A model in which every Boolean algebra has many subalgebras

James Cummings
Hebrew University of Jerusalem *

Saharon Shelah
Hebrew University of Jerusalem †

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Abstract

We show that it is consistent with ZFC (relative to large cardinals) that every infinite Boolean algebra B has an irredundant subset A such that $2^{|A|} = 2^{|B|}$. This implies in particular that B has $2^{|B|}$ subalgebras. We also discuss some more general problems about subalgebras and free subsets of an algebra.

The result on the number of subalgebras in a Boolean algebra solves a question of Monk from [6]. The paper is intended to be accessible as far as possible to a general audience, in particular we have confined the more technical material to a "black box" at the end. The proof involves a variation on Foreman and Woodin's model in which GCH fails everywhere.

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1 Definitions and facts

In this section we give some basic definitions, and prove a couple of useful facts about algebras and free subsets. We refer the reader to [3] for the definitions of set-theoretic terms. Throughout this paper we are working in ZFC set theory.

Definition 1: Let M be a set. F is a **finitary function on** M if and only if $F : M^n \longrightarrow M$ for some finite n.

Definition 2: $\mathcal{M}$ is an **algebra** if and only if $\mathcal{M} = (M, \mathcal{F})$ where M is a set, and $\mathcal{F}$ is a set of finitary functions on M which is closed under composition and contains the identity function.

In this case we say $\mathcal{M}$ is an **algebra on** M**.**

Definition 3: Let $\mathcal{M} = (M, \mathcal{F})$ be an algebra and let $N \subseteq M$. Then N is a **subalgebra of** $\mathcal{M}$ if and only if N is closed under all the functions in $\mathcal{F}$. In this case there is a natural algebra structure on N given by $\mathcal{N} = (N, \mathcal{F} \restriction \mathcal{N})$.

Definition 4: Let $\mathcal{M} = (M, \mathcal{F})$ be an algebra. Then

$$Sub(\mathcal{M}) = \{ \ N \mid N \text{ is a subalgebra of } \mathcal{M} \ \}.$$

Definition 5: Let $\mathcal{M} = (M, \mathcal{F})$ be an algebra. If $A \subseteq M$ then

$$Cl_{\mathcal{M}}(A) = \{ \ F(\vec{a}) \mid F \in \mathcal{F}, \vec{a} \in A \ \}.$$

$Cl_{\mathcal{M}}(A)$ is the least subalgebra of $\mathcal{M}$ containing A.

Definition 6: Let $\mathcal{M} = (M, \mathcal{F})$ be an algebra. If $A \subseteq M$ then A is **free** if and only if

$$\forall a \in A \ a \notin Cl_{\mathcal{M}}(A - \{a\}).$$

In the context of Boolean algebras, free subsets are more usually referred to as **irredundant.** We can use free sets to generate large numbers of subalgebras using the following well-known fact.

Fact 1: If $A \subseteq M$ is free for $\mathcal{M}$ then $|Sub(\mathcal{M})| \geq 2^{|A|}$.

Proof: For each $B \subseteq A$ let $N_B = Cl_{\mathcal{M}}(B)$. It follows from the freeness of A that $N_B \cap A = B$, so that the map $B \longmapsto N_B$ is an injection from $\mathcal{P}A$ into $Sub(\mathcal{M})$.

♦

The next fact (also well-known) gives us one way of generating irredundant subsets in a Boolean algebra.

Fact 2: If κ is a strong limit cardinal and B is a Boolean algebra with $|B| \geq \kappa$ then B has an irredundant subset of cardinality κ.

Proof: We can build by induction a sequence $\langle b_\alpha : \alpha < \kappa \rangle$ such that b_β is not above any element in the subalgebra generated by $\vec{b} \restriction \beta$. The point is that a subalgebra of size less than κ cannot be dense, since κ is strong limit. It follows from the property we have arranged that we have enumerated an irredundant subset of cardinality κ.

♦

We make the remark that the last fact works even if $\kappa = \omega$. The next fact is a technical assertion which we will use when we discuss free subsets and subalgebras in the general setting.

Fact 3: Let μ be a singular strong limit cardinal. Let $\mathcal{M} = (M, \mathcal{F})$ be an algebra and suppose that $\mu \leq |M| < 2^\mu$ and $|\mathcal{F}| < \mu$. Then $Sub(\mathcal{M})$ has cardinality at least 2^μ.

Proof: Let $\lambda = \mathrm{cf}(\mu) + |\mathcal{F}|$, then $2^\lambda < \mu$ since μ is singular and strong limit. Define

$$P = \{\ A \subseteq M \mid |A| = \mathrm{cf}(\mu)\ \}.$$

From the assumptions on M and μ,

$$2^\mu = \mu^{\mathrm{cf}\mu} \leq |M|^{\mathrm{cf}\mu} = |P| \leq 2^{\mu.\mathrm{cf}\mu} = 2^\mu,$$

so that $|P| = 2^\mu$.

Observe that if $A \in P$ then $|Cl_\mathcal{M}(A)| \leq \lambda$. So if we define an equivalence relation on P by setting

$$A \equiv B \iff Cl_\mathcal{M}(A) = Cl_\mathcal{M}(B),$$

then the classes each have size at worst 2^λ. Hence there are 2^μ classes and so we can generate 2^μ subalgebras by closing representative elements from each class.

♦

2 Pr(κ)

Definition 7: Let κ be a cardinal. $Pr(\kappa)$ is the following property of κ: for all algebras $\mathcal{M} = (M, \mathcal{F})$ with $|M| = \kappa$ and $|\mathcal{F}| < \kappa$ there exists $A \subseteq M$ free for $\mathcal{M}$ such that $|A| = \kappa$.

In some contexts we might wish to make a more complex definition of the form "$Pr(\kappa, \mu, D, \sigma)$ iff for every algebra on κ with at most μ functions there is a free set in the σ-complete filter D" but $Pr(\kappa)$ is sufficient for the arguments here.

We collect some information about $Pr(\kappa)$.

Fact 4: If κ is Ramsey then $Pr(\kappa)$.

Proof: Let $\mathcal{M} = (M, \mathcal{F})$ be an algebra with $|M| = \kappa$, $|\mathcal{F}| < \kappa$. We lose nothing by assuming $M = \kappa$. We can regard $\mathcal{M}$ as a structure for a first-order language $\mathcal{L}$ with $|\mathcal{L}| < \kappa$. By a standard application of Ramseyness we can get $A \subseteq \kappa$ of order type κ with A a set of order indiscernibles for $\mathcal{M}$.

