Evasion and prediction II

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Abstract

A subgroup $G \leq \mathbb{Z}^\omega$ exhibits the Specker phenomenon if every homomorphism $G \to \mathbb{Z}$ maps almost all unit vectors to 0. We give several combinatorial characterizations of the cardinal $\mathfrak{se}$, the size of the smallest $G \leq \mathbb{Z}^\omega$ exhibiting the Specker phenomenon. We also prove the consistency of $\mathfrak{b} < \mathfrak{e}$, where $\mathfrak{b}$ is the unbounding number and $\mathfrak{e}$ the evasion number. Our results answer several questions addressed by Blass.

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Introduction

Specker [Sp] proved that given a homomorphism h from $\mathbb{Z}^\omega$ to the infinite cyclic group $\mathbb{Z}$, where $\mathbb{Z}^\omega$ denotes the direct product of countably many copies of $\mathbb{Z}$, we have $h(e_n) = 0$ for all but finitely many unit vectors $e_n \in \mathbb{Z}^\omega$ (in other words, the n–th component of e_n is 1, and its other components are 0). Blass [Bl] studied the *Specker–Eda number* $\mathfrak{se}$, the size of the smallest subgroup $G \leq \mathbb{Z}^\omega$ containing all unit vectors which still has the property that every homomorphism $h : G \to \mathbb{Z}$ annihilates almost all unit vectors. We will give various (mostly less algebraic) characterizations of $\mathfrak{se}$ (some of which already play a prominent role in Blass' work); we will also study some related cardinal invariants of the continuum.

To be more explicit, let $\leq^*$ denote the *eventual domination order* on the Baire space ω^ω; i.e. $f \leq^* g$ iff $f(n) \leq g(n)$ *for all but finitely many* n. We shall usually abbreviate the statement in italics by $\forall^\infty n$; similarly we will write $\exists^\infty n$ for *there are infinitely many* n. The *unbounding number* $\mathfrak{b}$ is the smallest size of a $\leq^*$–unbounded family $\mathcal{F}$ of functions in ω^ω (i.e., given any $g \in \omega^\omega$, there is $f \in \mathcal{F}$ with $\exists^\infty n \; (f(n) > g(n))$). Given a σ–ideal $\mathcal{I}$ on ω^ω, the *additivity* $\mathsf{add}(\mathcal{I})$ is the least cardinality of a family $\mathcal{F}$ of members of $\mathcal{I}$ whose union is not in $\mathcal{I}$. We shall use this cardinal only in the cases $\mathcal{I} = \mathcal{M}$, the ideal of meager sets, and $\mathcal{I} = \mathcal{L}$, the ideal of Lebesgue null sets. — While the preceding invariants have been studied by a number of people in the last two decades, the following concept was introduced only recently by Blass [Bl]. Given an at most countable set S, an *S–valued predictor* is a pair $\pi = (D_\pi, \langle \pi_n;\; n \in D_\pi \rangle)$ where $D_\pi \subseteq \omega$ is infinite and for each $n \in D_\pi$, π_n is a function from S^n to S. π *predicts* $f \in S^\omega$ iff for all but finitely many $n \in D_\pi$, we have $f(n) = \pi_n(f{\restriction}n)$; otherwise f *evades* π. The *evasion number* $\mathfrak{e}$ is the smallest size of a family $\mathcal{F}$ of functions in ω^ω such that no ω–valued predictor predicts all $f \in \mathcal{F}$. A $\mathbb{Z}$–valued predictor is *linear* iff all $\pi_n : \mathbb{Z}^n \to \mathbb{Q}$ are $\mathbb{Q}$–linear maps. The corresponding *linear evasion number* shall be denoted by $\mathfrak{e}_\ell$ (i.e., $\mathfrak{e}_\ell = \min\{|\mathcal{F}|;\; \mathcal{F} \subseteq \mathbb{Z}^\omega$ and no linear $\mathbb{Z}$–valued predictor predicts all $f \in \mathcal{F}\}$). (Blass' definition of linear evading [Bl, section 4] is slightly different; however, it gives rise to the same cardinal; we use the present definition because we shall work with functions in $\mathbb{Z}^\omega$ in 2.2.)

These notions enable us to phrase our main results.

Theorem A. *It is consistent with ZFC to assume* $\mathfrak{b} < \mathfrak{e}$*.*

Theorem B. $\mathfrak{se} = \mathfrak{e}_\ell = \min\{\mathfrak{e}, \mathfrak{b}\}$*.*

They will be proved in sections 1 and 2 of our work. Section 2 also contains a further purely combinatorial characterization of the cardinal $\mathfrak{se}$ (subsections 2.4 and 2.5). To put our results into a somewhat larger context, we point out the following consequences which involve some earlier results, due mostly to Blass [Bl].

Corollary. *(a)* $\mathsf{add}(\mathcal{L}) \leq \mathfrak{se} \leq \mathsf{add}(\mathcal{M}) \leq \mathfrak{b}$*;*

(b) any of the inequalities in (a) can be consistently strict;

(c) it is consistent with ZFC to assume $\mathfrak{e}_\ell < \mathfrak{e}$*.*

Theorems A and B together with the Corollary give a complete solution to Questions (1) through (3) in [Bl, section 5]. Note in particular that the cardinals (2) through (5) in Corollary 8 in [Bl, section 3] are indeed equal.

