WAS SIERPIŃSKI RIGHT IV?
SH546

Saharon Shelah

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

Rutgers University
Department of Mathematics
New Brunswick, NJ USA

Abstract. We prove for any $\mu = \mu^{<\mu} < \theta < \lambda, \lambda$ large enough (just strongly inaccessible Mahlo) the consistency of $2^\mu = \lambda \rightarrow [\theta]^2_3$ and even $2^\mu = \lambda \rightarrow [\theta]^2_{\sigma,2}$ for $\sigma < \mu$. The new point is that possibly $\theta > \mu^+$.

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§0 Introduction

An important theme is modern set theory is to prove the consistency of "small cardinals" having "a large cardinal property". Probably the dominant interpretation concerns large ideals (with reflection properties or connected to generic embedding). But here we deal with another important interpretation: partition properties. We continue here [Sh 276, §2], [Sh 288], [Sh 289], [Sh 473], [Sh 481] but generally do not rely on them except in the end (of the proof of 1.19) when it becomes like the proof of [Sh 276, §2]. This work is continued in Rabus and Shelah [RbSh 585].

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Preliminaries

0.1. *Let* $<^*_\chi$ be a well ordering of $\mathscr{H}(\chi) = \{x :$ the transitive closure of x has cardinality $< \chi\}$ agreeing with the usual well ordering of the ordinals,
P (and Q, R) will denote forcing notions, i.e. quasi orders with a minimal element $\emptyset = \emptyset_P$.

A forcing notion P is λ-closed or λ-complete if every increasing sequence of members of P, of length less than λ, has an upper bound.

0.2. If $P \in \mathscr{H}(\chi)$, then for a sequence $\bar p = \langle p_i : i < \gamma \rangle$ of members of P (not necessarily increasing) let $\alpha = \alpha_{\bar p} =: \sup\Big\{j : \{p_i : i < j\}$ has an upper bound in $P\Big\}$ and define the canonical upper bound of $\bar p$, denoted by $\& \bar p$ as follows:

- (a) the least upper bound of $\{p_i : i < \alpha_{\bar p}\}$ in P if there exists such an element
- (b) the $<^*_\chi$-first upper bound of $\bar p$ if (a) can't be applied but there is an upper bound of $\{p_i : i < \alpha_{\bar p}\}$,
- (c) p_0 if (a), (b) fail, $\gamma > 0$,
- (d) $\emptyset_P$ if $\gamma = 0$.

Let $p_0 \ \& \ p_1$ be the canonical upper bound of $\langle p_\ell : \ell < 2 \rangle$.
Take $[a]^\kappa = \{b \subseteq a : |b| = \kappa\}$ and $[a]^{<\kappa} = \bigcup\limits_{\theta < \kappa} [a]^\theta$.

0.3. For sets of ordinals, A and B, define $\mathrm{OP}_{B,A}$ as the maximal order preserving 1-to-1 function between initial segments of A and B, i.e., it is the function with domain $\{\alpha \in A : \mathrm{otp}(\alpha \cap A) < \mathrm{otp}(B)\}$ and $OP_{B,A}(\alpha) = \beta$ if and only if $\alpha \in A, \beta \in B$ and $\mathrm{otp}(\alpha \cap A) = \ \mathrm{otp}(\beta \cap B)$.

If A, B are sets of ordinals, let $A \triangleleft B$ mean A is a proper initial segment of B. If η, ν are sequences let $\eta \triangleleft \nu$ mean ν is an initial segment of ν. If we write $\trianglelefteq$ (rather than $\triangleleft$) we allow equality.

Let $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.

Definition 0.4. $\lambda \to [\alpha]^n_\theta$ holds provided that whenever F is a function from $[\lambda]^n$ to θ, <u>then</u> there is $A \subseteq \lambda$ of order type α and $t < \theta$ such that $[w \in [A]^n \Rightarrow F(w) \neq t]$.

Definition 0.5. $\lambda \to [\alpha]^n_{\kappa,\theta}$ if for every function F from $[\lambda]^n$ to κ there is $A \subseteq \lambda$ of order type α such that $\{F(w) : w \in [A]^n\}$ has power $\leq \theta$. If we write "$< \theta$" instead of θ we mean that the set above has cardinality $< \theta$.

Definition 0.6. A forcing notion P satisfies the Knaster condition (has property K) if for any $\{p_i : i < \omega_1\} \subseteq P$ there is an uncountable $A \subseteq \omega_1$ such that the conditions p_i and p_j are compatible whenever $i, j \in A$.

What problems do [Sh 276], [Sh 288], [Sh 289], [Sh 473] and [Sh 481] raise? The most important "minimal open", as suggested in [Sh 481] were:

A Question. (1) Can we get e.g. CON($2^{\aleph_0} \to [\aleph_2]^2_3$) (generally raise μ^+ in part (3) below to higher cardinals). We solve it here.
(2) Can we get CON($\aleph_\omega > 2^{\aleph_0} \to [\aleph_1]^2_3$) (the exact $\aleph_n$ seems to me less exciting).
(3) Can we get e.g. CON($2^\mu > \lambda \to [\mu^+]^2_3$)?

Also

B Question. (1) Can we get the continuity on a non-meagre set for functions $f : {}^\kappa 2 \to {}^\kappa 2$? (Solved in [Sh 473].)
(2) What can we say on continuity of 2-place functions (dealt with in Rabus Shelah [RbSh 585])?
(3) What about n-place functions? (continuing in this respect [Sh 288] probably just combine [RbSh 585] with)

C Question. (1) [Sh 481] for $\mu > \aleph_0$.
(2) Can we get e.g. CON($2^{\aleph_0} \geq \aleph_2$, and if P is $2^{\aleph_0}$-c.c., Q is $\aleph_2$-c.c., then $P \times Q$ is $2^{\aleph_0}$-c.c.).
(3) Can we get e.g. CON($2^{\aleph_0} > \lambda > \aleph_0$, and if P is λ-c.c., Q is $\aleph_1$-c.c. then $P \times Q$ is λ-c.c.); more general is CON($\mu = \mu^{<\mu} > \aleph_0 +$ if P is 2^μ-c.c. Q is μ^+-c.c. then $P \times Q$ is 2^μ-c.c).
So a large number are solved. But, of course, solving two of those problems does not necessarily solve their natural combinations.

(0C) Forcing Axiom.
Though our aim is consistency of partition theorems, by the way we generalize a forcing axiom of [Sh 80], and in Fall 2012 find it helpful to state it explicitly (for a new version of [BKSh:927]).

0.7 Theorem. *Suppose $\mu = \mu^{<\mu} < \chi$ and ε is a limit ordinal $< \mu$.*
1) For some forcing notion P of cardinality χ, μ-complete neither collapsing cardinalities nor changing cofinalities we have:
$\Vdash_P$ *"$2^\mu = \chi$ and for our fixed ε the Axiom $Ax_{\mu,\varepsilon}$ where*
$Ax_{\mu,\varepsilon}$ if Q is a μ-complete forcing notion of cardinality 2^μ satisfying $*^\varepsilon_\mu$ defined in 1.1 below and $\mathscr{I}_\alpha \subseteq Q$ dense for $\alpha < \alpha^* < \chi$ <u>then</u> some directed $G \subseteq Q$ is not disjoint to any $\mathscr{I}_\alpha$.
2) We can replace "μ-complete" by "$(< \mu)$-strategically complete" (in the demand

on P and, in the axiom, on Q.
3) Assume $\langle(\mu_\alpha, \varepsilon_\alpha, \lambda_\alpha) : \alpha < \alpha(*)\rangle, \alpha(*)$ an ordinal, $\mu_\alpha = (\mu_\alpha)^{<\mu_\alpha} < \lambda_\alpha = (\lambda_\alpha)^{<\lambda_\alpha}, \varepsilon_\alpha$ a limit ordinal $< \mu_\alpha$ and $[\varepsilon \le \alpha(*)$ a limit ordinal $\Rightarrow 2^{\Sigma\{\mu_\alpha : \alpha < \varepsilon\}} = (\sum\limits_{\alpha<\varepsilon} \mu_\alpha)^+]$ and $[\alpha < \beta < \alpha(*) \Rightarrow \lambda_\alpha < \mu_\alpha]$. Then there is a forcing notion $\mathbb{P}$ such that:

(a) *$\mathbb{P}$ is $(< \mu_0)$-complete*
(b) *forcing with $\mathbb{P}$ collapse, no cardinal, change no cofinality*
(c) *$|\mathbb{P}| \le \prod\limits_{\alpha<\alpha(*)} \lambda_\alpha$; moreover*
   - *if $\alpha(*) = \beta + 1, \mathbb{P}$ satisfy the μ^+-c.c.*
   - *if $\alpha(*)$ is an inaccessible cardinal and $\alpha < \alpha(*) \Rightarrow \lambda_\alpha < \alpha(*)$ then P has cardinality $\alpha(*)$*
   - *if in addition $\alpha(*)$ is Mahlo $\mathbb{P}$ satisfies the $\alpha(*)$-c.c.*

(d) *in $\mathbf{V}^{\mathbb{P}}$, for every $\alpha < \alpha(*)$, the axiom $Ax_{\mu_\alpha, \varepsilon_\alpha}$ from part (1) holds.*

4) Similar to part (2) when $\alpha(*)$ is the class of ordinals, so $\langle(\mu_\alpha, \varepsilon_\alpha, \lambda_\alpha) : \alpha < \alpha(*)\rangle$ is a class.

Proof. 1) Obvious by the claims up to Theorem 1.19 and included in the proof of it.
2) By induction on $\gamma \le \alpha(*)$ we choose P_γ and if γ is a successor ordinal also $\underset{\sim}{Q}_{\gamma-1}$ such that $\langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \le \gamma, \beta < \gamma\rangle$ is an Easton support iteration and $\Vdash_{P_\beta}$ "$\underset{\sim}{Q}_\beta$ is a forcing notion of cardinality λ_β satisfying the $*^\varepsilon_{\mu_\beta}$ and forcing $\mathrm{Ax}^\varepsilon_{\mu_\alpha, \varepsilon_\alpha}$". The induction step is by part (1) and the relevant results on the iteration are well known, that is by induction on $\beta(*) \le \alpha(*)$ we prove that for every $\gamma(*) \le \beta(*)$ the iteration from $\gamma(*)$ to $\beta(*)$, i.e. $\langle P_{\gamma(*)+\alpha}, \underset{\sim}{Q}_{\gamma(*)+\beta} : \alpha \le \beta(*)-\gamma(*), \beta < \beta(*)-\gamma(*)\rangle$ satisfifes the conclusion for in the universe $\mathbf{V}^{P_{\gamma(*)}}$.
3) Similarly.