Now A must be free by an easy application of indiscernibility.

♦

Fact 5: If $Pr(\kappa)$ holds and $\mathbb{P}$ is a μ^+-c.c. forcing for some $\mu < \kappa$ then $Pr(\kappa)$ holds in $V^{\mathbb{P}}$.

Proof: Let $\dot{\mathcal{M}} = (\hat{\kappa}, \langle \dot{F}_\alpha : \alpha < \lambda \rangle)$ name an algebra on κ with λ functions, $\lambda < \kappa$. For each $\vec{a}$ from κ and $\alpha < \lambda$ we may (by the chain condition of $\mathbb{P}$) enumerate the possible values of $\dot{F}_\alpha(\vec{a})$ as $\langle G_{\alpha,\beta}(\vec{a}) : \beta < \mu \rangle$.

Now define in V an algebra $\mathcal{M}^* = (\kappa, \langle G_{\alpha,\beta} : \alpha < \lambda, \beta < \mu \rangle)$ and use $Pr(\kappa)$ to get $I \subseteq \kappa$ free for $\mathcal{M}^*$. Clearly it is forced that I be free for $\dot{\mathcal{M}}$ in $V^{\mathbb{P}}$.

♦

3 Building irredundant subsets

In this section we will define a combinatorial principle $(*)$ and show that if $(*)$ holds then we get the desired conclusion about Boolean algebras. We will also consider a limitation on the possibilities for generalising the result.

Definition 8 (The principle $(*)$): The principle $(*)$ is the conjunction of the following two statements:

S1. For all infinite cardinals κ, 2^κ is weakly inaccessible and

$$\kappa \leq \lambda < 2^\kappa \Longrightarrow 2^\lambda = 2^\kappa.$$

S2. For all infinite cardinals κ, $Pr(2^\kappa)$ holds.

Theorem 1: $(*)$ implies that every infinite Boolean algebra B has an irredundant subset A such that $2^{|A|} = 2^{|B|}$.

Proof: Observe that a Boolean algebra can be regarded as an algebra with $\aleph_0$ functions (just take the Boolean operations and close under composition).

Define a closed unbounded class of infinite cardinals by

$$C = \{ \ \mu \mid \exists \theta \ 2^\theta = \mu \text{ or } \mu \text{ is strong limit } \}$$

There is a unique $\mu \in C$ such that $\mu \leq |B| < 2^{\mu}$. By the assumption S1, $2^{\mu} = 2^{|B|}$, so it will suffice to find a free subset of size μ. Since μ is infinite we can find B_0 a subalgebra of B with $|B_0| = \mu$, and it suffices to find a free subset of size μ in B_0.
Case A: $\mu = 2^{\theta}$. In this case $\mu > \omega$, and S2 implies that $Pr(\mu)$ holds, so that B_0 has a free (irredundant) subset of cardinality μ.
Case B: μ is strong limit. In this case Fact 2 implies that there is an irredundant subset of cardinality μ.

♦

At this point we could ask about the situation for more general algebras. Could every algebra with countably many functions have a large free subset, or a large number of subalgebras? There is a result of Shelah (see Chapter III of [9]) which sheds some light on this question.

Theorem 2: If κ is inaccessible and not Mahlo (for example if κ is the first inaccessible cardinal) then $\kappa \not\rightarrow [\kappa]^2_{\kappa}$.

From this it follows immediately that if κ is such a cardinal, then we can define an algebra $\mathcal{M}$ on κ with countably many functions such that $|Sub(\mathcal{M})| = \kappa$.

In the next section we will prove that in the absence of inaccessibles some information about general algebras can be extracted from $(*)$. The model of $(*)$ which we eventually construct will in fact contain some quite large cardinals; it will be a set model of ZFC in which there is a proper class of cardinals α which are $\beth_3(\alpha)$-supercompact.

4 General algebras

In this section we prove that in the absence of inaccessible cardinals $(*)$ implies that certain algebras have a large number of subalgebras. By the remarks at the end of the last section, it will follow that the restriction that no inaccessibles should exist is essential.

Theorem 3: Suppose that $(*)$ holds and there is no inaccessible cardinal. Let κ and λ be infinite cardinals. If $\mathcal{M} = (M, \mathcal{F})$ is an algebra such that $|M| = \lambda$ and $|\mathcal{F}| = \kappa$, and $2^\kappa \leq \lambda$, then $Sub(\mathcal{M})$ has cardinality 2^λ.

Proof:

Let $\mathcal{M}$, κ and λ be as above. Define a class of infinite cardinals C by

$$C = \{ \mu \mid \exists \theta \; 2^\theta = \mu \text{ or } \mu \text{ is singular strong limit.} \}$$

There are no inaccessible cardinals, so C is closed and unbounded in the class of ordinals.

Let μ be the unique element of C such that $\mu \leq \lambda < 2^\mu$, where we know that μ exists because $\lambda \geq 2^\kappa$ and C is a closed unbounded class of ordinals. Since $2^\kappa \leq \lambda < 2^\mu$, $\kappa < \mu$.

Now let N be a subalgebra of $\mathcal{M}$ of cardinality μ; such a subalgebra can be obtained by taking any subset of size μ and closing it to get a subalgebra. S1 implies that $2^\lambda = 2^\mu$, so it suffices to show that the algebra $\mathcal{N} = (N, \mathcal{F} \restriction N)$ has 2^μ subalgebras.

We distinguish two cases:

Case A: μ is singular strong limit. In this case we may apply Fact 3 to conclude that $\mathcal{N}$ has 2^μ subalgebras.

Case B: $\mu = 2^\theta$. It follows from S2 that $Pr(\mu)$ is true, hence there is a free subset of size μ for $\mathcal{N}$. But now by Fact 1 we may generate 2^μ subalgebras.

This concludes the proof.

♦

For algebras with countably many functions the last result guarantees that there are many subalgebras as long as $|M| \geq 2^{\aleph_0}$. In fact we can do slightly better here, using another result of Shelah (see [1]).

Theorem 4: If C is a closed unbounded subset of $[\aleph_2]^{\aleph_0}$ then $|C| \geq 2^{\aleph_0}$.

Corollary 1: If $(*)$ holds and there are no inaccessibles, then for every algebra $\mathcal{M}$ with $|M| > \aleph_1$ and countably many functions $|Sub(\mathcal{M})| = 2^{|M|}$.

