\nu : \nu <^* \eta \text{ and } \nu \in S\}$. In particular from the assumption for λ, the conclusion for λ^+ (i.e. the existence of such S) was gotten. However, not all cases were done there. Our approach is more algebraic. In §2 we deal with generalizations to other varieties and in §3 we present relevant material from [Sh 161] (on λ-systems) to make the paper self-contained.

Explanation of the proof of the main theorem. It may be helpful to read this explanation if you are lost or stuck during the proof but it assumes some notations from the proof. We construct G that is freely generated by $x[a]$ (for $a \in \bigcup\limits_{\eta \in S_c} B_\eta$) and $y_{\eta,m}$ (for $\eta \in S_f$ and $m < \omega$) except the equation

$$(*)_{\eta,m} \qquad 2y_{\eta,m+1} = y_{\eta,m} + \Sigma\{x[a^\ell_{\eta,m}] : \ell < n\}.$$

Let $G = G_{I_0}$, $I_0 = I_{<>,\lambda}$.

Let $\alpha < \lambda$ and we want to show that if $\alpha < \lambda$ and $\langle \alpha \rangle \notin S$ then $G_{<>,\alpha}$ (which is essentially the subgroup generated by the $y_{\eta,m}$ and $x[a^\ell_{\eta,m}]$ satisfying $\eta(0) < \alpha$) is a free direct summand of $G = G_{<>,\lambda}$.
We do not see combinatorially why this holds, so we find $I_1 \supseteq I_{<>,\alpha}$, $I_1 \in K^+$ such that

$$(**) \qquad \eta \in S_f \backslash S_f[I_1] \Rightarrow \bigcup_{\ell<n} s^\ell_\eta \text{ is almost disjoint to } Y[I_1]$$

So let g_{I_0,I_1} be the natural homomorphism from G_{I_0} to G_{I_1}; well, why does it work? by $(**)$.
Also g_{I_0,I_1} is the identity on $G_{<>,\alpha}$ and $G_{I_1} \backslash G_{<>,\alpha}$ is $\cong G_{I_2}$ where $I_2 = I_1 \backslash I_{<>,\alpha}$, but $I_2 \in K^+$ so $G_{I_1}/G_{<>,\alpha}$ is free hence $G_{<>,\alpha}$ is a direct summand of G_{I_1}, so there is a projection f from G_{I_1} onto $G_{I_{\langle\rangle,\alpha}}$ so $f \circ g_{I_0,\alpha}$ is a projection from G onto $G_{I_{\langle\rangle,\alpha}}$ and we can complete the proof.

To accomplish $(**)$ we need good control over how e.g. s^ℓ_η $(\eta(0) > \alpha)$ intersect $B_{<\alpha>}$, and this is the information we put in the appendix on the λ-system (really old [Sh 161] is O.K., but we retain the appendix to ease reading).

0.1 Definition. For Ξ a set of variables, Γ set of equations in some variables (maybe outside Ξ) let $G(\Xi, \Gamma)$ be the (abelian) group freely generated by Ξ, except the equations in $\Gamma \restriction \Xi$, i.e. the equations from Γ mentioning only variables from Ξ.

0.2 Observation. 1) A sufficient condition (assuming $\Xi \subseteq \Xi'$ sets of variables) for

- $(*)$ $G(\Xi', \Gamma)$ is a free extension of $G(\Xi, \Gamma)$ (i.e. the mapping induced by id_Ξ from $G(\Xi, \Gamma)$ into $G(\Xi', \Gamma)$, which is always homomorphism, is an embedding, and $G(\Xi', \Gamma)$ divided by the range of this mapping is a free group),

is

- $(**)$ there is an increasing continuous sequence $\langle \Xi_\zeta : \zeta \leq \zeta^* \rangle$, $\Xi_0 = \Xi, \Xi_{\zeta^*} = \Xi'$, and $G(\Xi_{\zeta+1}, \Gamma)$ is a free extension of $G(\Xi_\zeta, \Gamma)$.

2) Another sufficient condition for $(*)$ of 0.2, is that by change of the variables in $\Xi' \backslash \Xi$, the set of equations $\Gamma \restriction \Xi'$ is only $\Gamma \restriction \Xi$.

§1 Proving λ-Separability

Here we prove the main theorem; the reader is advised to look at 3.6, 3.7 at least during reading the beginning of the proof, and also to look again at the explanation in §0 of the proof when arriving to read the middle of the proof.

1.1 Definition. A group G is λ-separable if:
$H \subseteq G, Rk(H) < \lambda \Rightarrow H$ included in a free direct summand of G.
(Remember: for an uncountable group H, its rank, $Rk(H)$ is equal to its cardinality, $|H|$.)

1.2 Main Theorem. *If there is a λ-free non λ^+-free (abelian) group ($\lambda > \aleph_0, \lambda$ necessarily regular) then there is a λ-free, λ-separable, not λ^+-free group.*

Proof. The hypothesis of the theorem on the existence of such groups is analyzed in detail in [Sh 161] (most relevant are [Sh 161],3.6,3.7), and in particular, it implies the existence of $n, S, \lambda(\eta, S), \langle B_\eta : \eta \in S_c\rangle$, $\langle s^\ell_\eta : \eta \in S_f, \ell < n\rangle$ with the properties as in [Sh 161],3.6,3.7 presented in 3.6, 3.7 of the Appendix here, and let $\langle a^\ell_{\eta,m} : m < \omega\rangle$ list s^ℓ_η in increasing order for the order of $B_{(\eta\restriction\ell)\hat{}\langle\lambda(\eta\restriction\ell)\rangle}$ (see clause (i) of 3.7) and without loss of generality we have in addition

- $(*)$ for $\eta \in S_f, \ell < n$, we have
  $\alpha_{\eta,\ell,m} =: \min\{\beta : a^\ell_{\eta,m} \in B_{(\eta\restriction\ell)\hat{}<\beta>}\} < \lambda(\eta \restriction \ell, S)$
  is non-decreasing in m,

and we call its limit $\beta^*(\eta, \ell)$ (so $s^\ell_\eta \subseteq B_{(\eta\restriction\ell)\hat{}\langle\beta^*(\eta,\ell)\rangle}$ and $\beta^*(\eta, \ell) \le \eta(\ell)$).

- $(**)$ if $\rho \in S_i, \nu \triangleleft \rho, k = \ell g(\nu)$ and $\mathrm{cf}(\rho(k)) = \lambda(\rho, S)$ then
  - (a) for $\beta < \lambda(\rho, S)$ we have $\sup\{\beta^*(\eta, k) : \rho\hat{}\langle\beta\rangle \trianglelefteq \eta \in S_f\}$ is $< \rho(k)$
  - (b) the sequence $\langle\min\{\beta^*(\eta, k) : \rho\hat{}\langle\beta\rangle \trianglelefteq \eta \in S_f\} : \beta < \lambda(\rho, S)\rangle$ is strictly increasing with limit $\rho(k)$.

(see Appendix, clauses $(f)(\alpha), (f)(\beta)$ and (g) of 3.6).
Let

$$K = \{I : I \subseteq S_c \text{ and } [\eta \ne \nu \ \ \& \ \ \eta \in I \ \ \& \ \ \nu \in I \Rightarrow \neg(\eta \trianglelefteq \nu)] \text{ and } [\eta\hat{} < \beta > \in I \ \ \& \ \ \alpha < \beta \Rightarrow \eta\hat{} < \alpha > \in I]\},$$

$$K^+ = \{I \in K : I \ne \emptyset \text{ and } \eta\hat{} < \alpha > \in I \Rightarrow \vee_{\beta < \lambda(\eta, I)}[\eta\hat{} < \beta > \notin I]\}.$$

For $I \in K$ let $J[I] =: \{\eta \in S_i$: for some $\alpha, \eta\hat{} < \alpha > \in I\}$, so for $\eta \in J[I]$ there is a unique $\alpha_I[\eta] \le \lambda(\eta, S)$ such that $[\eta\hat{} < \alpha > \in I \Leftrightarrow \alpha < \alpha_I[\eta]]$, note:
$I \in K^+, \eta \in J[I] \Rightarrow \alpha_I[\eta] < \lambda(\eta, S)$. For $I \in K$ let
$S_f[I] =: \{\eta \in S_f$: for some $k, \eta \restriction k \in I\}$; note that the k is unique and if

$I \neq \{<>\}$, then $k > 0$, so we choose to write $k = k_I(\eta) + 1$ (so for $I = \{<>\}, k_I(\eta) = -1$). Also let

$$Y[I] =: \cup\{B_\nu : \text{for some } \eta \in I \text{ we have } \eta \trianglelefteq \nu \in S_c\}$$

For $\eta \in S_f$ let $w_I(\eta) = \{\ell < n : \text{for every } m < \omega, a^\ell_{\eta,m} \in Y[I]\}$ (equivalently: for infinitely many $m < \omega, a^\ell_{\eta,m} \in Y[I]$).
For every $I \in K$ we define a group G_I, it is freely generated by $\Xi_I =: \{y_{\eta,m} : m < \omega \text{ and } \eta \in S_f[I]\} \cup \{x[a] : a \in Y[I]\}$ except the equations (we call this set Γ_I):

$(*)_I$ for $\eta \in S_f[I]$ and $m < \omega$, the equation $\varphi^m_{I,\eta}$ defined as:
$(*)^m_{I,\eta}$ $2y_{\eta,m+1} = y_{\eta,m} + \sum\{x[a^\ell_{\eta,m}] : \ell < n \text{ and } a^\ell_{\eta,m} \in Y[I]\}$

Note that ${}^1\lambda \in K$, and let $G = G_{({}^1\lambda)}$; this abelian group is the example as in [Sh 161], Lemma 5.3, in particular G is not free. Let $<_{\ell x}$ be lexicographic order of S, clearly it is a well ordering.