§1

We return here to consistency of statements of the form $\chi \rightarrow [\theta]^2_{\sigma,2}$ (i.e. for every $c : [\chi]^2 \rightarrow \sigma$ there is $A \in [\chi]^\theta$ such that on $[A], c$ has at most two values), (when $2^\mu \geq \chi > \theta^{<\mu} > \mu$, of course). In [Sh 276, §2] this was done for $\mu = \aleph_0, \chi = 2^\mu, \theta = \aleph_1, 2 < \sigma < \omega$ and χ quite large (in the original universe χ is an Erdös cardinal). Originally, [Sh 276, §2] was written for any $\mu = \mu^{<\mu}$ (χ measurable in the original universe) but because of the referee urging it is written up there for $\mu = \aleph_0$ only; though with an eye on the more general result which is only stated. In [Sh 288] the main objective is to replace colouring of pairs by colouring of n-tuples (and even $(< \omega)$-tuples) but we also say somewhat more on the $\mu > \aleph_0$ case (in [Sh 288, 1.4]) and using only k^2_2-Mahlo (for a specific natural number k^2_2)(an improvement for $\mu = \aleph_0$ too), explaining that it is like [Sh 289]. A side benefit of the present paper is giving a full self-contained proof of this theorem even for 1-Mahlo. The main point of this work is to increase θ, and this time write it for $\mu = \mu^{<\mu} > \aleph_0$, too.

The case $\theta = \mu^+$ is easier as it enables us to separate the forcing producing the sets admitting few colours: each appear for some $\delta < \chi, cf(\delta) = \mu^+$, is connected to a closed subset a_δ of δ unbounded in δ of order type μ^+, so that below $\alpha < \delta$ in P_α we get little information on the colouring on the relevant set. Here there is less separation, as names of such colouring can have long common initial segments, but they behave like a tree and in each node we divide the set to μ sets, each admitting only 2 colours.
As we would like to prove the theorem also for $\mu > \aleph_0$, we repeat material on μ^+-c.c., essentially from [Sh 80], [ShSt 154a], [Sh 288].

1.1 Definition. : 1) Let D be a normal filter on μ^+ to which $\{\delta < \mu^+ : \text{cf}(\delta = \mu\}$ belongs. A forcing notion Q satisfies $*^\epsilon_D$ where ϵ is a limit ordinal $< \mu$, if player I has a winning strategy in the following game $*^\epsilon_D[Q]$ defined as follows:
<u>Playing</u>: the play finishes after ϵ moves.

In the ζ-th move:

Player I — if $\zeta \neq 0$ he chooses $\langle q^\zeta_i : i < \mu^+\rangle$ such that $q^\zeta_i \in Q$
and $(\forall \xi < \zeta)(\forall^D i < \mu^+)p^\xi_i \leq q^\zeta_i$ and he chooses a function $f_\zeta : \mu^+ \rightarrow \mu^+$ such that for a club of $i < \mu^+, f_\zeta(i) < i$;
if $\zeta = 0$ let $q^\zeta_i = \emptyset_Q$, $f_\zeta =$ is identically zero.

Player II — he chooses $\langle p^\zeta_i : i < \mu^+\rangle$ such that $(\forall^D i)q^\zeta_i \leq p^\zeta_i$ and $p^\zeta_i \in Q$.

<u>The Outcome</u>: Player I wins provided that for some $E \in D$: if $\mu < i < j < \mu^+, i, j \in E$, $cf(i) = cf(j) = \mu$ and $\bigwedge\limits_{\xi<\epsilon} f_\xi(i) = f_\xi(j)$ <u>then</u> the set $\{p^\zeta_i : \zeta < \epsilon\} \cup \{p^\zeta_j : \zeta < \epsilon\}$ has an upper bound in Q.
1A) If D is $\{A \subseteq \mu^+ :$ for some club E of μ^+ we have $i \in E \;\; \& \;\; \text{cf}(i) = \mu \Rightarrow i \in A\}$ we may write μ instead of D (in $*^\varepsilon_D$ and in the related notions defined below and above).
2) A strategy for a player is a sequence $\bar F = \langle F_\zeta : \zeta < \epsilon\rangle, F_\zeta$ telling him what to do in the ζ-th move depending only on the previous moves of the other player. <u>But</u> here a play according to the strategy $\bar F$ will mean the player chooses in the

ζ-th move for each $i < \mu^+$ an element of Q which is possibly strictly above (in $\leq_Q$'s sense) of what F_ζ dictates and a function f_ζ such that on some $E \in D$, the equivalence relation $f_\zeta(\alpha) = f_\zeta(\beta)$ induce on E refine the one which the strategy induces (this change does not change the truth value of "player X has a winning strategy"). This applies to the game $\otimes^\varepsilon_Q$ in part (5) below.

3) We define $**^\varepsilon_\mu$ similarly but for ζ limit q^ζ_i is not chosen (so player II has to satisfy for limit ζ just $\forall \xi < \zeta \Rightarrow (\forall^D i)(p^\xi_i \leq p^\zeta_i)$).

4) We may allow the strategy to be non-deterministic, e.g. choose not f_ζ just f_ζ / D_{μ^+}.

5) We say a forcing notion Q is ε-strategically complete if for the following game, $\bigotimes^\varepsilon_Q$ player I has a winning strategy.

In the ζ-th move:

Player I - if $\zeta \neq 0$ he chooses $q_\zeta \in Q$ such that $(\forall \xi < \zeta) p_\xi \leq q_\zeta$ if $\zeta = 0$ let $q_\zeta = \emptyset_Q$.

Player II - he chooses $p_\zeta \in Q$ such that $q_\zeta \leq p_\zeta$.

<u>The Outcome</u>: In the end Player I wins provided that he always has a legal move.

6) We say Q is $(< \mu)$-strategically complete if for each $\varepsilon < \mu$ it is ε-strategically closed.

1.2 Remark. 1) In this paper, in the case $\mu = \aleph_0$ we can use the Knaster condition instead of $*^\varepsilon_\mu$.

2) We use below $*^\varepsilon_\mu$ and not $**^\varepsilon_\mu$ but $**^\varepsilon_\mu$ could serve as well.

3) We may consider omitting the strategic completeness (a weak version of it is hidden in player I winning $*^\varepsilon_D[Q]$), but no present use.

1.3 Definition. 1) Let $\bar{F}^\ell = \langle F^\ell_\zeta : \zeta < \varepsilon \rangle$ be a strategy for player I in the game $*^\varepsilon_D[Q]$ for $\ell = 1, 2$. We say $\bar{F}^1 \leq \bar{F}^2$ equivalently, $\bar{F}^2$ is above $\bar{F}^1$ if any play $\langle (\bar{q}^\zeta, f_\zeta, \bar{p}^\zeta) : \zeta < \varepsilon \rangle$ in which player I uses the strategy $\bar{F}^2$ (that is letting $(\langle q'_i : i < \mu^+ \rangle, f) = F_\zeta(\langle \bar{p}^\xi : \xi < \zeta \rangle)$ we have $i < \mu^+ \Rightarrow q'_i \leq q^\zeta_i$ and for some $E \in D, i \in E$ & $j \in E \wedge f(i) = f(j) \Rightarrow f_\zeta(i) = f_\zeta(i)$) is also a play in which player I uses the strategy $\bar{F}^1$.

2) Let $\alpha^* < \beta^* < \mu, \mathbf{St}$ be a winning strategy for player I in the game $\otimes^\beta_Q$. We say $\langle \bar{F}^\alpha : \alpha < \alpha^* \rangle$ is an increasing sequence of strategies of player I in $*^\varepsilon_D[Q]$ obeying $\mathbf{St}$ if:

- (a) $\bar{F}^\alpha$ is a winning strategy of player I in $*^\varepsilon_D[Q]$
- (b) for $\alpha < \beta < \alpha^*, \bar{F}^\beta$ is above $\bar{F}^\alpha$
- (c) if $\langle (\bar{q}^\zeta, f_\zeta, \bar{p}^\zeta) : \zeta < \varepsilon \rangle$ is a play of $*^\varepsilon_D[Q]$, Player I uses his strategy $\bar{F}^\beta$, <u>then</u> for any $i < \mu^+$, letting $F^\alpha(\langle \bar{p}^\xi : \xi < \zeta \rangle) = (\bar{q}^{\alpha,\xi}, f'_{\alpha,\zeta})$ we have:

$$Q \models \mathbf{St}(\langle q^{\alpha,\xi}_i : \xi < \zeta \rangle) \leq q^{\alpha,\zeta}_i.$$

3) Similarly to (1), (2) for the game $\otimes^\varepsilon_Q$ (instead $*^\varepsilon_D[Q]$).

1.4 Observation. 1) Assume Q is μ-complete. If $\delta < \mu$ and $\langle \bar{F}^\alpha : \alpha < \delta \rangle$ is an increasing sequence of winning strategies of player I in $*^\varepsilon_D[Q]$, <u>then</u> some winning

strategy $\bar{F}^\delta$ of player I in $*^\varepsilon_D[Q]$ is above every $\bar{F}^\alpha(\alpha < \delta)$.
2) Assume $\beta^* < \mu$ and Q is β^*-strategically complete with a winning strategy **St**. If $\beta < \beta^*$ and $\langle \bar{F}^\alpha : \alpha < \beta \rangle$ is an increasing sequence of winning strategies of player I in $*^\varepsilon_D[Q]$ obeying **St**, then for some $\bar{F}^\beta$, $\langle F^\alpha : \alpha < \beta+1 \rangle$ is an increasing sequence of winning strategies of player I in $*^\varepsilon_D[Q]$ obeying **St**.
3) Similarly with $\otimes^\varepsilon_Q$ instead of $*^\varepsilon_Q[D]$.

Proof. Straight.

1.5 Definition. Assume P, R are forcing notions, $P \subseteq R$, $P \lessdot R$.
1) We say $\restriction$ is a restriction operation for the pair (P, R) (or $(P, R, \restriction)$ is a strong restriction triple) if (P, Q are as above, of course, and) for every member $r \in R, r \restriction P \in P$ is defined such that:

(a) $r \restriction P \le r$,
(b) if $r \restriction P \le p \in P$ then r, p are compatible in R in fact have a lub
(c) $r^1 \le r^2 \Rightarrow r^1 \restriction P \le r^2 \restriction P$
(d) if $p \in P$ then $p \restriction P = p$ and $\emptyset_Q \restriction P = \emptyset_P$

(so this is a strong, explicit way to say $P \lessdot R$).
1A) We say weak restriction triple if we omit in clause (b) the "have a lub".
2) We say "$(P, R, \restriction)$ is ε-strategically complete" if

(α) $\restriction$ is a restriction operation for the pair (P, R)
(β) P is ε-strategically complete
(γ) if St_1 is a winning strategy for player I in the game, $\bigotimes^\varepsilon_p$, then in the game $\bigotimes^\varepsilon = \bigotimes^\varepsilon[P, R; St_1]$ the first player has a winning strategy St_2.

Playing: A play of $\bigotimes^\varepsilon$ is a play $\langle (p_\zeta, q_\zeta) : \zeta < \varepsilon \rangle$ of $\bigotimes^\varepsilon_R$ but

(α) $\langle (q_\zeta \restriction P, q_\zeta \restriction P) : \zeta < \varepsilon \rangle$ is a play of the game $\bigotimes^\varepsilon_P$ in which the first player uses the strategy St_1 (see 1.1(2)!).

Outcome: If condition $(\beta)_\zeta$ below fails in stage ζ for some $\zeta < \varepsilon$ then the first player loses immediately, and if not, then he wins.