PROOF OF COROLLARY. (a) This follows from Theorem B and Blass' results [Bl, Theorems 12 and 13]. The well–known inequality $\mathsf{add}(\mathcal{M}) \leq \mathfrak{b}$ is due to Miller [Mi].

(b) The consistency of $\mathsf{add}(\mathcal{M}) < \mathfrak{b}$ is well–known (it holds e.g. in the Mathias or Laver real models); for the consistency of $\mathsf{add}(\mathcal{L}) < \mathfrak{se}$ see [Bl] (in particular [Bl, Theorem 9]); the consistency of $\mathfrak{se} < \mathsf{add}(\mathcal{M})$ follows from Theorem B and [Br, Theorem A].

(c) This is immediate from Theorems A and B. $\square$

A set of reals predicted by a single predictor is small in various senses; it belongs, in particular, both to $\mathcal{M}$ and $\mathcal{L}$. This motivates us to introduce the σ–ideal $\mathcal{J}$ on ω^ω generated by such sets of reals (see [Br, section 4] for more on this). Clearly, the uniformity of $\mathcal{J}$ (i.e., the size of the smallest set of reals not in $\mathcal{J}$) is closely related to the evasion number. In fact, $\mathfrak{e} \leq \mathfrak{e}(\omega)$ where $\mathfrak{e}(\omega)$ denotes the former cardinal. We shall show in section 3 that these two cardinals are equal under some additional assumption, thus giving a partial answer to [Br, section 6, question (4)].

The results of this work are due to the second author. It was the first author's task to work them out and to write up the paper.

Notational remarks. A p.o. $\mathbb{P}$ is *σ–centered* iff there are $\mathbb{P}_n \subseteq \mathbb{P}$ $(n \in \omega)$ so that $\mathbb{P} = \bigcup_n \mathbb{P}_n$ and given $n \in \omega$, $F \subseteq \mathbb{P}_n$ finite, there is $p \in \mathbb{P}$ extending all $q \in F$. $\mathbb{P}$–names are denoted by symbols like $\dot{h}, \dot{\pi}, \dot{D}$, ... $|$ stands for *divides*; $\not|$ means *does not divide.*

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§ 1. Proof of Theorem A

1.1. We shall use a finite support iteration of *ccc* p.o.'s of length κ (where $\kappa \geq \omega_2$ is a regular cardinal) over a model V for CH to prove the consistency of $\mathfrak{e} > \mathfrak{b}$. In fact, in the resulting model, $\mathfrak{b} = \omega_1$ and $\mathfrak{e} = \kappa$. We start with defining the p.o. $\mathbb{P}$ we want to iterate. Notice that it is quite similar to the one used in [Br, 4.3.] for predicting below a given function.

$$\begin{aligned}\langle d, \pi, F\rangle \in \mathbb{P} \Longleftrightarrow\ & d \in 2^{<\omega} \text{ is a finite partial function,}\\ & \pi = \langle \pi_n;\ n \in d^{-1}(\{1\})\rangle \text{ and } \pi_n : \omega^n \to \omega \text{ is a finite partial function,}\\ & F \subseteq \omega^\omega \text{ is finite and } (f \neq g \in F \longrightarrow \max\{n;\ f{\restriction}n = g{\restriction}n\} < |d|).\end{aligned}$$

The order is given by:

$$\begin{aligned}\langle d', \pi', F'\rangle \leq \langle d, \pi, F\rangle \Longleftrightarrow\ & d' \supseteq d, \pi' \supseteq \pi, F' \supseteq F \text{ and}\\ & (f \in F, n \in (d')^{-1}(\{1\}) \setminus d^{-1}(\{1\}) \longrightarrow \pi'_n(f{\restriction}n) = f(n))\\ & \text{(in particular } \pi'_n(f{\restriction}n) \text{ is defined).}\end{aligned}$$

Notice that we use the convention that stronger conditions are smaller in the p.o. — The first two coordinates of a condition are intended as a finite approximation to a generic predictor; the third coordinate then guarantees that functions are predicted from some point on. Thus it is straightforward that $\mathbb{P}$ adjoins a predictor which predicts all ground–model functions. Hence iterating $\mathbb{P}$ increases $\mathfrak{e}$.

Furthermore $\mathbb{P}$ is σ–centered (and thus in particular *ccc*). To see this simply notice that conditions with the same initial segment in the first two coordinates are compatible.

So it remains to show that $\mathfrak{b} = \omega_1$ after iterating $\mathbb{P}$. For this it suffices to show the following:

(∗) whenever $G \in W$ is an unbounded family of functions from ω to ω, and $\mathbb{P} \in W$ is the p.o. defined above, then

$$\Vdash_{\mathbb{P}} \text{“}G \text{ is unbounded”}.$$

Using $(*)$ we can show that $\omega^\omega \cap V$ is still unbounded in the final model: $(*)$ guarantees that it stays unbounded in successor steps of the iteration; and one of the usual preservation results for finite support iterations (see, e.g., [JS, Theorem 2.2]) shows that it does so in limit steps of the iteration as well. Now, $V \models CH$; hence $\omega^\omega \cap V$ is an unbounded family of size ω_1 in the final model.