A Fact. If $I \in K^+$ then G_I is free.

Proof. We can find functions ℓ and m, where for $\eta \in S_f[I]$ we have $\ell(\eta) \in \{k_I(\eta), \ldots, n-1\}$ and $m(\eta) < \omega$ and we can find a list $\langle \eta_\zeta : \zeta < \zeta^* \rangle$ of $S_f[I]$ such that:

$(*)$ $\{a^{\ell(\eta_\zeta)}_{\eta_\zeta,m} : m \in [m(\eta_\zeta), \omega)\}$ is disjoint to $\{a^\ell_{\eta_\varepsilon,m} : a^\ell_{\eta_\varepsilon,m} \in Y[I], m < \omega, \varepsilon < \zeta, \ell < n\}$ <u>and</u> $\{\eta \in S_f[I] : \eta <_{\ell x} \nu\}$ is an initial segment of $\langle \eta_\zeta : \zeta < \zeta^* \rangle$ for each $\nu \in J[I]$

[why? for each $\nu \in J[I]$ well order $\{\eta \in S_f[I] : \nu \lhd \eta\}$ by [Sh 161], 3.10 (and 3.6 clause (c) and the definition of K^+), say by $<^*_\nu$, then order the blocks by $<_{\ell x}$]. Without loss of generality $m(\eta)$ is minimal such that $(*)$ holds.

For $\zeta \leq \zeta^*$ let H_ζ be the subgroup of G_I generated by $\Xi_\zeta = \{x[a^\ell_{\eta_\varepsilon,m}] : \varepsilon < \zeta, m < \omega, \ell \in [k(\eta), n)\} \cup \{y_{\eta_\varepsilon,m} : \varepsilon < \zeta, m < \omega\}$. Let $H_{\zeta^*+1} = G_I$.

Now $\langle H_\zeta : \zeta \leq \zeta^* + 1 \rangle$ is increasing continuous, $H_0 = \{0\}$, $H_{\zeta^*+1} = G_I$ and $H_{\zeta+1}/H_\zeta$ is free. Why? we use 0.2(1), so it is enough to prove $G(\Xi_{\zeta+1}, \Gamma_I)$ is a free extension of $G(\Xi_\zeta, \Gamma_I)$ for each $\zeta \leq \zeta^*$. For $\zeta = \zeta^*$, we just add variables $(\{x[a] : a \in \Xi_I \backslash \Xi_{\zeta^*}\})$ but no equations. For $\zeta < \zeta^*$, we can "forget" $y_{\eta_\zeta,m}$ for $m < m(\eta_\zeta)$ and replace/omit $x[a^{\ell(\eta_\zeta)}_{\eta_\zeta,m}]$ for $m \in [m(\eta_\zeta), \omega)$, so $G(\Xi_{\zeta+1}, \Gamma_I)$ is freely generated over $G(\Xi_\zeta, \Gamma_I)$ by
$\{y_{\eta_\zeta,m} : m \geq m(\eta_\zeta)\} \cup \{x[a] : x[a] \in \Xi_{\zeta+1} \backslash \Xi_\zeta \backslash \{x[a^{\ell(\eta_\zeta)}_{\eta_\zeta,m}] : m \geq m(\eta_\zeta)\}\}$.

B Notation. Let $I_{\eta,\alpha} =: \{\eta\hat{\ } < \beta >: \beta < \alpha\}$ (for $\eta \in S_i$). Let $G_{\eta,\alpha} =: G_{I_{\eta,\alpha}}$ so:

$G_{<>,\lambda}$ is the group G (which we shall prove exemplifies the conclusion of 1.2)
$J[I_{<>,\lambda}] = \{<>\}$

C Definition. 1) $I_1 \leq I_2$ (from K) if $S_f[I_1] \subseteq S_f[I_2]$ and $(\forall \eta \in S_f(I_1))[k_{I_1}(\eta) \geq k_{I_2}(\eta)]$. This implies $Y[I_1] \subseteq Y[I_2]$ and there is a clear relation between Γ_{I_1} and Γ_{I_2} : each equation $\varphi = \varphi^m_{I_1,\eta}$ in Γ_{I_1} "appears" in Γ_{I_2} as $\psi = \varphi^m_{I_2,\eta}$ but ψ is with more $x[a^\ell_{\eta,m}]$'s (for same old η but new ℓ's which appear "because" of some $\nu \in S_f[I_2] \backslash S_f[I_1]$) and Γ_{I_2} has members (not related to any equation from Γ_{I_1}) involving a new η. Another way to state this relation is
$(\forall \eta \in I_1)(\exists \nu \in I_2)[\nu \trianglelefteq \eta]$.
2) $I_1 \leq^d I_2$ if $I_1 \leq I_2$ and $J[I_1]$ is a $<_{\ell x}$-initial segment of $J[I_2]$.

D Fact. 1) $\leq$ and $\leq^d$ are partial orders (of K).
2) If $I \in K \backslash \{\{<>\}\}$ then $I = \bigcup\limits_{\eta \in J[I]} I_{\eta,\alpha_I(\eta)}$.
3) If $I_1 \leq^d J_2$ then $Y[I_1]$ is a subset of $Y[I_2]$.

Proof. Check.

E Definition. Assume $I_1 \leq^d I_2$ (both in K), let h_{I_1,I_2} be the homomorphism from G_{I_1} into G_{I_2} defined by $h(x[a]) = x[a], h(y_{\eta,m}) = y_{\eta,m}$ for $x[a] \in Y[I_1], \eta \in S_f[I_1]$.

F Fact. h_{I_1,I_2} is really a homomorphism.

Proof. Look at the relevant equations.

G Fact. If $I_1 \leq^d I_2$ are from K^+ and $(\forall \eta)\,[\eta \in J[I_1] \Rightarrow \eta\hat{\ }\langle \alpha_{I_1}(\eta)\rangle \notin S]$ then

- (α) $G_{I_2}/h_{I_1,I_2}(G_{I_1})$ is free and
- (β) h_{I_1,I_2} is one to one.
- (γ) $\mathrm{Rang}(h_{I_1,I_2}) = \langle y_{\eta,m}, x[a] : a \in Y[I_1], \eta \in S_f[I_1] \text{ and } m < \omega\rangle_{G_{I_2}}$ (i.e. the subgroup generated by this set)

so we look at h_{I_1,I_2} as the identity.

Proof. Like the proof of Fact A.

H Conclusion. If $I_1 \leq^d I_2$ (so are from K) then h_{I_1,I_2} is an embedding.

Proof. As a direct limit of ones satisfying the assumptions of Fact G.

I Fact.

- (α) $G = G_{({}^1\lambda)} = G_{<>,\lambda} = \bigcup\limits_{\alpha<\lambda} G_{<>,\alpha}$ (increasing continuous)
- (β) for $\alpha < \lambda$ the group $G_{<>,\alpha}$ is free.

Proof. For clause (α) as $\Gamma_{({}^1\lambda)} = \bigcup\limits_{\alpha<\lambda} \big(\Gamma \restriction \{y_{\eta,m}, x[a^\ell_{\eta,m}] : \eta(0) < \alpha\}\big)$, using Fact H (see Fact G last line).
For clause (β) see Fact A.

J Definition. For $I_1 \leq I_2$ (in K), satisfying $\otimes_{I_1,I_2}$ below, let g_{I_2,I_1} be the homomorphism from G_{I_2} into G_{I_1} defined by:

(i) if $a \in Y[I_1]$ then $g_{I_2,I_1}(x[a]) = x[a]$
(ii) if $a \in Y[I_2]\backslash Y[I_1]$ then $g_{I_2,I_1}(x[a]) = 0$
(iii) if $\eta \in S_f[I_1]$ then $g_{I_2,I_1}(y_{\eta,m}) = y_{\eta,m}$
(iv) if $\eta \in S_f[I_2]\backslash S_f[I_1]$ and $\{a^\ell_{\eta,k} : k \in [m,\omega)$ and $\ell \geq k_{I_2}[\eta]$ (equivalently $a^\ell_{\eta,k} \in Y[I_2])\}$ is disjoint to $Y[I_1]$ then $g_{I_2,I_1}(y_{\eta,m}) = 0$

(this is enough for defining g_{I_2,I_1})
where

$\otimes_{I_1,I_2}$ for $\eta \in S_f[I_2]\backslash S_f[I_1]$, $\bigcup_{\ell \in [k_{I_2}(\eta),n)} s^\ell_\eta$ is almost disjoint to $Y[I_1]$
(i.e. has finite intersection).

K Fact. Assume $I_1 \leq I_2$ are in K. Then

(α) g_{I_2,I_1} really defines a homomorphism which is onto (when $I_1 \leq I_2$ and $\otimes_{I_1,I_2}$ holds)
(β) Kernel(g_{I_2,I_1}) is the subgroup of G_{I_2} generated by the set of $x[a]$'s and $y_{\eta,m}$'s which by Definition J are sent by g_{I_2,I_1} to 0.