$(\beta)_\zeta$ for every $\zeta < \varepsilon$, if $p \in P$ is above $q_\zeta \restriction P$ then $\{p\} \cup \{q_\xi : \xi < \zeta\}$ has an upper bound. (Read second sentence in 1.1(2)).

2A) We say $(P, R, \restriction)$ is $(< \varepsilon)$-strategically complete if it is ζ-strategically complete for every $\zeta < \varepsilon$.
3) Let "$(P, R, \restriction)$ satisfy $*^\epsilon_\mu$" mean (usually $\restriction$ will be understood from context hence omitted):

(α) $\restriction$ is a restriction operation for the pair (P, R)
(β) P satisfies $*^\epsilon_\mu$
(γ) If St_1 is a winning strategy for player I in the game $*^\epsilon_\mu[P]$ then in the following game called $*^\epsilon_\mu[P, R; St_1]$ the first player has a winning strategy St_2.

Playing: As before in $(*)^{\varepsilon}_{\mu}[R]$, but $\langle < q^{\zeta}_i \restriction P : i < \mu^+ >, < p^{\zeta}_i \restriction P : i < \mu^+ >,$ $f_\zeta : \zeta < \epsilon\rangle$ is required to be a play of $*^{\epsilon}_{\mu}[P]$ in which first player uses the strategy St_1 (see the second sentence of 1.1(2)).

We also demand that if $\{p^{\zeta}_j : j < i\} \subseteq P$, then $q^{\zeta}_i \in P$.

The outcome: Player I wins provided that:

$(*)$ for some club E of μ^+ if $i < j$ are from E, $\mathrm{cf}(i) = cf(j) = \mu$, $\bigwedge\limits_{\xi<\varepsilon} f_\xi(i) = f_\xi(j)$ and $r \in P$ is a $\leq_P$-upper bound of $\{p^{\zeta}_i \restriction P : \zeta < \epsilon\} \cup \{p^{\zeta}_j \restriction P : \zeta < \epsilon\}$, then[1] $\{r\} \cup \{p^{\zeta}_i : \zeta < \epsilon\} \cup \{p^{\zeta}_j : \zeta < \epsilon\}$ has an upper bound in R.

In this case we say that St_2 projects to St_1 or is above St_1. If we omit the demand on the outcome (so maybe St_2 is not a winning strategy of player I in $*^{\varepsilon}_{\mu}[R]$), we say St_2 weakly projects to St_1.

Note: Naturally in St_2 the functions f_ζ code more information than St_1, we may use a function g to decode the "older" part.

3A) The game $*^{\varepsilon}_{D}[P, R, \restriction]$ and "$(P, R, \restriction)$ satisfies $*^{\varepsilon}_{D}$" are defined naturally and similarly projections of strategies.

4) We say $(P, R, \restriction)$ satisfies strongly $*^{\varepsilon}_{\mu}$ if (when $\restriction$ is clear from context, it is omitted):

(α) $\restriction$ is a restriction operation for the pair (P, R)

(β) P satisfies $*^{\varepsilon}_{\mu}$

(γ) the first player has a winning strategy in the game $*^{\varepsilon}_{\mu}[P, R, \restriction]$ where

Playing: Just like a play of $*^{\varepsilon}_{\mu}[R]$, except that

$\bigoplus$ in addition, for every limit ordinal $\zeta < \varepsilon$, in the ζ-the move first the second player is allowed to choose $\langle r^{\zeta}_i : i < \mu^+\rangle$ such that: $\bigwedge\limits_{\xi<\zeta} p^{\xi}_i \restriction P \leq r^{\zeta}_i \in P$ is an upper bound of $\{q^{\xi}_i \restriction P : \xi < \zeta\}$ and the first player choosing q^{ζ}_i has to satisfy also $(\forall^D i)(r^{\zeta}_i \leq q^{\zeta}_i)$.

Outcome: Player I wins if $(*)$ from part (3) holds or

$(*)^-$ in the play $\Big\langle \langle p^{\zeta}_i \restriction P : i < \mu^+\rangle, \langle q^{\zeta}_i \restriction P : i < \mu^+\rangle : \zeta < \varepsilon \Big\rangle$ of $*^{\varepsilon}_{\mu}[P]$ the first player loses, (note concerning the outcome, then now in $(*)$ in part (3), the existence of r is not (even essentially) guaranteed.)

5) If $\restriction_\ell$ is a restriction operation for $(P_\ell, P_{\ell+1})$ for $\ell = 1, 2, \restriction = \restriction_1 \circ \restriction_2$, then "a strategy St of first player in $*^{\varepsilon}_{\mu}[P_1, P_3]$ project to one for $*^{\varepsilon}_{\mu}[P_1, P_2]$" is defined naturally.

1.6 Remark. We may restrict ourselves to a suitable family of strategies St_1 (to work in the iteration this family has to be suitably closed).

[1]could let some strategy determine r, no need at present

1.7 Claim. *1) If the forcing notion P satisfies $*^\varepsilon_\mu$ <u>then</u> P satisfies the μ^+-c.c.*
2) If P satisfies $*^\varepsilon_\mu$ and R is the trivial forcing $\{\emptyset_P\}$ <u>then</u> the pair (R, P) satisfies $*^\varepsilon_\mu$ where $\restriction$ is defined by $p \restriction R = \emptyset$.
3) If $(P, R, \restriction)$ satisfies $*^\varepsilon_\mu$ <u>then</u> P and R satisfy $*^\varepsilon_\mu$.
4) If triples $(P_0, P_1, \restriction_0), (P_1, P_2, \restriction_1)$ satisfy $*^\varepsilon_\mu$ <u>then</u> $(P_0, P_2, \restriction_0 \circ \restriction_1)$ satisfies $*^\varepsilon_\mu$.
5) If P satisfies $*^\varepsilon_\mu$ and $\Vdash_P$ "$\underset{\sim}{Q}$ satisfies $*^\varepsilon_\mu$" <u>then</u> $P * \underset{\sim}{Q}$ satisfies $*^\varepsilon_\mu$ moreover the pair $(P, P * \underset{\sim}{Q})$ (with the natural $\restriction$) satisfies $*^\varepsilon_\mu$.

Proof. Should be clear.

1.8 Remark. 1) if D is a normal filter on μ^+ to which $\{\delta < \mu^+ : \mathrm{cf}(\delta) = \mu\}$ belongs, then in 1.7 we can repalce $*^\varepsilon_\mu$ by $*^\varepsilon_D$ (of course, in part (5), D in V^P means the normal filter it generates).
Similarly for the claim below.
2) Assume that in the game of choosing $A_i \in D^+$ for $i < \varepsilon$ (or $i < \mu$), with player I choosing A_{2i}, player II choosing A_{2i+1}, A_i decreasing, player II loses iff he sometime has no legal move; player I has a strategy guaranteeing that he has legal moves. (If κ in measurable V in $V^{\mathrm{Levy}(\mu<\kappa)}$ this holds for some D by [JMMP].) In fact assume more generally that $\mathscr{P}$ is a partial order and $\mathscr{F} : \mathscr{P} \to \{A : A \subseteq \mu^+\}$ is decreasing: $\mathscr{P} \models x \le y \Rightarrow \mathscr{F}(y) \subseteq \mathscr{F}(x)$ and $\mathscr{E}$ is a function with domain $\mathscr{P}$ where $\mathscr{E}(x)$ is a non-empty subset of $[\mathscr{F}(x)]^2$ and $\mathscr{P} \models x \le y \Rightarrow \mathscr{E}(y) \subseteq \mathscr{E}(y)$ (above $\mathscr{P} = (D^+, \supseteq), \mathscr{F}$ is the identity and we say that a forcing notion Q satisfies

$*^\varepsilon_{\mathscr{P},\mathscr{F},\mathscr{E}}$ if in the following game $*^\varepsilon_{\mathscr{P},\mathscr{F},\mathscr{E}}[Q]$, the first player has a winning strategy.

A play last ε moves, in the ζ-th move player I chooses $x_\zeta \in \mathscr{P}$ such that $\xi < \zeta \Rightarrow y_\xi \le_{\mathscr{P}} x_\zeta$ and $\langle q^\zeta_i : i \in \mathscr{F}(x_\zeta)\rangle$ such that $\xi < \zeta \;\;\&\;\; i \in \mathscr{F}(x_\zeta) \Rightarrow p^\xi_i \le q^\zeta_i$ and player II chooses $y_\zeta \in \mathscr{P}$ such that $x_\zeta \le y_\zeta$ and $\langle p_i : i \in \mathscr{F}(y_\zeta)\rangle$ such that $i \in \mathscr{F}(y_\zeta) \Rightarrow q^\zeta_i \le_Q P^\zeta_i$.
<u>Outcome</u>: Player I wins a play if

- (α) for every limit $\zeta < \varepsilon$ he has a legal move (this depends on having upper bounds in $\mathscr{P}$ and in Q)
- (β) for every $\{i, j\} \in \bigcap_{\zeta<\varepsilon} \mathscr{E}(x_\zeta)$, in Q there is an upper bound to $\{p^\zeta_i : \zeta < \varepsilon\} \cup \{p^\zeta_j : \zeta < \varepsilon\}$.

The natural generalizations of the relevant lemmas works for this notion.
3) We can systematically use the weak restriction triples, and/or use the strong version of $*^\varepsilon_\mu$ for triples in this paper.

1.9 Claim. *1) If the forcing notions P_1, P_2 are equivalent <u>then</u> P_1 satisfies $*^\varepsilon_\mu$ iff P_2 satisfies $*^\varepsilon_\mu$.*
2) Suppose $\restriction$ is a restriction operation for $(P_1, P_2), B_\ell$ the complete Boolean algebra

corresponding to P_ℓ *(so* $B_1 \lessdot B_2$*) and* $\restriction'$ *is the projection from* B_2 *to* B_1 *and* $P'_\ell = (B_\ell \backslash \{0\}, \geq)$ *then*

(a) $(P'_1, P'_2, \restriction')$ *is a restriction triple and*

(b) $(P_1, P_2, \restriction)$ *satisfies* $*^\varepsilon_\mu$ *iff* $(P'_1, P'_2, \restriction')$ *satisfies* $*^\varepsilon_\mu$.

2A) In part (2) it is enough to assume that $\restriction$ *is a weak restriction operation.*
3) If a forcing notion Q *satisfies* $*^\varepsilon_\mu$ *<u>then</u> player I has a winning strategy in the play even if we demand from him:* $\bigwedge_{\xi<\zeta} p^\xi_i = \emptyset_Q \Rightarrow q^\zeta_i = \emptyset_Q$ *for each* $i < \mu^+$.
4) Similarly for $(P, R, \restriction)$ *satisfying* $*^\varepsilon_\mu$ *demanding* $\bigwedge_{\xi<\zeta} p^\xi_i = \emptyset_R \Rightarrow q^\zeta_i = \emptyset_R$ *and* $\bigwedge_{\xi<\zeta} p^\xi_i \Rightarrow q^\zeta_i \in P$.

