On Finite Rigid Structures

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Abstract

The main result of this paper is a probabilistic construction of finite rigid structures. It yields a finitely axiomatizable class of finite rigid structures where no $L^{\omega}_{\infty,\omega}$ formula with counting quantifiers defines a linear order.

1 Introduction

In this paper, structures are finite and of course vocabularies are finite as well. A class is always a collection of structures of the same vocabulary which is closed under isomorphisms.

An r-ary *global relation* on a class K is a function ρ that associates an r-ary relation ρ_A with each structure $A \in K$ in such a way that every isomorphism from A to a structure B extends to an isomorphism from the structure (A, ρ_A) to the structure (B, ρ_B) [G].

Recall that a structure is *rigid* if it has no nontrivial automorphisms. If a binary global relation $<$ defines a linear order in a class K (that is, on each structure in K) then every structure in K is rigid. Indeed, suppose that θ is an automorphism of a structure $A \in K$ and let a be an arbitrary element of A. Since

$$\begin{aligned} A \models \theta(x) < \theta(a) \quad &\iff \quad A \models x < a, \\ A \models \theta(x) > \theta(a) \quad &\iff \quad A \models x > a, \end{aligned}$$

the number of elements preceding $\theta(a)$ in the linear order $<_A$ equals the number of elements preceding a. Hence $\theta(a) = a$.

Conversely, if every structure in a class K is rigid then some global relation ρ defines a linear order on each structure in K. Alex Stolboushkin constructed a finitely axiomatizable class of rigid structures such that no first-order formula defines a linear order in K [S]. Anuj Dawar conjectured that, for every finitely axiomatizable class K of rigid structures, some formula in the fixed-point extension of first-logic defines a linear order in K [D]. Using the probabilistic method, we refute the conjecture and construct a finitely axiomatizable class of structures where no $L^{\omega}_{\infty,\omega}$ formula with counting quantifiers defines a linear order (Theorem 4.1). At the end of Section 4, we answer a question of Scott Weinstein [W] related to rigid structure.

To make this paper self-contained, we provide a reminder in the rest of this section. As in a popular version of first-order logic, $L^{\omega}_{\infty,\omega}$ formulas are built from atomic formulas by

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means of negations, conjunctions, disjunctions, the existential quantifier and the universal quantifier. The only difference is that, in $L^\omega_{\infty,\omega}$, one is allowed to form the conjunction and the disjunction of an arbitrary set S of formulas provided that the total number of variables in all S-formulas is finite. $L^\omega_{\infty,\omega}(C)$ is the extension of $L^\omega_{\infty,\omega}$ by means of *counting quantifiers* $(\exists 2x)$, $(\exists 3x)$, etc. The semantics is obvious. $L^k_{\infty,\omega}$ (resp. $L^k_{\infty,\omega}(C)$) is the fragment of $L^\omega_{\infty,\omega}$ (resp. $L^\omega_{\infty,\omega}(C)$) where formulas use at most k variables.

There is a pebble game $G^k(A,B)$ appropriate to $L^k_{\infty,\omega}(C)$ [IL]. Here A and B are structures of the same purely relational vocabulary. The game is played by Spoiler and Duplicator on a board comprised by A and B. For each $i = 1,\dots,k$, there are two identical pebbles marked by i. Initially there are no pebbles on the board. After every round, either both i-pebbles are off the board or else one of them covers an element of A and the other covers an element of b; furthermore the pebbles on the board define a partial isomorphism from A to B. (This means that (i) an i-pebble and a j-pebble cover different elements of A if and only if their twins cover different elements of B, and (ii) the map that takes a pebble-covered element of A to the element of B covered by the pebble of the same number is a partial isomorphism.)

A round of $G^k(A,B)$ is played as follows.

1. Spoiler chooses a number i; if the i-pebbles are on the board, they are taken off the board. Then Spoiler chooses a structure $M \in \{A,B\}$ and a nonempty subset X of M.

2. Duplicator chooses a subset Y of the remaining structure N such that $\|Y\| = \|X\|$. If N has no subsets of cardinality $\|X\|$, the game is over; Spoiler has won and Duplicator has lost.

3. Spoiler puts an i-pebble on an element $y \in Y$.

4. Duplicator puts the other i-pebble on an element $x \in X$ in such a way that the pebbles define a partial isomorphism. If X has no appropriate element x, the game is over; Spoiler has won and Duplicator has lost. Otherwise Duplicator wins the round

Spoiler wins a play of the game if the number of rounds in the play is infinite.

Theorem 1.1 ([IL]) *If Duplicator has a winning strategy in $G^k(A,B)$ then no $L^k_{\infty,\omega}(C)$ sentence ϕ distinguishes between A and B.*

It is not hard to prove the theorem by induction on ϕ. The converse implication is true too [IL] but we will not use it.

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2 Hypergraphs

2.1 Preliminaries

In this paper, a *hypergraph* is a pair $H = (U,T)$ where $U = |H|$ is a nonempty set and T is a collection of 3-element subsets of U; elements of U are *vertices* of H, and elements of T are *hyperedges* of H. It can be seen as a structure with universe U and irreflexive symmetric ternary relation $\{(x, y, z) : \{x, y, z\} \in T\}$.

Every nonempty subset X of U gives a *sub-hypergraph*

$$H|X = (X, \{h : h \in T \wedge h \subseteq X\}$$

of H. The number of hyperedges in $H|X$ will be called the *weight* of X and denoted $[X]$. As usual, the number of vertices of X is called the cardinality of X and denoted $\|X\|$.