Proof: If $|M| \geq 2^{\aleph_0}$ we already have it. If $\aleph_2 \leq |M| < 2^{\aleph_0}$ then apply the theorem we just quoted and the fact that $2^{|M|} = 2^{\aleph_0}$.

♦

5 How to make a model of $(*)$.

In this section we sketch the argument of [2] and indicate how we modify it to get a model in which $(*)$ holds. We begin with a brief review of our conventions about forcing. For us $p \leq q$ means that p is stronger than q, a forcing is λ-closed if every decreasing sequence of length less than λ has a lower bound, a forcing is λ-dense if it adds no $< \lambda$-sequence of ordinals, and $Add(\kappa, \lambda)$ is the forcing for adding λ Cohen subsets of κ.

Foreman and Woodin begin the construction in [2] with a model V in which κ is supercompact, and in which for each finite n they have arranged that $\beth_n(\kappa)$ is weakly inaccessible and $\beth_n(\kappa)^{<\beth_n(\kappa)} = \beth_n(\kappa)$. They force with a rather complex forcing $\mathbb{P}$, and pass to a submodel of $V^{\mathbb{P}}$ which is of the form $V^{\mathbb{P}^\pi}$ for $\mathbb{P}^\pi$ a projection of $\mathbb{P}$.

$\mathbb{P}$ here is a kind of hybrid of Magidor's forcing from [5] to violate the Singular Cardinals Hypothesis at $\aleph_\omega$ and Radin's forcing from [8]. Just as in [5] the forcing $\mathbb{P}$ does too much damage to V and the desired model is an inner model of $V^{\mathbb{P}}$, but in the context of [2] it is necessary to be more explicit about the forcing for which the inner model is a generic extension.

The following are the key properties of $\mathbb{P}^\pi$.

P1. κ is still inaccessible (and in fact is $\beth_3(\kappa)$-supercompact) in $V^{\mathbb{P}^\pi}$.

P2. $\mathbb{P}^\pi$ adds (among other things) a generic club of order type κ in κ. In what follows we will assume that a generic G for $\mathbb{P}^\pi$ is given, and enumerate this club in increasing order as $\langle \kappa_\alpha : \alpha < \kappa \rangle$.

P3. **In V** each cardinal κ_α reflects to some extent the properties of κ. In particular in V each κ_α is measurable, and for each n we have that $\beth_n(\kappa_\alpha)$ is weakly inaccessible and $\beth_n(\kappa_\alpha)^{<\beth_n(\kappa_\alpha)} = \beth_n(\kappa_\alpha)$.

P4. For each α, if $p \in \mathbb{P}^\pi$ is a condition which determines κ_α and $\kappa_{\alpha+1}$ then $\mathbb{P}^\pi \restriction p$ (the suborder of conditions which refine p) factors as

$$\mathbb{P}_\alpha \times Add(\beth_4(\kappa_\alpha), \kappa_{\alpha+1}) \times \mathbb{Q}_\alpha,$$

where

(a) $\mathbb{P}_\alpha$ is κ_α^+-c.c. if α is limit and $\beth_4(\kappa_\beta)^+$-c.c. if $\alpha = \beta + 1$.

(b) All bounded subsets of $\beth_4(\kappa_{\alpha+1})$ which occur in $V^{\mathbb{P}^\pi \restriction p}$ have already appeared in the extension by $\mathbb{P}_\alpha \times Add(\beth_4(\kappa_\alpha), \kappa_{\alpha+1})$.

Given all this it is routine to check that if we truncate $V^{\mathbb{P}^\pi}$ at κ then we obtain a set model of ZFC in which GCH fails everywhere, and moreover exponentiation follows the pattern of clause 1 in $(*)$.

The reader should think of $\mathbb{P}^\pi$ as shooting a club of cardinals through κ, and doing a certain amount of work between each successive pair of cardinals. Things have been arranged so that GCH will fail for a long region past each cardinal κ_α on the club, so that what needs to be done is to blow up the powerset of a well-chosen point in that region to have size $\kappa_{\alpha+1}$.

For our purposes we need to change the construction of [2] slightly, by changing the forcing which is done between the points of the generic club to blow up powersets. The reason for the change is that we are trying to get $Pr(\lambda)$ to hold whenever λ is of the form 2^η, and in general it will not be possible to arrange that $Pr(\lambda)$ is preserved by forcing with $Add(\lambda, \rho)$.

The solution to this dilemma is to replace the Cohen forcing defined in V by a Cohen forcing defined in a well chosen inner model. We need to choose this inner model to be small enough that Cohen forcing from that model preserves $Pr(\lambda)$, yet large enough that it retains some degree of closure sufficient to make the arguments of [2] go through. For technical reasons we will be adding subsets to $\beth_5(\kappa_\alpha)$, rather than $\beth_4(\kappa_\alpha)$ as in the case of [2].

To be more precise we will want to replace clauses P3 and P4 above by

P3*. For each $n < 6$ it is the case in V that κ_α is measurable, $\beth_n(\kappa_\alpha)$ is weakly inaccessible, and $\beth_n(\kappa_\alpha)^{<\beth_n(\kappa_\alpha)} = \beth_n(\kappa_\alpha)$. Also $Pr(\beth_n(\kappa_\alpha))$ holds for each such n.

There exists for all possible values of κ_α, $\kappa_{\alpha+1}$ a notion of forcing $Add^*(\beth_5(\kappa_\alpha), \kappa_{\alpha+1})$ (which we will denote by $\mathbb{R}_\alpha$ in what follows) such that

(a) $\mathbb{R}_\alpha$ is $\beth_4(\kappa_\alpha)$-closed, $\beth_4(\kappa_\alpha)^+$-c.c. and $\beth_5(\kappa_\alpha)$-dense.

(b) $\mathbb{R}_\alpha$ adds $\kappa_{\alpha+1}$ subsets to $\beth_5(\kappa_\alpha)$.

(c) In $V^{\mathbb{R}_\alpha}$ the property $Pr(\beth_n(\kappa_\alpha))$ still holds [1] for $n < 6$.

P4*. If p determines κ_α and $\kappa_{\alpha+1}$ then $\mathbb{P}^\pi \restriction p$ factors as

$$\mathbb{P}_\alpha \times \mathbb{R}_\alpha \times \mathbb{Q}_\alpha,$$

where

(a) $\mathbb{P}_\alpha$ is κ_α^+-c.c. if α is limit, and $\beth_5(\kappa_\beta)^+$-c.c. if $\alpha = \beta + 1$.

