

*ALMOST FREE GROUPS BELOW  
THE FIRST FIX POINT EXIST*  
**SH523**

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**Anotated Content**

§1 -  $\lambda$ -free does not imply free for  $\lambda <$  first fix point

[We represent material from [MgSh:204], assuming knowledge of [Sh:g, Ch.II].  
Just enough to make §2 intelligible.]

§2 - Nicely compact sets and non-reflection

[We answer a question of Foreman and Magidor on reflection of stationary subsets  
of  $\mathcal{S}_{<\aleph_2}(\lambda) = \{a \subseteq \lambda : |a| < \aleph_2\}$ . We answer a question of Mekler Eklof on the  
closure operations of the incompactness spectrum.]

§3 - *NPT* is not transitive

[We prove the consistency of  $NPT(\lambda, \mu) + NPT(\mu, \kappa) \not\Rightarrow NPT(\lambda, \kappa)$ ].

§1  $\lambda$ -FREE DOES NOT IMPLY FREE FOR  $\lambda < \text{FIRST FIX POINT}$ 

**1.1 Definition.**  $NPT(\lambda, \kappa)$  means that there is a family  $\{A_i : i < \lambda\}$  of subsets of  $\lambda$ ,  $|A_i| \leq \kappa$ , the family has no transversal but for  $\alpha < \lambda$  we have  $\{A_i : i < \alpha\}$  has a transversal. Let  $PT(\lambda, \kappa)$  be  $\neg NPT(\lambda, \kappa)$ .

Now by [Sh:161],  $NPT(\lambda, \aleph_0)$  is equivalent to the existence of  $\lambda$ -free not  $\lambda^+$ -free abelian group. Also “canonical” examples of “incompactness” are given there.

Magidor Shelah [MgSh:204] proved

**1.2 Theorem (ZFC).** *If  $\lambda$  is smaller than the first  $\alpha = \aleph_\alpha > \aleph_0$  then there is a  $\lambda$ -free not free abelian group.*

*This was done by providing an induction step from incompactness in  $\kappa$ ,*

*$\aleph_0 < \kappa = cf(\kappa) < \aleph_\kappa$  to incompactness in some  $\lambda \geq \aleph_\kappa$ , in fact  $\aleph_{\kappa+1}$ .*

*However, subsequently, to answer questions of Eklof and Mekler (in 2.2, 2.3) and of Magidor and Foreman (in 2.8), we do more.*

*Explanation of the History: Generally on the history of this problem see the book of Eklof and Mekler [EM]. The existence of such group for  $\lambda = \aleph_{\omega+1}$  is proved in [Sh:108] assuming  $\aleph_\omega$  strong limit or just  $2^{\aleph_0} < \aleph_\omega$ ; this is based on investigating  $I[\aleph_{\omega+1}]$  (where for a regular uncountable  $\lambda$ ,*

$I[\lambda] = \{S \subseteq \lambda : \text{for some club } E \text{ of } \lambda \text{ and } \bar{a} = \langle a_\alpha : \alpha < \lambda \rangle \text{ we have } a_\alpha \subseteq \alpha, \text{otp}(a_\alpha) < \alpha \text{ and } \alpha \in S \cap E \Rightarrow \alpha = \sup(a_\alpha)\}$ ), see on it [Sh:g, AG §1]. In the earlier version of [MgSh:204], for  $\lambda > cf(\lambda) = \kappa$  when  $2^\kappa < \lambda$  we replace a division to cases defined from properties  $I[\lambda]$  by one defined from  $\bar{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle <_I$ -increasing cofinal in  $\prod_{i < \kappa} \lambda_i / I$ .

*It is shown there that the set of bad (for  $\bar{f}$ ) ordinals  $< \aleph_{\kappa+1}$  is “small”, (but assuming something like  $2^\kappa < \lambda$ , then an example in  $\lambda$  is gotten from one on  $\kappa^+$ . In the present version of [MgSh:204] this was eliminated by showing: for good  $\delta < \lambda$  there is a club of good ordinals  $< \delta$  (for  $\bar{f}$ ); and similarly for bad (we ignore ordinals of small cofinality).*

In this section we represent analyses from [MgSh:204] needed in §2; it is organized around the proof of

**1.3 Lemma.** *If  $NPT(\kappa, \theta)$ , and  $\kappa \geq \theta = cf(\theta)$  then  $NPT(\kappa^{+(\kappa+1)}, \theta)$ .*

*Remark.* 1) Similarly  $NPT_I$ , see on it [[§6]Sh:355 (for  $I$  an ideal on  $\theta$  extending  $J_\theta^{bd}$ ).

Till end of the proof of 1.2 we make:

**1.4 Convention.**  $\kappa$  is a fixed regular cardinal.  $\lambda$  is a cardinal  $> \kappa$  of cofinality  $\kappa$ .

(For proving 1.2, we need just  $\lambda = \aleph_\kappa$ , but latter we will be interested in other values too).  $J$  denotes ideals on  $\kappa$  extending  $J_\kappa^{bd} = \{A \subseteq \kappa : A \text{ bounded}\}$ .

$D, E$  denotes filters on  $\kappa$  containing the co-bounded subsets of  $\lambda$ .

The rest of the section proves 1.3.

**1.5 Claim.** *Suppose  $\lambda = \kappa^{+\kappa}$ ,  $\mu = \lambda^+$ ;  $J = J_\kappa^{bd}$ , then there are  $\langle \lambda_i : i < \kappa \rangle$ ,  $\langle f_\alpha : \alpha < \mu \rangle$  such that the following holds:*

- (\*) (a)  $\langle \lambda_i : i < \kappa \rangle$  strictly increasing sequence of regular cardinals  $> \kappa^+$  with limit  $\lambda$ ,
- (b) for  $\alpha < \mu$ ,  $f_\alpha \in \prod \lambda_i$ ,  $[\alpha < \beta < \mu \Rightarrow f_\alpha <_J f_\beta]$ ,
- (c)  $\left( \forall f \in \prod_{i < \kappa} \lambda_i \right) (\exists \alpha < \mu) [f <_J f_\alpha]$ ,
- (d)  $\mu = cf(\mu) > \lambda$  (what is trivial here),
- (e)  $J$  is an ideal on  $\kappa$  containing  $J_\kappa^{bd}$ .

*Proof.* By [1.5]Sh:355.

\* \* \*

For awhile (till we finish proving 1.15) we shall assume (\*) of 1.5.

1.6 Context. (\*) of 1.5.

Let us quote [1.2]Sh:355, for the reader's convenience.

**1.7 Claim.** *Assume  $cf(\delta) > \kappa^+$ ,  $I$  is an ideal on  $\kappa$  and suppose  $\langle f_\alpha : \alpha < \delta \rangle$  is a  $<_I$ -increasing sequence of members of  ${}^\kappa \text{Ord}$ . Then exactly one of the following holds:*

- (i) for some ultrafilter  $D$  on  $\kappa$  disjoint from  $I$  we have:
  - (\*) $_D$  there are sets  $s_i \subseteq \text{Ord}$ ,  $|s_i| \leq \kappa$  and  $\langle \alpha_\zeta : \zeta < cf(\delta) \rangle$  increasing continuous with limit  $\delta$ , such that for each  $\zeta < cf(\delta)$  for some  $h_\zeta \in \prod_{i < \kappa} s_i$  we have:
 
$$f_{\alpha_\zeta} / D < h_\zeta / D < f_{\alpha_{\zeta+1}} / D$$
- (ii) (\*\*) $_I$  some  $f \in {}^\kappa \text{Ord}$  is a  $<_I$ -eub of  $\langle f_\alpha : \alpha < \delta \rangle$ , (i.e.  $f$  satisfies  $(\alpha) + (\beta)$  below) and  $(\gamma)$  holds:
  - ( $\alpha$ ) for  $\alpha < \delta$ ,  $f_\alpha <_I f$
  - ( $\beta$ ) if  $g \in {}^\kappa \text{Ord}$ ,  $g <_I f$  then for some  $\alpha$ ,  $g <_I f_\alpha$  and
  - ( $\gamma$ )  $cf[f(i)] > \kappa$  for  $i < \kappa$
- (iii) condition (i) fails and (\*\*\*) $_I$  for some unbounded  $A \subseteq \delta$  and  $t_\alpha \subseteq \kappa$  for  $\alpha \in A$  and  $g \in {}^\kappa \text{Ord}$  we have:
  - ( $\alpha$ ) for  $\alpha < \beta$  in  $A$ ,  $t_\beta \setminus t_\alpha \in I$  but  $t_\alpha \setminus t_\beta \notin I$  (i.e.  $\langle t_\alpha / I : \alpha \in A \rangle$  is strictly decreasing in  $\mathcal{P}(\kappa) / I$ )
  - ( $\beta$ )  $t_\alpha = \{i < \kappa : f_\alpha(i) \leq g(i)\}$ .