Vertices x, y of a hypergraph H are *adjacent* if there is a hyperedge $\{x, y, z\}$; the vertex z *witnesses* that x and y are adjacent.

Definition 2.1.1 A vertex set X is *dense* if $\|X\| \leq 2[X]$. A hypergraph is l*-meager* if it has no dense vertex sets of cardinality $\leq 2l$. $\square$

Lemma 2.1.1 *In a 2-meager hypergraph, the intersection of any two distinct hyperedges contains at most one vertex.*

Proof If $\|h_1 \cap h_2\| = 2$ then $h_1 \cup h_2$ is 2-dense. $\square$

Definition 2.1.2 A vertex set X is *super-dense* or *immodest* if $\|X\| < 2[X]$. A hypergraph is l*-modest* if it has no super-dense sets of cardinality $\leq 2l$. $\square$

It follows that if X is a dense vertex set of cardinality $\leq 2l$ in an l-modest hypergraph then $\|X\| = 2[X]$ and in particular $\|X\|$ is even.

2.2 Cycles

Definition 2.2.1 A sequence $x_1, \ldots, x_k$ of $k \geq 3$ distinct vertices is a *weak cycle* of length k if it satisfies the following two conditions where the subscripts are viewed as numbers modulo k:

1. Each x_i is adjacent to x_{i+1}.
2. Either $k > 3$ or else $k = 3$ but $\{x_1, x_2, x_3\}$ is not a hyperedge.

$\square$

We will index elements of a weak cycle of length k with numbers modulo k.

Definition 2.2.2 A weak cycle $x_1, \ldots, x_k$ is a *cycle* of length $k \geq 3$ if no triple x_i, x_{i+1}, x_{i+2} forms a hyperedge. A corresponding *witnessed cycle* of length k is a vertex sequence $x_1, \ldots, x_k, y_1, \ldots, y_k$ where each y_i witnesses that x_i is adjacent to x_{i+1}. $\square$

Definition 2.2.3 A vertex sequence x_1, x_2 is a *cycle* of length 2 if there are distinct vertices y_1, y_2 different from x_1, x_2 such that $\{x_1, x_2, y_1\}$ and $\{x_2, x_1, y_2\}$ are hyperedges; the sequence x_1, x_2, y_1, y_2 is a corresponding *witnessed cycle* of length 2. $\square$

Lemma 2.2.1 *Every weak cycle includes a cycle. More exactly, some (not necessarily contiguous) subsequence of a weak cycle is a cycle. Thus, an acyclic hypergraph (that is, a hypergraph without any cycles) has no weak cycles.*

Proof We prove the lemma by induction on the length. Let $x_1, \ldots, x_k$ be a weak cycle that is not a cycle, so that some x_i, x_{i+1}, x_{i+2} is a hyperedge; without loss of generality, $i = 1$. Then the sequence $x_1, x_3, \ldots, x_k$ of length $k - 1$ is a weak cycle or a hyperedge. In the first case, use the induction hypothesis. In the second, $k = 4$ and x_1, x_3 form a cycle witnessed by x_2 and x_4. $\square$

Theorem 2.2.1 *In any l-modest graph,*

- *every minimal dense set of cardinality $2k \le 2l$ is a witnessed cycle of length k, and*
- *every witnessed cycle of length $k \le l$ is a minimal dense set of cardinality $2k$.*

The theorem clarifies the structure of minimal dense sets of cardinality $\le 2l$ which play an important role in our probabilistic construction. However the theorem itself will not be used and can be skipped. The rest of this subsection is devoted to proving the theorem.

Proof Fix some number $l \ge 2$ and restrict attention to l-modest hypergraphs.

Lemma 2.2.2 *For every vertex set X, the following statements are equivalent:*

1. *X is a dense set of cardinality 4.*
2. *X is a minimal dense set of cardinality 4*
3. *Vertices of X form a witnessed cycle of length 2.*

Proof It is easy to see that (1) is equivalent to (2) and that (3) implies (1). It remains to check that (1) implies (3). Suppose (1). By l-modesty $[X] = 2$. Thus, X includes two hyperedges h_1 and h_2. Clearly, $h_1 \cup h_2 = X$ and $\|h_1 \cap h_2\| = 2$. It is easy to see that the vertices of $h_1 \cap h_2$ form a cycle and the vertices of X form a corresponding witnessed cycle. $\square$

In the rest of this subsection, $3 \le k \le l$.

Lemma 2.2.3 *Every witnessed cycle $x_1, \ldots, x_k, y_1, \ldots, y_k$ forms a dense set of cardinality $2k$.*

Proof Let $W = \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$. It suffices to check that the k hyperedges $\{x_i, x_{i+1}, y_i\}$ are all distinct. For then, using l-modesty, we have

$$2k \le 2[W] \le \|W\| \le 2k.$$

If $i \ne j$ but $\{x_i, x_{i+1}, y_i\} = \{x_j, x_{j+1}, y_j\}$ then either $x_j = x_{i+1}$ or else $x_j = y_i$ in which case $x_{j+1} = x_i$. Without loss of generality, $x_j = x_{i+1}$ and therefore $j = i + 1$ modulo k. If also $x_{j+1} = x_i$ then $i = j + 1 = i + 2$ modulo k which contradicts the fact that $k > 2$. Thus $x_{j+1} = y_i$, so that $y_i = x_{i+2}$ and therefore $\{x_i, x_{i+1}, x_{i+2}\}$ is a hyperedge which contradicts the definition of cycles. $\square$

Lemma 2.2.4 *Every minimal dense vertex set of cardinality $2k$ forms a witnessed cycle of length k.*

Proof Without loss of generality, the given minimal vertex set contains all vertices of the given hypergraph H; if not, restrict attention to the corresponding sub-hypergraph of H.