(b) $\mathbb{R}_\alpha = Add^*(\beth_5(\kappa_\alpha), \kappa_{\alpha+1})$ is as above.

(c) All bounded subsets of $\beth_5(\kappa_{\alpha+1})$ in $V^{\mathbb{P}^\pi \restriction p}$ are already in the extension by $\mathbb{P}_\alpha \times \mathbb{R}_\alpha$.

We'll show that given a model in which P1, P2, P3* and P4* hold we can construct a model of $(*)$.

Lemma 1: Let G be generic for a modified version of $\mathbb{P}^\pi$ obeying P1, P2, P3* and P4*. If we define $V_1 = V[G]$, then V_1 is a model of ZFC in which $(*)$ holds.

Proof: S1 is proved just as in [2].

It follows from our assumptions that in V_1 we have the following situation:

- μ is strong limit if and only if $\mu = \kappa_\lambda$ for some limit λ.
- Cardinal arithmetic follows the pattern that for $n < 5$

$$\beth_n(\kappa_\alpha) \leq \theta < \beth_{n+1}(\kappa_\alpha) \Longrightarrow 2^\theta = \beth_{n+1}(\kappa_\alpha),$$

while

$$\beth_5(\kappa_\alpha) \leq \theta < \kappa_{\alpha+1} \Longrightarrow 2^\theta = \kappa_{\alpha+1}.$$

[1] The hard work comes in the case $n = 5$, density guarantees that the property survives for $n < 5$.

We need to show that for all infinite λ we have $Pr(2^\lambda)$. The proof divides into two cases.

Case 1: 2^λ is of the form $\beth_m(\kappa_\eta)$ where $1 \leq m \leq 5$ and η is a limit ordinal. By the factorisation properties in P4* above it will suffice to show that $Pr(2^\lambda)$ holds in the extension by $\mathbb{P}_\eta \times \mathbb{R}_\eta$.

Certainly $Pr(2^\lambda)$ holds in the extension by $\mathbb{R}_\eta$, because P3* says just that. Also we have that $\kappa_\eta^+ < \beth_1(\kappa_\eta) \leq 2^\lambda$, and $\mathbb{P}_\eta$ is κ_η^+-c.c. so that applying Fact 5 we get that $Pr(2^\lambda)$ holds in the extension by $\mathbb{P}_\eta \times \mathbb{R}_\eta$.

Case 2: 2^λ is of the form $\beth_m(\kappa_{\beta+1})$ where $0 \leq m \leq 5$. Again it suffices to show that $Pr(2^\lambda)$ holds in the extension by $\mathbb{P}_{\beta+1} \times \mathbb{R}_{\beta+1}$.

$Pr(2^\lambda)$ holds in $V^{\mathbb{R}_{\beta+1}}$, $\beth_5(\kappa_\beta)^+ < \kappa_{\beta+1} \leq 2^\lambda$ and $\mathbb{P}_{\beta+1}$ is $\beth_5(\kappa_\beta)^+$-c.c. so that as in the last case we can apply Fact 5 to get that $Pr(2^\lambda)$ in the extension by $\mathbb{P}_{\beta+1} \times \mathbb{R}_{\beta+1}$.

♦

6 The preparation forcing

Most of the hard work in this paper comes in preparing the model over which we intend (ultimately) to do our version of the construction from [2].

We'll make use of Laver's "indestructibility" theorem from [4] as a labour-saving device. At a certain point below we will sketch a proof, since we need a little more information than is contained in the statement.

Fact 6 (Laver): Let κ be supercompact, let $\eta < \kappa$. Then there is a κ-c.c. and η-directed closed forcing $\mathbb{P}$, of cardinality κ, such that in the extension by $\mathbb{P}$ the supercompactness of κ is indestructible under κ-directed closed forcing.

Now we will describe how to prepare the model over which we will do Radin forcing. Let us start with six supercompact cardinals enumerated in increasing order as $\langle \kappa_i : i < 6 \rangle$ in some initial model V_0.

6.1 Step One

We'll force to make each of the κ_i indestructibly supercompact under κ_i-directed closed forcing. To do this let $\mathbb{S}_0 \in V_0$ be Laver's forcing to make κ_0 indestructible under κ_0-directed closed forcing, and force with $\mathbb{S}_0$. In $V_0^{\mathbb{S}_0}$ the cardinal κ_0 is supercompact by construction, and the rest of the κ_i are still supercompact because $|S_0| = \kappa_0$ and supercompactness survives small forcing.

Now let $\mathbb{S}_1 \in V_0^{\mathbb{S}_0}$ be (as guaranteed by Fact 6) a forcing which makes κ_1 indestructible and is κ_0-directed closed. In $V_0^{\mathbb{S}_0 * \mathbb{S}_1}$ we claim that all the κ_i are supercompact and that both κ_0 and κ_1 are indestructible. The point for the indestructibility of κ_0 is that if $\mathbb{Q} \in V_0^{\mathbb{S}_0 * \mathbb{S}_1}$ is κ_0-directed closed then $\mathbb{S}_1 * \mathbb{Q}$ is κ_0-directed closed in $V_0^{\mathbb{S}_0}$, so that κ_0 has been immunised against its ill effects and is supercompact in $V_0^{\mathbb{S}_0 * \mathbb{S}_1 * \mathbb{Q}}$.

Repeating in the obvious way, we make each of the κ_i indestructible. Let the resulting model be called V.

6.2 Step Two

Now we will force over V with a rather complicated κ_0-directed closed forcing. Broadly speaking we will make κ_0 have the properties that are demanded of κ_α in P3* from the last section.

Of course κ_0 will still be supercompact after this, and we will use a reflection argument to show that we have many points in κ_0 which are good candidates to become points on the generic club.

To be more precise we are aiming to make a model W in which the following list of properties holds:

K1. κ_0 is supercompact.

K2. For each $i < 5$, $2^{\kappa_i} = \kappa_{i+1}$ and κ_{i+1} is a weakly inaccessible cardinal with $\kappa_{i+1}^{<\kappa_{i+1}} = \kappa_{i+1}$.

For any $\mu > \kappa_5$ there is a forcing $\mathbb{Q}_\mu \subseteq V_\mu$ such that (if we define $\mathbb{Q}_\eta = \mathbb{Q}_\mu \cap V_\eta$ for η with $\kappa_5 < \eta < \mu$)

K3. $|\mathbb{Q}_\eta| = \eta^{<\kappa_5}$, $\mathbb{Q}_\eta$ is κ_4-directed closed, κ_5-dense and κ_4^+-c.c.