**1.8 Claim.** 1) For  $\langle f_\alpha : \alpha < \delta \rangle, \kappa, I$  as in the hypothesis of Claim 1.3 such that  $cf(\delta) > \kappa^+$ , the following are equivalent:

- (a) there are  $A \subseteq \delta$  unbounded, and  $s_\alpha \in I$  for  $\alpha \in A$  such that:  
 $\langle f_\alpha(i) : \alpha \in A, i \in \kappa \setminus s_\alpha \rangle$  is strictly increasing in  $\alpha$  for each  $i < \kappa$
- (b)  $(**)_I$  of Claim 1.3 holds for some  $f$  and

$$\{i < \kappa : cf[f(i)] \neq cf(\delta)\} \in I$$

(note:  $f/I$  is unique as a  $<_I$ -lub of  $\langle f_\alpha : \alpha < \delta \rangle$ ).

2) If  $cf(\delta) > gen(I)$ , in clause (a) above without loss of generality  $s_\alpha = s$  for  $\alpha \in A$ ; if  $I = J_\kappa^{bd}$  without loss of generality  $s_\alpha = s = [i(*), \kappa)$  for some fixed  $i(*)$ .

**1.9 Definition.** We define (for  $\bar{f}$  and  $J, \kappa, \lambda, \mu$ ):

1)  $S^{gd}$  is the set of  $\bar{f}$ -good ordinals. An ordinal  $\delta$  is weakly  $\bar{f}$ -good if  $\delta$  is a limit ordinal,  $\delta < lg(\bar{f})$  is of cofinality  $> \kappa$  but  $< \lambda$  such that for some  $t \subseteq \kappa$ ,  $t \notin J$  and for some unbounded  $A \subseteq \delta$  and  $\langle s_\alpha : \alpha \in A \rangle$  such that  $s_\alpha \in J$  and for each  $i < \kappa$ ,  $\langle f_\alpha(i) : \alpha \in A, i \in t \setminus s_\alpha \rangle$  is strictly increasing (we may say  $\delta$  is  $\bar{f}$ -good by  $t$ ); so  $\bar{f} \upharpoonright \delta$  has a  $<_J$ -eub (see below)  $f_\delta^*$ ,  $f_\delta^*(i) = \sup\{f_\alpha(i) : \alpha \in A \text{ and } i \in s_\alpha\}$ . An ordinal  $\delta$  is  $\bar{f}$ -good if we could have  $t = \kappa$ .

2)  $S^{bd}$ , a set of bad points for  $\bar{f}$ , is the set of limit ordinals  $\delta < lg(\bar{f})$  of cofinality  $> \kappa$  but  $< \lambda$  such that  $\bar{f} \upharpoonright \delta$  satisfies  $(**)_J$  of Claim [1.2]Sh:355 - for some  $f_\delta^*$  (i.e.  $\bar{f} \upharpoonright \delta$  has as a  $<_J$ -eub  $f_\delta^*$  (i.e.  $<_J$ -exact upper bound, this means (here):  $\alpha < \delta \Rightarrow f_\alpha <_J f_\delta$  and  $g <_J f_\delta \Rightarrow \bigvee_{\alpha < \delta} g <_J f_\alpha$ ) such that  $\{i : cf[f_\delta^*(i)] \leq \kappa\} \in J$ ),

but  $\delta \notin S^{gd}$ ; without loss of generality  $f_\delta^* \in \prod_{i < \kappa} \lambda_i$  (as  $f_\delta^* \leq_J f_\delta$ ).

3)  $S^{ch}$ , a set of  $\delta$  chaotic points for  $\bar{f}$ , is the set of limit ordinals  $\delta < lg(\bar{f})$  of cofinality  $> \kappa$  but  $< \lambda$  such that:

- (a)  $\bar{f} \upharpoonright \delta$  satisfies  $(*)_{D_\delta}$  of Claim [1.2]Sh:345 i.e. for some ultrafilter  $D_\delta$  on  $\kappa$  disjoint to  $J$  and  $\langle a_i : i < \kappa \rangle$ ,  $a_i$  a set of  $\leq \kappa$  ordinals, for every  $i < \delta$  for some  $j < \delta$  (but  $> i$ ) and  $g \in \prod_{i < \kappa} a_i$  we have  $f_i/D < g/D < f_j/D$

or

- (b)  $\bar{f} \upharpoonright \delta$  satisfies  $(***)_I$  of Claim [1.2(iii)]Sh:355; (i.e. for some  $A, g$  we have  $A \subseteq \delta$  unbounded,  $g \in {}^\kappa Ord$  and  $\langle \{i : f_\alpha(i) \leq g(i)\} / J : \alpha \in A \rangle$  is strictly decreasing in  $\mathcal{P}(\kappa)/J$ . (Hence  $cf(\delta) \leq 2^\kappa$  and if  $J$  is a maximal ideal, this does not occur).

4)  $S^{wg}(\bar{f})$  is the set of weakly good  $\delta < \mu$  satisfying  $\delta \notin S^{ch}$ ;  $S^{vbd}$  is the set  $S^{bd} \setminus S^{wg} \setminus S^{ch}$ .

**1.10 Remark.** 1) For the specific  $J = J_\kappa^{bd}$  we could redefine  $S^{gd}$  as follows: for some unbounded  $A \subseteq \delta$  and  $i(*) < \kappa$  for  $\alpha \in A$ , for each  $i \in (i(*), \kappa)$   $\langle f_\alpha(i) : \alpha \in A \rangle$  is strictly increasing. We use:  $J$  is generated by a family of  $< cf(\delta)$  subsets of  $\kappa$ .

2) More accurately, we should have written  $S^{gd}[\bar{f}]$  or even  $S_J^{gd}[\bar{f}, \lambda]$ , (as  $\kappa$  can be

reconstructed), similarly for  $bd$  and  $ch$ .

3)  $gd, bd, ch$  stands for good, bad, chaotic, respectively.

- 1.11 Fact.** 1)  $\langle S^{gd}, S^{bd}, S^{ch} \rangle$  is a partition of  $\{\delta < \ell g(\bar{f}) : \kappa < cf(\delta) < \lambda\}$ . Also  $\langle S^{wgd}, S^{vbd}, S^{ch} \rangle$  is a partition of that set.  
 2)  $\delta \in S^{gd} \cup S^{bd}$  iff  $\bar{f} \upharpoonright \delta$  satisfies  $(*)_I$  of Claim [\[\[1.2\]Sh:355](#), for  $I = J$ ; i.e.  $\bar{f} \upharpoonright \delta$  has a  $<_J$ -eub.  
 3)  $S^{gd} \subseteq S^{wg}$  and  $S^{vbd} \subseteq S^{bd}$ .

*Proof.* If  $\delta \in S^{gd}$ , then  $\bar{f} \upharpoonright \delta$  satisfies  $(*)_I$  of Claim [\[\[1.2\]Sh:355](#) (for  $I = J$ ), by 1.9. Now Part 2) holds by the definition of  $S^{bd}$ . Now Part (1) is immediate by Claim [\[\[1.2\]Sh:345](#) (and Definition 1.9).