1.10 Convention. Strategies are as in 1.9(3),(4).

1.11 Definition/Claim. *Assume for* $\ell = 1, 2$ *that* $(P, R_\ell, \restriction_\ell)$ *is a restriction triple,* $(P, R_\ell, \restriction_\ell)$ *satisfies* $*^\varepsilon_\mu$*, and we let*
$R = \{(p, r_1, r_2) : p \in P, r_1 \in R_1, r_2 \in R_2, P \models \text{“}r_1 \restriction P \leq p\text{”} \text{ and } P \models \text{“}r_2 \restriction P \leq p\text{”}\}$
identifying $r_1 \in R_1$ *with* $(r_1 \restriction P, r_1, \emptyset_{R_2})$*, and identifying* $r_2 \in R_2$ *with* $(r_2 \restriction P, \emptyset_{R_1}, r_2)$.
Under the quasi order

$$\begin{aligned}(p, r_1, r_2) \leq (p'_1, r'_1, r'_2) \text{ iff } & p \leq_P p' \\ \& \quad & lub_{R_1}\{p, r_1\} \leq_{R_1} lub_{R_1}\{p, r'_1\} \\ \& \quad & lub_{R_2}\{p, r_2\} \leq_{R_2} lub_{R_2}\{r''_2\}.\end{aligned}$$

<u>Then</u> $R_\ell \lessdot R$ *(for* $\ell = 1, 2$*) and* $(R_\ell, R, \restriction'_\ell)$ *is a restriction triple and it satisfies* $*^\varepsilon_\mu$*, where* $(p, r_1, r_2) \restriction'_\ell R_\ell =$ *the lub of* p, r_ℓ *in* R_ℓ *(see clause (b) of Definition 1.5(1)).*

1.12 Definition/Lemma. Let $\mu = \mu^{<\mu} < \kappa = \text{cf}(\kappa) \leq \lambda \leq \chi$. (Usually fixed hence suppressed in the notation). We define and prove the following by induction on (the ordinal) α:
1) [Def]. Let $\mathscr{K}^\alpha = \mathscr{K}^\alpha_{\mu,\kappa,\lambda,\chi}$ be the family of sequences $\bar{Q} = \langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha \rangle$ such that:

(a) $\langle P_\beta, \underset{\sim}{Q}_\beta : \beta < \alpha \rangle$ is a $(< \mu)$-support iteration (so $P_\alpha = \text{Lim}_\mu \bar{Q}$ denotes the natural limit)

(b) $a_\beta \subseteq \beta$, $|a_\beta| < \kappa$, $[\gamma \in a_\beta \Rightarrow a_\gamma \subseteq a_\beta]$

(c) $\underset{\sim}{Q}_\beta$ is strategically $(< \mu)$-complete, has cardinality $< \lambda$ and is a $P^*_{a_\beta}$-name (see parts 1.12(2)(b) and 1.12(5)(b) below).

1A) [Def] $\bar{Q}$ is called standard <u>if</u>: for every $\beta < \ell g(\bar{Q})$ each element of $\underset{\sim}{Q}_\beta$ is from V, even from $\mathscr{H}(\chi)$, and the order is a fixed quasi order from V such that any chain of length $< \mu$ which has an upper bound has a lub (we can use less), but note that the set of elements is not necessarily from V.

2) [Def]. For $\bar{Q}$ as above:

(a) $a \subseteq \alpha$ is called $\bar{Q}$-closed if $[\beta \in a \Rightarrow a_\beta \subseteq a]$; we also call it $\langle a_\beta : \beta < \alpha\rangle$-closed and let $\bar{a}^{\bar{Q}} = \langle a_\beta : \beta < \alpha\rangle$

(b) for a $\bar{Q}$-closed subset a of α we let

$$\begin{aligned} P_a = \{p \in P_\alpha :& \mathrm{Dom}(p) \subseteq a \text{ and for each } \beta \in \mathrm{Dom}(p) \\ & \text{we have: } p(\beta) \text{ is a } P_{a\cap\beta}\text{-name} \\ & \text{(i.e. involving only } G_{P_\beta} \cap P_{a\cap\beta} \\ & \text{so necessarily } Q \in V[G_{P_\beta} \cap P_{a\cap\beta}])\} \end{aligned}$$

$$\begin{aligned} P^*_a = \{p \in P_\alpha :& \mathrm{Dom}(p) \subseteq a \text{ and for each } \beta \in \mathrm{Dom}(p) \text{ we have: } p(\beta) \\ & \text{is a } P^*_{a_\beta}\text{-name and: if } \underset{\sim}{Q}_\beta \subseteq V \text{ and } \bar{Q} \text{ is standard, then} \\ & p(\beta) \text{ is from } V\}. \end{aligned}$$

On both P_a and P^*_a, the order is inherited from P_α. Note that P^*_a is defined by induction on $\sup(a)$.

3) [Lemma] For $\bar{Q}$ as above, $\beta < \alpha$

(a) $\bar{Q} \restriction \beta \in \mathscr{K}^\beta$

(b) if $a \subseteq \beta$ then: a is $\bar{Q}$-closed <u>iff</u> a is $(\bar{Q} \restriction \beta)$-closed

(c) if $a \subseteq \alpha$ is $\bar{Q}$-closed, <u>then</u> so is $a \cap \beta$, in fact β is $\bar{Q}$-closed and the intersection of a family of $\bar{Q}$-closed subsets of α is $\bar{Q}$-closed.

4) [Lemma]. For $\bar{Q}$ as above, and $\beta < \alpha$,

(a) $P_\beta \lessdot P_\alpha$, moreover, if $p \in P_\alpha, p \restriction \beta \le q \in P_\beta$ then $(p \restriction (\alpha \backslash \beta)) \cup q \in P_\alpha$ is a lub of p, q

(b) P_α / P_β is strategically $(< \mu)$-complete (hence does not add new sequences of length $< \mu$ of old elements).

5) [Lemma]. For $\bar{Q}$ as above

(a) P^*_α is a dense subset of P_α

(b) if a is $\bar{Q}$-closed <u>then</u> $P_a \lessdot P_\alpha$ and P^*_a is a dense subset of P_a.

(c) if a is $\bar{Q}$-closed, $p \in P_\alpha, p \restriction a \le q \in P_a$ <u>then</u> $(p \restriction (\alpha \backslash a)) \cup q$ belongs to P_α and is a lub of p, q in P_α

(d) if a is $\bar{Q}$-closed, <u>then</u> $\bar{Q} \restriction a \in \mathscr{K}^{\mathrm{otp}(a)}$ (up to renaming of indexes)

(e) if $a \subseteq b \subseteq \ell g(\bar{Q})$ are $\bar{Q}$-closed, <u>then</u> $(P^*_a, P^*_b, \restriction)$ is a restriction triple (where $p \restriction P^*_b = p \restriction a$)

6) [Lemma]. The sequence $\bar{Q} = \langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha \rangle$ belongs to $\mathscr{K}^\alpha$ if α is a limit ordinal and $\bigwedge_{\gamma<\alpha} \bar{Q} \restriction \gamma \in \mathscr{K}^\gamma$.

7) [Lemma]. The sequence $\bar{Q} = \langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha \rangle$ belongs to $\mathscr{K}^\alpha$ if $\alpha = \gamma + 1$, $a_\gamma \subseteq \gamma$ is a $(\bar{Q} \restriction \gamma)$-closed set of cardinality $< \kappa, \underset{\sim}{Q}_\gamma$ is a $P^*_{a_\gamma}$-name of a $(< \mu)$-strategically complete forcing notion of cardinality $< \lambda$.

8) [Def]. $\mathscr{K}^{<\alpha} = \bigcup_{\beta<\alpha} \mathscr{K}^\beta$.

Proof. Straightforward.

1.13 Definition. Let $\mu = \mu^{<\mu} < \kappa = \operatorname{cf}(\kappa) \leq \lambda \leq \chi$ (usually fixed hence suppressed in the notation) and ε a limit ordinal $< \mu$. We define the following by induction on (the ordinal) α:
1) We let $\mathscr{K}^{\varepsilon,\alpha} = \mathscr{K}^{\varepsilon,\alpha}_{\mu,\kappa,\lambda,\chi}$ be the family of sequences $\bar{Q} = \langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta, I_\beta : \beta < \alpha \rangle$ such that:

(α) $\langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha \rangle \in \mathscr{K}^\alpha$

(β) I_β is a family of $\bar{Q}$-closed (see part (2) below, it is not what was defined in 1.12(2)(a)) subsets of a_β, closed under finite unions, increasing unions of length $< \mu$ and such that $\emptyset \in I_\beta$

(γ) each a_β is $(\bar{Q} \restriction \beta)$-closed (see part (2) below, this is not as in 1.12)

(δ) if $b \in I_\beta$ then the pair $(P^*_b, P^*_{a_\beta \cup \{\beta\}})$ satisfies $*^\varepsilon_\mu$, of course, for the natural restriction operation.

(2) For $\bar{Q} \in \mathscr{K}^{\varepsilon,\alpha}$ (even satisfying just 1.13(1)(α) + (β)) we say that a set a is $\bar{Q}$-closed in b (or is $\langle a_\beta, I_\beta : \beta < \alpha \rangle$-closed) if $a \subseteq b \subseteq \alpha, [\beta \in a \Rightarrow a_\beta \subseteq a]$ and $[\beta \in b \backslash a \Rightarrow a \cap a_\beta \in I_\beta]$. If we omit "in b" we mean $b = \alpha$.

(3)

(a) $\bar{Q}$ is simple if for all $\beta < \alpha$
$I_\beta = \{b \subseteq a_\beta : b$ is $\bar{a}^{\bar{Q}}$-closed and: for every $\gamma \in a_\beta \cup \{\beta\}$, if $\operatorname{cf}(\gamma) = \mu^+$ and $\gamma = \sup(\gamma \cap b)$, then $\gamma \in b\}$.

(b) $\bar{Q}^- = \langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta : \beta < \alpha \rangle, a^{\bar{Q}} = \langle a_\beta : \beta < \alpha \rangle$, and $\bar{I}^Q = \langle I_\beta : \beta < \alpha \rangle$

(c) $\bar{Q}$ is standard if $\bar{Q}^-$ is standard

(d) $\mathscr{K}^{\varepsilon,<\alpha} = \bigcup_{\beta<\alpha} \mathscr{K}^{\varepsilon,\beta}$.