K4. In $W^{\mathbb{Q}_\eta}$, $2^{\kappa_5} \geq \eta$ and $Pr(\kappa_i)$ holds for $i < 6$.

K5. If $\eta < \zeta \leq \mu$ then $\mathbb{Q}_\eta$ is a complete suborder of $\mathbb{Q}_\zeta$.

For each $i < 5$ let $\mathbb{P}_i$ be the Cohen conditions (as computed by V) for adding κ_{i+1} subsets of κ_i. We will abbreviate this by $\mathbb{P}_i = Add(\kappa_i, \kappa_{i+1})_V$. Let $\mathbb{P}_{i,j} = \prod_{i \leq n \leq j} \mathbb{P}_n$.

Let $V_1 = V[\dot{\mathbb{P}}_{0,3}]$. In V_1 we know that $2^{\kappa_i} = \kappa_{i+1}$ for $i < 4$, and that κ_5 is still supercompact since $|\mathbb{P}_{0,3}| = \kappa_4$. In V_1 define a forcing as in Fact 6 for making κ_5 indestructible. More precisely choose $\mathbb{R} \in V_1$ such that

- In V_1, $\mathbb{R}$ is κ_5-c.c. and κ_4^{+50}-directed closed with cardinality κ_5.
- In $V_1[\dot{\mathbb{R}}]$ the supercompactness of κ_5 is indestructible under κ_5-directed closed forcing.

Let $V_2 = V_1[\dot{\mathbb{R}}]$. Define $\mathbb{Q}_\mu = Add(\kappa_5, \mu)_{V_2}$, and observe that for every η $\mathbb{Q}_\eta = Add(\kappa_5, \eta)_{V_2}$. This is really the key point in the construction, to choose $\mathbb{Q}_\mu$ in exactly the right inner model (see our remarks in the last section just before the definition of P3*).

Finally let $W = V_2[\dot{\mathbb{P}}_4]$, where we stress that $\mathbb{P}_4$ is Cohen forcing as computed in V.

We'll prove that we have the required list of facts in W.

K1. κ_0 is supercompact.

Proof: Clearly $\mathbb{P}_{0,3} * \mathbb{R}$ is κ_0-directed closed in V. $\mathbb{P}_4$ is κ_4-directed closed in V so is still κ_0-directed closed in V_2, hence $\mathbb{P}_{0,3} * \mathbb{R} * \mathbb{P}_4$ is κ_0-directed closed in V. By the indestructibility of κ_0 in V, κ_0 is supercompact in W.

♦

K2. For each $i < 5$, $2^{\kappa_i} = \kappa_{i+1}$ and κ_{i+1} is a weakly inaccessible cardinal with $\kappa_{i+1}^{<\kappa_{i+1}} = \kappa_{i+1}$.

Proof: In V_1 we have this for $i < 4$. $\mathbb{P}_4$ is κ_4-closed and κ_4^+-c.c. in V, $\mathbb{P}_{0,3}$ is κ_3^+-c.c. even in $V[\dot{\mathbb{P}}_4]$ so by the usual arguments with Easton's lemma $\mathbb{P}_4$ is κ_4-dense and κ_4^+-c.c. in V_1. $\mathbb{R}$ is κ_4^{+50}-closed in V_1 so that $\mathbb{P}_4$ is κ_4-dense and κ_4^+-c.c. in V_2. Moreover, by the closure of $\mathbb{R}$ we still have the claim for $i < 4$ in V_2. κ_5 is inaccessible (indeed supercompact) in V_2 so that after forcing with $\mathbb{P}_4$ we have the claim for $i < 5$.

♦

Recall that $\mathbb{Q}_\eta = Add(\kappa_5, \eta)_{V_2}$.

K3. $|\mathbb{Q}_\eta| = \eta^{<\kappa_5}$, $\mathbb{Q}_\eta$ is κ_4-directed closed, κ_5-dense and κ_4^+-c.c.

Proof: The cardinality statement is clear. $\mathbb{Q}_\eta$ is κ_5-directed closed in V_2, so that (by what we proved about the properties of $\mathbb{P}_4$ in V_2 during the proof of K2) $\mathbb{Q}_\eta$ is κ_4-directed closed and κ_5-dense in W. Since $\mathbb{Q}_\eta$ is κ_5-closed in V_2, $\mathbb{P}_4$ is κ_4^+-c.c. in $V_2[\dot{\mathbb{Q}}_\eta]$. $\mathbb{Q}_\eta$ is κ_5^+-c.c. in V_2 so that $\mathbb{Q}_\eta \times \mathbb{P}_4$ is κ_5^+-c.c. in V_2, so that finally $\mathbb{Q}_\eta$ is κ_5^+-c.c. in W.

♦

K4. In $W^{\mathbb{Q}_\eta}$, $2^{\kappa_5} \geq \eta$ and $Pr(\kappa_i)$ holds for $i < 6$.

Proof: Easily $2^{\kappa_5} \geq \eta$. We break up the rest of the proof into a series of claims.

Claim 1: For $i < 4$, $Pr(\kappa_i)$ holds in $W[\dot{\mathbb{Q}}_\eta]$.

Proof: κ_i is supercompact in $V[\dot{\mathbb{P}}_{i,3}]$, because $\mathbb{P}_{i,3}$ is κ_i-directed closed. $\mathbb{P}_{0,i-1}$ has κ_{i-1}^+-c.c. in that model so that $Pr(\kappa_i)$ holds in V_1. Also we know that $W[\dot{\mathbb{Q}}_\eta] = V_1[\dot{\mathbb{R}}][\dot{\mathbb{P}}_4][\dot{\mathbb{Q}}_\eta]$ is an extension of V_1 by κ_4-dense forcing, so $Pr(\kappa_i)$ is still true in $W[\dot{\mathbb{Q}}_\eta]$.

♦

Claim 2: $Pr(\kappa_4)$ holds in $W[\dot{\mathbb{Q}}_\eta]$.