2), 3) Easy, too.  $\square_{1.11}$

**1.12 Claim.** 1) If  $\delta \in S^{bd}$  then letting  $\lambda_i^\delta = cf[f_\delta^*(i)]$  (see 1.9(2)), we have:

- (i)  $\{i : \lambda_i^\delta \leq \kappa \text{ or } \lambda_i^\delta > cf(\delta)\} \in J$  (so without loss of generality it is empty)
- (ii)  $\lambda_i^\delta < \lambda_i$  and
- (iii)  $(\prod_{i < \kappa} \lambda_i^\delta, <_J)$  has true cofinality  $cf(\delta)$  and
- (iv) for no  $\lambda'$  is  $\{i : \lambda_i^\delta \neq \lambda'\} \in J$
- (v) If  $\delta$  is not weakly good, then for every  $\lambda'$  we have  $\{i : \lambda_i^\delta = \lambda'\} \in J$
- (vi) for every  $\lambda' \neq cf(\delta)$  we have  $\{i : \lambda_i^\delta = \lambda'\} \in J$ .

2) If  $J$  is  $\theta$ -complete,  $\lambda = \chi^{+\zeta}$ ,  $\zeta \leq \theta$  a limit ordinal and  $\kappa \leq \chi$ , then:

- (i)  $\delta \in S^{gd} \Rightarrow \{i < \kappa : \lambda_i^\delta \neq cf(\delta)\} \in J$ .
- (ii)  $\delta \in S^{vbd} \Rightarrow \{i < \kappa : \lambda_i^\delta \geq \chi\} \in J$ .
- (iii)  $\delta \in S^{wg} \Rightarrow \{i < \kappa : \lambda_i^\delta \geq \chi, \lambda_i^\delta \neq cf(\delta)\} \in J$ .
- (iv)  $cf(\delta) \geq \chi$  &  $\delta \in S^{wg} \setminus S^{gd} \Rightarrow \{i < \kappa : \lambda_i^\delta \geq \chi \text{ and } \lambda_i^\delta = cf(\delta)\} \in J^+$ .

3) If  $J$  is  $\theta$ -complete,  $\zeta \leq \theta$  (a limit ordinal) and  $\lambda = \kappa^{+\zeta}$  then  $S^{bd} = \emptyset$ .

*Proof.* 1) The first phrase holds as  $f_\delta^*$  is  $<_J$ -exact upper bound of  $\bar{f} \upharpoonright \delta$  (see Claim [\[\[1.2\]Sh:355](#)). For the rest see [\[\[1.6\]Sh:355](#).

2) Suppose  $\delta \in S^{bd}$  is not weakly good and  $t = \{i : \lambda_i^\delta > \chi\}$  is not in  $J$ ; as  $\delta \in S^{bd}$  trivially  $\{i : \lambda_i^\delta > cf(\delta)\} \in J$ , so  $t_1 = \{i : \chi < \lambda_i^\delta \leq cf(\delta)\}$  is not in  $J$ . As  $cf(\delta) < \chi^{+\zeta}$ ,  $\zeta \leq \theta$ , the set  $\{\sigma : \chi \leq \sigma = cf(\sigma) < \chi^{+\zeta}\}$  has  $< |\zeta| \leq \theta$  members but  $J$  is  $\theta$ -complete, hence for some  $s \subseteq t_1$ ,  $s \notin J$  and  $\langle \lambda_i^\delta : i \in s \rangle$  is constant, say  $\lambda^*$ , now also  $(\prod_{i \in s} \lambda_i^\delta, <_J)$  has true cofinality  $cf(\delta)$  (by 1.12(1)(iii)), but this implies  $\delta$

is weakly  $f$ -good by “ $\Leftarrow$ ” of [\[\[1.6\(1\)\]Sh:355](#). So we have proved (ii). For (i), (iii), (iv) the proof is similar. (In fact for clause (i) the assumption of part (2) is not needed).

3) If  $\delta \in S^{bd}$ , by 1.12(2)(i) we have  $\{i < \kappa : \lambda_i^\delta > \kappa\} \in J$ , so one of the clauses of Definition 1.9(2) fails by (ii), (iii) of part (2) of our Claim, so  $\delta \notin S^{bd}$ , contradiction.

$\square_{1.12}$

**1.13 Claim.** *If  $\delta \in S^{gd}$ , then for some club  $C$  of  $\delta$  we have:*

$$\alpha \in C \ \& \ cf(\alpha) > \kappa \Rightarrow \alpha \in S^{gd}.$$

*Similarly for  $S^{ugd}$  (by  $t$ ).*

*Proof.* Straightforward by its definition.

**1.14 Claim.** *If  $\delta \in S^{ch}$ , then for some club  $C$  of  $\delta$*

$$\alpha \in C \ \& \ cf(\alpha) > \kappa \Rightarrow \alpha \in S^{ch}.$$

*Proof.* If  $\delta \in S^{ch}$ , then

Case a: For some ultrafilter  $D$  on  $\kappa$  disjoint to  $J$  (say  $D_\delta$ ) and club  $C$  of  $\delta$ , there are  $\langle g_\alpha : \alpha \in C \rangle$  such that:

- (a) for  $\alpha < \beta$  in  $C$  we have  $f_\alpha/D < g_\alpha/D < f_\beta/D$
- (b) for each  $i < \kappa$  the set  $s_i = \{g_\alpha(i) : \alpha \in C\}$  has power  $\leq \kappa$ .

Now for every  $\delta' \in C$  such that  $\delta' = \sup(\delta' \cap C)$  and  $cf(\delta') > \kappa$  clearly  $\langle g_\alpha : \alpha \in C \cap \delta' \rangle, C \cap \delta', \langle s_i : i < \kappa \rangle$  witness that  $\langle f_\alpha : \alpha < \delta' \rangle$  also satisfies  $(*)_D$  of [Claim 1.2]Sh:355, hence  $\delta' \in S^{ch}$ .

Case b: (Not case (a)). (See 1.9(3)). Also trivial proof. □<sub>1.14</sub>

**1.15 Claim.** *For every regular  $\chi, \kappa < \chi < \lambda, S_\chi^{gd} =: \{\delta \in S^{gd} : cf(\delta) = \chi\}$  is stationary.*

*Proof.* Suppose  $C^*$  is a club of  $\mu$ . We choose by induction on  $\zeta < \chi$ ,  $\alpha_\zeta, g_\zeta, s_\zeta$  such that:

- (1)  $\alpha_\zeta \in C^*$
- (2)  $\bigcup_{\xi < \zeta} \alpha_\xi < \alpha_\zeta$
- (3)  $g_\zeta \in \prod_{i < \kappa} \lambda_i, g_\zeta(i) =: \sup \{f_{\alpha_\xi}(i) + 1 : \xi < \zeta\}$
- (4)  $s_\zeta \in J$
- (5)  $i \in \kappa \setminus s_\zeta \Rightarrow g_\zeta(i) < f_{\alpha_\zeta}(i)$ .

In stage  $\zeta$ , first define  $g_\zeta$  by (3) then find  $\beta_\zeta < \mu$  such that  $g_\zeta <_J f_{\beta_\zeta}$  (possible as  $\langle \lambda_j : j < \kappa \rangle$  is the  $<_J$ -eub of  $\langle f_\alpha : \alpha < \mu \rangle$ ) then choose  $\alpha_\zeta, \beta_\zeta \cup \bigcup_{\xi < \zeta} \alpha_\xi < \alpha_\zeta \in C^*$  (possible as  $|\zeta| < \lambda, C^*$  unbounded in  $\mu$ ). Lastly choose  $s_\zeta$  to satisfy (5) by the definition of  $<_J$ . Clearly (1)-(5) holds.

Let  $\delta = \bigcup_{\zeta < \mu} \alpha_\zeta$ , clearly  $\delta \in C^*$  (being closed) and  $cf(\delta) = \chi$  and clearly  $\delta \in S_\chi^{gd}$ .