Proof. Check the equations.

Main Fact L. If $\alpha < \lambda$ and $< \alpha > \notin S$ then $G_{<>,\alpha}$ is a direct summand of $G = G_{\{<>\}}$.

Proof. We can define by induction on k a number $\ell_k \leq n : \ell_k = 0$, if ℓ_k is defined and $< n$, let ℓ_{k+1} be the unique ℓ such that $\ell_k < \ell \leq n$ and $\eta \in S_f \Rightarrow cf(\eta(\ell_k)) = \lambda(\eta \restriction \ell, S)$ (exists by 3.3(f), all $\eta \in S_f$ behave the same by 3.6(a) (and see 3.2(6)(d)), note: if $\eta \in S_f \Rightarrow$ cf$(\eta(\ell_k)) = \aleph_0$ then $\ell_{k+1} = n$. Clearly if ℓ_k is defined and $< n$ then $\ell_k < \ell_{k+1} \leq n$. So for some k^*, $\ell_{k^*} = n$.
We shall define by induction on $k \leq k^*$ the following J_k and, when $k < k^*$, $\langle \alpha_\eta : \eta \in J_k \rangle$ such that:

(0) $J_k \subseteq S \cap {}^{\ell_k}\lambda$
(1) $\alpha_\eta < \lambda(\eta, S)$ and $\eta\hat{\ }\langle \alpha_\eta \rangle \notin S$ for $\eta \in J_k$
(2) $J_{k+1} = \{\eta : \eta \in S \cap {}^{\ell_{k+1}}\lambda$ and $\eta \restriction \ell_k \in J_k$ but $\eta(\ell_k) > \alpha_{\eta \restriction \ell_k}\}$
(3) if $\eta \in J_{k+1}, k+1 < k(*), \alpha \in [\alpha_\eta, \lambda(\eta, S))$ and $\eta\hat{\ } < \alpha > \trianglelefteq \ \nu \in S_f$ then $s^{\ell_k}_\nu \cap B_{\eta \restriction \ell_k \hat{\ } \langle \alpha_{\eta \restriction \ell_k} \rangle}$ is finite
(4) $J_0 = \{<>\}, \alpha_{<>} = \alpha$.

For $k = 0$ use clause (4). For $k+1$ we define J_{k+1} by clause (2), now if $k+1 < k(*)$ for $\eta \in J_{k+1}$ we have to find α_η to satisfy clauses (1), (3), this is possible by $(*)$,$(**)$ in the beginning of the proof of Theorem 1.2.

Let $I_0 = \{< \beta >: \beta < \lambda\}$,
$I_1 = \{\eta\hat{\ } < \beta >$: for some $k < k^*$ we have $\eta \in J_k$ and $\beta < \alpha_\eta\}$,
$I_2 = I_1 \backslash \{< \beta >: \beta < \alpha_{<>}\}$,

$I_3 = \{<\beta>: \beta < \alpha = \alpha_{<>}\}$.
Note that by the inductive choice of the J_k's:

> $\otimes$ if $\eta \in S_f \backslash S_f[I_1]$ then $\{a^\ell_{\eta,m} : \ell < n \text{ and } m < \omega\}$ has a finite intersection with $Y[I_1]$.

(Use (3) noting that if $\eta \in S_f \backslash S_f[I_1]$ then $\eta(\ell_k) > \alpha_{\eta \restriction \ell_k}$ for every $k < k^*$ such that $\eta \restriction \ell_k \in J_k$).
Note also that: $I_0 \in K, I_1 \in K^+, I_2 \in K^+, I_3 \in K^+$. Also $I_3 \le^d I_1$ and $I_3 \le^d I_0$ and $I_2 \le I_1 \le I_0$ (see Definition C(1)) and $G_{I_0} = G$.

Note that g_{I_0,I_1} is well defined (see Definition J and Fact K).
[Why? We have to check $\otimes_{I_1,I_0}$ as defined there, but $\otimes$ above says this]. Note also that g_{I_1,I_2} is well defined (again we have to check $\otimes_{I_2,I_1}$ as defined in Definition J, but for $\eta \in S_f(I_1) \backslash S_f(I_2)$ by their definitions, $\eta(0) < \alpha_{<>}$ so easily $\bigcup_{\ell<n} s^\ell_\eta$ is disjoint to the required set). Look at the sequence $G = G_{I_0} \underset{g_{I_0,I_1}}{\longrightarrow} G_{I_1} \underset{g_{I_1,I_2}}{\longrightarrow} G_{I_2}$.

We know that G_{I_2} is free (by Fact A as $I_2 \in K^+$), g_{I_1,I_2} is a homomorphism from G_{I_1} onto G_{I_2} (see above, by Fact K, clause (α) and $\otimes$ above) hence $Ker(g_{I_1,I_2})$ is a direct summand of G_{I_1}, so there is a projection g^* of G_{I_1} onto $Ker(g_{I_1,I_2})$. Also h_{I_3,I_1}, h_{I_3,I_0} are embeddings (by conclusion H) as $I_3 <^d I_1, I_3 <^d I_0$, (check or see above). Also $h_{I_3,I_1}(G_{I_3}) = Ker(g_{I_1,I_2})$ (compare Fact G clause (γ) and Fact K clause (β)). Hence $h_{I_3,I_0} \circ h^{-1}_{I_3,I_1} \circ g^* \circ g_{I_0,I_1}$ is a projection from $G = G_{<>} = G_{I_0}$ onto Rang(h_{I_3,I_0}) i.e. essentially $G_{<>,\alpha}$. This finishes the proof of the main fact, hence the theorem 1.2. $\square_{1.2}$

[Question: here we can increase α_η; can we make it exact? (See Appendix 3.6)].

1.3 Claim. *We can strengthen the conclusion of 1.2 to: for any given $W \subseteq \lambda$ we can demand: there is a λ-free non-free group G with set of elements λ such that*

$$\{\delta \in W : G \restriction \delta \text{ is a subgroup of } G, \text{ and is a free direct summand of } G\}$$

is a stationary subset of λ.

Proof. In the proof of 1.2;

(A) for any $W_0 \subseteq \{\alpha < \lambda : \langle \alpha \rangle \in S\}$ stationary subset of λ, we can replace S by $\{\eta : \eta \in S \text{ and } \ell g(\eta) > 0 \Rightarrow \eta(0) \in W_0\}$
(B) assuming that the set of member of G is λ then $\{\delta < \lambda : \delta \text{ is the set of elements of } G_{<>,\delta}\}$ is a club of λ.

Together with Main Fact L and Fact I, we are done. $\square_{1.3}$

1.4 Discussion. We can rephrase the proof of 1.1 combinatorially; i.e. explicitly write a set of generators X such that $G = G_\alpha \oplus \langle X \rangle_G$, do not think it is clearer. To some extent this is done in Fact A of the proof of 2.2.

§2 The General Case: for a variety

We note here that a parallel theorem holds for any suitable variety considering two variants of λ-separable (see Definition 2.1(2) and Definition 2.4). We do the general case in less details.

2.1 Definition. 1) T is a variety if T is a theory (in a vocabulary τ) all whose axioms are equations or just has the form $\forall x_1, \dots, x_n \varphi$, φ an atomic formula. Without loss of generality every member of τ (function symbol or predicate) appears in some axiom of T.
2) A model M of T is called λ-separable if for every $A \subseteq M, |A| < \lambda$ we can represent M as a free product $M_1 * M_2$ such that $A \subseteq M_1$ and M_2 is free.
3) T has the n-th h-construction principle if we can find N, $b_{\ell,m}$ (for $\ell < n, m < \omega$) and $N_{\bar m}$ (for $\bar m \in {}^n\omega$) such that:

(i) N a model of T of cardinality $\le |T| + \aleph_0$
(ii) N is free, moreover, for each $\ell^* < n$ and $m^* < \omega$ we can complete $\{b_{\ell,n} : \ell < n, m < \omega$ and $[\ell = \ell^* \Rightarrow m < m^*]\}$ to a free basis of N, call the set of additional elements $C_{\ell,m}$
(*iii*)(α) if $\bar m^i = \langle m^i_\ell : \ell < n\rangle \in {}^n\omega$ (for $i = 1, 2$) and $\bar m^1 \le \bar m^2$ (i.e. $(\forall \ell < n)(m^1_\ell \le m^2_\ell)$ then $N_{\bar m^2} \subseteq N_{\bar m^1} \subseteq N$,
(β) $b_{\ell,m} \in N_{\langle m_k : k<n\rangle} \Leftrightarrow m \ge m_\ell$ and
(γ) N is the free product $N_{\bar m} * \langle\{b_{\ell,m} : \ell < n, m < m_\ell\}\rangle_N$.

(iv) for no free model F of T, is $N * F/\langle b_{\ell,m} : \ell < n, m < \omega\rangle_N$ free (equivalently $N * F$ has a free basis extending $\{b_{\ell,m} : \ell < n, m < \omega\}$).