Proof: $W = V_2[\dot{\mathbb{P}}_4] = V[\dot{\mathbb{P}}_{0,4}][\dot{\mathbb{R}}]$. Arguing as in the previous claim, $Pr(\kappa_4)$ holds in $V[\mathbb{P}_{0,4}]$. $\mathbb{P}_4$ is κ_4^+-c.c. in V_1 so $\mathbb{R}$ is κ_4^+-dense in $V[\dot{\mathbb{P}}_{0,4}]$,hence $Pr(\kappa_4)$ holds in W. Finally $\mathbb{Q}_\eta$ is κ_5-dense in W, so $Pr(\kappa_4)$ holds in $W[\dot{\mathbb{Q}}_\eta]$.

♦

Claim 3: $Pr(\kappa_5)$ holds in $W[\dot{\mathbb{Q}}_\eta]$.

Proof: $W[\dot{\mathbb{Q}}_\eta] = V_2[\dot{\mathbb{P}}_4][\dot{\mathbb{Q}}_\eta] = V_2[\dot{\mathbb{Q}}_\eta][\dot{\mathbb{P}}_4]$. As $\mathbb{Q}_\eta$ is κ_5-directed closed in V_2, κ_5 is supercompact in $V_2[\mathbb{Q}_\eta]$. $\mathbb{P}_4$ is κ_4^+-c.c. in $V_2[\dot{\mathbb{Q}}_\eta]$, so that $Pr(\kappa_4)$ is still true in $W[\dot{\mathbb{Q}}_\eta]$.

♦

This finishes the proof of K4.

♦

K5. If $\eta < \zeta \leq \mu$ then $\mathbb{Q}_\eta$ is a complete suborder of $\mathbb{Q}_\zeta$.

Proof: This is immediate by the uniform definition of Cohen forcing.

♦

We finish this section by proving that in W we can find a highly supercompact embedding $j : W \longrightarrow M$ with critical point κ_0 such that in M the cardinal κ_0 has some properties resembling K1–K5 above. The point is that there will be many $\alpha < \kappa_0$ which have these properties in W, and eventually we'll ensure that every candidate to be on the generic club added by the Radin forcing has those properties.

For the rest of this section we will work in the model W unless otherwise specified.

Definition 9: A pair of cardinals (α, κ) is *sweet* iff (setting $\alpha_n = \beth_n(\alpha)$)

1. α is measurable.

2. For each $i < 6$, α_i is weakly inaccessible and $\alpha_i^{<\alpha_i} = \alpha_i$.

   There is a forcing $\mathbb{Q}^\alpha_\kappa \subseteq V_\kappa$ such that (setting $\mathbb{Q}^\alpha_\eta = \mathbb{Q}^\alpha_\kappa \cap V_\eta$)

3. $|\mathbb{Q}^\alpha_\eta| = \eta^{<\alpha_5}$, $\mathbb{Q}^\alpha_\eta$ is α_4-directed closed, α_5-dense and α_4^+-c.c.

4. In the generic extension by $\mathbb{Q}^\alpha_\eta$, $2^{\alpha_5} \geq \eta$ and $Pr(\alpha_i)$ holds for $i < 6$.

5. If $\eta < \zeta \leq \kappa$ then $\mathbb{Q}^\alpha_\eta$ is a complete suborder of $\mathbb{Q}^\alpha_\zeta$.

Theorem 5: Let $\lambda = \beth_{50}(\kappa_5)$. In W there is an embedding $j : W \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa_0$, ${}^\lambda M \subseteq M$, and in the model M the pair $(\kappa_0, j(\kappa_0))$ is sweet.

Proof: Recall that we started this section with a model V_0, and forced with some $\mathbb{S}_0$ to make κ_0 indestructible. Let $\mathbb{S}$ be the forcing that we do over $V_0^{\mathbb{S}_0}$ to get to the model V_2, and recall that we force over V_2 with $\mathbb{P}_4$ to get W. $\mathbb{S} * \mathbb{P}_4$ is κ_0-directed closed in $V_0^{\mathbb{S}_0}$.

Fix a cardinal μ much larger than λ, such that $\mu^{<\kappa} = \mu$. As in [4], fix in V_0 an embedding $j_0 : V_0 \longrightarrow N$ such that

1. $\mathrm{crit}(j_0) = \kappa_0$, $\mu < j_0(\kappa_0)$.

2. $V_0 \models {}^\mu N \subseteq N$.

3. $j_0(\mathbb{S}_0) = \mathbb{S}_0 * \mathbb{S} * \mathbb{P}_4 * \mathbb{R}$, where $\mathbb{R}$ is μ^+-closed in $N^{\mathbb{S}_0 * \mathbb{S} * \mathbb{P}_4}$.

Let G_0 be $\mathbb{S}_0$-generic over V_0, let $g_1 * g_2$ be $\mathbb{S} * \mathbb{P}_4$-generic over $V_0[G_0]$. Let $W = V_0[G_0][g_1][g_2]$ and let $N^+ = N[G_0][g_1][g_2]$.

Now N^+ is closed under μ-sequences in W so that the factor forcing $\mathbb{R}$ is μ^+-closed in W. Let H be $\mathbb{R}$-generic over W, then it is easy to see that j_0 lifts to

$$j_1 : V_0[G_0] \longrightarrow N^+[H].$$

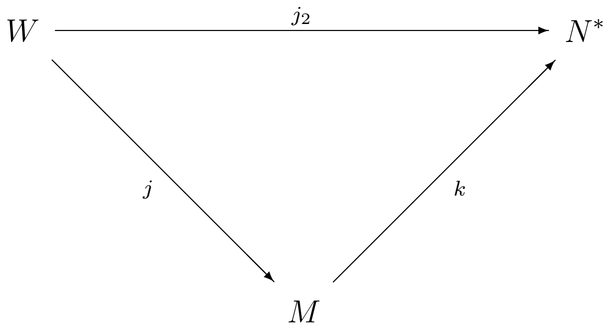
Now we know that $j_1 \restriction \mathbb{S} * \mathbb{P}_4 \in N[G_0]$, and $g_1 * g_2 \in N^+[H]$, so that $j[g_1 * g_2] \in N^+[H]$. By the directed closure of $j(\mathbb{S} * \mathbb{P}_4)$ in $N^+[H]$ we can find a "master condition" p (a lower bound in $j(\mathbb{S} * \mathbb{P}_4)$ for $j[g_1 * g_2]$) and then force over $W[H]$ with $j(\mathbb{S} * \mathbb{P}_4) \restriction p$ to get a generic X and an embedding

$$j_2 : W \longrightarrow N^* = N^+[H][X].$$

Of course this embedding is not in W but a certain approximation is. Observe that $\mathbb{R} * j(\mathbb{S} * \mathbb{P}_4)$ is μ^+-closed in W. Now factor j_2 through the ultrapower by

$$U = \{ \ A \subseteq \mathcal{P}_\kappa \mu \mid j[\mu] \in j_2(A) \ \},$$

to get a commutative triangle



By the closure $U \in W$, so j is an **internal** ultrapower map. It witnesses the μ-supercompactness of κ_0 in the model W, because (as can easily be checked) U is a fine normal measure on $\mathcal{P}_\kappa \mu$.