[Note: if  $gen(J) < \chi$ , as  $\chi = cf(\chi)$  for some  $t \in J$ ,  $\{\zeta < \mu : s_\zeta \subseteq t\}$  is unbounded in  $\chi$ . Also it is natural to deduce it from the existence of a stationary  $S \subseteq \{\delta < \mu : cf(\delta) = \chi\}, S \in I[\mu]$  (see [§1]Sh:420)].

We shall not use

**1.16 Lemma.** *Suppose  $\lambda = \kappa^{+\kappa}$ ,  $\langle \lambda_i : i < \kappa \rangle$ ,  $\mu, J$  are as in (\*) of 1.5 and in addition  $S^{bd} = \emptyset$ . Let  $C^* = \{\delta < \lambda^+ : \delta \text{ divisible by } \lambda^2\}$ . Then we can find  $\langle \alpha_{i,j}^\delta : \delta \in C^* \cap S^{gd}, i < cf(\delta), j < \kappa \rangle$  such that:*

- (a)  $\alpha_{i,j}^\delta < \delta$  (and, if you want,  $\langle \alpha_{i,j}^\delta : i < cf(\delta) \rangle$  is increasing continuous with limit  $\delta$ ,  $\langle \alpha_{i,j}^\delta : j < \kappa \rangle$  increasing with  $j$ ).
- (b) for every  $\delta(*) < \lambda^+$  we can find  $\langle C^\delta, s_i^\delta : \delta \in \delta(*) \cap C^* \cap S^{gd}, i < cf(\delta) \rangle$  such that:
  - ( $\alpha$ )  $C^\delta$  a club of  $cf(\delta)$
  - ( $\beta$ )  $s_i^\delta \in J$
  - ( $\gamma$ )  $\langle \{\alpha_{i,j}^\delta : i \in C^\delta, j \notin s_i^\delta\} : \delta \in \delta(*) \cap C^* \cap S^{gd} \rangle$  are pairwise disjoint.

This we refer to [MgSh:204], anyhow see proof of 2.6.

## §2 NICELY COMPACT SET AND NON-REFLECTION

This section answers the following two questions.

One question of Foreman and Magidor asks for consistency of the following reflection principle:

- $(*)_\lambda$  if  $S \subseteq \mathcal{S}_{<\aleph_2}(\lambda)$  is a stationary set (of  $\mathcal{S}_{<\aleph_2}(\lambda)$ , not of  $\lambda$ ), each  $a \in S$  is  $\omega$ -closed (i.e.  $\text{cf}(\delta) = \aleph_0$  &  $\delta = \sup(\delta \cap a) < \lambda \Rightarrow \delta \in a$ ) then for some  $B \subseteq \lambda$ ,  $|B| = \aleph_2$  and  $S \cap \mathcal{S}_{<\aleph_2}(B)$  is stationary.

We shall show that it is very hard to get them (in a way defeating the application they have in mind:  $\theta < \aleph_2$  follows from large cardinals). E.g. if  $\mu$  is strong limit singular of cofinality  $\aleph_0$ ,  $2^\mu = \mu^+$  then  $(*)_{\mu^+}$  is false.

Another question of Mekler and Eklof, improve somewhat [MgSh:204] showing in particular that if we have gotten in the (regular)  $\lambda, \kappa$  examples for incompactness then we can get one for  $\lambda^{+(\kappa+1)}$ .

To do this (see below) if  $\lambda > \kappa^{+\kappa}$ , we need a counterexample for  $\lambda$ , which without loss of generality is as in [Sh:161] canonical examples in which only cardinals in which examples exist appear (see below). For this we define when a set  $K$  of cardinals is nicely incompact and prove that the family of such sets is closed under the required operation.

**2.1 Definition.** A set  $K$  of regular cardinals is called nicely incompact if:

- (a) for each  $\lambda \in K$  there is a  $\lambda$ -free not  $\lambda^+$ -free abelian group or  $\lambda = \aleph_0$   
 (b) if  $\lambda$  is in  $K$ ,  $\kappa_1, \dots, \kappa_m \in K \cap \lambda$  then there is an example  $\langle s_\eta^\ell : \eta \in \mathfrak{S}_f, \ell < m \rangle$  of incompactness (see [§3]Sh:161 or [Sh:521], we assume knowledge of e.g. [AP]Sh:521) such that:

$$(\lambda_{<} = \lambda \text{ and}) \text{ for some } \langle k_1, \dots, k_m \rangle \text{ we have} \\ \eta \in S \ \& \ \ell g(\eta) = k_\ell \Rightarrow \lambda_\eta = \kappa_\ell.$$

**2.2 Claim.** 1) If  $K$  is nicely incompact,  $\lambda = \max(K)$ ,  $\lambda$  regular then  $K \cup \{\lambda^+\}$  is nicely incompact.

2) Assume  $K = \{\lambda_i : i < \delta\}$  increasing and  $j < \delta \Rightarrow \{\lambda_i : i \leq j\}$  nicely incompact. Then  $K$  is nicely incompact.

3)  $\{\aleph_0\}$  is nicely incompact.

*Proof.* Easy.

*Remark.* If proof of 2.2(1) is not clear - read proof of 2.3 and throw away most (use  $\{\delta < \lambda^+ : \text{cf}(\delta) = \lambda\}$  here instead  $\{\delta \in A_g : \text{cf}(\delta) = \lambda'\}$  there.



**2.3 Theorem.** *If  $K$  is nicely incompact,  $\chi, \kappa \in K$ ,  $K$  has no last element and  $\chi^{+\kappa} \geq \sup(K)$  then  $K \cup \{\chi^{+(\kappa+1)}\}$  is nicely incompact.*

*Remark.* By the closure properties 2.2(1), 2.2(2), 2.3 (and as a starting point 2.2(3):  $\{\aleph_0\}$  being nicely incompact) we get a family of  $\aleph_0$  cardinals which probably is the minimal set of incompactness.

*Proof.* If  $\chi < \kappa = \aleph_\kappa$ , then  $\chi^{+(\kappa+1)} = \kappa^+$  and the desired conclusion holds is by Claim 2.2(1).

If  $\chi < \kappa < \aleph_\kappa$  then  $\chi^{+\kappa} = (\kappa^+)^{\kappa} = \aleph_\kappa$  so without loss of generality  $\chi = \kappa^+$ .

So without loss of generality  $\chi \geq \kappa^+$ ,  $\lambda = \chi^{+\kappa}$ ,  $\mu = \lambda^+$ ,  $\lambda > \kappa$  and  $J$ ,  $\langle \lambda_i : i < \kappa \rangle$ ,  $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$  be as in (\*) of 1.5 and for simplicity  $f_{\alpha_1}(i_1) = f_{\alpha_2}(i_2) \Rightarrow i_1 = i_2$ , e.g.  $f_\alpha(i) > \sum_{j < i} \lambda_j$ ; and let  $S^{gd}, S^{bd}, S^{ch}$  be from

Definition 1.9. Remember that by 1.13, 1.14 we have

(\*) if  $x \in \{gd, wgd, ch\}$ ,  $\delta \in S^x$  then for some club  $E$  of  $\delta$  we have:  
 $\alpha \in E$  &  $\text{cf}(\alpha) \geq \kappa^+ \Rightarrow \alpha \in S^x$ .

In the definition of “ $K$  is nicely incompact” we have to check only for  $\mu = \lambda^+$ . Let  $m < \omega$  and  $\kappa_1, \dots, \kappa_m \in K \cap \lambda^+$ . So  $\langle \kappa_1, \dots, \kappa_m \rangle \in K \cap \chi^{+\kappa}$ , without loss of generality  $\kappa_1 < \kappa_2 < \dots < \kappa_m$ .

Let  $\lambda' \in K$  be such that  $\kappa < \lambda, \chi < \lambda'$  and  $\kappa_m < \lambda'$  (there is such a  $\lambda'$  as  $K$  has no last member).

Let  $S = \{\delta \in S^{gd}(\bar{f}) : \text{cf}(\delta) = \lambda'\}$  (by 1.15 it is stationary) and for  $\delta \in S$  choose  $E_\delta \subseteq \delta$  an unbounded subset of order type  $\lambda'$ .