2.1A Remark. On the n-th construction principle see Eklof Mekler [EM2] and then Mekler Shelah [MkSh 366]. The difference (between the n-th construction principle and the n-th h-construction principle) is clause (iii), it is not clarified here if it adds anything. In all cases the hope is that the analysis of [Sh 161],§3,§4 exhausts the reasons of the existence of the desired complicated object in λ, and the crucial parameter of the system S (see beginning of the proof of 1.2 or §3) is $n = n(S)$. So the hope is that for each T, the class of cardinals λ where we have an example is, for some $\alpha^* \le \omega$

$$\mathfrak{C}_{\alpha^*} = \Big\{\lambda : \text{there are } n, S, \langle\lambda(\eta, S) : \eta \in S_i\rangle, \langle B_\eta : \eta \in S_c\rangle$$
$$\langle s^\ell_\eta : \eta \in S_f, \ell < n\rangle \text{ as in 3.6, 3.7 and } n < \alpha^*\Big\}.$$

Usually we deal with varieties with countable vocabulary.

2.2 Theorem. *Assume there is a λ-free not λ^+-free abelian group exemplified by n, S, $\langle s^\ell_\eta : \ell < n$ and $\eta \in S_f\rangle$ as in the proof of 1.2 and the theory T has the n-th h-construction principle and $|T| < \lambda$.*

Then T has a λ-separable model of cardinality λ which is not free.

2.2A Conclusion. If there is a λ-free not λ^+-free abelian group then for the variety of groups (not the abelian one) there is a λ-free, λ-separable group G of cardinality λ which is not free. (I.e. G is a non-free group of cardinality λ, G can be represented as $\bigcup_{\alpha<\lambda} G_\alpha, G_\alpha$ increasing continuously of cardinality $< \lambda$, each G_α free and G is the free product (for the variety of groups) of $G_{\alpha+1}$ and some $H_{\alpha+1}$ for each $\alpha < \lambda$).

Proof of 2.2A. We should just check the condition of 2.1(3) which is straight as in [Sh 161].
[I.e. let N be the group freely generated by
$\{b_{\ell,m} : \ell \in [1,m)$ and $m < \omega\} \cup \{y_m : m < \omega\}$, let:

(a) $b_{0,0} =: y_0$
(b) $b_{0,m+1}$ is the product $b_{1,m+1}\, b_{2,m+1} \dots b_{n-1,m+1}\, b_{0,m}(y_{0,m+1})^2$
(c) $C_{\ell,m} = \{y_k : k \in [m,\omega)\} \cup \{b_{\ell,0}\}$
(d) for $\bar m \in {}^n\omega$ clearly
$\{b_{\ell,m} : \ell \in [1,n)$ and $m < \omega\} \cup \{b_{0,n} : n < m_0\} \cup C_{0,m}$ is a free basis of N and let $N_{\bar m}$ be the subgroup of N generated by
$\{b_{\ell,m} : \ell < n$ and $m \in [m_\ell,\omega)\} \cup \{y_m : m \in [m_0,\omega)\}$.

Now check].

Proof of 2.2. Let $\langle N, b_{\ell,m}, N_{\bar m} : \ell < n, m < \omega$ and $\bar m \in {}^n\omega\rangle$ exemplify the n-th h-construction principle. We choose $n, S, \dots$ as in the proof of 1.2.

Let M be freely generated by $x[a]$ (for $a \in \bigcup_{\eta\in S_c} B_\eta$) and $y_{\eta,c}$ (for $\eta \in S_f$ and $c \in N$) except that:

(i) $y_{\eta,c} = x[a]$ if $c = b_{\ell,m}$ and $a = a^\ell_{\eta,m}$
(ii) $\varphi(y_{\eta,c_1},\dots,y_{\eta,c_k})$ whenever $N \models$ "$\varphi(c_1,\dots,c_k)$" and φ is a τ-atomic formula.

Fact A. For $\alpha < \lambda$ such that $\langle\alpha\rangle \notin S$ we can find $Y_0, Y_1, Y_2, S_0, S_1, S_2$ such that:

(a) $S_2 = S_f, Y_2 = \bigcup_{\eta\in S_c} B_\eta$
(b) $S_0 = \{\eta \in S_f : \eta(0) < \alpha\}$ and $Y_0 = \cup\{B_\eta : \eta \in S_c$ and $\eta(0) < \alpha\}$
(c) $S_0 \subseteq S_1 \subseteq S_2$ and $Y_0 \subseteq Y_1 \subseteq Y_2$ and Y_1 is downward closed (remember Y_2 is a tree, see 3.6) so $a^\ell_{\eta,m} \in Y_1$ & $m_1 < m \Rightarrow a^\ell_{\eta,m_1} \in Y_1$
(d) for $\eta \in S_2 \backslash S_1$ the set $\{a^\ell_{\eta,m} : \ell < n, m < \omega\} \cap Y_1$ is finite

(e) there is a list $\langle \eta_\zeta : \zeta < \zeta^* \rangle$ of $S_1 \backslash S_0$ without repetitions and $\langle \ell(\zeta) : \zeta < \zeta^* \rangle$ such that $0 \le \ell(\zeta) < n$ and $\langle m(\zeta) : \zeta < \zeta^* \rangle, m(\zeta) < \omega$ such that:

(α) $\{a^{\ell(\zeta)}_{\eta_\zeta,m} : m \in [m(\zeta),\omega)\}$ is disjoint to $Y_0 \cup \{a^\ell_{\eta_\varepsilon,m} : \ell < n, \varepsilon < \zeta, m < \omega\}$

(β) $\{a^{\ell(\zeta)}_{\eta_\zeta,m} : m < \omega\} \subseteq Y_1$.

Proof. Included in the proof of Theorem 1.2.

Remark B. We can add

(f) S_1 is $S_f[I_1]$ from the proof of Theorem 1.2 so for some function k from $S_1 \backslash S_0$ to $n = \{0,\dots,n-1\}$ we have
$Y_1 = Y_0 \cup \{a^\ell_{\eta_\zeta,m} : \eta \in S_1, m < \omega$ and $\ell \in [k(\eta),n)\}$.

Fact C. Under the conclusion of Fact A, letting
$M_0 =: \langle \{x[a] : a \in Y_0\} \cup \{y_{\eta,c} : \eta \in S_0, c \in N\}\rangle_M$ we have: M_0 is free and for some M_2, $M = M_0 * M_2$.

Proof. Clearly M_1 is free (for T) as in the proof of Fact A in the proof of 1.2. The new point is to find M_2.
For each $\ell < n, m < \omega$, let $C_{\ell,m} \subseteq N$ be such that
$C_{\ell,m} \cup \{b_{\ell_1,m_1} : \ell_1 \ne \ell, m_1 < \omega$ <u>or</u> $\ell_1 = \ell, m_1 < m\}$ is a free basis of N with no repetitions.

We let M_2 be the submodel of M generated by:

(A) $y_{\eta_\zeta,c}$ <u>if</u> $\zeta < \zeta^*$ and $c \in C_{\ell(\zeta),m(\zeta)}$
(B) $x[a^\ell_{\eta_\zeta,m}]$ <u>if</u> $\zeta < \zeta^*, a^\ell_{\eta_\zeta,m} \in Y_1 \backslash Y_0$ and for no $\varepsilon < \zeta^*$ do we have $a^\ell_{\eta_\zeta,m} \in \{a^{\ell(k)}_{\eta_\varepsilon,k} : k \in [m(\varepsilon),\omega)\}$
(C) $x[a]$ <u>if</u> $a \in Y_2 \backslash Y_1$
(D) $y_{\eta,c}$ <u>if</u> $\eta \in S_2 \backslash S_1, c \in N_{\langle m_\ell(\eta):\ell<n\rangle}$ where $m_\ell(\eta) = \min\{m : a^\ell_{\eta,m} \notin Y_1\}$.

Now

$(*)_1$ $M = \langle M_0, M_2 \rangle$.

First we prove by induction on $\zeta < \zeta^*$ that $\{x[a^\ell_{\eta_\zeta,m}] : \ell < n$ and $m < \omega\} \subseteq \langle M_0, M_2\rangle$ and $\{y_{\eta_\zeta,c} : c \in N\} \subseteq \langle M_0, M_2 \rangle$. Arriving to ζ we split the proof to cases.

<u>Case 1</u>: $a^\ell_{\eta_\zeta,m} \in Y_0$.
Then $x[a^\ell_{\eta_\zeta,m}] \in Y_0 \subseteq M_0 \subseteq \langle M_0, M_2\rangle$.

<u>Case 2</u>: $a^\ell_{\eta_\zeta,m} \in Y_1 \backslash Y_0$ and for some $\varepsilon < \zeta, a^\ell_{\eta_\zeta,m} \in \{a^{\ell(\varepsilon)}_{\eta_\varepsilon,k} : k \in [m(\varepsilon),\omega)\}$.
We use the induction hypothesis on ε.

<u>Case 3</u>: $a^\ell_{\eta_\zeta,m} \in Y_1 \backslash Y_0$ and $\ell \ne \ell(\zeta) \vee [\ell = \ell(\zeta)\ \&\ m < m(\zeta)]$ and for no $\varepsilon < \zeta$, do we have $a^\ell_{\eta_\zeta,m} \in \{a^{\ell(\varepsilon)}_{\eta_\varepsilon,k} : k \in [m(\varepsilon),\omega)\}$.