It suffices to prove that H includes a weak cycle of length $\leq k$. For then, by Lemma 2.2.1, H includes a cycle of length $\leq k$. If a witnessed version of the cycle contains less than $2k$ vertices then, by the previous lemma, H contains a proper dense subset.

By contradiction suppose that H does not include a weak cycle of length k.

Claim 2.2.1 *A hypergraph of cardinality $2k$ is acyclic if no proper vertex set is dense and there is no weak cycles of length $\leq k$.*

Proof By contradiction suppose that there is a cycle of length $m > k$ and choose the minimal possible m. Consider a witnessed cycle $x_1, \ldots, x_m, y_1, \ldots, y_m$.

Since the hypergraph has $< 2m$ vertices, some y_i occurs in $x_1, \ldots, x_m$. Without loss of generality, $y_1 = x_j$ for some j, so that $\{x_1, x_2, x_j\}$ is a hyperedge and therefore j differs from $1, 2$ and 3. But then the sequence $x_2, \ldots, x_j$ is a weak cycle and thus includes a cycle of length $< m$. This contradicts the choice of m. $\square$

Claim 2.2.2 *Any acyclic hypergraph of positive weight contains a hyperedge Y such that at most one vertex of Y belongs to any other hyperedge.*

Proof Let $s = (x_1, \ldots, x_k)$ be a longest vertex sequence such that (i) for every $i < k$, x_i is adjacent to x_{i+1}, and (ii) for no $i < k-1$, the triple x_i, x_{i+1}, x_{i+2} forms a hyperedge. Since the hypergraph has hyperedges, $k \geq 2$. If $k = 2$ then all hyperedges are disjoint and the claim is obvious. Suppose that $k \geq 3$.

Pick a vertex y such that $Y = \{x_{k-1}, x_k, y\}$ is a hyperedge. We prove that neither x_k nor y belongs to any other hypergraph. Since there are no cycles of length 2, y is uniquely defined. We prove that neither x_k nor y belongs to any other hypergraph. Vertex y does not occur in $x_1, \ldots, x_k$; otherwise $x_i, \ldots, x_{k_1}$ is a weak cycle. Notice that y can replace x_k in s. Thus it suffices to prove that x_k does not belong to any other hyperedge.

By contradiction, suppose that a hyperedge $Z \neq Y$ contains x_k and let $z \in Z - Y$. By the maximality of s, it contains z; otherwise s can be extended by z. But then the final segment $S = [z, x_k]$ of s forms a weak cycle. $\square$

Claim 2.2.3 *No acyclic hypergraph is dense.*

Proof Induction on the cardinality of the given hypergraph I. The claim is trivial if $[I] = 0$. Suppose that $[I] > 0$. By the previous claim, I has a hyperedge $X = \{x, y, z\}$ such that neither y nor z belongs to any other hyperedge. Let J be the sub-hypergraph of I obtained by removing vertices y and z. Using the induction hypothesis, we have

$$\|I\| = \|J\| + 2 > 2[J] + 2 = 2([I] + 1) = 2[I].$$

$\square$

Now we are ready to prove the lemma. By Claim 2.2.1, H is acyclic. By Claim 2.2.3, H is not dense which gives the desired contradiction. $\square$

Lemma 2.2.5 *Every witnessed cycle of length k forms a minimal dense set.*

Proof Let W be the set of the vertices of the given witnessed cycle of length k. By Lemma 2.2.3, W is a dense set of cardinality $2k$. By the l-modesty of the hypergraph, W contains precisely k hyperedges. It is easy to see now that every proper subset X of W is acyclic; by Claim 2.2.3, X is not dense. □

Lemmas 2.2.2–2.2.5 imply the theorem. □

2.3 Green and Red Vertices

Fix $l \geq 2$ and consider a sufficiently modest hypergraph. More precisely, we require that the hypergraph is $(2l+2)$-modest. It follows that, for every dense set V of cardinality $\leq 4l+4$, $\|V\| = 2[V]$.

For brevity, we use the following terminology. A minimal dense vertex set of cardinality $\leq 2l$ is a *red block*. A vertex is *red* if it belongs to a red block; otherwise it is *green*. A hyperedge is *green* if it consists of green vertices. The *green sub-hypergraph* is the sub-hypergraph of green vertices.

Lemma 2.3.1 *Distinct red blocks are disjoint.*

Proof We suppose that distinct red blocks X and Y have a nonempty intersection Z and prove that the union $V = X \cup Y$ is immodest. Indeed, Z is a proper subset of X; otherwise Y is not a minimal dense set. Therefore Z is not dense and

$$\|V\| = \|X\|+\|Y\|-\|Z\| = 2[X]+2[Y]-\|Z\| < 2[X]+2[Y]-2[Z] = 2([X]+[Y]-[Z]) \leq 2[V].$$

□

Lemma 2.3.2 *Adjacent red vertices belong to the same red block.*

Proof Suppose that adjacent red vertices x and y belong to different red blocks X and Y respectively, and let h be a hyperedge containing x and y. We show that the set $V = X \cup Y \cup h$ is immodest. Indeed,

$$\|V\| \leq \|X\| + \|Y\| + 1 = 2[X] + 2[Y] + 1 < 2([X] + [Y] + 1) \leq 2[V].$$

□

Lemma 2.3.3 *No green vertex is adjacent to two different red vertices.*

Proof By contradiction suppose that a green vertex b is adjacent to distinct red vertices x and x'. Let X, X' be the red blocks of x, x' respectively, h be a hyperedge containing b and x, and h' be a hyperedge containing b and x'. We show that the set $V = X \cup X' \cup h \cup h'$ is immodest. By the previous lemma, $h = h'$ implies $X = X'$.