1.14 Claim. *Let $\bar{Q} \in \mathscr{K}^{\varepsilon,\alpha}$.*
1) If $\beta < \alpha$ then $\bar{Q} \restriction \beta =: \langle P_\gamma, \underset{\sim}{Q}_\gamma, a_\gamma, I_\gamma : \gamma < \beta\rangle$ belongs to $\mathscr{K}^{\varepsilon,\beta}$; moreover, if $b \subseteq \alpha$ is $\bar{a}^{\bar{Q}}$-closed <u>then</u> $\bar{Q} \restriction b \in \mathscr{K}^{\varepsilon,\mathrm{otp}(b)}$ (up to renaming of index sets) understanding $I_\beta^{\bar{Q}\restriction b} = I_\beta^{\bar{Q}} \restriction b$.
2) If $a \subseteq b \subseteq \beta \le \alpha$ and a is $\bar{Q}$-closed in b <u>then</u>: a is $(\bar{Q} \restriction \beta)$-closed in b.
3) If $\beta < \alpha, a \subseteq \alpha$ is $\bar{Q}$-closed and $\gamma \in \alpha \backslash \beta \Rightarrow a \cap a_\gamma \in I_\gamma$, <u>then</u> $a \cap \beta$ is $\bar{Q}$-closed.
4) If $\bar{Q}$ is simple, $\beta < \alpha, a \subseteq \alpha$ is $\bar{Q}$-closed and $cf(\beta) \ne \mu^+ \vee (\forall \gamma \in \alpha\backslash\beta)(a_\gamma \cap a \cap \beta$ is bounded in $\beta)$, <u>then</u> $a \cap \beta$ is $\bar{Q}$-closed.
5) The family of $\bar{Q}$-closed $a \subseteq \alpha$ is closed under increasing union of length $< \mu$ and $\emptyset$ belongs to it.
6) If a, b are $\bar{Q}$-closed, then so is $a \cup b$.
7) If $a \subseteq b \subseteq c \subseteq \ell g(\bar{Q}), a$ is $\bar{Q}$-closed in c, <u>then</u> a is $\bar{Q}$-closed in b.
8) If $a \subseteq b \subseteq \alpha, a$ is $\bar{Q}$-closed in b, <u>then</u> $a \cap \alpha$ is $(Q \restriction \beta)$-closed in $b \cap \beta$.

Proof. Straight.

1.15 Remark. Simple $\bar{Q}$ is what we shall use.

1.16 Lemma. *Assume $\bar{Q} \in \mathscr{K}^{\varepsilon,\alpha}$ and a, b are $\bar{Q}^-$-closed subsets of α and a is a $\bar{Q}$-closed subset of $b\,(\subseteq \alpha)$ and $\bar{Q}$ is simple or at least*

$(*)$ $\gamma < \beta < \alpha \Rightarrow a_\beta \cap (\gamma + 1) \in I_\beta$
(hence $\gamma < \beta < \alpha$ & $cf(\gamma) < \mu \Rightarrow a_\beta \cap \gamma \in I_\beta$).

<u>Then</u> the pair (P^*_a, P^*_b) satisfies $*^\varepsilon_\mu$.

Proof. We can assume by 1.14(1) that $b = \alpha$. By induction on α we shall show that for all $\bar{Q}$-closed subsets a of α the pair (P^*_a, P^*_α) satisfies $*^\varepsilon_\mu$ (see Definition 1.5(3)) and this is proved first when $a = \emptyset$ and then when $a \ne \emptyset$. So we fix a strategy St_a for the first player in $*^\varepsilon_\mu[P^*_a]$; why it exists? If $a = \emptyset$, trivially, if $a \ne \emptyset$ by the way the proof is arranged we know the conclusion for $(a', b') = (\emptyset, a)$, and as $\mathrm{otp}(a) \le \alpha$ clearly St_a exists. Next we shall choose a strategy for the first player in the game $*^\varepsilon_\mu[P^*_a, P^*_\alpha, St_a]$, where at stage $\zeta < \varepsilon$ the first player chooses $\{q^\zeta_\xi : \xi < \mu^+\}$, a regressive function f_ζ from μ^+ to μ^+ and the second player replies with suitable $\{p^\zeta_\xi : \xi < \mu^+\}$.

For simplicity the reader may assume that the $\underset{\sim}{Q}_\beta$ are μ-complete (which is the case used; otherwise we have to use the $(< \mu)$-strategic completeness (and remember 1.1(2) second sentence).

<u>Case 1</u>: $\alpha = \beta + 1, \beta \in a$.

So $a_\beta \subseteq a$, now $a \cap \beta$ is $(\bar{Q} \restriction \beta)$-closed (by 1.14(2)) hence by the induction hypothesis $(P^*_{a\cap\beta}, P^*_\beta)$ satisfies $*^\varepsilon_\mu$. Apply 1.11 with $P^*_{a\cap\beta}, P^*_\beta, P^*_a$ here standing for P, R_1, R_2 there and we get that (R_2, R) satisfies $*^\varepsilon_\mu$, which (translating) is the desired conclusion.

Case 2: $\alpha = \beta + 1, \beta \notin a$.

We know that $a \cap a_\beta \in I_\beta$.

By Definition 1.13(1)(δ) we know that $(P^*_{a \cap a_\beta}, P^*_{a_\beta \cup \{\beta\}})$ satisfies $*^\varepsilon_\mu$. By 1.11 we get that $(P^*_a, P^*_{a_\beta \cup \{\beta\} \cup a})$ satisfies $*^\varepsilon_\mu$. Now $a' =: a_\beta \cup \{\beta\} \cup a$ is $\bar{Q}$-closed by 1.14(6) and $\beta \in a'$ so by Case 1 we have: $(P^*_{a'}, P^*_\alpha)$ satisfies $*^\varepsilon_\mu$. Together by 1.7(4) we have: (P^*_a, P^*_α) satisfies $*^\varepsilon_\mu$.

Case 3: α a limit ordinal, $\mathrm{cf}(\alpha) \leq \mu$.

Here we use 1.9(3) (i.e. 1.9(A)).

We can find an increasing continuous sequence $\langle \gamma_\Upsilon : \Upsilon < \mathrm{cf}(\alpha) \rangle$ of ordinals $< \alpha$ with limit $\alpha, \gamma_0 = 0$ and $\gamma_{\Upsilon+1}$ a successor ordinal. Note that $(a \cap \gamma_{\Upsilon+1}) \cup \gamma_\Upsilon$ is $(\bar{Q} \restriction \gamma_{\Upsilon+1})$-closed as $[\gamma_\Upsilon \text{ limit } \Rightarrow \Upsilon \text{ limit } \ \& \ \mathrm{cf}(\Upsilon) < \mu]$ moreover $a \cup \Upsilon_\gamma$ is $\bar{Q}$-closed. We define by induction on $\Upsilon \leq \mathrm{cf}(\alpha)$ a strategy St^*_Υ of player I in the game $*^\varepsilon_\mu[P^*_a, P^*_{a \cup \gamma_\Upsilon}]$ such that for $\Upsilon_1 < \Upsilon$ we have that St^*_Υ projects to $St^*_{\Upsilon_1}$ (see Definition 1.5(4)) and St^*_0 is St_a.

If we do not assume that all the $\underset{\sim}{Q}_\beta$ are μ-complete, then we demand that, moreover, they satisfy:

$\boxtimes$ if $\left\langle \langle q^\zeta_i : i < \mu^+ \rangle, f_\zeta, \langle p^\zeta_i : i < \mu^+ \rangle : \zeta < \varepsilon \right\rangle$ is a play of $*^\varepsilon_\mu[P^*_a, P^*_{a \cup \gamma \Upsilon}, St_a]$, then for any ordinal β, looking at $\langle q^\zeta_i(\beta), p^\zeta_i(\beta) : \zeta < \varepsilon \rangle$ letting $\zeta(\beta, \emptyset) = \mathrm{Min}\{\zeta : q^\zeta_i * \beta) \neq \emptyset_Q\}$ if $\zeta \in [\zeta(\beta,0), \zeta(\beta,1))$ and $q^\zeta_i \restriction \beta$ forces that $\langle q^\xi_i(\beta) : \xi \in [\zeta, (\beta, 0), \zeta] \rangle$ is increasing, then $q^\zeta_i \restriction \beta$ forces that some $\langle q'_\xi, p'_\xi : \xi < \zeta - \zeta(\beta, 0) + 1 \rangle$ is a play of $\otimes^\varepsilon_{\underset{\sim}{Q}_\beta}$ in which player I uses a fix winning strategy (as in 1.1(2)!) and $p'_0 = q^{\zeta(\beta,0)}_i(\beta)$, (remember q'_0 not chosen) and $0 < \xi < \zeta - \zeta(\beta, 0) + 1 \Rightarrow q'_\xi = q^{\zeta(\beta,0)+\xi}_i(\beta)$ and $0 < \xi < \zeta - \zeta(\beta, 0) \Rightarrow p'_\xi = p^\xi_i(\beta)$.

This, of course, puts on us a burden also in successor γ just to increase the condition. The inductive step is done by 1.11, the limit stage is straight (using $\boxtimes$ to show we can).

Case 4: α limit ordinal, $\mathrm{cf}(\alpha) > \mu^+$.

During the play, player I in the ζ-th move also chooses an ordinal $\gamma_\zeta, \gamma_\zeta$ increases continuously with $\zeta, \gamma_0 = 0$ as follows:

$$\gamma_{\zeta+1} = \min\{\gamma < \alpha : (\forall i < \mu^+)(\forall \xi \leq \zeta)(p^\xi_i, q^\xi_i \in P_\gamma)\}$$

and he will make $q^\zeta_i \in P_{\gamma_\zeta}$, and the rest is as in Case 3.

Case 5: $\mathrm{cf}(\alpha) = \mu^+$.

Let $\langle \gamma_\Upsilon : \Upsilon < \mu^+ \rangle$ be increasing continuously with limit $\alpha, \gamma_0 = 0$, $\mathrm{cf}(\gamma_\Upsilon) \leq \mu$, and we imitate Case 4, separating to different plays according to the value of $j^\zeta_i = \mathrm{Min}\{j < i : \text{for each } \xi < \zeta \text{ we have } p^\xi_i \restriction \gamma_i \in P_{\gamma_j} \text{ and } q^\xi_i \restriction \gamma_i \in P_{\gamma_j}\}$. $\square_{1.16}$

1.17 Claim. *Assume*

(a) $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\alpha, a_\alpha, I_\alpha : \alpha < \delta \rangle$

(b) δ *a limit ordinal*

(c) *for every* $\alpha < \delta$ *we have* $\bar{Q} \restriction \alpha \in \mathscr{K}^{\varepsilon,\alpha}$.

Then $\bar{Q} \in \mathscr{K}^{\varepsilon,\delta}$.

Proof. Check.

1.18 Claim. *Assume*

(a) $\bar{Q} \in \mathscr{K}^{\varepsilon,\alpha}$

(b) $a_\alpha \subseteq \alpha$ *is* $\bar{Q}$*-closed,* $|a_\alpha| < \kappa$

(c) $I_\alpha \subseteq \{b \subseteq a_\alpha : b$ *is* $\bar{Q}$*-closed* $\}$

(d) I_α *is closed under finite unions,* I_α *is closed under increasing unions of length* $< \mu$ *and* $\emptyset \in I_\alpha$

(e) $\underset{\sim}{Q}_\alpha$ *is a* $P^*_{a_\alpha}$*-name of a forcing notion of cardinality* $< \lambda$

(f) *if* $b \in I_\alpha$ *then* $(P_b, P^*_{a_\alpha} * \underset{\sim}{Q}_\alpha)$ *satisfies* $*^\varepsilon_\mu$

(g) $P_\alpha = \ Lim_\mu \bar{Q}$.

Then $\bar{Q}\hat{\ }\langle P_\alpha, \underset{\sim}{Q}_\alpha, a_\alpha, I_\alpha \rangle$ *belongs to* $\mathscr{K}^{\varepsilon,\alpha+1}$.