Now $\varepsilon < \zeta^*$ implies $a^\ell_\eta \notin \{a^{\ell(\varepsilon)}_{\eta_\varepsilon,k} : k \in [m(\varepsilon), w)\}$.
[Why? If $\varepsilon < \zeta$ this is assumed in the case, if $\varepsilon = \zeta$ this is follows by $\ell \neq \ell(\zeta)$, and if $\varepsilon \in (\zeta, \zeta^*)$ this follows by clause $(e)(\alpha)$ (with ε's here standing for ζ, ε there). Hence the assumption of clause (B) holds.]

By clause (B), $x[a^\ell_{\eta_\zeta,m}] \in M_2 \subseteq \langle M_0, M_2\rangle$.

Case 4: $a^\ell_{\eta_\zeta,m} \in Y_2 \backslash Y_1$.
By clause (C), $x[a^\ell_{\eta_\zeta,m}] \in M_2 \subseteq \langle M_0, M_2\rangle$.

Case 5: No previous cases.

By the earlier cases $\ell = \ell(\zeta)$ and
$\{x[a^{\ell^*_1}_{\eta_\zeta,m_1}] : \ell^*_1 < n, m^*_1 < \omega$ and $[\ell^*_1 \neq \ell(\zeta) \Rightarrow m^*_1 < m(\zeta)]\} \subseteq \langle M_0, M_2\rangle$.

Let $N' =: \{c \in N : y_{\eta_\zeta,c} \in \langle M_0, M_2\rangle\}$, so by the previous sentence
$\{b_{\ell_1,m_1} : \ell_1 < n, m_1 < \omega$ and $\ell_1 = \ell(\zeta) \Rightarrow m_1 < m(\zeta)\} \subseteq N'$, and by clause (A) also $C_{\ell(\zeta),m(\zeta)} \subseteq N'$ hence (see clause (ii) in Definition 2.1) clearly $N' = N$, so $x[a^{\ell_1}_{\eta_\zeta,m}] \in \langle M_0, M_2\rangle$ and $y_{\eta_\zeta,c} \in \langle M_0, M_2\rangle$.

We have proved $\{x[a] : a \in Y_1 \backslash Y_0\} \subseteq \{x[a^\ell_{\eta_\zeta,m}] : \ell < n, m < \omega, \zeta < \zeta^*\} \subseteq \langle M_0, M_2\rangle$. As $\{x[a] : a \in Y_0\} \subseteq M_0 \subseteq \langle M_0, M_2\rangle$ and by clause (C) we have $\{x[a] : a \in Y_2 \backslash Y_1\} \subseteq \langle M_0, M_2\rangle$ we conclude $\{x[a] : a \in Y_2\} \subseteq \langle M_0, M_2\rangle$.

Also we have proved $\{y_{\eta_\zeta,c} : c \in N, \zeta < \zeta^*\} \subseteq \langle M_0, M_2\rangle$ (this was done during the proof of case 5) so $\{y_{\eta,c} : \eta \in S_1 \backslash S_0$ and $c \in N\} \subseteq \langle M_0, M_2\rangle$.
Also for $\eta \in S_2 \backslash S_1$, letting $N^\eta = \{c \in N : y_{\eta,c} \in \langle M_0, M_2\rangle\}$,
$m_\ell = \min\{m : a^\ell_{\eta,m} \notin Y_1\}$ we have: by clause (D), $N_{\langle m_\ell : \ell < n\rangle} \subseteq N^\eta$, and
$\{a^\ell_{\eta,m} : \ell < n, m < \omega\} \subseteq Y_2$ so $b_{\ell,m} \in N^\eta$ hence $N^\eta = N$ so
$\{y_{\eta,c} : \eta \in S_2 \backslash S_1, c \in N\} \subseteq \langle M_0, M_2\rangle$. Lastly if $\eta \in S_0$ we have
$\{y_{\eta,c} : c \in N\} \subseteq M_0 \subseteq \langle M_0, M_2\rangle$. Together $\{y_{\eta,c} : \eta \in S_2$ and $c \in N\} \subseteq \langle M_0, M_2\rangle$; and also we note above $\{x[a] : a \in Y_2\} \subseteq \langle M_0, M_2\rangle$; we can conclude $M = \langle M_0, M_2\rangle$, i.e. $(*)_1$.

So to finish the proof we need

$(*)_2$ $M = M_0 * M_2$

(i.e. they generate M freely).
Look at the equations in the definition of M and together with the proof of $(*)_1$ rewrite them in terms of the generators of M_0 and of M_2. The equations either trivialized or speak on generators of M_0 or speak on generators of M_2. [more?] $\square_{2.2}$

Note that as the variety of abelian groups is very nice, e.g. a subgroup of a free abelian group is free, distinct definitions for general varieties become identified for it; so Theorem 1.2 has various generalizations and Theorem 2.2 is not the unique one. Another generalization is presented below.

2.3 Theorem. *Assume λ is as in 1.2 with $n, S, \langle s^\ell_\eta : \ell < n, \eta \in S_f\rangle$ such that T has the n-th construction principle (i.e. in Definition 2.1 we omit clause (iii), but demanding each $C_{\ell,n}$ is infinite; this holds without loss of generality by clause (iv) of Definition 2.1). Then there is a model M of T, not free of cardinality λ, but is λ-proj-separable, where:*

2.4 Definition. For a variety T and a model M of T and cardinality λ we say M is λ-proj-separable, if for every $A \subseteq M, |A| < \lambda$ there is a free $M' \subseteq M$ including A and a projection h from M onto M'.

Proof of 2.3. We define $M, x[a], y_{\eta,c}$ as in the proof of 2.2. For every $\ell(*) < n$ and $m(*) < \omega, m(*) > 0$ there is a homomorphism $g_{\ell(*),m(*)}$ from N onto $\langle b_{\ell,n} : \ell < n, m < \omega$ and $[\ell = \ell(*) \Rightarrow m < m(*)]\rangle_N$ which is the identity on $\langle b_{\ell,m} : \ell < n, m < \omega$ and $[\ell = \ell(*) \Rightarrow m < m(*)]\rangle_N$ (maps the members of $C_{\ell(*),m(*)}$ onto $\{b_{\ell(*),0}\}$.) Let Γ be the set of equations which we make the generators satisfy. We choose $Y_0, Y_1, Y_2, S_0, S_1, S_2$ as in Fact A from the proof of 2.2 and without loss of generality $\zeta < \zeta^* \Rightarrow m(\zeta) > 0$. Let $\{\eta_\zeta : \zeta \in [\zeta^*, \zeta^{**})\}$ list $S_2 \backslash S_1$.

For each $\eta_\zeta \in S_2 \backslash S_1$ we can choose

$$\ell(\zeta) = \ell_{k^*-1}, m(\zeta) = \text{ Min}\{m : 0 < m < \omega \text{ and } a^{\ell(\zeta)}_{\eta,m} \notin Y_1\}.$$

Let M_1 be the model of T generated by
$\Xi_1 = \{x[a^\ell_{\eta,m}] : \ell < n, \eta \in S_1, m < \omega\} \cup \{y_{\eta,c} : c \in N, \eta \in S_2\}$ freely except

$$\Gamma_1 = \Big\{ y_{\eta,c} = x[a] : c = b_{\ell,m}, a = a^\ell_{\eta,m} \text{ and } \eta \in S_1 \Big\} \cup$$
$$\Big\{ \varphi(y_{\eta,c_1}, \dots, y_{\eta,c_k}) : N \models \varphi(c_1, \dots, c_k), \varphi \text{ a } T\text{-atomic formula} \Big\}.$$

Let M^-_2 be the model of T generated by (note: I_1, J_k are from the proof of 1.2)

$$\Xi^-_2 =: \Big\{ x[a] : a \in Y_2 \text{ but if } a \in B_{\eta\char`\^\langle\lambda(\eta,S)\rangle},$$
$$\ell g(\eta) = \ell_{k^*-1} \text{ and } \eta \in J_{k^*-1} \subseteq J[I_1] \text{ then } a \text{ is in the first level}$$
$$\text{(i.e. like } a^{\ell_{k^*}}_{\eta,0}) \text{ or } a \in B_{\eta\char`\^\langle\alpha_\eta\rangle}$$
$$(\alpha_\eta \text{ from the choice of } I_1) \Big\}$$

freely except the equations

$$\Gamma^-_2 = \Gamma_1 = \Big\{ y_{\eta,c} = x[a] : c = b_{\ell,m}, a = a^\ell_{\eta,m}, \eta \in S_1, \ell < n, m < \omega \text{ and } x[a] \in \Xi^-_2 \Big\} \cup$$
$$\Big\{ \varphi(y_{\eta,c_1}, \dots, y_{\eta,c_k}) : N \models \varphi[c_1, \dots, c_k], \varphi \text{ a } T\text{-atomic formula}, \eta \in S_1 \Big\}.$$

(Note that if $\eta \in J_{k-1}$ and $\eta \triangleleft \nu \in S_f$ then $\mathrm{cf}(\nu(\ell_{k^*-1})) = \aleph_0$).
Clearly $M_0 \subseteq M_1 \subseteq M^-_2 \subseteq M$.
We define a homomorphism h_2 from M into $M^-_2 : h_2 \restriction M^-_2$ is the identity, and for $\eta = \eta_\zeta \in S_2 \backslash S_1$ and $c \in N$ we let:

$$h_2(y_{\eta,c}) = y_{\eta,g_{\ell(\zeta),m(\zeta)}(c)}.$$

Note: $h_2(x[a^\ell_{\eta_\zeta,m}]) = x[a^\ell_{\eta_\zeta,m}]$ when $\ell \neq \ell(\zeta) \vee m < m(\zeta)$ by the tree structure of $\bigcup\limits_{\eta \in S_c} B_\eta$, the cases of the definition of h_2 are compatible and the equations are preserved. So h_2 is a homomorphism and even a projection from M onto M_2^-.