It remains to check that $(\kappa_0, j(\kappa_0))$ is sweet in M. We'll actually check that $(\kappa_0, j_2(\kappa_0))$ is sweet in N^*, this suffices because $\mathrm{crit}(k) > \mu > \kappa_0$.

Clauses 1 and 2 are immediate by the closure of M inside the model W. Now to witness sweetness let us set

$$\mathbb{Q}^{\kappa_0}_{j_2(\kappa_0)} = Add(\kappa_5, j_2(\kappa_0))_{V_0[G_0][g_1]} = Add(\kappa_5, j_2(\kappa_0))_{N[G_0][g_1]}.$$

Clause 3 is true in N^+ by the same arguments that we used for W above. N^* is an extension of N^+ by highly closed forcing so that Clause 3 is still true in N^*. Clause 5 is easy so we are left with Clause 4.

Let h be $\mathbb{Q}_\eta^{\kappa_0}$-generic over N^* for $\mathbb{Q}_\eta^{\kappa_0}$. Then h is generic over N^+. By the chain condition of $\mathbb{Q}_\eta^{\kappa_0}$ and the closure of N^+ in W, W and N^+ see the same set of antichains for $\mathbb{Q}_\eta^{\kappa_0}$. Hence h is $\mathbb{Q}_\eta^{\kappa_0}$-generic over W, and we showed that $Pr(\kappa_i)$ holds in $W[h]$. But then by closure again $Pr(\kappa_i)$ holds in $N^+[h]$, by Easton's lemma $H * X$ is generic over $N^+[h]$ for highly dense forcing so that finally $Pr(\kappa_i)$ holds in $N^*[h]$.

♦

7 The final model

In this section we will indicate how to modify the construction of [2], so as to force over the model W obtained in the last section and produce a model of $(*)$. The modification that we are making is so minor that it seems pointless to reproduce the long and complicated definitions of $\mathbb{P}$ and $\mathbb{P}^\pi$ from [2]; in this section we just give a sketch of what is going on in [2], an indication of how the construction there is to be modified, and then some hints to the diligent reader as to how the proofs in [2] can be changed to work for our modified forcing.

As we mentioned in Section 5, the construction of [2] involves building a forcing $\mathbb{P}$ and then defining a projection $\mathbb{P}^\pi$; the idea is that the analysis of $\mathbb{P}$ provides information which shows that $\mathbb{P}^\pi$ does not too much damage to the cardinal structure of the ground model. We start by giving a sketchy account of $\mathbb{P}^\pi$ as constructed in [2].

Recall the the aim of $\mathbb{P}^\pi$ is to add a sequence $\langle (\kappa_\alpha, F_\alpha) : \alpha < \kappa \rangle$ such that:

1. $\vec{\kappa}$ enumerates a closed unbounded subset of κ.
2. F_α is $Add(\beth_4(\kappa_\alpha), \kappa_{\alpha+1})$-generic over V.
3. In the generic extension, cardinals are preserved and κ is still inaccessible.

A condition will prescribe finitely many of the κ_α, and for each κ_α that it prescribes it will give some information about F_α; it will also place some constraint on the possibilities for adding in new values of κ_α and for giving information about the corresponding F_α. The idea goes back to Prikry forcing, where a condition prescribes an initial segment of the generic ω-sequence and puts constraint on the possibilities for adding further points.

To be more specific $\mathbb{P}^\pi$ is built from a pair $(\vec{w}, \vec{\mathcal{F}})$ where $w(0) = \kappa$ and $w(\alpha)$ is a measure on V_κ for $\alpha > 0$. $w(\alpha)$ will concentrate on pairs $(\vec{u}, \vec{h})$ that resemble $(\vec{w} \restriction \alpha, \vec{\mathcal{F}} \restriction \alpha)$; in particular if we let $\kappa_{\vec{u}} =_{def} u(0)$, $\kappa_{\vec{u}}$ will be a cardinal that resembles κ. $\mathcal{F}(\alpha)$ will be an ultrafilter on a Boolean algebra $\mathbb{Q}(\vec{w}, \alpha)$, which consists of functions f such that $\mathrm{dom}(f) \in w(\alpha)$ and $f(\vec{u}, \vec{k}) \in RO(Add(\beth_4(\kappa_{\vec{u}}), \kappa))$, with the operations defined pointwise.

Conditions in $\mathbb{P}^\pi$ will have the form

$$\langle (\vec{u}_0, \vec{k}_0, \vec{A}_0, \vec{f}_0, s_0), \ldots, (\vec{u}_{n-1}, \vec{k}_{n-1}, \vec{A}_{n-1}, \vec{f}_{n-1}, s_{n-1}), (\vec{w}, \vec{\mathcal{F}}, \vec{B}, \vec{g}) \rangle$$

where

1. $(\vec{u}_i, \vec{k}_i) \in V_\kappa$, $\kappa_{\vec{u}_0} < \ldots \kappa_{\vec{u}_{n-1}} < \kappa$.
2. $\mathrm{dom}(f_i(\alpha)) = A_i(\alpha) \in u_i(\alpha)$, $\mathrm{dom}(g(\alpha)) = B(\alpha) \in w(\alpha)$.
3. $f_i(\alpha) \in k_i(\alpha)$, $g(\alpha) \in \mathcal{F}(\alpha)$.
4. $s_i \in Add(\beth_4(\kappa_{u_i}), u_{i+1})$ for $i < n-1$, $s_{n-1} \in Add(\beth_4(\kappa_{u_{n-1}}), \kappa)$.

The aim of this condition is to force that the cardinals κ_{u_i} are on the generic club, and that the conditions s_i are in the corresponding generics. It is also intended to force that if we add in a new cardinal α and Add condition s between κ_{u_i} and $\kappa_{u_{i+1}}$ then there is a pair $(\vec{v}, \vec{h})$ in some $A_{i+1}(\beta)$ such that $\alpha = \kappa_{\vec{v}}$ and $s \leq k_{i+1}(\beta)(\vec{v}, \vec{h})$. This motivates the ordering (which we do not spell out in detail here); a condition is refined either by refining the "constraint parts" or by adding in new quintuples that obey the current constraints and which impose constraints compatible with the current ones.