If $h = h'$ then

$$\|V\| = \|X\| + 1 = 2[X] + 1 < 2([X] + 1) \leq [V].$$

If $h \neq h'$ but $X = X'$ then

$$\|V\| \leq \|X\| + 3 = 2[X] + 3 < 2([X] + 2) \leq 2[V].$$

If $X \neq X'$ then

$$\|V\| \leq \|X\| + \|X'\| + 3 = 2[X] + 2[X'] + 3 < 2([X] + [X'] + 2) \leq [V].$$

□

Definition 2.3.1 A hypergraph is *odd* if, for every nonempty vertex set X, there is a hyperedge h such that $\|h \cap X\|$ is odd. □

For future reference, some assumptions are made explicit in the following theorem.

Theorem 2.3.1 *Suppose that a hypergraph H of cardinality n satisfies the following conditions where $n' < n$.*

- *H is $(2l+2)$-modest.*
- *The number of red vertices is $< n'$.*
- *Every vertex set of cardinality $\geq n'$ includes a hyperedge.*
- *For every nonempty vertex set X of cardinality $< n'$, there exist a vertex $x \in X$ and distinct hyperedges h_1, h_2 such that $h_1 \cap X = h_2 \cap X = \{x\}$.*

Then the green sub-hypergraph of H is an odd, l-meager hypergraph of cardinality $> n - n'$.

Proof Since the green sub-hypergraph G is obtained from H by removing all dense vertex sets of cardinality $\leq 2l$, G is l-meager. By the second condition, $\|G\| > n-n'$. To check that G is odd, let X be a nonempty set of green vertices. If $\|X\| \geq n'$, use the third condition. Suppose that $\|X\| < n'$ and let x, h_1, h_2 be as in the fourth condition; both $\|h_1 \cap X\|$ and $\|h_2 \cap X\|$ are odd. By Lemma 2.3.3, at least one of the two hyperedges is green. □

2.4 Attraction

Definition 2.4.1 In an arbitrary hypergraph, a vertex set X *attracts* a vertex y if there are vertices x_1, x_2 in X such that $\{x_1, x_2, y\}$ is a hyperedge. X is *closed* if it contains all elements attracted by X. As usual, the *closure* $\bar{X}$ of X is the least closed set containing X. □

Lemma 2.4.1 *In an l-meager hypergraph, if X is a vertex set of cardinality $k \leq l$ then $\|\bar{X}\| < 2k$.*

Proof Construct sets $X_0, \ldots, X_m$ as follows. Set $X_0 = X$. Suppose that sets $X_0, \ldots, X_i$ have been constructed. If X_i is closed, set $m = i$ and terminate the construction process. Otherwise pick a hyperedge h such that $\|h \cap X_i\| = 2$ and let $X_{i+1} = h \cup X_i$. We show that $m < k$.

By contradiction suppose that $m \geq k$. Check by induction on i that $\|X_i\| = k + i$ and $[X_i] \geq i$. Since the hypergraph is l-meager, we have: $2[X_k] < \|X_k\| = 2k \leq 2[X_k]$. This gives the desired contradiction. □

Lemma 2.4.2 *Suppose that Y is a vertex set of cardinality $\leq k$ in a $2k$-meager hypergraph and $p = \|\bar{Y} - Y\|$. Then $p < n$ and there is an ordering $z_1, \ldots, z_p$ of $\bar{Y} - Y$ such that each z_j is attracted by $Y \cup \{z_i : i < j\}$.*

Proof By the previous lemma, $\|\bar{Y}\| < 2\|Y\|$. Hence $p = \|\bar{Y} - Y\| < \|Y\| \leq n$. Choose elements z_j by induction on j. Suppose that $1 \leq j \leq p$ and all elements z_i with $i < j$ have been chosen. Since $\|\bar{Y}\| = \|\|Y\|\| + p$ vertices, the set $Z_{j-1} = Y \cup \{z_i : i < j\}$ is not closed. Let z_j be any element in $\bar{Y} - Y$ attracted by Z_{j-1}. $\square$

Theorem 2.4.1 *Suppose that X is a vertex set of cardinality $< k$ in a $2k$-meager hypergraph, $z_0 \notin \bar{X}$, $Y = \bar{X} \cup \{z_0\}$, $Z = \bar{Y}$ and $p = \|Z - Y\|$. Then $p < k$ and there is an ordering $z_1, \ldots, z_p$ of $Z - Y$ such that, for every $j > 0$, z_j is attracted by $Y \cup \{z_i : 1 \leq i < j\}$ and there is a unique hyperedge h_j witnessing the attraction.*

Proof By the previous lemma, $p < k$. Construct sequence $z_1, \ldots, z_p$ as in the proof of the previous lemma. For any $j > 0$, let h_j be a hyperedge witnessing that $Z_{j-1} = Y \cup \{z_i : 1 \leq i < j\}$ attracts y_j.

By contradiction suppose that, for some positive $j \leq p$, some hyperedge $h'_j \neq h_j$ witnesses that z_j is attracted by Z_{j-1}. Let $S = \{h_1, \ldots, h_j, h'_j\}$. We show that $V = \bigcup S$ is a dense set of cardinality $\leq 2k$ which contradicts the $2k$-meagerness of the hypergraph.