Trivially, we can find a projection h_1 from M_2^- onto M_1.

Next note that M_1 is a free extension of M_0 (a free basis is $\{y_{\eta_\zeta,c} : c \in C_{\ell(\zeta),m(\zeta)}$ and $\zeta < \zeta^*\} \cup \{x[a] : a \in Y_1 \backslash Y_0$ and for no $\zeta < \zeta^*$ is $a \in \{a^{\ell(\varepsilon)}_{\eta_\zeta,m} : m \in [m(\varepsilon),\omega)\}\}$.

So we can find a projection h_0 from M_1 onto M_0. So $h_0 \circ h_1 \circ h_0$ is a projection as required. $\square_{2.3}$

2.5 Claim. *Theorems 2.2, 2.3 can be strengthened as in 1.3.*

2.6 Discussion. Implicit in the proof of 2.3 is an alternative criterion sufficient for the conclusion of 2.2.

§3 Appendix: characterizing the existence in λ of an almost free abelian group

To make the main theorem 1.2 more easily read we present part of [Sh 161], more exactly a variant to [Sh 161],3.6,3.7,p.212.
Numbers are as in [Sh 161].

3.1 Definition.
(1) For a regular uncountable cardinal $\lambda(>\aleph_0)$ we call S a λ-set if:

(a) S is a set of strictly decreasing sequences of ordinals $<\lambda$.
(b) S is closed under initial segments and is nonempty.
(c) For $\eta \in S$ if we let $W(\eta, S) =: \{i : \eta\hat{\ } <i> \in S\}$ and $\lambda(\eta, S) =:$ Sup $W(\eta, S)$ <u>then</u> whenever $W(\eta, S)$ is not empty, $\lambda(\eta, S)$ is a regular uncountable cardinal and $W(\eta, S)$ is a stationary subset of $\lambda(\eta, S)$. Also $\lambda(<>, S) = \lambda$ (and by clause (a) we know $\lambda(\eta\hat{\ } <\alpha>, S) \leq |\alpha|$).

(2) For a λ-set S, let S_f(= set of final elements of S) be $\{\eta \in S : (\forall i)\eta\hat{\ } <i> \notin S\}$ and S_i(= set of initial elements of S) be $S \backslash S_f$ so ($S_f = \{\eta \in S : \lambda(\eta, S) = 0\}$).
We sometimes allow $\lambda = 0$. Then the only λ-set is $\{<>\}$.
(3) For λ-sets S^1, S^2 we say $S^1 \leq S^2$ (S^1 a sub-λ-set of S^2) if $S^1 \subseteq S^2$ and $\lambda(\eta, S^1) = \lambda(\eta, S^2)$ for every $\eta \in S^1$ (so $S^1_i = S^1 \cap S^2_i$). Clearly $\leq$ is transitive.

Notation: In this section S will be used to denote λ-sets.

3.1A Remark. Many of the properties below holds also if we waive the "decreasing" demand in clause (a) but not all, and for what we want to analyze we can get such S, so we have concentrated on this family of sets.

3.2 Claim.
(1) If S is a λ-set, $\lambda(\eta, S) > \kappa$ for every $\eta \in S_i$ (holds always for $\kappa = \aleph_0$) and G is a function from S_f to κ <u>then</u> for some $S^1 \leq S$ we have: G is constant on S^1_f.
(2) If S is a λ-set and $\eta \in S_i$, <u>then</u> $S^{[\eta]} = \{\nu : \eta\hat{\ }\nu \in S\}$ is a $\lambda(\eta, S)$-set, and $\lambda(\nu, S^{[\eta]}) = \lambda(\eta\hat{\ }\nu, S)$ for every $\nu \in S^{[\eta]}$.
(3) If S is a λ-set, κ a regular cardinal, $(\forall \eta \in S)\,(\lambda(\eta, S) \neq \kappa)$ and G is a function from S to κ <u>then</u> for some $S^1 \leq S$ and $\gamma < \kappa$ for every $\eta \in S^1$ we have $G(\eta) < \gamma$.
(4) If $\lambda > \aleph_0$ is regular, $W \subseteq \lambda$ stationary, for $\delta \in W$, S^δ is a λ_δ-set for some $\lambda_\delta \leq \delta$ or $S^\delta = \{<>\}$ <u>then</u> $S =: \{<>\} \cup \{\langle\delta\rangle\hat{\ }\eta : \eta \in S^\delta, \delta \in W\}$ is a λ-set and $\lambda(<\delta>\hat{\ }\eta, S) = \lambda(\eta, S^\delta)$ for $\delta \in W, \eta \in S^\delta$ and $S_i = \{\langle\rangle\} \cup \bigcup\limits_{\delta \in W} S^\delta_i$.
(5) If S is a λ-set, F a function with domain $S\backslash\{<>\}$, $F(\eta\hat{\ }\langle\alpha\rangle) < 1 + \alpha$ <u>then</u> F is essentially constant for some $S^1 \leq S$ which means $F \restriction \{\eta \in S^1 : \ell g(\eta) = m\}$ is constant for each m.
(6) For any λ-set S there is a λ-set $S^1 \leq S$ such that:

(a) *all $\eta \in S_f$ has the same length n*
(b) *for each $\ell < n$ either*
  (i) *every $\eta(\ell)\,(\eta \in S_f)$ is an inaccessible cardinal (not necessarily strong limit), <u>or</u>*
  (ii) *every $\eta(\ell)\,(\eta \in S_f)$ is a singular limit ordinal,*

(c) *for each* $\ell < n$, *either*

  (i) $\lambda(\eta \restriction (\ell+1), S) = \eta(\ell)$ *for every* $\eta \in S_f$ *or*

  (ii) $\lambda(\eta \restriction (\ell+1), S) = \lambda_S^{\ell+1}$ *for every* $\eta \in S_f$ *(for a fixed* $\lambda_S^{\ell+1}$*).*

(d) *The truth value of "*$cf(\eta(\ell)) = \lambda(\eta \restriction m, S)$*" is the same for all* $\eta \in S_f$ *(for constant* $\ell, m < n$*).*

Proof. Straightforward, e.g.
(5) In first glance we get only: if $\rho \in S_i$ then $F \restriction \{\rho\hat{\ }\langle\alpha\rangle : \alpha \in W(\rho, S)\}$ is constant (by Fodor's lemma and the demand "$W(\rho, S)$ is a stationary subset of $\lambda(\rho, S)$". However, as every $\eta \in S$ is (strictly) decreasing sequence of ordinals we can iterate this (simpler if we first apply part (6) clause (a)). $\square_{3.2}$

3.3 Claim. *Suppose* P *is a family of sets which exemplify the failure of* $PT(\lambda, \kappa^+)$ *(where* $\lambda > \kappa$*) i.e.* $a \in P = |a| \leq \kappa, P$ *has no transversal (= one to one choice function) but every* $P' \subseteq P$ *of cardinality* $< \lambda$ *has a transversal.* Then *there is a* λ*-set* S *and function* F *with domain* S_f *such that:*

(a) *For each* $\eta \in S_f, F(\eta)$ *is a subfamily of* P *of power* $\leq \kappa$*.*
(b) *For* $\eta \in S_i$ *we have* $\lambda(\eta, S) > \kappa$*.*
(c) *For* $\eta \in {}^{\omega >}(\lambda+1)$*, let* $F^0(\eta) =: \cup\{F(\tau) : \tau <_{\ell x} \eta \text{ and } \tau \in S_f\}$*, where* $<_{\ell x}$ *is the lexicographic order,* $F^1(\eta) =: \cup\{F(\tau) : \eta \trianglelefteq \tau \in S_f\}$ *and* $F^2(\eta) =: \cup\{A : A \in F^0(\eta\hat{\ }\langle\lambda(\eta, S)\rangle)\} \setminus \cup\{A : A \in F^0(\eta)\}$*.*

Note that for $\eta \in S$ *we have* $F^2(\eta\hat{\ }\langle\lambda(\eta, S)\rangle) = F^0(\eta) \cup F^1(\eta)$*.*

(d)

  (α) *For* $\eta \in S_f, F^1(\eta)/F^0(\eta)$ *is not free, (that is* $F^1(\eta)$ *has no one to one choice function with range disjoint to* $\cup\{A : A \in F^0(\eta)\}$*).*

  (β) *For* $\eta \in S_i, F^1(\eta)/F^0(\eta)$ *is* $\lambda(\eta, S)$*-free not free and* $|F^1(\eta)| = \lambda(\eta, S)$ *(this follows as* $|\{\tau : \eta \trianglelefteq \tau \in S\}|) = \lambda(\eta, S)$*).*

(e) *If* $\eta\hat{\ } < \alpha > \in S$ then α *is a limit ordinal,* $cf(\alpha) \leq \lambda(\eta\hat{\ } < \alpha >, S) + \kappa \leq |\alpha|$ *and if* $\beta < \lambda(\eta, S)$ *is an inaccessible cardinal* $(> \aleph_0)$ *then* $\beta \cap W(\eta, S)$ *is not a stationary subset of* β*.*
(f) *If* $\eta\hat{\ } < \alpha > \triangleleft \nu \in S_f$ *and* $cf(\alpha) > \kappa$ *then for some natural number* k *we have* $\eta\hat{\ } < \alpha > \trianglelefteq \nu \restriction k$ *and* $\lambda(\nu \restriction k, S) = cf(\alpha)$ *(so if* α *is an inaccessible cardinal then* $k = \ell g(\eta)$*).*

Proof. See [Sh 161].