To start with the proof of $(*)$, let $\dot{h}$ be a $\mathbb{P}$–name for a function in ω^ω. For each $d \in 2^{<\omega}, \pi = \langle \pi_n;\ n \in d^{-1}(\{1\}) \rangle$ an initial segment of a predictor (as in the definition of $\mathbb{P}$), $k \in \omega$ and $\bar{f}^* = \langle f^*_\ell \in \omega^{|d|};\ \ell < k \rangle$ we define $h = h_{d,\pi,\bar{f}^*} \in (\omega+1)^\omega$ by

$$h(n) := \min\{m \leq \omega;\ \text{ for no } p \in \mathbb{P} \text{ with } p = \langle d, \pi, F \rangle,\ F = \{f_\ell;\ \ell < k\},\ f_\ell \restriction |d| = f^*_\ell, \\ \text{do we have } p \Vdash_{\mathbb{P}} \text{“}\dot{h}(n) > m\text{”}\}.$$

1.2. Main Claim. *$h \in \omega^\omega$.*

1.3. Proof of $(*)$ from the Main Claim. Let $h^* \in \omega^\omega$ such that for all $d, \pi, \bar{f}^*$ as above we have $h_{d,\pi,\bar{f}^*} \leq^* h^*$. As G is unbounded we can find $f \in G$ such that there are infinitely many n with $f(n) > h^*(n)$. We claim that $\Vdash_{\mathbb{P}}$“$\exists^\infty n(f(n) > \dot{h}(n))$”. This will show $(*)$.

Assume $m \in \omega$ and $p \in \mathbb{P}$ are such that

$$p \Vdash_{\mathbb{P}} \text{“}\forall n \geq m\ (f(n) \leq \dot{h}(n))\text{”}.$$

Find $d, \pi, \bar{f}^*$ such that $p = \langle d, \pi, F \rangle$ where $F = \{f_\ell;\ \ell < k\}$ and $f_\ell \restriction |d| = f^*_\ell$. Find $n \geq m$ such that $f(n) > h^*(n)$ and $h^*(n) \geq h_{d,\pi,\bar{f}^*}(n)$. Then

$$p \Vdash_{\mathbb{P}} \text{“}h_{d,\pi,\bar{f}^*}(n) < f(n) \leq \dot{h}(n)\text{”},$$

contradicting the definition of $h_{d,\pi,\bar{f}^*}$. $\square$

1.4. Proof of the Main Claim (1.2.). Let $d, \pi, k, \bar{f}^* = \langle f^*_\ell;\ \ell < k \rangle$ as above and $n \in \omega$ be fixed. Now assume that we have $p_i = \langle d, \pi, \{f^i_\ell;\ \ell < k\} \rangle$ with $f^i_\ell \restriction |d| = f^*_\ell$ and

$$p_i \Vdash_{\mathbb{P}} \text{“}\dot{h}(n) > i\text{”}.$$

We shall reach a contradiction. As we can replace $\langle p_i;\ i \in \omega \rangle$ by a subsequence, if necessary, we may assume that for all $\ell < k$:

either $(a)_\ell$ for some $g_\ell \in \omega^\omega$ for all i $(f^i_\ell \restriction i = g_\ell \restriction i)$

or $(b)_\ell$ for some $i_\ell \in \omega$ and $\hat{g}_\ell \in \omega^{i_\ell}$ $(f^i_\ell \restriction i_\ell = \hat{g}_\ell \ \wedge\ f^i_\ell(i_\ell) > i)$.

Notice that $i_\ell \geq |d|$ in the latter case. — Let $d^* := d \cup 0_{[|d|, \max(i_\ell;\ (b)_\ell \text{ holds})+1)}$; i.e. the function d^* takes value 0 between $|d|$ and the maximum of the i_ℓ. Put $F^* := \{g_\ell;\ (a)_\ell \text{ holds}\}$. Then clearly $p^* = \langle d^*, \pi, F^* \rangle \in \mathbb{P}$. Now choose ℓ^* and $q \leq p^*$ such that

$$q \Vdash_{\mathbb{P}} \text{“}\dot{h}(n) = \ell^*\text{”}.$$

We shall find $i > \ell^*$ so that q and p_i are compatible; this is a contradiction because q and p_i force contradictory statements.

Assume $q = \langle d^q, \pi^q, F^q \rangle$. Choose $i \geq \ell^*$ large enough such that:

(A) $i \geq |d^q|$;

(B) $i \geq \max\{\max\{\sigma(j);\ \sigma \in dom(\pi^q_m)\ \wedge\ j \in m\};\ m \in (d^q)^{-1}(\{1\})\}$.

Notice that (A) implies that $f^i_\ell \restriction |d^q| = g_\ell \restriction |d^q|$ whenever $(a)_\ell$ holds, while $f^i_\ell(i_\ell) > \max\{\max\{\sigma(j);\ \sigma \in dom(\pi^q_m)\ \wedge\ j \in m\};\ m \in (d^q)^{-1}(\{1\})\}$ by (B) in case $(b)_\ell$ holds. For such i let $q^i = \langle d^i, \pi^i, F^i \rangle$ where

— $d^i = d^q \cup 0_{[|d^q|, a)}$, where a is large enough such that all functions in F^i disagree before a;

— $\pi^i \supseteq \pi^q$ such that for all $m \in (d^q)^{-1}(\{1\}) \setminus d^{-1}(\{1\})$ and all f^i_ℓ so that $(b)_\ell$ holds, we have

$$f^i_\ell(m) = \pi^i_m(f^i_\ell \restriction m). \qquad (\star)$$

(This can be done because, by (B), π^q_m was not defined yet for sequences of the form $f^i_\ell \restriction m$.)

— $F^i = F^q \cup \{f^i_\ell;\ \ell < k\}$.