Let  $E_\delta = \{\alpha_\zeta^\delta : \zeta < \lambda'\}$  (increasing).

Now choose an example for  $\lambda' : \langle s_\eta^\ell : \eta \in \mathfrak{S}'_f \text{ and } \ell < n(*) \rangle$ ,  $\mathfrak{S}'$  a  $\lambda'$ -set (see [Sh:161] or [AP:Sh:521]), such that for some sequence  $\langle k_1, \dots, k_n, k \rangle$  of natural number  $\leq \lg(\eta)$  for  $\eta \in \mathfrak{S}'$

$$\eta \in \mathfrak{S}' \text{ \& } \lg(\eta) = k_i \Rightarrow \lambda_\eta = \kappa_\ell$$

$$\eta \in \mathfrak{S}' \text{ \& } \lg(\eta) = k \Rightarrow \lambda_\eta = \kappa.$$

(possible by Definition 2.1).

Lastly, let

$$\mathfrak{S} = \{ \langle \rangle \} \cup \{ \langle \delta \rangle \hat{\eta} : \delta \in S \subseteq \lambda^+ \text{ and } \eta \in \mathfrak{S}' \}.$$

For  $\eta \in \mathfrak{S}_f$  let (denoting  $\delta = \eta(0)$  which necessarily is in  $S$  and letting  $\eta = \langle \delta \rangle \hat{\nu}$  so  $\nu = \eta \upharpoonright [1, \lg(\eta))$ ):

$$s_\eta^{\ell+1} = (s_\nu^\ell \times \{\delta\}) \text{ if } \ell < n(*), \text{ and } s_\eta^0 = \left\{ \langle f_\delta(\eta(k)), \eta \upharpoonright [1, \lg(\eta)), \alpha_{\eta(k)}^\delta \rangle : k < \omega \right\}.$$

$$A_\eta = \cup \{ s_\eta^\ell : \ell \leq n(*) \}$$

First note that  $\langle A_\eta : \eta \in \mathfrak{S}_f \rangle$  has the right form hence has no one-to-one choice function (see [§3]Sh:161 or [AP]Sh:521).

Secondly, to prove every subfamily of cardinality  $< \mu$  has a one-to-one choice function it is enough to prove 2.6 below. It applies with  $\lambda', \mu$  here standing for  $\chi, \mu^*$  there. The no reflection is by 2.4 below.

*2.4 Observation.* Let  $\bar{\lambda}, J, \mu, \lambda$  be as in (\*) of 1.5.

1) If  $J$  is  $\theta$ -complete,  $\lambda = \chi^{+\zeta}$ ,  $\zeta \leq \theta$  a limit ordinal,  $\text{cf}(\chi) > \kappa$ , then:

- (i)  $\delta \in S^{gd} \Rightarrow \{i < \kappa : \lambda_i^\delta \neq \text{cf}(\delta)\} \in J$
- (ii)  $\delta \in S^{vbd} \Rightarrow \{i < \kappa : \lambda_i^\delta \neq \text{cf}(\delta)\} \in J$
- (iii)  $\delta \in S^{wg} \Rightarrow \{i : \lambda_i^\delta \geq \chi, \lambda_i^\delta \neq \text{cf}(\delta)\} \in J$
- (iv)  $\text{cf}(\delta) \geq \chi$  &  $\delta \in S^{wg} \setminus S^{gd} \Rightarrow \{i < \kappa : \lambda_i^\delta \geq \chi \text{ and } \lambda_i^\delta = \text{cf}(\delta)\} \in J^+$ .

2) If  $J$  is  $\theta$ -complete,  $\zeta \leq \theta$  (a limit ordinal) and  $\lambda = \chi^{+\zeta}$  then for no  $\delta \in S^{bd}$  do we have:  $\{\alpha \in S^{gd} : \alpha < \delta, \text{cf}(\alpha) \geq \chi\}$  is stationary.

*Proof.* 1) Like 1.12.

2) Let  $\delta$  be a counterexample, let  $f_\delta^*$  be a  $<_J$ -eub of  $\bar{f} \upharpoonright \delta$  and by part (1) without loss of generality  $s^* = \{i < \kappa : \lambda_i^\delta = \text{cf}[f_\delta^*(i)] < \chi\} \in J^+$ . Choose  $E_i \subseteq f_\delta^*(i)$  a club of order type  $\lambda_i^\delta$ .

Choose  $\langle \alpha_\zeta : \zeta < \text{cf}(\delta) \rangle$  is a strictly increasing sequence of ordinals with limit  $\delta$ . Now we can choose by induction on  $\zeta < \text{cf}(\delta)$  a pair  $(\beta_\zeta, g_\zeta)$  such that:

- (a)  $g_\zeta \in \prod_{i < \kappa} E_i$  such that  
 $\varepsilon < \zeta \Rightarrow f_{\beta_\varepsilon} \upharpoonright s^* <_J g_\zeta \upharpoonright s^*$
- (b)  $\beta_\zeta \in \left( \bigcup_{\varepsilon < \zeta} \beta_\varepsilon \cup \alpha_\zeta, \delta \right)$  is such that  $g_\zeta \upharpoonright s^* <_J f_{\beta_\zeta} \upharpoonright s^*$ .

So if  $\delta$  is a counterexample, then for some  $\zeta < \text{cf}(\delta)$  we have:  $\text{cf}(\zeta) \geq \chi$  and  $\beta^* = \bigcup_{\varepsilon < \zeta} \beta_\varepsilon \in S^{gd}$ . Hence there is an increasing sequence  $\langle \gamma_j : j < \text{cf}(\beta^*) \rangle$  of

ordinals with limit  $\beta^*$ , and  $t_j \in J$  for  $j < \text{cf}(\beta^*)$  such that for each  $i, \langle f_{\gamma_j}(i) : j < \text{cf}(\beta^*), i \notin t_j \rangle$  is strictly increasing. For each  $i \in s^*$  there is  $j(i) < \text{cf}(\beta^*)$  such that all the ordinals  $\{f_{\gamma_j}(i) : j(i) \leq j < \text{cf}(\beta^*) \text{ and } i \notin t_j\}$  realize the same Dedekind cut of  $E_i$ . Let  $j(*) = \sup_{i < \kappa} j(i) < \text{cf}(\beta^*)$  (as  $\text{cf}(\beta^*) \geq \text{cf}(\chi) > \kappa$ ). Now

the contradiction should be clear.  $\square_{2.4}$

*2.5 Remark.* 1) Alternatively, in the proof of 2.3 redefine  $S^{ch}(\bar{f})$  replacing “ $|s_i| \leq \kappa$ ” by “ $|s_i| \leq \chi$ ”. We call it  $S_\chi^{ch}(\bar{f})$  (similarly the others).

2) Note: if  $\lambda$  belongs to some nicely incompact  $K^*$ , then there is a finite nicely incompact  $K \subseteq K^*$  to which  $\lambda$  belongs.

**2.6 Claim.** *In the context of (\*) of 1.5, suppose  $\kappa < \chi = cf(\chi) < \mu^* \leq \mu$  and  $S \subseteq \{\delta \in S^{gd} : cf(\delta) = \chi\}$  is stationary but reflect in no  $\delta \in S^{bd}$  of cofinality  $< \mu^*$  and  $f_{\alpha_1}(i_1) = f_{\alpha_2}(i_2) \Rightarrow i_1 = i_2$  (e.g.  $f_\alpha(i) > \sup_{j < i} \lambda_j$ ). For  $\delta \in S$  let  $E_\delta$  be an unbounded subset of  $\delta$ . Let  $A_\delta = E_\delta \times (Rang f_\delta) \subseteq \delta \times \lambda$ . Then  $\langle A_\delta : \delta \in S \rangle$  has no one-to-one choice function but is  $\mu^*$ -free in the following sense:*

(\*) *if  $B \subseteq S$ ,  $|B| < \mu^*$  then we can find  $\alpha_\delta < \delta$  and  $s_\delta \in J$  for  $\delta \in B$  such that the sets  $\langle (E_\delta \setminus \alpha_\delta) \times Rang(f_\delta \upharpoonright (\kappa \setminus s_\delta)) : \delta \in B \rangle$  are pairwise disjoint.*

*Remark.* 1) When we use it in 2.3,  $\mu^* = \mu$  and the non-reflection is guaranteed by 2.4.