Since V contains all hyperedges in S, $[V] \geq j + 1$. Since none of the vertices $z_1, \ldots, z_j$ is attracted by $\bar{X}$, $\|h \cap \bar{X}\| \leq 1$ for all $h \in S$ and thus $\|V \cap \bar{X}\| \leq j + 1$. We have

$$\|V\| = \|(V \cap \bar{X}) \cup \{z_0, \ldots, z_j\}\| \leq (j+1) + (j+1) \leq 2 \cdot [V].$$

Thus V is a dense set of cardinality $\|V\| \leq 2(j+1) \leq 2(p+1) \leq 2k$. $\square$

3 Existence

Theorem 3.1 *For any integers $l \geq 2$ and $N > 0$, there exists an odd l-meager hypergraph of cardinality $> N$.*

In fact, there exists an odd l-meager hypergraph of cardinality precisely N but we do not need the stronger result here.

Proof Now fix $l \geq 2$ and $N > 0$ and choose a positive real $\varepsilon < 1/(2l+3)$. Let n range over integers $\geq 2N$ divisible by 4 and U be the set of positive integers $\leq n$. For each 3-element subset a of U, flip a coin with probability $p = n^{-2+\varepsilon}$ of heads, and let T is the collection of triples a such that the coin comes up heads. This gives a random graph $H = (U, T)$.

We will need the following simple inequality. In this section, $\exp \alpha = e^{\alpha}$ and $\log \alpha = \log_e \alpha$.

Claim 3.1 *For all positive reals q, r, s such that $p^r < 1/2$,*

$$\exp(-2qn^{s-2r+r\varepsilon}) < (1-p^r)^{qn^s} < \exp(-qn^{s-2r+r\varepsilon}) \tag{1}$$

Proof Suppose that $0 < \alpha < 1/2$. By Mean Value Theorem applied to function $f(t) = -\log(1-t)$ on the interval $[0, \alpha]$, there is a point $t \in (0, \alpha)$ such

$$f(\alpha) - f(0) = -\log(1-\alpha) = (\alpha - 0)f'(t) = \alpha/(1-t).$$

Since $\alpha < \alpha/(1-t) < \alpha/(1-\alpha) < a/(1-1/2) = 2\alpha$, we have $\alpha < -\log(1-\alpha) < 2\alpha$ and therefore $e^{-2\alpha} < 1-\alpha < e^{-\alpha}$. Now let $\alpha = p^r$ and raise the terms to power qn^s. $\square$

Call an event $E = E(n)$ *almost sure* if the probability $\mathbf{P}[E]$ tends to 1 as n grows to infinity. We prove that, almost surely, H satisfies the conditions of Theorem 2.3.1 with $n' = n/4$ and therefore the green subgraph of H is an odd l-meager graph of cardinality $> N$.

Lemma 3.1 *Almost surely, H is $(2l+2)$-modest.*

Proof It suffices to prove that, for each particular $m \le 4l+4$, the probability q_m that there is a super-dense vertex sets of cardinality m is $o(1)$. A vertex set X of cardinality m is super-dense if $m < 2[X]$, that is, if X includes more than $m/2$ hyperedges. Let k be the least integer that exceeds $m/2$. Then $m \le 2k-1$ and therefore $n^{m-2k} \le n^{-1}$. Also $2k-2 \le m \le 4l+4$, so that $k \le 2l+3$ and $k\varepsilon < 1$. Let $M = \binom{m}{3}$ and $c = \binom{M}{k}$. We have

$$q_m < \binom{n}{m} \cdot c \cdot p^k < c \cdot n^m \cdot n^{(-2+\varepsilon)k} = c \cdot n^{m-2k+k\varepsilon} \le c \cdot n^{-1+k\varepsilon} = o(1).$$

$\square$

Lemma 3.2 *Almost surely, the number of red vertices is $< n/4$.*

Proof It suffices to prove that the expected number of red vertices is $o(n)$. Indeed, let r be the number of red vertices and s ranges over the integer interval $[n/4, n]$. Then

$$\mathrm{E}[r] \ge \sum s \cdot \mathbf{P}[r = s] \ge \frac{n}{4} \sum \mathbf{P}[r = s] = \frac{n}{4}\mathbf{P}[r \ge \frac{n}{4}]$$

and thus $\mathbf{P}[r \ge \frac{n}{4}]$ tends to 0 if $\mathrm{E}[r] = o(n)$.

Furthermore, it suffices to show that, for each particular $m \le 2l$, the expected number $f(m)$ of vertices v such that v belongs to a dense set X of cardinality m is $o(n)$. Let $k = \lceil m/2 \rceil$. Then $m \le 2k$ and therefore $n^{m-2k} \le 1$. Also, $2k \le m-1 < 2l$ and therefore $k < l$ and $k\varepsilon < 1$. Let $M = \binom{m}{3}$ and $c = \binom{M}{k}$. We have

$$f(m) \le n \cdot \binom{n-1}{m-1} cp^k < n \cdot n^{m-1} cp^k = c \cdot n^m p^k = c \cdot n^{m-2k+k\varepsilon} \le c \cdot n^{k\varepsilon} = o(n).$$

$\square$

Lemma 3.3 *Almost surely, every vertex set of cardinality $\ge n/4$ includes a hyperedge.*

Proof Chose a real $c > 0$ so small that $cn^3 \le \binom{n/4}{3}$ and let q be the probability that there exists a vertex set of cardinality $\ge n/4$ which does not include any hyperedges. Using inequality (1), we have

$$q < 2^n \cdot (1-p)^{\binom{n/4}{3}} < e^n \cdot (1-p)^{cn^3} < e^n \cdot \exp(-cn^{1+\varepsilon}) = o(1).$$

$\square$

Lemma 3.4 *For every nonempty vertex set X of cardinality $< n/4$, there exist a vertex $x \in X$ and hyperedges h_1, h_2 such that*

$$h_1 \cap X = h_2 \cap X = h_1 \cap h_2 = \{x\}.$$

Proof Let X range over nonempty vertex sets of cardinality $< n/4$, Y be the collection of even numbers $y \in U - X$, and Z be the collection of odd numbers $z \in U - X$. Clearly, $\|Y\| \geq n/4$ and $\|Z\| \geq n/4$.