Proof. Check.

1.19 Theorem. *Suppose* $\mu = \mu^{<\mu} < \kappa = \lambda < \chi$ *and* χ *is measurable.*
1) For some forcing notion P *of cardinality* χ*,* μ*-complete not collapsing cardinalities not changing cofinalities we have:*
$\Vdash_P$ *"*$2^\mu = \chi$ *and for every* $\sigma < \mu$ *and* $\theta < \kappa$ *we have* $\chi \rightarrow [\theta]^2_{\sigma,2}$*" (and for a fixed* ε *the Axiom: if* Q *is a* μ*-complete forcing notion of cardinality* $< \kappa$ *satisfying* $*^\varepsilon_\mu$ *and* $\mathscr{I}_\alpha \subseteq Q$ *dense for* $\alpha < \alpha^* < \kappa$ *then some directed* $G \subseteq Q$ *is not disjoint to any* $\mathscr{I}_\alpha$*).*
2) We can replace "μ*-complete" by "*$(< \mu)$*-strategically complete" (in the demand on* P *and, in the axiom, on* Q*.*

1.20 Remark. We can add "P satisfies $*^\varepsilon_\mu$" if the appropriate squared diamond holds which is true in reasonable inner models.

Proof. We concentrate on part (2). If we would like to do part (1), we should just demand all the $\underset{\sim}{Q}_i$ are μ-complete.

Stage A: Fix $\varepsilon < \mu$ and let $\mathscr{K}_*^\alpha = \{\bar{Q} \in \mathscr{K}^{\varepsilon,\alpha} : \bar{Q}$ is simple and standard$\}$, $\mathscr{K}_* = \bigcup_{\alpha<\chi} \mathscr{K}_*^\alpha$. (Note: $\bar{Q}$-closed will mean as in 1.13(3)(a),1.13(2).) By preliminary forcing without loss of generality "χ measurable" is preserved by forcing with $({}^{\chi>}2, \trianglelefteq)$ (= adding a Cohen subset of χ), see Laver [L]. Let us define a forcing notion R:
$R = \{\bar{Q} : \bar{Q} \in \mathscr{K}_*^\alpha$ for some $\alpha < \chi$ and $\bar{Q} \in \mathscr{H}(\chi)\}$
ordered by: $\bar{Q}^1 \leq \bar{Q}^2$ iff $\bar{Q}^1 = \bar{Q}^2 \restriction \ell g(\bar{Q}^1)$.
As R is equivalent to $({}^{\chi>}2, \trianglelefteq)$ we know that in V^R, χ is still measurable. Let $\bar{Q}^\chi = \langle P_\beta, \underset{\sim}{Q}_\beta, a_\beta : \beta < \chi\rangle$ be $\bigcup G_R$ and P_χ be the limit so $P^* = P^*_\chi \subseteq P_\chi$ is a dense subset, those are R-names. Now $R * \underset{\sim}{P}^*$ is the forcing P we have promised. The non-obvious point is $\Vdash_{R*\underset{\sim}{P}^*_\chi}$ "$\chi \to [\theta]^2_{\sigma,2}$" (where $\theta < \kappa, \sigma < \mu$). So suppose $(r^*, \underset{\sim}{p}^*) \in R * \underset{\sim}{P}^*_\chi$ and $(r^*, \underset{\sim}{p}^*) \Vdash$ "the colouring $\underset{\sim}{\tau} : [\chi]^2 \to \sigma$ is a counterexample". Let $\chi_1 = (2^\chi)^+$. Let $G_R \subseteq R$ be generic over V, $r^* \in G_R$. By [Sh 289], but the meaning is explained below in V^R we can find an end extension strong $(\chi_1, \chi, \chi, 2^{\kappa+\lambda+2^\mu}, (\kappa+\lambda+2^\mu)^+, \omega)$-system $\bar{M} = \langle M_s : s \in [B]^{<\aleph_0}\rangle$ such that $M_s \prec (\mathscr{H}(\chi_1)^{V[G_R]}, \mathscr{H}(\chi_1), \in)$, for $x = \{\chi, G_R, p^*, \underset{\sim}{\tau}\}$, (i.e. $x \in \bigcap_s M_s$ and $B \in [\chi]^\chi$). We do not define this as for helping to prove the next theorem (1.13) we assume less in $V[G_R], M_s \prec (\mathscr{H}(\chi_1)^{V[G_R]}, \in, \mathscr{H}(\chi_1), G_R)$ and:

$(*)_0$ $\bar{M} = \langle M_s : s \in [B]^{<(1+n^*)}\rangle$ is an end extension $(\chi_1, \chi, \chi, 2^{\kappa+\lambda+2^\mu}$, $(\kappa+\lambda+2^\mu)^+, n^*)$-system for x, for some $2 \leq n^* \leq \omega$.

where $(*)_0$ means:

$(*)'$ $B \in [\chi]^\chi$ and $M_s \prec (\mathscr{H}(\chi_1), \in), x \in \bigcap_s M_s, M_s \cap M_t = M_{s\cap t}$.

Furthermore, $\|M_s\| = 2^{\kappa+\lambda+2^\mu}$ and $[M_s]^{\kappa+\lambda+2^\mu} \subseteq M_s$. In addition, for $v_1, v_2 \in [B]^n, n < 1+n^*$ there is f_{v_1,v_2}, the unique isomorphism from M_{v_1} onto M_{v_2}, and: $|v_1 \cap \varepsilon_1| = |v_2 \cap \varepsilon_2|, \varepsilon_1 \in v_1, \varepsilon_2 \in v_2 \Rightarrow f_{v_1,v_2}(\varepsilon_1) = \varepsilon_2$. Finally, $s \triangleleft t \Rightarrow M_s \cap \chi \triangleleft M_t \cap \chi$.

We meanwhile concentrate on case $n^* = 2$.

Stage B: We assume $(*)$.
Let $C = \{\delta < \chi : \delta = \sup(B \cap \delta)$ and $(s \in [B \cap \delta]^n$ for some
$n < 1+n^* \Rightarrow M_s \cap \chi \subseteq \delta)\}$.
Let $\gamma(*) = \operatorname{Min}(B)$. Now for $p \in P^*_\chi \cap M_{\{\gamma(*)\}}$ and $\bar{c} = \langle c_1, c_2\rangle \in \sigma \times \sigma$ let us define the statement

$(*)^{\bar{c}}_p$ if $p \leq p^0 \in P^* \cap M_{\{\gamma(*)\}}$ then we can find $p^1, p^2 \in P^*_\chi \cap M_{\{\gamma(*)\}}, p^0 \leq p^1$, $p^0 \leq p^2$ such that for $\ell = 1, 2$:
for $\gamma_1 < \gamma_2$, $\gamma_1 \in B$, $\gamma_2 \in B$, we can find r_1, $r_2 \in P^* \cap M_{\{\gamma_1,\gamma_2\}}$ (so $\operatorname{Dom}(r_\ell) \subseteq M_{\{\gamma_1,\gamma_2\}} \cap \chi$) such that:

$$r_\ell \Vdash \text{"}\underset{\sim}{\tau}(\{\gamma_1, \gamma_2\}) = c_\ell\text{"}$$

$$r_\ell \restriction (\chi \cap M_{\{\gamma_\ell\}}) \leq f_{\{\gamma(*)\},\{\gamma_\ell\}}(p^1) \text{ (for strong system: equality)}$$

$$r_\ell \restriction (\chi \cap M_{\{\gamma_{3-\ell}\}}) \leq f_{\{\gamma(*)\},\{\gamma_{3-\ell}\}}(p^2) \text{ (for strong system: equality) .}$$

As $|\sigma \times \sigma| < \mu$ and the relevant forcing notions are $(< \mu)$-strategically complete, easily $\mathscr{I} = \{p \in P^* \cap M_{\{\gamma(*)\}} :$ for some $\bar{c}, (*)_p^{\bar{c}}$ hold$\}$ is a dense subset of $P^*_\chi \cap M_{\{\gamma(*)\}}$, but this partial forcing satisfies the μ^+-c.c. Hence we can find $\mathscr{I}^* = \{p_\zeta : \zeta < \mu\} \subseteq \mathscr{I}$, a maximal antichain of $P^*_\chi \cap M_{\{\gamma(*)\}}$ hence of P^*_χ (as ${}^{\mu \geq}(M_{\{\gamma(*)\}})$ is a subset of $M_{\{\gamma(*)\}}$). For $p \in \mathscr{I}^*$ we can choose $c_1(p), c_2(p) \in \sigma$ such that: $(*)_p^{(c_1(p),c_2(p))}$ hold.

Stage C: As G_R was any subset of R generic over V to which r^* belongs, there are R-names $\underset{\sim}{\gamma}(*), \langle(\underset{\sim}{p}_\xi, \underset{\sim}{c}_1(\underset{\sim}{p}_\xi), \underset{\sim}{c}_2(\underset{\sim}{p}_\xi)) : \xi < \mu\rangle$, $\langle \underset{\sim}{M}_s : s \in [\underset{\sim}{B}]^{<\aleph_0}\rangle$, $\langle \underset{\sim}{f}_{s,t} : (s,t) \in \bigcup\limits_{n<1+n^*} \Big([\underset{\sim}{B}]^n \times [\underset{\sim}{B}]^n\Big)\rangle$ forced by r^* to be as above. As R is χ-complete, $\chi > 2^{\kappa+\lambda+2^\mu}$, without loss of generality r^* forces values $\gamma(*), M_\emptyset, M_{\{\gamma(*)\}}, \langle(p^*_\zeta, c^*_1(p^*_\zeta), c^*_2(p^*_\zeta)) : \zeta < \mu\rangle$.

We now try to choose by induction on $\zeta \leq \theta + 1$, $\bar{Q}^\zeta, \alpha^\zeta, \gamma^\zeta$ such that:

$(A)(a)$ $\bar{Q}^\zeta \in R$

(b) $\bar{Q}^0 = \{r^*\}$

(c) $\ell g(\bar{Q}^\zeta) = \alpha^\zeta$

(d) $\xi < \zeta \Rightarrow \bar{Q}^\xi = \bar{Q}^\zeta \restriction \alpha^\xi$

(e) $\langle \alpha^\zeta : \zeta \leq \theta + 1\rangle$ is (strictly) increasing continuous

(f) $\alpha^\zeta < \gamma_\zeta < \alpha^{\zeta+1}$

(g) $\bar{Q}^{\zeta+1} \Vdash_R$ "$\gamma^\zeta \in \underset{\sim}{B}$"

(h) $\bar{Q}^{\zeta+1}$ forces ($\Vdash_R$) a value to $\langle \underset{\sim}{M}_s \cap V : s \in [\underset{\sim}{B} \cap (\gamma_\zeta + 1)]^{<1+n^*}\rangle$ which we call $\langle M_s : s \in [B_\zeta]^{<1+n^*}\rangle$.