Now we clearly have $q^i \in \mathbb{P}$ and $q^i \leq q$. So we are left with checking $q^i \leq p_i$. The inclusion relations are all satisfied. Hence it suffices to see that for all $\ell < k$ and $m \in (d^i)^{-1}(\{1\}) \setminus d^{-1}(\{1\})$, we have

$$f^i_\ell(m) = \pi^i_m(f^i_\ell \restriction m). \qquad (+)$$

In case $(b)_\ell$ holds this is true by $(\star)$. In case $(a)_\ell$ holds we have $f^i_\ell \restriction (m+1) = g_\ell \restriction (m+1)$ for all such m. As $q \leq p^*$ we have $\pi^i_m(g_\ell \restriction m) = \pi^q_m(g_\ell \restriction m) = g_\ell(m)$ for such m, and $(+)$ holds again. This completes the proof of the Main Claim. $\square$

§ 2. Proof of Theorem B

2.1. Theorem. $\mathfrak{se} \leq \mathfrak{e}$.

PROOF. Let $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| < \mathfrak{se}$. By Blass' result $\mathfrak{se} \leq \mathfrak{b}$ [Bl, Theorem 2], there is $g \in \omega^\omega$ such that for all $f \in \mathcal{F}$ $\forall^\infty n$ $(f(n) < g(n))$. Without loss g is strictly increasing. We let $\langle p_n;\ n \in \omega \rangle$ be a sequence of distinct primes such that $p_n >> g(n)$ and $p_n >> \prod_{\ell < n} p_\ell$. For $f \in \mathcal{F}$, let $a_f \in \omega^\omega$ be defined by

$$a_f(n) := f(n) \cdot \textstyle\prod_{\ell \leq n} p_\ell.$$

Let $G \leq \mathbb{Z}^\omega$ be the pure closure of the subgroup generated by the unit vectors e_n, $n \in \omega$, and the a_f, $f \in \mathcal{F}$. Clearly $|G| < \mathfrak{se}$. Hence there is $h : G \longrightarrow \mathbb{Z}$ a homomorphism such that $W := \{n;\ h(e_n) \neq 0\}$ is infinite.

Let us define

$$W^* := \{n \in \omega;\ \exists i > n\ (p_i | h(e_m) \text{ whenever } m \in \{n+1, ..., i-1\} \text{ but } p_i \not| h(e_n))\}.$$

We claim that W^* is an infinite subset of W. To see this, first note that trivially $W^* \subseteq W$, by the clause $p_i \not| h(e_n)$. Next, given $n_0 \in W$, find $i > n_0$ so that $p_i \not| h(e_{n_0})$. Then clearly there is $n \geq n_0$ so that $n \in W$ and $p_i \not| h(e_n)$ and for all $m \in \{n+1, ..., i-1\}$, $p_i | h(e_m)$. Thus $n \in W^*$. This shows that W^* is infinite.

We introduce a predictor $\pi = (W^*, \langle \pi_n;\ n \in W^* \rangle)$ as follows. Given $n \in W^*$ and $s \in \omega^n$ so that $\max rng(s) < g(n-1)$, if there is $f \in \mathcal{F}$ with $s \subseteq f$ and $f(n) < g(n)$ and $|h(a_f)| < p_{n-1}$, then let $\pi_n(s) = f(n)$ for some f with the above property. Otherwise $\pi_n(s)$ is arbitrary.

We claim that π predicts all $f \in \mathcal{F}$. This clearly finishes the proof. Assume this were false, i.e. there is $f \in \mathcal{F}$ which evades π. Let $n \in W^*$ be large enough, such that $\max rng(f{\restriction}n) < g(n-1)$, $f(n) < g(n)$, $|h(a_f)| < p_{n-1}$ and $\pi_n(f{\restriction}n) \neq f(n)$. Then, by the definition of π, there must be $f' \in \mathcal{F}$ with $f'{\restriction}n = f{\restriction}n$, $f'(n) < g(n)$, $|h(a_{f'})| < p_{n-1}$ and $\pi_n(f'{\restriction}n) = f'(n) \neq f(n)$. Now, for $k \in \{f, f'\}$, we let

$$a_k^0 = (a_k(0), ..., a_k(n-1), 0, ...)$$

$$a_k^1 = (0, ..., 0, a_k(n), 0, ...)$$

$$a_k^2 = (0, ..., 0, a_k(n+1), ..., a_k(i-1), 0, ...)$$

$$a_k^3 = (0, ..., 0, a_k(i), a_k(i+1), ...)$$

where i witnesses that $n \in W^*$. So we have $a_k = a_k^0 + a_k^1 + a_k^2 + a_k^3$. Thus

$$h(a_{f'} - a_f) = h(a_{f'}^0 - a_f^0) + h(a_{f'}^1 - a_f^1) + h(a_{f'}^2 - a_f^2) + h(a_{f'}^3 - a_f^3). \qquad (\star)$$

Clearly $h(a_{f'}^0 - a_f^0) = h(0) = 0$. Next, $p_i \cdot \prod_{\ell \le n} p_\ell$ divides $h(a_{f'}^3 - a_f^3)$ by definition of the a_k; it also divides $h(a_{f'}^2 - a_f^2)$ by definition of the a_k and because $p_i | h(e_m)$ for $m \in \{n+1, ..., i-1\}$ as i witnesses $n \in W^*$. Thus $(\star)$ yields the equation

$$h(a_{f'} - a_f) = h(a_{f'}^1 - a_f^1) \quad \text{in} \quad \mathbb{Z}/(p_i \cdot \prod_{\ell \le n} p_\ell)\mathbb{Z}. \qquad (\star\star)$$