Remark. Note clause (f), it is crucial; without it we won't be able to prove the desired conclusion.

3.4 Definition.
(1) A λ-system is $\langle B_\eta : \eta \in S_c\rangle$ where:

(a) S is a λ-set, and we let $S_c =: \{\eta\hat{}\langle i\rangle : \eta \in S_i \text{ and } i < \lambda(\eta, S)\}$
(b) $B_{\eta\hat{}\langle i\rangle} \subseteq B_{\eta\hat{}\langle j\rangle}$ when $\eta \in S_i$ and $i < j < \lambda(\eta, S)$
(c) If δ is a limit ordinal $< \lambda(\eta, S)$ then $B_{\eta\hat{}\langle\delta\rangle} = \cup\{B_{\eta\hat{}\langle i\rangle} : i < \delta\}$
(d) $|B_{\eta\hat{}\langle i\rangle}| < \lambda(\eta, S)$ for $i < \lambda(\eta, S)$.

(2) The λ-system $\langle B_\eta : \eta \in S_c\rangle$ is called disjoint if the sets $\{B_{\eta\hat{}\langle\lambda(\eta,S)\rangle} : \eta \in S_i\}$ (see (3) below) are pairwise disjoint.
(3) We let $S_m =: S\backslash\{<>\}$, $B_{\eta\hat{}\langle\lambda(\eta,S)\rangle} =: B^*_\eta =: \cup\{B_{\eta\hat{}\langle i\rangle} : i < \lambda(\eta, S)\}$ for $\eta \in S_i$.

3.5 Claim. *Suppose λ is a regular uncountable cardinal, $\langle B_\eta : \eta \in S_c\rangle$ a λ-system, and for $\eta \in S_f, s_\eta \subseteq \bigcup\limits_{\ell<\ell(\eta)} B_{\eta\restriction(\ell+1)}$. Then $\{s_\eta : \eta \in S_f\}$ has no transversal.*

Proof. Straightforward (or see [Sh 161]).

3.6 Claim. *Suppose $PT(\lambda, \kappa^+)$ fails (see 3.3).*[1] *Then there is a disjoint λ-system $\langle B_\eta : \eta \in S_c\rangle$ and sets s^ℓ_η (for $\eta \in S_f$ and $\ell < \ell g(\eta)$), and C_δ (for $\delta < \lambda$ a limit ordinal) and $\varepsilon_{\eta,\ell}$ (for $\eta \in S$ and $\ell < \ell g(\eta)$) such that:*

(a) *S satisfies the conclusion of Claims 3.2(6),3.3(e) and 3.3(f), in particular $\eta \in S_f \Rightarrow \ell g(\eta) = n$.*
(b) $s^\ell_\eta \subseteq B_{\eta\restriction(\ell+1)}, 0 < |s^\ell_\eta| \leq \kappa$.
(c) *For every $I \subseteq S_f$: if $|I| < \lambda$ then $\{\bigcup\limits_\ell s^\ell_\eta : \eta \in I\}$ has a transversal (as as indexed set). Moreover, for every $\rho \in S_i$ if $I \subseteq \{\nu : \rho \trianglelefteq \nu \in S_f\}$ and $|I| < \lambda(\rho, S)$ then the family $\{\bigcup\limits_{\ell\geq\ell g(\rho)} s^\ell_\eta : \eta \in I\}$ has a transversal.*
(d) *If $s^\ell_\eta \cap s^m_\nu \neq \emptyset$ then*
  (α) *$\ell = m$ and the sequences η, ν are different only at the ℓ-th place i.e. $\rho =: \eta \restriction \ell = \nu \restriction \ell$ and $\eta \restriction [\ell+1, n) = \nu \restriction [\ell+1, n)$ and*
  (β) *$\lambda(\eta \restriction i, S) = \lambda(\nu \restriction i, S)$ when $\ell + 1 < i < n$ and*
  (γ) *either $\lambda(\eta \restriction (\ell+1), S) = \eta(\ell)$ and $\lambda(\nu \restriction (\ell+1), S) = \nu(\ell)$ are both inaccessible cardinals or $\lambda(\eta \restriction (\ell+1), S) = \lambda(\nu \restriction (\ell+1), S)$.*

(e) *For $\eta\hat{} < \delta >\in S$ we have*
  (α) *C_δ is a closed unbounded subset of δ, $C_\delta = \{\zeta(\delta, i) : i < cf(\delta)\}$, $\zeta(\delta, i)$ increasing continuously with i*
  (β) *In addition if $\nu = \eta \restriction \ell, \nu \in S_i, \eta \in S_i, \lambda(\eta, S) = cf[\eta(\ell)] > \aleph_0$ then $\varepsilon_{\eta,\ell}$ is a strictly increasing function from $\lambda(\nu, S)$ to $\lambda(\nu, S)$*
  (γ) *in clause (β) if $\delta =: \eta(\ell)$ is an inaccessible cardinal (hence necessarily $\ell g(\eta) = \ell + 1$) then $\emptyset = W(\nu, S) \cap \{\zeta(\delta, i) : i$ belong to the range of $\varepsilon_{\eta,\ell}\}$*

[1] we are interested mainly in the case $\kappa = \aleph_0$

(f)

(α) *If* $\ell < m < n, \eta \in S_f$, $cf[\eta(\ell)] = \lambda(\eta \restriction m, S) > \kappa$ *then* $s^\ell_\eta \subseteq B_{(\eta\restriction\ell)\char`^\langle\zeta+1\rangle} \backslash B_{(\eta\restriction\ell)\char`^\langle\zeta\rangle}$ *where* $\zeta = \zeta(\eta(\ell), \varepsilon_{\eta,\ell}(\eta(m)) + 2)$*; i.e.* ζ *is the* $(\varepsilon_\eta(\eta(m)) + 2)$*-th member of* $C_{\eta(\ell)}$*. Moreover if* $s^\ell_\eta \cap s^\ell_\nu \neq \emptyset$, $\eta \neq \nu$ *then* $\zeta(\eta(\ell), \eta(m)) = \zeta(\nu(\ell), \nu(m))$.

(β) *If* $\ell < m < n = \ell g(\eta), \eta \in S_f, cf[\eta(\ell)] = \lambda(\eta \restriction m, S) \leq \kappa$ *then* $s^\ell_\eta \subseteq B_{\eta\restriction(\ell+1)} \backslash B_{\eta\restriction\ell\char`^\langle\zeta\rangle}$ *where* $\zeta = \zeta(\eta(\ell), \eta(m))$*; i.e.* $\zeta$ *is the* $(\eta(m) + 1)$*-th member of* $C_{\eta(\ell+1)}$ *and* $\xi < \eta(\ell) \Rightarrow |s^\ell_\eta \backslash B_{(\eta\restriction\ell)\char`^\langle\xi\rangle}| = \kappa$*. Moreover if* $s^\ell_\eta \cap s^\ell_\nu \neq \emptyset, \eta \neq \nu$ *then* $\zeta(\eta(\ell), \eta(m)) = \zeta(\nu(\ell), \nu(m))$*.*

(g) *If* $\ell < \ell g(\eta), \eta \in S_f, cf[\eta(\ell)] \leq \kappa$ *then for no* $\zeta < \eta(\ell)$ *is* $s^\ell_\eta \subseteq B_{\eta\restriction\ell\char`^\langle\zeta\rangle}$.

(h) *For some well ordering* $<^*_\eta$ *of* B^*_η $(\eta \in S_i)$ *if* $\eta\char`^\langle i\rangle \trianglelefteq \nu \in S_f$, *then* $[cf(i) \geq \kappa \Rightarrow s^{\ell g(\eta)}_\nu$ *has order type* $\kappa]$ *and* $cf(i) < \kappa \Rightarrow s^{\ell(\eta)}_\nu$ *has order type* $\kappa \times (cf|s^\ell_\eta|)]$*. (This is not really used.)*

Proof. Straightforward and in the most important case see 3.7's proof.

Remark. In the proof we get that each s^ℓ_ν has order type ω.