2) If we want to replace  $A_\delta$  by a subset of  $\delta$ , assume every  $\delta \in S$  is divisible by  $\lambda^2$ , and every  $\alpha \in E_\delta$  is divisible by  $\lambda$ , and let

$$A_\delta = \{\alpha_\zeta^\delta + f_\delta(i) : i < \kappa, \theta < \chi\} \text{ where } S_\delta = \{\alpha_\zeta^\delta : \zeta < \chi\} \text{ (increasing) .}$$

3) If  $J = J_\kappa^{bd}$  it may look nicer to assume  $\langle Rang f_\alpha \upharpoonright [i, \kappa) : \alpha \in E_\delta \rangle$  (for suitable  $i = i_\delta < \kappa$ ) are pairwise disjoint (not hard to get as  $\delta \in S^{gd}$ ). Then use  $f_\alpha \upharpoonright [i, \kappa)$  only.

4) It seems reasonable to demand (and it causes no problem)  $i < \kappa$  &  $\alpha < \mu \Rightarrow f_\alpha(i) > \sup\{f_\beta(j) : j < i, \beta < \mu\}$ .

*Proof.* This is proved by induction on  $\cup\{\delta + 1 : \delta \in B\}$ .

Without loss of generality  $B \neq \emptyset$ .

*Case 1.*  $B$  has a last element  $\delta(*)$ .

By the induction hypothesis there are  $\langle (\alpha_\delta^1, s_\delta^1) : \delta \in B \setminus \{\delta(*)\} \rangle$  as required for  $B \setminus \{\delta(*)\}$ . For each  $\delta \in B \cap \delta(*)$  we have  $t_\delta = \{i < \kappa : f_\delta(i) \geq f_{\delta(*)}(i)\} \in J$ . Let us define for  $\delta \in B$ :

$$\alpha_\delta = \begin{cases} \alpha_\delta^1 & \text{if } \delta \in B, \delta \neq \delta(*) \\ 0 & \text{if } \delta = \delta(*) \end{cases}$$

$$s_\delta = \begin{cases} s_\delta \cup t_\delta & \text{if } \delta \in B, \delta \neq \delta(*) \\ \emptyset & \text{if } \delta = \delta(*) \end{cases}$$

*Case 2.*  $B$  has no last element,  $\delta(*) = \sup(B)$  has cofinality  $\leq \chi$ .

Let  $\langle \gamma_\zeta : \zeta < cf \delta(*) \rangle$  be increasing continuous sequence with limit  $\delta(*)$ ,  $\gamma_0 = 0$  and  $cf(\gamma_\zeta) \neq \chi$ , hence  $\gamma_\zeta \notin S$ . For each  $\zeta$ , apply the induction hypothesis to  $B_\zeta =: B \cap (\gamma_\zeta, \gamma_{\zeta+1})$  and get  $\langle (\alpha_\delta^\zeta, s_\delta) : \delta \in B_\zeta \rangle$ .

We let for  $\delta \in B$  (note:  $B$  is the disjoint union of  $B_\zeta$  for  $\zeta < \chi$ ):

$$\alpha_\delta = \text{Max}\{\alpha_\delta^\zeta, \gamma_\zeta\} \text{ for } \delta \in B_\zeta$$

$$s_\delta = s_\delta^\zeta \text{ for } \delta \in B_\zeta.$$

Easy to check that they are as required.

*Case 3.*  $B$  has no last element,  $\delta(*) = \sup(B)$  has cofinality  $> \chi$  and  $S^{gd}$  (hence  $S$ ) does not reflect in  $\delta(*)$ .

Same proof as Case 2.

*Case 4.*  $B$  has no last element,  $\delta(*) = \sup(B)$  has cofinality  $> \chi$  and  $S^{gd}$  reflects in  $\delta(*)$ .

As  $|B| < \mu^*$ , clearly  $cf(\delta(*)) < \mu^*$  hence (by assumption)  $\delta(*) \notin S^{bd}$ . Also  $\delta(*) \notin S^{ch}$  (by 1.14 as:  $S^{gd} \cap \delta(*)$  is stationary and  $S^{gd} \cap S^{ch} = \emptyset$  (which holds by 1.11)). So  $\delta(*) \in S^{gd}$ , hence we can find  $\langle \gamma_{\zeta+1} : \zeta < cf(\delta(*)) \rangle$  increasing with limit  $\delta(*)$ , and  $s_{\zeta} \in J$  for  $\zeta < cf(\delta(*))$  be such that:

- (\*)  $i < \kappa$  &  $\zeta < \xi < cf(\delta(*))$  &  $i \notin s_{\zeta} \cup s_{\xi} \Rightarrow f_{\gamma_{\zeta}}(i) < f_{\gamma_{\xi}}(i)$
- (\*\*)  $\gamma_{\zeta} \in S \Rightarrow cf(\zeta) = \chi$ .

For limit  $\zeta < cf(\delta(*))$  let  $\gamma_{\zeta} =: \bigcup_{\xi < \zeta} \gamma_{\xi+1}$  and let  $\gamma_0 = 0$ .

Now if  $\zeta < cf(\delta(*))$  is limit and  $\gamma_{\zeta} \in S$  (so  $cf(\zeta) = \chi$ ), let

$$t_{\zeta}^0 =: \{i : \text{for arbitrarily large } \xi < \zeta, i \in \kappa \setminus s_{\xi+1} \text{ and } f_{\gamma_{\xi+1}}(i) \geq f_{\gamma_{\zeta}}(i)\}$$

now  $t_{\zeta}^0$  is in  $J$  [otherwise, for each  $i \in t_{\zeta}^0$  let  $Y_{\zeta}^i = \{\xi < \zeta : i \in \kappa \setminus s_{\xi+1} \text{ and } f_{\gamma_{\xi+1}}(i) \geq f_{\gamma_{\zeta}}(i)\}$  by the choice of  $\langle s_{\xi+1} : \xi < cf(\delta(*)) \rangle$ , clearly

$$\xi_1 < \xi_2 < \zeta \text{ \& } \xi_1 \in Y_{\zeta}^i \text{ \& } i \in \kappa \setminus s_{\xi_2} \Rightarrow \xi_2 \in Y_{\zeta}^i.$$

Let  $\xi(i) \leq \zeta$  be  $\text{Min}(Y_{\zeta}^i \cup \{\zeta\})$ , and  $\xi(*) = \text{Min}\{\xi(i) : i < \kappa, \xi(i) < \zeta\}$ , now  $\xi(*) < \zeta$  as  $cf(\zeta) = \chi > \kappa$ ; and look at what occurs for  $\xi = \xi(*) + 1$ ].

Let  $t_{\xi}^1 = \{i : \text{for every large enough } \xi < \zeta, i \notin s_{\xi+1}\}$ . Again easily  $t_{\xi}^1 \in J$ . Lastly let  $t_{\xi}^2 = \{i : f_{\gamma_{\xi}}(i) \geq f_{\gamma_{\xi+1}}(i)\}$  now as  $\gamma_{\xi} < \gamma_{\xi+1}$  clearly  $t_{\xi}^2 \in J$  and  $t_{\zeta}^3 = s_{\zeta+1} \in J$ .

Now for each  $\zeta < cf(\delta(*))$ ,  $B_{\zeta} =: B \cap (\gamma_{\zeta}, \gamma_{\zeta+1})$  satisfies the induction hypothesis, hence we have  $\langle (\alpha_{\delta}^{\zeta}, s_{\delta}^{\zeta}) : \delta \in B_{\zeta} \rangle$  as required.