The right–hand side in $(\star\star)$ must be non–zero, because $p_i \not| h(e_n)$ (as i witnesses $n \in W^*$) and $p_i \not| (a_{f'}(n) - a_f(n)) = \prod_{\ell \le n} p_\ell \cdot (f'(n) - f(n))$ (as $f'(n), f(n) < g(n) << p_n << p_i$). However, it certainly is divisible by $\prod_{\ell \le n} p_n$, whereas the left–hand side in $(\star\star)$ is not unless it is zero (as $|h(a_f)|, |h(a_{f'})| < p_{n-1} << p_n$). This shows that the equation $(\star\star)$ cannot hold, the final contradiction. □

Note that this result improves [Br, Theorem 3.2].

2.2. Lemma. $\mathfrak{e}_\ell \ge \min\{\mathfrak{e}, \mathfrak{b}\}$.

Proof. Let $\mathcal{F} \subseteq \mathbb{Z}^\omega$, $|\mathcal{F}| < \min\{\mathfrak{e}, \mathfrak{b}\}$. Find $g \in \omega^\omega$ strictly increasing so that for all $f \in \mathcal{F}$, we have $|f| <^* g$, where $|f|(n) = |f(n)|$. We partition ω into intervals I_n, $n \in \omega$, so that $\max(I_n) + 1 = \min(I_{n+1})$, as follows. $I_0 = \{0\}$. Assume I_n is defined; choose I_{n+1} so that $|I_{n+1}| > [2 \cdot g(\max(I_n))]^{\sum_{i \le n} |I_i|}$. For $f \in \mathcal{F}$, define $\bar f$ by $\bar f(n) := f \restriction I_n$, and let $\bar{\mathcal{F}} = \{\bar f;\ f \in \mathcal{F}\}$. Use $|\bar{\mathcal{F}}| < \mathfrak{e}$ to get a single predictor $\bar\pi = (\bar D, \langle \bar\pi_n;\ n \in \bar D\rangle)$ predicting all the $\bar f \in \bar{\mathcal{F}}$. For $n \in \bar D$, let $\Gamma_n := rng(\bar\pi_n \restriction (-g(\max(I_{n-1})), g(\max(I_{n-1})))^{\bigcup_{i<n} I_i}) \cap \mathbb{Z}^{I_n}$. So $|\Gamma_n| < |I_n|$; hence for some $i_n \in I_n$, the vector $\bar x_{i_n} = \langle t(i_n);\ t \in \Gamma_n\rangle$ depends on the vectors $\{\bar x_i = \langle t(i);\ t \in \Gamma_n\rangle;\ \min(I_n) \le i < i_n\}$. Say $\bar x_{i_n} = \sum_{\min(I_n) \le i < i_n} q_i^n \bar x_i$, where $q_i^n \in \mathbb{Q}$. In particular, for fixed $t \in \Gamma_n$, we have $t(i_n) = \sum_{\min(I_n) \le i < i_n} q_i^n t(i)$. This allows us to define a linear predictor $\pi = (D, \langle \pi_n;\ n \in D\rangle)$ with $D = \{i_n;\ n \in \omega\}$ and $\pi_{i_n}(s) = \sum_{\min(I_n) \le i < i_n} q_i^n s(i)$. Note that if $n \in \omega$ is such that $\max rng(|f| \restriction \cup_{i<n} I_i) < g(\max(I_{n-1}))$ and $\bar\pi_n(\bar f \restriction n) = \bar f(n)$, then $\pi_{i_n}(f \restriction i_n) = f(i_n)$. Hence, as $\bar\pi$ predicts all $\bar f \in \bar{\mathcal{F}}$, π predicts all $f \in \mathcal{F}$. □

2.3. Clearly, Theorem B follows from 2.1., 2.2. and Blass' results $\mathfrak{e}_\ell \le \mathfrak{se} \le \mathfrak{b}$ [Bl, Theorem 2, Corollary 8 and Theorem 10]. □

2.4. Definition. Given $D \subseteq \omega$ infinite and $\bar{a} = \langle a_n \in [\omega]^{\leq n};\ n \in D\rangle$, the *slalom* $S_D^{\bar{a}}$ is the set of all functions f in ω^ω with $f(n) \in a_n$ for almost all $n \in D$.

Using this notion we can give a combinatorial characterization of the cardinal $\mathfrak{e}_\ell = \mathfrak{se}$.

2.5. Lemma. $\min\{\mathfrak{e}, \mathfrak{b}\} = \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \omega^\omega$ *and for all* $D \subseteq \omega$ *and* $\bar{a} = \langle a_n \in [\omega]^{\leq n};\ n \in D\rangle$ *there is* $f \in \mathcal{F} \setminus S_D^{\bar{a}}\}$.

Note. It is immediate that the cardinal on the right–hand side is bigger than or equal to the additivity of Lebesgue measure $\mathsf{add}(\mathcal{L})$, by Bartoszyński's characterization of that cardinal ([Ba 1], [Ba 2]). We also note that the original proof of $\mathsf{add}(\mathcal{L}) \leq \mathsf{add}(\mathcal{M})$ [Ba 1] shows in fact that this cardinal is $\leq \mathsf{add}(\mathcal{M})$ as well. This gives an alternative proof of Blass' $\min\{\mathfrak{e}, \mathfrak{b}\} \leq \mathsf{add}(\mathcal{M})$ [Bl, Theorem 13].