Let x range over X, $\sigma(x, X)$ mean that there exist vertices $y_1, y_2 \in Y$ such that $\{x, y_1, y_2\}$ is a hyperedge, and $\tau(x, X)$ mean that there exist vertices $z_1, z_2 \in Z$ such that $\{x, z_1, z_2\}$ is a hyperedge. Call X *bad* if and $\sigma(x, X) \wedge \tau(x, X)$ fails for all x. We prove that, almost surely, there are no bad vertex sets.

Choose a real $c > 0$ so small that $cn^2 < \binom{n/4}{2}$. For given X and x,

$$\mathbf{P}[\neg\sigma(x, X)] = (1-p)^{\binom{\|Y\|}{2}} \leq (1-p)^{\binom{n/4}{2}} < (1-p)^{cn^2} < \exp[-cn^{\varepsilon}].$$

The last inequality follows from inequality (1). Similarly, $\mathbf{P}[\neg\tau(x, X)] < \exp[-cn^{\varepsilon}]$. Hence

$$\mathbf{P}[\neg\sigma(x, X) \vee \neg\tau(x, Y)] \leq \mathbf{P}[\neg\sigma(x, X)] + \mathbf{P}[\neg\tau(x, X)] < 2\exp[-cn^{\varepsilon}] = \exp[\log 2 - cn^{\varepsilon}].$$

If $\|X\| = m$ then

$$\mathbf{P}[X \text{ is bad }] < (\exp[\log 2 - cn^{\varepsilon}])^m = \exp[m(\log 2 - cn^{\varepsilon})].$$

For each $m < n/4$, let q_m be the probability that there is a bad vertex set of cardinality m. For sufficiently large n, $\log 2n - cn^{\varepsilon} < 0$ and therefore $\exp(\log 2n - cn^{\varepsilon}) < 1$. Thus

$$q_m \leq n^m \cdot \exp[m(\log 2 - cn^{\varepsilon})] = \exp[m(\log 2n - cn^{\varepsilon})] \leq \exp[\log 2n - cn^{\varepsilon}].$$

Finally, let q be the probability of the existence of a bad set. We have

$$q < \frac{n}{4}\exp[\log 2n - cn^{\varepsilon}] = o(1).$$

□

Theorem 3.1 is proved. □

4 Multipedes

The domain $\{x : \exists y(xEy)\}$ and the range $\{y : \exists x(xEy)\}$ of a binary relation E will be denoted $D(E)$ and $R(E)$ respectively.

Definition 4.1 A 1-*multipede* is a directed graph (U, E) such that $D(E) \cap R(E) = \emptyset$, $D(E) \cup R(E) = U$, every element in $D(E)$ has exactly one outgoing edge and every element in $R(E)$ has exactly two incoming edges. □

If xEy holds then x is a *foot* of y and y is the *segment* $S(x)$ of x. We extend function S as follows. If x is a segment then $S(x) = x$. If X is a set of segments and feet then $S(X) = \{S(x) : x \in X\}$.

Definition 4.2 A 2^--*multipede* is a structure (U, E, T) such that (U, E) is a 1-multipede and (U, T) is a hypergraph where each hyperedge h satisfies the following conditions:

- Either all elements of h are segments or else all elements of h are feet.
- If h is a foot hyperedge then $S(h)$ is a hyperedge as well.

□

If $X = \{x, y, z\}$ is a segment hyperedge then every 3-element foot set A with $S(A) = X$ is a *slave* of X. A slave A of X is *positive* if A is a hyperedge; otherwise it is *negative*. Two slaves of X are *equivalent* if they are identical or one can be obtained from the other by permuting the feet of two segments. In other words, if a, a' are different feet of x and b, b' are different feet of y and c, c' are different feet of z then the eight slaves of X split into the following two equivalence classes

$$\{a, b, c\}, \{a, b', c'\}, \{a', b, c'\}, \{a', b', c\}$$

and

$$\{a', b, c\}, \{a, b', c\}, \{a, b, c'\}, \{a', b', c'\}$$

Definition 4.3 A 2-*multipede* is a 2^--multipede where, for each segment hyperedge X, exactly four slaves of X are positive and all four positive slaves are equivalent. □

A 2-multipede (U, E, T) is *odd* if the segment hypergraph $(R(E), T)$ is so.

Lemma 4.1 *If an automorphism θ of an odd 2-multipede does not move any segment then it does not move any foot either.*

Proof By contradiction suppose that θ moves a foot a of a segment x. Clearly, $\theta(a)$ is the other foot of x. Let X be the collection of segments x such that θ permutes the feet of x. Since the multipede is odd, there exists a segment hyperedge h such that $\|h \cap X\|$ is odd. It is easy to see that θ takes positive slaves of X to negative ones and thus is not an automorphism. □

Lemma 4.2 *Let M is a $2k$-meager 2-multipede and Υ be the extension of the vocabulary of M by means of individual constants for every segment of M. No $L^k_{\infty,\omega}(C)$ sentence in the vocabulary Υ distinguishes between M and the 2-multipede N obtained from M by permuting the feet of one segment.*

To be on the safe side, let us explain what it means that N is obtained from M by permuting the feet of one segment. To obtain N, choose a segment x and perform the following transformation for every segment hyperedge h that contains x: Make all positive slaves of h negative and the other way round.