Notice that  $B$  is partitioned to  $B_{\zeta}$  (for  $\zeta < cf(\delta(*))$ ) and  $\{\gamma_{\zeta} : \zeta \text{ limit} < cf(\delta(*))\} \cap S$ . We define

$$\alpha_{\delta} = \begin{cases} \text{Max}\{\alpha_{\delta}^{\zeta}, \gamma_{\zeta}\} & \text{if } \delta \in B_{\zeta}, \zeta < cf(\delta(*)) \\ 0 & \text{if } \delta \in \{\gamma_{\zeta} : \zeta \text{ limit}\} \cap S \end{cases}$$

for  $\zeta < cf(\delta(*))$  and  $\delta \in B_{\zeta}$  let

$$t_{\zeta, \delta} = s_{\zeta} \cup s_{\zeta+1} \cup \{i < \kappa : \neg[f_{\gamma_{\zeta}}(i) < f_{\delta}(i) < f_{\gamma_{\zeta+1}}(i)]\} \in J$$

$$s_\delta = \begin{cases} s_\delta^\zeta \cup t_{\delta,\zeta} & \text{if } \delta \in B_\zeta, \zeta < \text{cf}(\delta(*)) \\ t_\zeta^0 \cup t_\zeta^1 \cup t_\zeta^2 \cup t_\zeta^3 & \text{if } \delta = \gamma_\zeta, \zeta < \text{cf}(\delta(*)) \text{ limit}, \delta \in B(\subseteq S) \end{cases}$$

It is easy to check that  $\langle (\alpha_\delta, s_\delta) : \delta \in B \rangle$  is as required: let  $\delta_1, \delta_2 \in B$  be distinct, without loss of generality  $\delta_1 < \delta_2$ , now

Case 1: If  $\delta_1, \delta_2 \in B_\zeta$  for some  $\zeta$ ; then use the choice of  $\langle \alpha_{\delta_1}^\zeta, s_{\delta_1}^\zeta \rangle, \langle \alpha_{\delta_2}^\zeta, s_{\delta_2}^\zeta \rangle$ .

Case 2: If  $\delta_1 \in B_{\zeta_1}, \delta_2 \in B_{\zeta_2}, \zeta_1 \neq \zeta_2$  then note  $\alpha_{\delta_2} \geq \gamma_{\zeta_2}$ .

Case 3: If  $\delta_1, \delta_2 \in \{\gamma_\zeta : \zeta < \text{cf}(\delta(*))\} \cap S$ , then let  $\delta_1 = \gamma_{\zeta_1}, \delta_2 = \gamma_{\zeta_2}$  hence by (\*\*) we have  $\zeta_1 + 1 < \zeta_2$ , so  $i \in \kappa \setminus t_{\zeta_1}^2 \Rightarrow f_{\gamma_{\zeta_1}}(i) < f_{\gamma_{\zeta_1+1}}(i)$  and  $i \in \kappa \setminus t_\zeta^3 \setminus t_\zeta^0 \setminus t_\zeta^1 \Rightarrow f_{\gamma_{\zeta_1+1}}(i) < f_{\gamma_{\zeta_2}}(i)$  so the conclusion is easy.

Case 4: If  $\delta_2 \in B_{\zeta_2}, \delta_1 \in \{\gamma_\zeta : \zeta < \text{cf}(\delta(*))\} \cap S$  then use  $\alpha_{\delta_2} \geq \gamma_{\zeta_2}$ .

Case 5: If  $\delta_1 \in B_{\zeta_1}, \delta_2 = \gamma_{\zeta_2} \in \{\gamma_\zeta : \zeta < \text{cf}(\delta(*))\} \cap S$  then  $\zeta_2 \geq \zeta_1$  hence by (\*\*) we have  $\zeta_2 > \zeta_1 + 1$ , hence  $i \in \kappa \setminus s_{\zeta_1+1} \setminus t_{\zeta_2}^0 \setminus t_{\zeta_2}^1 \Rightarrow f_{\gamma_{\zeta_1+1}}(i) < f_{\gamma_{\zeta_2}}(i)$  hence  $i \in \kappa \setminus s_{\delta_1} \setminus s_{\delta_2} \Rightarrow i \in \kappa \setminus t_{\zeta,\delta} \setminus s_{\zeta_1+1} \setminus t_{\zeta_2}^0 \setminus t_{\zeta_2}^1 \Rightarrow f_{\delta_1}(i) < f_{\gamma_{\zeta_1+1}}(i) < f_{\delta_2}(i)$ .  $\square_{2.6}$

**2.7 Conclusion.** Suppose  $\text{cf}(\lambda) = \kappa < \lambda, \kappa < \chi = \text{cf}(\chi) < \lambda$ . Then there is a stationary  $S \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) = \chi\}$  and  $\langle A_\delta : \delta \in S \rangle$  such that:

- ( $\alpha$ )  $A_\delta = \{\alpha_{i,\zeta}^\delta : i < \chi, \zeta < \kappa\} \subseteq \delta$
- ( $\beta$ )  $\alpha_{i_1,\zeta_1}^{\delta_1} = \alpha_{i_2,\zeta_2}^{\delta_2} \Rightarrow i_1 = i_2$  &  $\zeta_1 = \zeta_2$
- ( $\gamma$ ) for each  $\delta \in S$ ,  $\zeta < \kappa$  we have  $\bigwedge_\gamma$  [the sequence  $\langle \alpha_{i,\zeta} : i < \chi \rangle$  is increasing with limit  $\delta$ ]
- ( $\delta$ )  $\langle A_\delta : \delta \in S \rangle$  is  $\chi^{+\kappa+1}$ -free in the strong sense of 2.6; i.e. if  $E \subseteq S, |E| \leq \chi^{+\kappa}$  then we can find  $\langle \langle i_\delta, \zeta_\delta \rangle : \delta \in E \rangle$  such that:
  - (a)  $i_\delta < \chi, \zeta_\delta < \kappa$
  - (b) the sets  $A'_\delta = \{\alpha_{i,\zeta}^\delta : \zeta_\delta < \zeta < \kappa \text{ and } i_\delta < i < \chi\}$  (for  $\delta \in E$ ) are pairwise disjoint.

*Proof.* Combine the previous claims.

(Note:  $\lambda$  necessarily is  $\geq \chi^{+\kappa+1}$  but can be strictly bigger).

**2.8 Conclusion.** 1) If  $\lambda > \text{cf}(\lambda) = \kappa, \kappa < \chi = \text{cf}(\chi) < \lambda$ , then we can find a stationary  $S \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) = \chi\}$  and  $A_\delta \subseteq \delta = \sup(A_\delta), |A_\delta| = \chi$  for  $\delta \in S$ , such that:

- (\*) if  $B \subseteq S, |B| \leq \chi^{+\kappa}$  then  $\{A_\delta : \delta \in B\}$  has a one-to-one choice function.

- 2) If in (1)  $\diamond_S$  holds, without loss of generality  $\{A_\delta : \delta \in S\}$  is a stationary subset of  $\mathcal{S}_{<\chi}(\lambda^+)$  each  $A_\delta$  is  $\omega$ -closed (and even  $(<\chi)$ -closed).
- 3)  $\diamond_S$  holds if  $2^\lambda = \lambda^+$  &  $\lambda \geq \beth_\omega$  (also if  $\lambda < \beth_\omega$  this is normally O.K.).
- 4) Condition (\*) of (1) implies that for no  $A^* \subseteq \lambda$  of power  $\leq \chi^{+\kappa}$ , is  $\{A_\delta : A_\delta \subseteq A^*\}$  a stationary subset of  $\mathcal{S}_{<\chi^+}(A^*)$ .

*Proof.* 1) By 2.7.

2) We can replace  $A_\delta$  by any  $A'_\delta$ ,  $A_\delta \subseteq A'_\delta \subseteq \delta$ ,  $|A'_\delta| = \chi$ , and preserve the conclusion of (1); by  $\diamond_S$  we can do it as to get stationary set.

3) By [Sh:460] (see more there).