Proof. " $\geq$ ". By Theorem B, it suffices to show that $\mathfrak{e}_\ell$ is bigger than or equal to the cardinal on the right–hand side. However, this is exactly like Blass' original proof of $\mathsf{add}(\mathcal{L}) \leq \mathfrak{e}_\ell$ [Bl, Theorem 12], and we therefore leave details to the reader.

" $\leq$ ". This argument is very similar to the one in Lemma 2.2. So we just stress the differences.

Take $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| < \min\{\mathfrak{e}, \mathfrak{b}\}$. Find g strictly increasing and eventually dominating all functions from $\mathcal{F}$. As before, partition ω into intervals I_n, $n \in \omega$; this time we require that $i_{n+1} := g(\max(I_n))^{\sum_{i \leq n} |I_i|} \in I_{n+1}$. $\bar{f}$, $\bar{\mathcal{F}}$ and $\bar{\pi}$, $\bar{D}$ are defined as before.

We put $D := \{i_n;\ n \in \bar{D}\}$ and $a_{i_n} = \{\bar{\pi}_n(s)(i_n);\ s \in g(\max(I_{n-1}))^{\bigcup_{i<n} I_i}\} \in [\omega]^{\leq i_n}$, and leave it to the reader to check that $\mathcal{F} \subseteq S_D^{\bar{a}}$. $\square$

2.6. The notion of linear predicting can be generalized as follows (see [Br, section 4] for details). Let $\mathbb{K}$ be an at most countable field. A $\mathbb{K}$–valued predictor $\pi = (D_\pi, \langle \pi_n;\ n \in D_\pi\rangle)$ is *linear* iff all $\pi_n : \mathbb{K}^n \to \mathbb{K}$ are linear. $\mathfrak{e}_\mathbb{K}$ is the corresponding *linear evasion number*. We easily see $\mathfrak{e}_\mathbb{Q} = \mathfrak{e}_\ell$. Rewriting the proof of 2.2. in this more general context gives $\mathfrak{e}_\mathbb{K} \geq \min\{\mathfrak{e}, \mathfrak{b}\}$ for arbitrary $\mathbb{K}$ and $\mathfrak{e}_\mathbb{K} \geq \mathfrak{e}$ in case $\mathbb{K}$ is finite. As $\mathfrak{e}_\mathbb{K} \leq \mathfrak{b}$ for infinite $\mathbb{K}$ [Br, 5.4.], we get $\mathfrak{e}_\mathbb{K} = \min\{\mathfrak{e}, \mathfrak{b}\}$ for such fields — in particular all $\mathfrak{e}_\mathbb{K}$ for $\mathbb{K}$ a countable field are equal. We do not know whether this is true for finite $\mathbb{K}$. Note that $\mathfrak{e}_\mathbb{K} > \mathfrak{e}, \mathfrak{b}$ is consistent for such fields [Br, section 4].

§ 3. Some results on evasion ideals

3.1. Definition. We say a predictor $\pi = (D, \langle \pi_n;\ n \in D \rangle)$ *predicts* a function $f \in \omega^\omega$ *everywhere* if $\pi_n(f{\restriction}n) = f(n)$ holds for all $n \in D$. We put $\mathfrak{e}(\omega) := \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \omega^\omega \wedge$ for all countable families of predictors Π there is $f \in \mathcal{F}$ evading all $\pi \in \Pi\}$, the *uniformity of the evasion ideal* $\mathcal{J}$. — As usual, $\mathsf{cov}(\mathcal{M})$ denotes the *covering number* of the ideal $\mathcal{M}$, i.e. the smallest size of a family $\mathcal{F} \subseteq \mathcal{M}$ so that $\bigcup \mathcal{F} = \omega^\omega$.

3.2. Observation. *Assume $\langle D^n;\ n \in \omega \rangle$ is a decreasing sequence of infinite subsets of ω, and $\langle \pi^n = (D^n, \langle \pi^n_k;\ k \in D^n \rangle);\ n \in \omega \rangle$ is a sequence of predictors. Then there are a set $D \subseteq \omega$, almost included in all D^n, and a predictor $\pi = (D, \langle \pi_k;\ k \in D \rangle)$ predicting all functions which are predicted by one of the π^n.*

Proof. We can assume that each function which is predicted by some π^n is predicted everywhere by some π^m — otherwise go over to sequences $\langle E^n;\ n \in \omega \rangle$ and $\langle \bar\pi^n = (E^n, \langle \bar\pi^n_k;\ k \in E^n \rangle);\ n \in \omega \rangle$ such that (i) for all $n \in \omega$ there is $m \in \omega$ so that $E^m \subseteq D^n$ and $\bar\pi^m_k = \pi^n_k$ for $k \in E^m$ and (ii) for all $n, m \in \omega$ there is $\ell \in \omega$ so that $E^\ell \subseteq E^n \setminus m$ and $\bar\pi^\ell_k = \bar\pi^n_k$ for $k \in E^\ell$.

Choose $d^n \in D^n$ minimal with $d^n > d^{n-1}$, and put $D = \{d^n;\ n \in \omega\}$. Fix $n \in \omega$ and $s \in \omega^{d^n}$. To define $\pi_{d^n}(s)$, choose, if possible, $i \leq n$ minimal so that for all $k \in D^i \cap d^n$, we have $\pi^i_k(s{\restriction}k) = s(k)$, and let $\pi_{d^n}(s) = \pi^i_{d^n}(s)$. If this is impossible, let $\pi_{d^n}(s)$ be arbitrary.