The key fact about $\mathbb{P}^\pi$ is that given a condition p, any question about the generic extension can be decided by refining the constraint parts of p (this is modelled on Prikry's well-known lemma about Prikry forcing). It will follow from this that bounded subsets of κ are derived from initial segments of the generic as in P4 b) from Section 5, which in turn will imply that $\mathbb{P}^\pi$ has the desired effect of causing the GCH to fail everywhere below κ. It remains to be seen that forcing with $\mathbb{P}^\pi$ preserves the large cardinal character of κ; this is done via a master condition argument which succeeds because $\vec{w}$ is quite long (it has a so-called *repeat point*) and the forcing $Add(\beth_4(\lambda), \mu)$ is $\beth_4(\lambda)$-directed-closed.

We can now describe how we modify the construction of [2]. To bring our notation more into line with [2], let $\kappa = \kappa_0$, let $V = W$, let $j : V \longrightarrow M$ be the embedding constructed at the end of the last section. For each $\alpha < \kappa$ such that (α, κ) is sweet let us fix $\langle \mathbb{Q}^\alpha_\eta : \eta \leq \kappa \rangle$ witnessing this. Our modification of [2] is simply to restrict attention to those pairs $(\vec{u}, \vec{k})$ such that $(\kappa_{\vec{u}}, \kappa)$ is sweet (there are sufficiently many because $(\kappa, j(\kappa))$ is sweet in M) and then to replace the forcing $Add(\beth_4(\kappa_{\vec{u}}), \kappa)$ by the forcing $\mathbb{Q}^{\kappa_{\vec{u}}}_\kappa$. It turns out that the proofs in [2] go through essentially unaltered, because they only make appeal to rather general properties (chain condition, distributivity, directed closure) of Cohen forcing.

There is one slightly subtle point, which is that is we will need a certain uniformity in the dependence of $\mathbb{Q}^\alpha_\eta$ on η, as expressed by the equation $\mathbb{Q}^\alpha_\eta = \mathbb{Q}^\alpha_\zeta \cap V_\eta$. The real point of this is that when we consider a condition in $j(\mathbb{P}^\pi)$ which puts onto the generic club a point $\alpha < \kappa$ followed by the point κ, the forcing which is happening between α and κ is $j(\mathbb{Q})^\alpha_\kappa$; it will be crucial that $j(\mathbb{Q})^\alpha_\kappa = j(\mathbb{Q}^\alpha_\kappa) \cap V_\kappa = \mathbb{Q}^\alpha_\kappa$, that is the same forcing that would be used in $\mathbb{P}^\pi$ between α and κ.

We conclude this section with a short discussion of the necessary changes in the proofs. We assume that the reader has a copy of [2] to hand; all references below are to theorem and section numbers from that paper.

Section Three: We will use $j : V \longrightarrow M$ to construct the master sequence $(\vec{M}, \vec{g})$. Recall that we chose j to witness $\beth_{50}(\kappa_5)$-supercompactness of κ_0, it can be checked that this is enough to make all the arguments below go through.

When we build $(\vec{M}, \vec{g})$ we will choose each g_α so that $\mathrm{dom}(g_\alpha)$ contains only $(\vec{u}, \vec{h})$ with $(\kappa_{\vec{u}}, \kappa)$ sweet, and then let $g_\alpha(\vec{u}, \vec{h}) \in \dot{\mathbb{Q}}^{\kappa_{\vec{u}}}_\kappa$. As in [2], if $j_\alpha : V \longrightarrow N_\alpha$ is the ultrapower by M_α then $W \models {}^{\kappa_3}N_\alpha \subseteq N_\alpha$. If we let F

be the function given by

$$F : (\vec{u}, \vec{h}) \longmapsto \mathbb{Q}_{\kappa}^{\kappa_{\vec{u}}}$$

then $[F]_{M_\alpha}$ is κ_4-closed in N_α, hence is κ_3^+-closed in V.

This closure will suffice to make appropriate versions of 3.3, 3.4 and 3.5 go through.

Section Four: 4.1 is just as in [2]. A version of 4.2 goes through because $\mathbb{Q}_{\kappa}^{\kappa_a} \subseteq V_\kappa$.

Section Five: In the definition of "suitable quintuple" we demand that $k_\delta(b) \in \mathbb{Q}_{\kappa_u}^{\kappa_b}$ and $s \in \mathbb{Q}_{\kappa}^{\kappa_u}$.

The definition of "addability" is unchanged. Clauses d) and e) of that definition still make sense because (e.g.) $\mathbb{Q}_{\kappa_v}^{\kappa_a}$ is a complete suborder of $\mathbb{Q}_{\kappa_w}^{\kappa_a}$.

Versions of 5.1 and 5.2 go through without change.

In the definition of "condition in $\mathbb{P}$" we demand that $k_\delta(b) \in \mathbb{Q}_{\kappa_w}^{\kappa_b}$, $s \in \mathbb{Q}_{\kappa_{u_{i+1}}}^{\kappa_{u_i}}$. The ordering on $\mathbb{P}$ is as before.

The definitions of "canonical representatives" and of "upper and lower parts" are unchanged, as is 5.3.

We can do a "local" version of our argument for 3.3 above to show 5.4 goes through. Similarly 5.5, 5.6 and 5.8 go through.

Section Six: We need to make the obvious changes in the definitions of "suitable quintuple" and of the projected forcing $\mathbb{P}^\pi$. The proofs all go through because the forcing $\mathbb{Q}_\eta^\alpha$ has the right chain condition and closure.

Section Seven: The master condition argument of this section goes through because the forcing notion $j(\alpha \longmapsto \mathbb{Q}_\kappa^\alpha)(\kappa)$ is κ_4-directed closed.

8 Conclusion

Putting together the results of the preceding sections we get the following result.

Theorem 6 (Main Theorem): If Con(ZFC + GCH + six supercompact cardinals) then Con(ZFC + every infinite Boolean algebra B has an irredundant subset A such that $2^{|A|} = 2^{|B|}$.

Shelah has shown that the conclusion of this consistency result implies the failure of weak square, and therefore needs a substantial large cardinal hypothesis.

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