Proof Call a collection X of segments and feet *closed* if it satisfies the following conditions:

- The segments of X form a closed set in the sense of Definition 2.4.1.
- If a is foot of x then $a \in X \leftrightarrow x \in X$.

Call a partial isomorphism α from M to N *regular* if α leaves segments intact and takes any foot to a foot of the same segment. The domain of a partial isomorphism α will be denoted $D(\alpha)$. A regular partial isomorphism α is *safe* if there is a regular extension of α to the closure $\overline{D(\alpha)}$.

Claim 4.1 *Each safe partial isomorphism α from M to N has a unique regular extension to $\overline{D(\alpha)}$.*

Proof Let $X = D(\alpha)$ and suppose that β and γ are regular extension of α to $\bar{X}$. Let $Y = S(X)$ and $Z = S(\bar{Y})$. By Lemma 2.4.2, there exists a linear order $z_1, \ldots, z_p$ of the elements of $Z - Y$ such that each z_j is attracted by the set $Z_{i-1} = Y \cap \{y_i : i < j\}$. We need to prove that, for every j, either both β and γ leave the feet of z_j intact or else both of them permute the feet. We proceed by induction on j. Suppose that β and γ coincide on the feet of every y_i with $i < j$ and let h witness that Z_{j-1} attracts z_j. Let $\{a, b, c\}$ be any positive slave of h where c is a feet of z_j. By the induction hypothesis, $\beta(a) = \gamma(a)$ and $\beta(b) = \gamma(b)$; let $a' = \beta(a)$ and $b' = \beta(b)$. Since β and γ are partial isomorphisms, both $\{a', b', \beta(c)\}$ and $\{a', b', \gamma(c)\}$ are hyperedges in N. Since N is a 2-multipede, $\beta(c) = \gamma(c)$. $\Box$

The unique regular extension of α will be denoted $\bar{\alpha}$.

Claim 4.2 *Suppose that α is a safe partial isomorphism from M to N with domain X of cardinality $< n$. For every element $a \in |M| - \bar{X}$, there is a safe extension of α to $X \cup \{a\}$ which leaves a intact.*

Proof We construct a regular extension β of $\bar{\alpha}$ to $\overline{X \cup \{a\}}$. Let z_0 be the segment of a, $Y = S(\bar{X}) \cup z_0$, $Z = S(\bar{Y})$ and $p = \|Z - Y\|$. By Theorem 2.4.1, there is a linear ordering $z_1, \ldots, z_p$ on the vertices of $Z - Y$ such that, for every $j > 0$, z_j is attracted by $Y \cup \{z_i : 1 \le i < j\}$ and there is a unique hyperedge h_j witnessing the attraction.

The desired β leaves intact all segments in Z and the feet of z_0. It remains to define β on the feet of segments z_j, $1 \le j \le k$. We do that by induction on j. Suppose that β is defined on the feet of all z_i with $i < j$ and let h_j be as above. Let d be a foot of y_j and pick a positive slave $\{b, c, d\}$ of h_j in M; β is already defined at b and c. The slave $\{\beta(b), \beta(c), \beta(d)$ of h_j should be positive in N. This defines uniquely whether $\beta(d)$ equals d or the other foot of y_j.

We need to check that β is a partial isomorphism from M to N. The only nontrivial part is to check that if A is a slave of a segment hyperedge h then A is positive in M if and only if $\beta(A)$ is positive in N. Without loss of generality, $A \not\subseteq \bar{X}$. Let j be the least number such that $S(\bar{X}) \cup \{z_0, \ldots, z_j\}$ includes h. Since $\bar{X}$ does not attract z_0, $\bar{X}$ includes all hyperedges in $S(\bar{X}) \cup \{z_0\}$; thus $j > 0$. By the uniqueness property of h_j, $h = h_j$. By the construction of β, A is positive in M if and only if $\beta(A)$ is positive in N. $\Box$

The desired winning strategy of Duplicator is to ensure that, after each round, pebbles define a safe partial isomorphism. Suppose that pebbles define a safe partial isomorphism α and Spoiler starts a new round. By the symmetry between M and N, we may suppose that Spoiler chooses M and a subset X of elements of M. Duplicator chooses N and a subset $\{f(x) : x \in X\}$ where f is as follows. If $x \in \overline{D(\alpha)}$ then $f(x) = \bar{\alpha}(x)$; otherwise $f(x) = x$. Now use the previous Lemma. $\Box$

Definition 4.4 A 3-*multipede* is a structure $(M, <)$ where M is a 2-multipede and $<$ is a linear order on the set of segments of M. $\square$

Definition 4.5 A 4-*multipede* is a 3-multipede together with (i) additional elements representing uniquely all sets of segments and (ii) the corresponding containment relation ε. $\square$

We skip the details of the definition of 4-multipedes. The additional elements are called *super-segments*.

A 4-multipede is *odd* if the hypergraph of segments is so.

Lemma 4.3 *The collection of odd 4-multipedes is finitely axiomatizable.*

Proof We give only three axioms which express that every set of segments is represented by a unique super-segment:

- There is a super-segment Y such that there is no x with $x\varepsilon Y$.
- For every super-segment Y and every segment x, there exists a super-segment Y' such that, for every y, $y\varepsilon Y' \leftrightarrow (y\varepsilon Y \vee y = x)$.
- Super-segments Y and Y' are equal if $x\varepsilon Y \leftrightarrow x\varepsilon Y'$ for all x.