(B) if $\zeta \leq \theta + 1$, $\mathrm{cf}(\zeta) > \mu$ then:

(a) $a_{\alpha_\zeta}^{\bar{Q}^{\zeta+1}} = \bigcup\{\chi \cap M_{\{\xi_1,\xi_2\}} : \{\xi_1, \xi_2\} \in [\{\gamma_\epsilon : \epsilon < \zeta\}]^{<1+n^*}\}$

(b) $I_{\alpha_\zeta}^{\bar{Q}^{\zeta+1}} = \{b : b$ an initial segment of $a_{\alpha_\zeta}^{\bar{Q}^{\zeta+1}}$ and $\mathrm{cf}(\mathrm{otp}(b)) \neq \mu^+\}$

[explanation: this satisfies the simplicity demands]

(c) $\underset{\sim}{Q}^{\bar{Q}^{\zeta+1}}_{\alpha_\zeta} = \{h : h$ a function, $\mathrm{Dom}(h) \subseteq \mu, |\mathrm{Dom}(h)| < \mu, h(i) \in Q^{\bar{Q}^\zeta}_{\alpha_\zeta,*}$ when defined $\}$ (see (d) below)

order $h_1 \leq h_2$ if $i \in \mathrm{Dom}(h_1) \Rightarrow h_1(i) \subseteq h_2(i)$ where $Q^{\bar{Q}^\zeta}_{\alpha_\zeta,*}$ is defined in clause (d) below

[explanation: the forcing notion in clause (d) adds a subset $\underset{\sim}{u}$ of ζ such that on $\{\gamma_\zeta : \zeta \in \underset{\sim}{u}\}$ the colouring $\underset{\sim}{\tau}$ get only two values; the forcing notion from clause (c) makes ζ the union of $\leq \mu$ such sets and this induces a representation of B_ζ as a union of μ sets on each $\underset{\sim}{\tau}$ get at most two colours]

(d) $Q^{\bar{Q}^\zeta}_{\alpha_\zeta,*} = \Big\{ u : u \in [\zeta]^{<\mu}$, and for some $\xi < \mu$ we have:

for every $j_1 < j_2$ from u, we can find p^1, p^2, r_1, r_2 such that for $\ell = 1, 2$ we have:

$$p^*_\xi \leq p^\ell \in M_{\{\gamma(*)\}} \cap P^*_\chi,$$

$$r_\ell \in P^*_\chi \cap M_{\{\gamma_{j_1}, \gamma_{j_2}\}},$$

$$r_\ell \Vdash \text{“}\underset{\sim}{\tau}(\{\gamma_{j_1}, \gamma_{j_2}\}) = c^*_\ell(p^*_\xi),$$

$$r_\ell \restriction (\chi \cap M_{\{\gamma_{j_\ell}\}}) \leq f_{\{\gamma(*)\},\{\gamma_{j_1}\}}(p^1),$$

$$r_\ell \restriction (\chi \cap M_{\{\gamma_{j_{3-\ell}}\}}) \leq f_{\{\gamma(*)\},\{\gamma_{j_{3-\ell}}\}}(p^2),$$

$$\text{and } r_1 \in \underset{\sim}{G}_{P_{\alpha_\zeta}} \text{ or } r_2 \in \underset{\sim}{G}_{P_{\alpha_\zeta}} \Big\}.$$

Stage D: Again we shall use less than obtained for later use.

The point is to verify that we can carry the induction. Now there is no problem to do this for $\zeta = 0$ and for ζ limit. So we deal with $\zeta + 1, \zeta \leq \theta$ and we are assuming that $\bar{Q}^\zeta$ is already defined. If $\mathrm{cf}(\zeta) \leq \mu$ clause (B) is empty and it is easy to satisfy clause (A) is easy. So assume $\mathrm{cf}(\zeta) \geq \mu^+$. Now as before clause (A) is easy. The point is to choose $\bar{Q}^{\zeta+1}$ or just $\bar{Q}^{\zeta+1} \restriction (\alpha_\zeta + 1)$ to satisfy clause (B). Now $\underset{\sim}{Q}_{\alpha_\zeta}$ is chosen by clause (B) so $\bar{Q}^{\zeta+1} \restriction (\alpha_\zeta + 1)$ is now fixed.

The point is to prove that the condition concerning $*^\epsilon_\mu$ from Definition 1.5 holds as required in Definition 1.13(1)(d). From now on we may omit the superscript $\bar{Q}^{\zeta+1}$ or $\bar{Q}^{\zeta+1} \restriction (\alpha_\zeta + 1)$ so $P^*_{\alpha_\zeta} = P^{\bar{Q}^{\zeta+1}}_{\alpha_\zeta} \restriction (\alpha_\zeta + 1)$, etc.

That is, we assume $b \in I_{\alpha_\zeta}$ and we will prove that $(P^*_b, P^*_{a_{\alpha_\zeta} \cup \{\alpha_\zeta\}})$ satisfies $*^\varepsilon_\mu$. Note

$(*)_1$ if $\bar{Q}^{\xi+1}$ is well defined (or just $\bar{Q}^{\xi+1} \restriction (\alpha_\xi + 1) \in R$) and $\mathrm{cf}(\xi) > \mu$ then (P_{α_ξ} is well defined and) in $V^{P_{\alpha_{\xi+1}}}$, $\{\gamma_\Upsilon : \Upsilon < \xi\}$ is well defined and it can be represented as $\bigcup_{i<\mu} \mathscr{U}_i$, such that each $u \in [\mathscr{U}_i]^{<\mu}$ belongs to $Q^{\bar{Q}^\xi}_{\alpha_\xi,*}$

$(*)_2$ if $\zeta(1) < \zeta(2) \le \zeta$ and $\mathrm{cf}(\zeta(1)), cf(\zeta(2)) > \mu$ then $Q^{\bar{Q}^{\zeta(1)}}_{\alpha_{\zeta(1)},*} \subseteq Q^{\bar{Q}^{\zeta(2)}}_{\alpha_{\zeta(2)},*}$, also for the compatibility relation

$(*)_3$ the elements of $Q^{\bar{Q}^\zeta}_{\alpha_\zeta,*}$ are from V, in fact are sets of ordinals of cardinality $< \mu$ ordered by $\subseteq$ and the lub of set of cardinality $< \mu$ members is the union (if there is an upper bound), so $Q^{\bar{Q}^\zeta}_{\alpha_\zeta,*}$ is μ-complete

$(*)_4$ $\bar{Q}^\zeta$ is well defined <u>and</u>
$\Vdash_{P_{\alpha_\zeta}}$ "for $\xi < \zeta$, if $\mathrm{cf}(\xi) > \mu$ then,
$Q^{\bar{Q}^\xi}_{\alpha_\xi,*}$ is the union of μ sets, each set $(< \mu)$-directed
and with any two elements having a lub".

Hence

$(*)_5$ if $\mathrm{cf}(\zeta) > \mu^+$, then in $V^{P_{\alpha_\zeta}}$, each subset of $Q^{\bar{Q}^{\zeta+1}}_{\alpha_\zeta,*}$ of cardinality $\le \mu^+$ is included in the union of μ sets, each directed and with any two elements having a lub.

Note that by the definition of $Q^{\bar{Q}^\zeta}_{\alpha_\zeta,*}$ we have

$(*)_6$ a family of $< \mu$ members of $Q^{\bar{Q}^\zeta}_{\alpha_\zeta,*}$ has a common upper bound iff any two of them are compatible, and then the union is a lub of the family.

So if $\mathrm{cf}(\zeta) > \mu^+$, we are done as by $(*)_5 + (*)_6$ we have $\Vdash_{P_\zeta}$ "Q_ζ satisfies $*^\varepsilon_\mu$" and can use 1.7(4).

So we can assume $\zeta = \Upsilon(*) \le \theta + 1$ and $\mathrm{cf}(\zeta) = \mathrm{cf}(\alpha_\zeta) = \mu^+$, and let $\langle \Upsilon(i) : i < \mu^+ \rangle$ be increasing continuous with limit ζ and $\mathrm{cf}(\Upsilon(i)) \le \mu$ for $i < \mu^+$. Let $b \in I_{\alpha_\zeta}$, hence b is a bounded subset of a_ζ. So by the induction hypothesis and 1.7(4) without loss of generality $b = \bigcup\{M_{\{\gamma_{\Upsilon_0},\gamma_{\Upsilon_1}\}} \cap \alpha_\zeta : \Upsilon_0 < \Upsilon_1 < \Upsilon(0)\}$.

Define $c_0 = b_0 = b$ and for $\Upsilon \in [\Upsilon(0), \Upsilon(*))$ let
$b_{1,\Upsilon} = b_0 \cup \big(M_{\{\gamma_\Upsilon\}} \cap \alpha_\zeta\big) \cup \bigcup\limits_{\Upsilon_1 < \Upsilon(0)} \Big(M_{\{\gamma_{\Upsilon_1},\gamma_\Upsilon\}} \cap \alpha_\zeta\Big)$
(the third term could be waived with minor changes),

$$b_1 = b_{1,\Upsilon(0)}, b_2 = b_1 \cup b_{1,\Upsilon(0)+1}, c_2 = \bigcup\{b_{1,\Upsilon} : \Upsilon \in [\Upsilon(0), \zeta)\}$$

$c_3 = a_{\alpha_{\Upsilon(*)}} = \bigcup\{M_{\{\gamma_{\Upsilon_1},\gamma_{\Upsilon_2}\}} \cap \alpha_{\Upsilon(*)} : \Upsilon_1 < \zeta, \Upsilon_2 < \zeta\}$
and $c_4 = a_{\alpha_{\Upsilon(*)}} \cup \{\alpha_\zeta\}$.

<u>Note</u>: There is no c_1.
All these sets are $\bar{Q}^{\alpha_\zeta+1}$-closed. We now choose several winning strategies which exist by the induction hypothesis on ζ.

Let St_0 be a winning strategy of the first player in a game above $*^{\varepsilon}_{\mu}[P^*_{b_0}]$. Let St_1 be a winning strategy of the first player in $*^{\varepsilon}_{\mu}[P^*_{b_0}, P^*_{b_1}]$ which projects to St_0. For every $\Upsilon \in [\Upsilon(0), \Upsilon(*))$ let $St_{1,\Upsilon}$ be a winning strategy of the first player in $*^{\varepsilon}_{\mu}[P^*_{b_0}, P^*_{b_{1,\Upsilon}}]$ conjugate to St_1 (by $\mathrm{OP}_{b_{1,\Upsilon}, b_1}$).