To see that this works, take $f \in \omega^\omega$ and $i \in \omega$ minimal so that π^i predicts f everywhere. As the set of functions predicted everywhere by a single predictor is closed, there are $n \geq i$ and $s \in \omega^{d^n}$ so that $s \subseteq f$ and s is not predicted everywhere by any of the π^j where $j < i$. Then $\pi_{d^m}(f{\restriction}d^m) = \pi^i_{d^m}(f{\restriction}d^m)$ for all $m \geq n$, as required. $\square$

3.3. Theorem. $\mathfrak{e} \geq \min\{\mathfrak{e}(\omega), \mathsf{cov}(\mathcal{M})\}$*; thus either* $\mathfrak{e} < \mathsf{cov}(\mathcal{M})$ *or* $\mathfrak{e}(\omega) \leq \mathsf{cov}(\mathcal{M})$ *imply* $\mathfrak{e} = \mathfrak{e}(\omega)$.

Remark. The statement is very similar to a recent result of Kamburelis who proved $\mathfrak{s} \geq \min\{\mathfrak{s}(\omega), \mathsf{cov}(\mathcal{M})\}$, where $\mathfrak{s}$ is the splitting number and $\mathfrak{s}(\omega)$ the $\aleph_0$–splitting number.

Proof. The second statement easily follows from the first. To prove the latter, let $\mathcal{F} \subseteq \omega^\omega$, $|\mathcal{F}| < \min\{\mathfrak{e}(\omega), \mathsf{cov}(\mathcal{M})\}$. We shall show $|\mathcal{F}| < \mathfrak{e}$. For $\sigma \in \omega^{<\omega} \setminus \{\langle\rangle\}$, we construct recursively sets $D^\sigma \subseteq \omega$ and predictors $\pi^\sigma = (D^\sigma, \langle \pi^\sigma_n;\ n \in D^\sigma \rangle)$ such that:

(i) $D^{\sigma\restriction i} \supseteq D^\sigma$ for $i \in |\sigma|$;
(ii) for all $f \in \mathcal{F}$ and all $\sigma \in \omega^{<\omega}$ there is $i \in \omega$ so that f is predicted by $\pi^{\sigma\hat{}\langle i\rangle}$.

First construct $\pi^{\langle i\rangle} = (D^{\langle i\rangle}, \langle \pi_n^{\langle i\rangle};\ n \in D^{\langle i\rangle}\rangle)$ satisfying (ii) by applying $|\mathcal{F}| < \mathfrak{e}(\omega)$.

To do the recursion, assume $\pi^\sigma = (D^\sigma, \langle \pi_n^\sigma;\ n \in D^\sigma\rangle)$ is constructed for some fixed $\sigma \in \omega^{<\omega}$. Given $f \in \omega^\omega$, define f^σ by:

$$f^\sigma(i) := f(k_i^\sigma),$$

where $\{k_i^\sigma;\ i \in \omega\}$ is the increasing enumeration of the set D^σ. Let $\mathcal{F}^\sigma = \{f^\sigma;\ f \in \mathcal{F}\}$. Again we get ω many predictors $\bar\pi^{\sigma\hat{}\langle i\rangle} = (\bar D^{\sigma\hat{}\langle i\rangle}, \langle \bar\pi_n^{\sigma\hat{}\langle i\rangle};\ n \in \bar D^{\sigma\hat{}\langle i\rangle}\rangle)$, $i \in \omega$, so that every $f^\sigma \in \mathcal{F}^\sigma$ is predicted by some $\bar\pi^{\sigma\hat{}\langle i\rangle}$. Let $D^{\sigma\hat{}\langle i\rangle} = \{k_j^\sigma;\ j \in \bar D^{\sigma\hat{}\langle i\rangle}\}$. Fix $j \in \bar D^{\sigma\hat{}\langle i\rangle}$ and $s \in \omega^{k_j^\sigma}$. Let $\bar s \in \omega^j$ be defined by $\bar s(\ell) = s(k_\ell^\sigma)$. Put $\pi_{k_j^\sigma}^{\sigma\hat{}\langle i\rangle}(s) := \bar\pi_j^{\sigma\hat{}\langle i\rangle}(\bar s)$. Now it is easy to see that $\pi^{\sigma\hat{}\langle i\rangle}$ predicts f whenever $\bar\pi^{\sigma\hat{}\langle i\rangle}$ predicts f^σ. Thus (i) and (ii) hold. This completes the recursive construction.