$\square$

Lemma 4.4 *Every odd 4-multipede is rigid.*

Proof Let θ is an automorphism of a 4-multipede M. Because of the linear order on segments, θ leaves intact all segments. Therefore it leaves intact all super-segments. By Lemma 4.1, it leaves intact all feet as well. $\square$

A 4-multipede is l-meager if the hypergraph of segments is so.

Lemma 4.5 *Let M is a $2k$-meager 4-multipede and Υ be the extension of the vocabulary of M by means of individual constants for every segment of M. No $L^k_{\infty,\omega}(C)$ sentence in the vocabulary Υ distinguishes between M and the 4-multipede N obtained from M by permuting the feet of a segment.*

Proof The proof is similar to that of Theorem 4.1. We use the terminology and notation of the proof of Theorem 4.1. Call a collection of segments, feet and super-segments *closed* if the subcollection of segments and feet is so. Lemma 4.2 remains true. Lemma 4.3 remains true as well; if a is a super-segment, then $\bar{X} \cup \{a\}$ is closed and the desired β is the extension of $\bar{\alpha}$ by means of $\gamma(a) = a$. The remainder of the proof is as above. $\square$

Lemma 4.6 *There exists j such that no $L^k_{\infty,\omega}(C)$ formula defines a linear order in any $2(j+k)$-meager 4-multipede.*

Proof Let M be any structure in the vocabulary of 4-multipedes, M' be an extension of M with individual constants for all elements of M, and $N = M''$ be an extension of M' with a linear order $<$. There exists an $L^{\omega}_{\infty,\omega}$ sentence ψ_N which describes N up to isomorphism: For each basic relation R of N and each tuple $\bar{x}$ of elements of M of appropriate length, ψ_N says whether $\bar{x}$ belongs to R or not. *Cf.* [HKL]. The number j of variables in ψ does not depend on M.

By contradiction suppose that an $L^{k}_{\infty,\omega}(C)$ formula ϕ defines a linear order in an $2(j+k)$-meager 4-multipede M. Define M' as above and let M be the extension of M' by means of the linear order $<$ defined by ϕ. Replace each atomic formula $t_1 < t_2$ in ψ_N with $\phi(t_1, t_2)$; here each t_i is a variable or an individual constant. The resulting $L^{j+k}_{\infty,\omega}(C)$ formula describes M' up to isomorphism. This contradicts the preceding lemma. $\Box$

Theorem 4.1 *There exists a finitely axiomatizable class of rigid structures such that no $L^{\omega}_{\infty,\omega}(C)$ sentence that defines a linear order in every structure of that class.*

Proof Consider the class K of odd 4-multipedes. By Lemmas 4.3 and 4.4, K is a finitely axiomatizable class of rigid structures. By Lemma 4.7, for every $L^{\omega}_{\infty,\omega}(C)$ sentence ϕ, there exists l such that ϕ does not define a linear order in any l-meager 4-multipede. It remains to show that K contains an l-meager 4-multipede. By Theorem 3.1, there exists an odd l-meager 4-hypergraph H. Extend H to a 4-multipede by attaching two feet to each vertex of H, choosing positive slaves in any way consistent with the definition of 2-multipedes, ordering the segments in an arbitrary way and finally adding representations of subsets of segments. The result is an l-meager 4-multipede. $\Box$

Call two structures k-equivalent if there is no $L^{k}_{\infty,\omega}$ sentence which distinguishes between them. We answer negatively a question of Scott Weinstein [W].

Theorem 4.2 *There exist k and a structure M such that every structure k-equivalent to M is rigid but not every structure k-equivalent to M is isomorphic to M.*

Theorem remains true even if $L^{k}_{\infty,\omega}$ is replaced with $L^{k}_{\infty,\omega}(C)$ in the definition of k-equivalence.

Proof By Lemma 4.3, there exists k such that a first-order sentence with k variables axiomatizes the class of odd 4-multipedes. By Theorem 3.1, there exists a $2k$-meager odd hypergraph, and therefore there exists a $2k$-meager odd 4-multipede M. By the choice of k, every structure isomorphic to M is rigid. By Lemma 4.5, there a structure k-equivalent to M (even if counting quantifiers are allowed) but not isomorphic to M. $\Box$

References

D Anuj Dawar, "Feasible Computation through Model Theory", PhD Thesis, Institute for Research in Cognitive Science, University of Pennsylvania, Philadelphia, 1993.

G Yuri Gurevich, "Logic and the challenge of computer science", In "Current Trends in Theoretical Computer Science" (Ed. E. Börger), Computer Science Press, 1988, 1–57.

HKL Lauri Hella, Phokion G. Kolaitis and Kerkko Luosto, "How to Define a Linear Order on Finite Models", Symposium on Logic in Computer Science, IEEE Computer Society Press, 1994, 40–49.

IL Neil Immerman and E.S. Lander, "Describing Graphs: A First-Order Approach to Graph Canonization", in "Complexity Theory Retrospective", Ed. Alan Selman, Springer Verlag, 1990, 59–81.

S Alexei Stolboushkin, "Axiomatizable Classes of Finite Models and Definability of Linear Order", in Proc. 7th IEEE Annu. Symp. on Logic in Computer Sci., Santa Cruz, CA, 1992, 64–70.

W Scott Weinstein, Private Correspondence, Oct. 1993.