4) Check. □<sub>2.7</sub>

*2.9 Observation.* Suppose  $\text{cf}(\lambda) = \kappa < \lambda$  and there is  $\langle A_\alpha : \alpha < \lambda^+ \rangle$ ,  $A_\alpha \in [\lambda]^\kappa$ , such that:

for each  $\beta < \lambda^+$  we can find  $\langle A'_\alpha : \alpha < \beta \rangle$  pairwise disjoint with

$$A'_\alpha \subseteq A_\alpha \text{ \& } |A_\alpha \setminus A'_\alpha| < \kappa.$$

Then:

- (A) for every regular  $\chi \in (\kappa, \lambda)$  we can find  $\langle \langle \alpha_{i,\zeta}^\delta : i < \chi, \zeta < \kappa \rangle : \delta < \lambda^+, \text{cf}(\delta) = \chi \rangle$  as in 2.7,  $\lambda^+$ -free, in 2.6's sense; i.e. see clause ( $\delta$ ) of 2.7.
- (B) If  $K$  is a nicely incompact set,  $\kappa = \max(K)$ ,  $\lambda \geq \sup(K)$  and then  $K \cup \{\lambda^+\}$  is nicely incompact.

*Proof.* Trivial. For (A) just replace every ordinal of  $< \lambda$  by  $\chi$  ordinals; and copy them on each interval  $[\lambda\alpha, \lambda\alpha + \lambda)$ . Also (B) is immediate. □<sub>2.9</sub>

*2.10 Conclusion.* If  $\kappa = \text{cf}(\lambda) < \lambda$  and  $pp_{\Gamma(\kappa)}(\lambda) > \lambda^+$  then the assumption of 2.9 hence the conclusions of 2.9 and 2.7 holds.

*Proof.* By [1.6(2)]Sh:355 the assumptions of 2.9 holds.

## §3 NPT IS NOT NECESSARILY TRANSITIVE

**3.1 Lemma.** *Assume the consistency of ZFC+ “there are two weakly compact cardinals  $> \aleph_0$ ”. Then it is consistent with ZFC that for some regular  $\lambda > \mu > \aleph_0$ ,  $NPT(\lambda, \mu) + NPT(\mu, \aleph_0)$  does not imply  $NPT(\lambda, \aleph_0)$ ; (i.e. in some generic extension of universe  $V$  this holds provided that  $V \models “\aleph_0 < \kappa < \lambda$ ,  $\kappa$  and  $\lambda$  are weakly compact”).*

*3.2 Remark.* We give more specific information in the three cases below.

*Proof.* First assume  $\kappa < \mu < \lambda$  are such that:

- (\*) (1)  $PT(\kappa, \aleph_0)$ ,  $\kappa$  regular
- (2)  $NPT(\mu, \aleph_0)$ , ( $\mu$  regular)
- (3)  $PT(\lambda, \aleph_0)$ ,  $\lambda = \lambda^{<\lambda}$ , even if we add a Cohen subset to  $\lambda$
- (4)  $\lambda \in I[\lambda]$  (see on it and references in the discussion after 2.2; if  $\lambda$  is strongly inaccessible or  $\lambda = \chi^+$  &  $\chi = \chi^{<\chi}$  this always holds).

(So all three are regular uncountable cardinals).

Shoot a non-reflecting stationary subset  $\tilde{S}$  of  $\{\delta < \lambda : cf(\delta) = \kappa\}$  by the forcing notion

$$Q_0 = \left\{ h : h \text{ a function from } (\alpha + 1) \text{ to } \{0, 1\}, \alpha < \lambda, \right. \\ \left. \begin{array}{l} [\delta \leq \alpha \ \& \ h(\delta) = 1 \Rightarrow cf(\delta) = \kappa], \\ h^{-1}(\{1\}) \text{ is non-reflecting} \end{array} \right\}.$$

In  $V^{Q_0}$  no bounded subset of  $\lambda$  is added (so  $NPT(\mu, \aleph_0)$ ,  $PT(\kappa, \aleph_0)$  continue to hold). Also trivially  $NPT(\lambda, \kappa)$  hence  $NPT(\lambda, \mu)$  (as  $\mu \geq \kappa$ , e.g. for  $\delta \in \tilde{S}$  choose  $A_\delta \subseteq \delta$ ,  $|A_\delta| = otp(A_\delta) = \kappa$ ,  $\sup(A_\delta) = \delta$ ).

So it is enough to prove  $PT(\lambda, \aleph_0)$ . If not, there is a  $\lambda$ -set  $\mathfrak{S}$  and  $\{A_\eta : \eta \in \mathfrak{S}_f\}$  witnessing it as in [§3]Sh:161 or see [AP]Sh:521. As  $PT(\kappa, \aleph_0)$  (also in  $V^Q$ ) we have  $\eta \in S \Rightarrow \lambda_\eta \neq \kappa$ , hence (see [§3]Sh:161 or [AP]Sh:521)

$$(*) \ \langle i \rangle \in S \Rightarrow cf(i) \neq \kappa.$$

So if we kill the stationary set  $\tilde{S}$  by a forcing notion

$$Q_1 = \{g : a \text{ function for } \alpha + 1 < \lambda \text{ to } \{0, 1\}, h^{-1}(\{1\}) \text{ closed disjoint to } \tilde{S}\},$$

then in the universe  $(V^{Q_0})^{Q_1}$ , still  $NPT(\lambda, \aleph_0)$  by the same example.

[Why? The set  $S = \{i < \lambda : \langle i \rangle \in \mathfrak{S}\}$  is stationary (as  $\mathfrak{S}$  is a  $\lambda$ -set). Now (see [AP]Sh:88) the set  $S$  is in  $I[\lambda]$  in  $V^Q$  (as  $\lambda$  is) and is stationary and disjoint to  $\tilde{S}_0$

hence is stationary also in  $(V^{Q_0})^{Q_1}$ , and no bounded subset of  $\lambda$  is added.]

But  $Q_0 * Q_1$  is just adding a Cohen subset of  $\lambda$  hence by  $(*)$ (3) we know  $(V^{Q_0})^{\ast Q_1} \models PT(\lambda, \aleph_0)$ . So we have gotten an example as required in the lemma, provided we start with a universe satisfying  $(*)$ . So when  $(*)$  holds?

*Case 1.*  $\lambda$  supercompact,  $\kappa$  first measurable, choose any  $\mu$  such that  $\kappa < \mu < \lambda$ ,  $\mu = \text{cf}(\mu)$ , we can add to  $\mu$  a set  $S \subseteq \{\delta < \mu : \text{cf}(\delta) = \aleph_0\}$  stationary non-reflecting and then make  $\lambda$  Laver indestructible by forcing notions not adding subsets of  $\lambda$ . So G.C.H. is O.K.

*Case 2 (Small Cardinals).* By the forcing in [MgSh:204] (but for  $\lambda$  regular adding  $\lambda$ -Cohen sets) the following case is O.K.

$$GCH + \kappa = \aleph_{\omega^2+1} + \mu = \aleph_{\omega_1+1} + \lambda = \aleph_{\omega_1+\omega^2+1}.$$

*Case 3.*  $V = L$ ,  $\kappa < \lambda$  are uncountable weakly compact cardinals in  $L$ ,  $\mu = \kappa^+$ ,  $P$  is a  $\mu^+$ -complete forcing notion making  $(*)$ (3) hold without adding any subset to  $\mu$ . (E.g.  $P$  is  $P_\lambda$ , where  $\langle P_i, Q_j : i \leq \lambda, j < \lambda \rangle$  is an iteration with Easton support,  $\lambda_j$ -the  $j$ th inaccessible cardinal in  $L$  which is  $> \kappa$ ,  $Q_j$  is the  $\lambda_j$ -Cohen forcing in  $V^{P_j}$ ; i.e.  $Q_j = \{h : h \text{ a function from some } \alpha < \lambda_j \text{ to } \lambda_j \text{ in } V^{P_j}\}$ . Now in  $V^P$ ,  $\kappa, \mu, \lambda$  are as required.  $\square_{3.1}$

By work of Jensen there is a non-reflecting stationary  $S \subseteq \lambda$  for  $\lambda$  regular non weakly compact



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