3.7 Claim. *Suppose in Claim 3.6 that* $\kappa = \aleph_0$*. Then we can add*

(i) *for* $\eta \in S_i, B_\eta$ *has the structure of a tree with* ω *levels (e.g., is a family of finite sequences, closed under initial segments except that* $\langle\rangle \notin B_\eta$*), and* $\eta \triangleleft \nu \in S_f$ *implies* $s^\ell_\eta = \{a^\ell_{\eta,m} : m < \omega\}$ *is a branch (of order type* $\leq \omega$*) (a branch is a maximal linearly ordered subset), and for* $m < \ell$*, and* $k < \omega$*, the k'th element of* s^m_ν*, together with* $\nu \restriction \ell$ *determines the k-th element of* s^ℓ_ν*. Also if* $\ell < m < n = \ell g(\eta), \eta \in S_f, cf[\eta(\ell)] = \lambda(\eta \restriction m) = \aleph_0$ *then* $\langle Min\{\xi :$ *in* $s^\ell_\eta \cap B_{(\eta\restriction\ell)\char`^\langle\xi\rangle}$ *there are at least k elements*$\} : k < \omega\rangle$ *is strictly increasing with limit* $\eta(\ell)$*.*

Proof of 3.7. Without loss of generality let P exemplify $PT(\lambda, \kappa)$ fails, so there are S (a λ-set) and F, F^0, F^1, F^2 as in claim 3.3. As we can shrink S, we can assume that it satisfies the conclusion of 3.2(6). Without loss of generality $\eta \in S_f \Rightarrow lg(\eta) = n$. Choose C_δ, $\zeta(\delta, i)$ as required in clause (e) (for subclauses (c), (α), (β) totally straight and for subclause (c)(γ) we use clause (*e*) of 3.3). For $\eta \in S_i$, $\alpha < \lambda(\eta, S)$, we let $D_{\eta\char`^\langle\alpha\rangle} =: \cup\{F^2(\eta\char`^ <\beta>: \beta < \alpha, \eta\char`^ <\beta> \in S\}$ so $\langle D_\eta : \eta \in S_c\rangle$ is a disjoint λ-system, without loss of generality disjont to S.
For $\eta \in S_f$ and $\ell = 0, \ldots, n-1$, we define $t^\ell_\eta =: D_{\eta\restriction(\ell+1)} \cap \cup\{A : A \in F(\eta)\}$.
For $\eta \in S_i$ and $\alpha \leq \lambda(\eta, S)$ we let

$$B_{\eta\char`^\langle\alpha\rangle} = \Big\{\rho : \rho \text{ is a finite sequence, of length } \geq 3 + (n - \ell g(\eta)), \quad \text{Rang } \rho \subseteq D_{\eta\char`^\langle\alpha\rangle} \cup \alpha \cup \{\eta\} \text{ but Rang}(\rho) \nsubseteq \alpha\Big\}.$$

Let

$$R = \Big\{(\ell, m, \eta) : \eta \in S_i, lg(\eta) = m, \ell \leq lg(\eta) \text{ and } \lambda(\eta, S) = \mathrm{cf}[\eta(\ell)] > \kappa\Big\}.$$

For $(\ell, m, \eta) \in R$ clearly $\langle \cup\{t^\ell_\nu : \eta \triangleleft \nu \in S_f \text{ and } \nu(m) < \alpha\} : \alpha < \lambda(\eta, S)\rangle$ is an increasing continuous sequence of subsets of B which may have cardinality $> \lambda(\eta, S)$, each of cardinality $< \lambda(\eta, S)$. But $\langle B_{(\eta\restriction\ell)\hat{}\langle\zeta(\eta(\ell),i)\rangle} : i < \lambda(\eta, S)\rangle$ is an increasing continuous sequence of sets with union $B_{\eta\restriction(\ell+1)}$ (remember $\langle\zeta(\eta(\ell), i) : i < \lambda(\eta, S)\rangle$ is an increasing continuous sequence of ordinals with limit $\eta(\ell)$ which has cofinality $\lambda(\eta, S)$). Hence

$$E_{\eta,\ell} =: \Big\{i < \lambda(\eta, S) : i \text{ is a limit ordinal such that } \cup\{s^\ell_\nu : \eta \triangleleft \nu \in S_f\} \cap B_{\eta\restriction(\ell)\hat{}\langle\zeta(\eta(\ell),i)\rangle} = \cup\{s^\ell_\nu : \eta \triangleleft \nu \in S_f \text{ and } \nu(m) < i\Big\}$$

is a club of $\lambda(\eta, S)$, so let $\varepsilon_{\eta,\ell} : \lambda(\eta, S) \to \lambda(\eta, S)$ be a strictly increasing continuous function with range $E_{\eta,\ell}$.
It is clear that $\langle B_\eta : \eta \in S_c\rangle$ is a disjoint λ-system (note $|B_{\eta\hat{}\langle i\rangle}| < \lambda(\eta, S)$ as $\lambda(\eta, S)$ is uncountable). Let $t^\ell_\eta = \{a(\eta, \ell, i) : i < \omega\}$ (possibly with repetitions).

We define s^ℓ_η by cases:

(α) if there is m such that $\ell < m < lg(\eta)$, $(\ell, m, \eta \restriction m) \in R$ and $\lambda(\eta \restriction m, S) > \aleph_0$ (there is at most one such m, and then $0 \leq \ell < m, cf(\eta(\ell)) = \lambda(\eta \restriction m, S) > \aleph_0$) we let
$\rho^\ell_\eta =: \langle\zeta(n(\ell), \varepsilon_\eta, \ell(\eta(n)) + 1), \ell, m\rangle\hat{}(\eta \restriction [\ell + 1, n))$,
$t^\ell_\eta =: \{\rho^\ell_\eta\hat{}\langle a(\eta, \ell, j) : j \leq m\rangle : m < \omega \text{ and } m > 0\}$

(β) $\rho^\ell_\eta = \langle 0, \ell, n\rangle\hat{}\eta \restriction [\ell + 1, n)$, if $cf(\eta(\ell)) \leq \kappa$ we let

$$s^\ell_\eta = \Big\{\rho^\ell_\eta\hat{}\langle y_0, \ldots, y_{2m-1}\rangle : m < \omega, m > 0, \text{ for each } k < m, \; y_{2k} = \min\{\zeta \in C_{\eta(\ell)} : a(\eta, \ell, 0), \ldots, a(\eta, \ell, k) \in B_{\eta\restriction\ell\hat{}\langle\zeta\rangle}\} \text{and } y_{2k+1} = a(\eta, \ell, k)\Big\}$$

Note that by clause (f) of 3.3, exactly one of those cases occurs.

Now $\langle B_\eta : \eta \in S_c\rangle$, s^ℓ_η (for $\eta \in S_f, \ell < \ell g(\eta)$) are as required in 3.6. The least trivial is (c). Suppose $I \subseteq S_f, |I| < \lambda$, so $\{\bigcup_{\ell<n} t^\ell_\eta : \eta \in I\}$ has a transversal, so there is a one-to-one function g, Dom $g = I$ and $g(\eta) \in \bigcup_\ell t^\ell_\eta$. Let $g(\eta) = a(\eta, h(\eta), g(\eta))$.

Now we define a function $g^* : \text{Dom } g^* = I$, $g^*(\eta) = \rho^\ell_\eta{}^\frown\langle a(\eta, h(\eta), i) : 0 \le i \le g(\eta)\rangle$. Clearly g^* is one-to-one, $g^*(\eta) \in \bigcup_{\ell<n} s^\ell_\eta$.

Let for $\eta \in S_i$, $<_\eta$ be a well ordering of $\{\eta\} \cup D_{\eta^\frown<\lambda(\eta,S)>}$ of order type $\lambda(\eta, S)$ such that η is first, and each $\{\eta\} \cup D_{\eta^\frown\langle\alpha\rangle}$ is an initial segment defined by α. Now $<^*_\eta$ will be $\rho_1 <^*_\eta \rho_2$ iff $\langle \max_{<_\eta} \text{Rang } \rho_1\rangle^\frown\rho_1 <_{lx} \langle \max_{<\eta} \text{Rang } \rho_2\rangle^\frown\rho_2 <_{lx}$ is lexicographically according to $<_\eta$.
It is also obvious that (i) holds, except possibly the last phrase; but the correction needed is small so we finish. $\square_{3.7}$

3.8 Claim. *Suppose $\langle B_\eta : \eta \in S_c\rangle, s^\ell_\eta(\eta \in S_f, \ell < \ell(\eta))$ are as in Claims 3.6, 3.7; we can omit 3.6(h)).*

Then for any $\rho \in S_i, m = \ell(\rho)$, and $I \subseteq \{\eta \in S_f : \rho \le \eta\}$ the following are equivalent:

$(A)_{\rho,I}$ *The family $\{\bigcup_{\ell\ge m} s^\ell_\eta : \eta \in I\}$ has a transversal.*

$(B)_{\rho,I}$ *There are a well ordering $<^*$ of I and $\{u_\eta : \eta \in I\}$ such that:*
- *(i) for $\eta <^* \nu$ (both in I), $u_\nu \cap (\bigcup_{\ell\le m} s^\ell_\eta) = \emptyset$.*
- *(ii) For every $\eta \in I$ for some $\ell, m \le \ell < \ell(\eta), u_\eta$ is an end-segment of s^ℓ_η.*
- *(iii) If $\xi <$ Min$\{\eta(m) : \eta \in I\}$ is given, we can demand that each $u_\eta(\eta \in I)$ is disjoint to $B_{\rho^\frown(\xi)}$.*

$(C)_{\rho,I}$ *There is no $\lambda(\rho, S)$-set S^* such that $\eta \in S^*_f \Rightarrow \rho^\frown\eta \in I$.*

$(D)_{\rho,I}$ *Suppose $\xi <$ Min$\{\eta(m) : \eta \in I\}$, there are $u_\eta(\eta \in I)$ where*
- *(i) the u_η are pairwise disjoint*
- *(ii) u_η is an end segment of some $s^\ell_\eta m \le \ell < \ell(\eta)$*
- *(iii) u_η is disjoint to $B_{\rho^\frown\langle\xi\rangle}$.*

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