Given $f \in \omega^\omega$, let $T_f = \{\sigma \in \omega^{<\omega};\ \text{ for all } i \le |\sigma|\ (\pi^{\sigma\restriction i} \text{ does not predict } f \text{ everywhere})\}$. By the above construction, the sets $[T_f]$ are nowhere dense for $f \in \mathcal{F}$. As $|\mathcal{F}| < \mathsf{cov}(\mathcal{M})$, there must be $g \in \omega^\omega \setminus \bigcup_{f\in\mathcal{F}}[T_f]$. Now use the Observation (3.2.) to construct a new predictor from the $\langle \pi^{g\restriction n};\ n \in \omega\rangle$ which will predict all $f \in \mathcal{F}$. $\square$

3.4. It is unclear whether $\mathfrak{e} = \mathfrak{e}(\omega)$ can be proved in ZFC. In view of Theorem 3.3 it seems reasonable to ask first

Question. *Is $\mathfrak{e} > \mathsf{cov}(\mathcal{M})$ consistent?*

Of course, we may also consider the cardinal $\mathfrak{e}_\ell(\omega)$, the smallest size of a family $\mathcal{F}$ of functions from ω to ω such that no countable family of linear predictors predicts all $f \in \mathcal{F}$. However, it is now easy to see that $\mathfrak{e}_\ell(\omega) = \mathfrak{e}_\ell$. This is so because $\mathfrak{e}_\ell(\omega) \le \min\{\mathfrak{e}(\omega), \mathfrak{b}\} \le \min\{\mathfrak{e}, \mathfrak{b}\} \le \mathfrak{e}_\ell$. To see the first inequality, note that the argument for $\mathfrak{e}_\ell \le \mathfrak{b}$ gives $\mathfrak{e}_\ell(\omega) \le \mathfrak{b}$ as well (see [Br, section 5.4] for a stronger result); for the second inequality, $\min\{\mathfrak{e}(\omega), \mathfrak{b}\} \le \mathsf{cov}(\mathcal{M})$ by rewriting Blass' $\min\{\mathfrak{e}, \mathfrak{b}\} \le \mathsf{cov}(\mathcal{M})$ [Bl, Theorem 13] and thus $\min\{\mathfrak{e}(\omega), \mathfrak{b}\} = \min\{\mathfrak{e}(\omega), \mathsf{cov}(\mathcal{M}), \mathfrak{b}\} \le \min\{\mathfrak{e}, \mathfrak{b}\}$ by Theorem 3.3; the third inequality is Lemma 2.2.

3.5. Duality. Most of the cardinal invariants of the continuum come in pairs and results about them usually can be dualized (see [Br, section 4.5] for details). In our situation, the dual cardinals are: the *dominating number* $\mathfrak{d}$ (dual to $\mathfrak{b}$), the smallest size of a family

$\mathcal{F} \subseteq \omega^\omega$ such that given any $g \in \omega^\omega$ there is $f \in \mathcal{F}$ with $g \leq^* f$; the *(linear) covering number* $\mathsf{cov}(\mathcal{J})$ ($\mathsf{cov}(\mathcal{J}_\ell)$) of the ideal $\mathcal{J}$ ($\mathcal{J}_\ell$) (the first being dual to both $\mathfrak{e}$ and $\mathfrak{e}(\omega)$, the second being dual to $\mathfrak{e}_\ell$), the least cardinality of a family of (linear) predictors Π such that every function $f \in \omega^\omega$ ($\mathbb{Z}^\omega$) is predicted by some $\pi \in \Pi$. Then we get:

THEOREM. *(a) It is consistent with ZFC to assume* $\mathfrak{d} > \mathsf{cov}(\mathcal{J})$.

(b) $\mathsf{cov}(\mathcal{J}_\ell) = \max\{\ \mathsf{cov}(\mathcal{J}), \mathfrak{d}\} = \min\{|\mathcal{S}|;\ \mathcal{S}$ *consists of slaloms* $S^{\bar{a}}_D$ *where* $\bar{a} = \langle a_n \in [\omega]^{\leq n};\ n \in D\rangle$ *and* $D \subseteq \omega$ *is infinite and* $\forall f \in \omega^\omega\ \exists S^{\bar{a}}_D \in \mathcal{S}\ \forall^\infty n \in D\ (f(n) \in a_n)\}$.

PROOF. These dualizations are standard, and we therefore refrain from giving detailed proofs. The model for (a) is gotten by iterating the p.o. $\mathbb{P}$ from § 1 ω_1 times with finite support over a model for $MA + \neg CH$. (b) is the dual version of Theorem B and Lemma 2.5. $\square$

We close our work with a diagram showing the relations between the cardinal invariants considered in this work (in particular, the Specker–Eda number $\mathfrak{se}$ and the evasion number $\mathfrak{e}$) and some other cardinal invariants of the continuum (in particular, those of Cichoń's diagram). We refer the reader to [Bl], [Br] or [Fr] for the cardinals not defined here. A similar diagram was drawn in [Br, section 4].

$\mathsf{cov}(\mathcal{L})$		$\mathsf{unif}(\mathcal{M})$	$\mathsf{cof}(\mathcal{M})$	$\mathsf{cov}(\mathcal{J}_\ell)$	$\mathsf{cof}(\mathcal{L})$
			$\mathfrak{d}$	$\mathsf{cov}(\mathcal{J})$	
	$\mathfrak{e}$	$\mathfrak{b}$			
$\mathsf{add}(\mathcal{L})$	$\mathfrak{se}$	$\mathsf{add}(\mathcal{M})$	$\mathsf{cov}(\mathcal{M})$		$\mathsf{unif}(\mathcal{L})$
	$\mathfrak{p}$				

In the diagram, cardinals increase as one moves up and to the right. To enhance readability, we omitted the relations $\mathfrak{e} \leq \mathsf{unif}(\mathcal{L})$, and its dual $\mathsf{cov}(\mathcal{L}) \leq \mathsf{cov}(\mathcal{J})$. The dotted lines give the relations $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\ \}$, $\mathfrak{se} = \min\{\mathfrak{e}, \mathfrak{b}\}$, and their dual versions.

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