For $\bar\Upsilon = \langle \Upsilon_1, \Upsilon_2 \rangle, \Upsilon_1 < \Upsilon_2, \{\Upsilon_1, \Upsilon_2\} \subseteq [\Upsilon(0), \Upsilon(*))$ let $b_{2,\bar\Upsilon} = b_{1,\Upsilon_1} \cup b_{1,\Upsilon_2} \cup \big(M_{\{\Upsilon_1,\Upsilon_2\}} \cap \alpha_\zeta\big)$ and let $St_{2,\bar\Upsilon}$ be a winning strategy in $*^{\varepsilon}_{\mu}[P^*_{b_{1,\Upsilon_1} \cup b_{1,\Upsilon_2}}, P^*_{b_{2,\bar\Upsilon}}]$ which is above $St_{1,\Upsilon_1} \times St_{1,\Upsilon_2}$ (remember that both project to St_0); also note as long as the second player uses conditions in $P^*_{b_{\ell,\Upsilon_\ell}}$ then so does the first player (for each $i < \mu^+$ separately).
Also, the first player has a winning strategy in $*^{\varepsilon}_{\mu}[P^*_{c_0}, P^*_{c_2}]$ but we want a very special winning strategy St_2: (letting g_2 be a fixed pairing function on μ^+) in a play $\big\langle \langle p^\xi_i : i < \mu^+ \rangle, \langle q^\xi_i : i < \mu^+ \rangle, f^\xi : \xi < \varepsilon \big\rangle$ where the first player uses the strategy St_2 we demand that clauses $(a) - (d)$ below holds:

(a) $\big\langle \langle p^\xi_i \restriction b_0 : i < \mu^+ \rangle, \langle q^\xi_i \restriction b_0 : i < \mu^+ \rangle, f^{1,\xi} : \xi < \varepsilon \big\rangle$
is a play of $*^{\varepsilon}_{\mu}[P^*_{b_0}]$ in which the first player uses the strategy St_0)

(b) for each $\Upsilon \in [\Upsilon(0), \Upsilon(*))$ defining

$$p^{2,\Upsilon,\xi}_i = \begin{cases} p^\xi_i \restriction b_{1,\Upsilon} & \underline{\text{if}} \quad \Upsilon(i) > \Upsilon \\ p^\xi_i \restriction b_0 & \underline{\text{if}} \quad \Upsilon(i) \le \Upsilon \end{cases}$$

$$q^{2,\Upsilon,\xi}_i = \begin{cases} q^\xi_i \restriction b_{1,\Upsilon} & \underline{\text{if}} \quad \Upsilon(i) > \Upsilon \\ q^\xi_i \restriction b_0 & \underline{\text{if}} \quad \Upsilon(i) \le \Upsilon \end{cases}$$

we have: $\big\langle \langle p^{2,\Upsilon,\xi}_i : i < \mu^+ \rangle, \langle q^{2,\Upsilon,\xi}_i : i < \mu^+ \rangle, f^{2,\Upsilon,\xi} : \xi < \varepsilon \big\rangle$ is a play of $*^{\varepsilon}_{\mu}[P^*_{b_0}, P^*_{b_{1,\Upsilon}}]$ in which the first player uses the strategy $St_{1,\Upsilon}$.

(c) For any pair $\bar\zeta = (\zeta_1, \zeta_2)$ of ordinals in $\mu \times \varepsilon$, let

$$\Upsilon(i, \bar\zeta) = \Upsilon_{\bar\zeta}(i) \text{ is the } \zeta_1\text{-th member of } \mathrm{Dom}(q^{\zeta_2}_i) \backslash \Upsilon(i)$$

$$p^{3,\bar\zeta,\xi}_i = \mathrm{OP}_{b_{1,\Upsilon(0)}, b_{1,\Upsilon_{\bar\zeta}(i)}}(p^\xi_i \restriction b_{1,\Upsilon_{\bar\zeta}(i)})$$

$$q^{3,\bar\zeta,\xi}_i = \mathrm{OP}_{b_{1,\Upsilon(0)}, b_{1,\Upsilon_{\bar\zeta}(i)}}(q^\xi_i \restriction b_{1,\Upsilon_{\bar\zeta}(i)}),$$

we demand that $\big\langle \langle p^{3,\bar\zeta,\xi}_i : i < \mu^+ \rangle, \langle q^{3,\bar\zeta,\xi}_i : i < \mu^+ \rangle, f^{3,\bar\zeta,\xi} : \xi < \varepsilon \big\rangle$ is a play of $*^{\varepsilon}_{\mu}[P^*_{b_0}, P^*_{b_{1,\Upsilon(0)}}]$ in which the first player uses the strategy $St_{1,\Upsilon(0)}$.

So for each $i < \mu$, for $\zeta_1 < \mu$ too large $\Upsilon(i, \bar\zeta)$ is not well defined and we stipulate the forcing conditions are $\emptyset$.

(d) $f^\xi(i)$ codes $f^{1,\xi}(i), \langle f^{2,\Upsilon,\xi}(i) : \Upsilon \in [\Upsilon(0), \Upsilon(*))$ and $(\exists \beta \in b_{1,\Upsilon} \backslash b_0)[p_i^\xi(\beta) \neq 0_{\underset{\sim}{Q}_\beta}]\rangle$ and $\langle f^{3,\bar{\zeta},\xi}(i) : \bar{\zeta} \in \mu \times \varepsilon,$ and $\Upsilon_{\bar{\zeta}}(i)$ is well defined$\rangle$ and the information on $p_i^\xi(\alpha_{\Upsilon(*)})$ and it codes

$$\begin{aligned}\big\{\langle j_1, \zeta_1, \zeta_2\rangle :&\beta, \text{ the } \zeta_2\text{-th member of } \mathrm{Dom}(p_i^\xi)\\ &\text{satisfies} : j_1 = \ \mathrm{Min}\{j : \beta \in \ \mathrm{Dom}(p_j^\xi)\},\\ &\beta \text{ is the } \zeta_1\text{-th member of } \mathrm{Dom}(p_{j_1}^\xi)\big\}\end{aligned}$$

and

$$\begin{aligned}\big\{\langle j, \zeta_1, \zeta_2\rangle : &\text{ for some } \Upsilon, \beta, \text{ the } \zeta_1\text{-th member of } \mathrm{Dom}(p_i^\xi),\\ &\text{belongs to } b_{1,\Upsilon} \backslash b_0 \text{ and satisfies} :\\ &j = \ \mathrm{Min}\{j' : (\mathrm{Dom}(p_{j'}^\xi) \cap (b_{1,\Upsilon} \backslash b_0) \neq \emptyset\}\\ &\text{and the } \zeta_2\text{-th member of } \mathrm{Dom}(p_j^\xi) \text{ belongs to } b_{1,\Upsilon} \backslash b_0\big\}\end{aligned}$$

(note: for each $\zeta_2 < \varepsilon, i < \mu^+$ we have:
$\{\zeta_1 < \mu : \Upsilon_{(\zeta_1,\zeta_2)}(i)$ is well defined$\}$ is a bounded subset of μ).

Check that such St_2 exists, (note that the number of times we have to increase $p_i \restriction b_0$ is $< \mu$).

Clearly $c_2 \subseteq c_3$ are $\bar{Q}$-closed, hence there is a winning strategy St_3 of the first player in $*_\mu^\varepsilon[P_{c_2}^*, P_{c_3}^*]$ above St_2.

(e) For any $\bar{\Upsilon} = (\Upsilon_1, \Upsilon_2)$ such that $\Upsilon(0) \leq \Upsilon_1 < \Upsilon_2 < \Upsilon(*)$, and defining
$p^{4,\bar{\Upsilon},\xi} = p_i^\xi \restriction b_{2,\bar{\Upsilon}}$,
$q_i^{4,\bar{\Upsilon},\xi} = q_i^\xi \restriction b_{2,\bar{\Upsilon}}$
(can behave similarly in clause (b)),
<u>we have</u>: $\big\langle \langle p_i^{4,\bar{\Upsilon},\xi} : i < \mu^+\rangle, \langle q_i^{4,\bar{\Upsilon},\xi} : i < \mu^+\rangle, f^{4,\bar{\Upsilon},\xi} : \xi < \varepsilon\big\rangle$
is a play of $*_\mu^\varepsilon[P_{b_{2,\bar{\Upsilon}}}^*]$ in which the first player uses the strategy $St_{2,\bar{\Upsilon}}$.

Lastly, let St_4 be a strategy of the first player in $*_\mu^\varepsilon[P_{c_3}^*, P_{c_4}^*]$ which is above St_3 and it guarantees:

$(*)$ if $\big\langle \langle p_i^\xi : i < \mu^+\rangle, \langle q_i^\xi : i < \mu^+\rangle, f_\xi^4 : \xi < \varepsilon\big\rangle$ is a play of the game in which the first player uses his strategy St_4 <u>then</u>:
- (α) $q_i^\xi \restriction a_{\alpha_\Upsilon}$ forces a value to $p_i^\xi(\alpha_{\Upsilon(*)})$
- (β) if $\Upsilon_1 \neq \Upsilon_2$ are from (the value forced on) $q_i^\xi(\alpha_{\Upsilon(*)})$ <u>then</u> $q_i^\xi \restriction a_\Upsilon$ is above the relevant parts of witnesses to this.

Clearly St_4 is (essentially) a strategy of the first player in $*_\mu^\varepsilon[P_{b_0}^*, P_{c_4}^*]$ (for the almost $*_\mu^\varepsilon$ case above St_0). All we have to prove is that St_4 is a winning strategy. So let $\big\langle \langle p_i^\xi : i < \mu^+\rangle, \langle q_i^\xi : i < \mu^+\rangle, f_\xi^4 : \xi < \varepsilon\big\rangle$ be a play of $*_\mu^\varepsilon[P_{b_0}^*, P_{c_4}^*]$ in which the first player uses the strategy St_4.

By the definition of the game $*^\varepsilon_\mu[P^*_{b_0}, P^*_{c_4}]$ without loss of generality for some club E_1 of μ^+ (see clause (a)):

$(**)_1$ if $\{i,j\} \subseteq S^{\mu^+}_\mu \cap E_1$ and
$\bigwedge\limits_{\xi<\varepsilon} f^4_\xi(i) = f^4_\xi(j)$ <u>then</u>
$\{p^\xi_i \restriction b_0, p^\xi_j \restriction b_0 : \xi < \varepsilon\}$ has an upper bound in $P^*_{b_0}$.

By clause (b) in the demands on $St_{1,\Upsilon}$ for some club E_2 of μ^+ we have:

$(**)_2$ if $\{i,j\} \subseteq S^{\mu^+}_\mu \cap E_2$ and
$\Upsilon \in [\Upsilon(0), \Upsilon(*))$ and
$\bigwedge\limits_{\xi<\varepsilon}[(b_{1,\Upsilon}\backslash b_0) \cap \text{ Dom}(p^\xi_i) \neq \emptyset \ \ \& \ \ (b_{1,\Upsilon}\backslash b_0) \cap \text{ Dom}(p^\xi_j) \neq \emptyset \Rightarrow f^{2,\Upsilon,\xi}(i) = f^{2,\Upsilon,\xi}(j)]$
(which holds if $\bigwedge\limits_{\xi<\varepsilon} f^4_\xi(i) = f^4_\xi(j)$), and r is an upper bound of
$\{p^\xi_i \restriction b_0, p^\xi_j \restriction b_0 : \xi < \varepsilon\}$ <u>then</u>
$\{p^\xi_i \restriction b_{1,\Upsilon}, p^\xi_j \restriction b_{1,\Upsilon} : \xi < \varepsilon\} \cup \{r\}$ has an upper bound in $P^*_{b_{1,\Upsilon}}$.

By clause (c) in the choice of St_2 we know that there is a club E_3 of μ^+ such that:

$(**)_3$ if $\bar\zeta \in \mu \times \varepsilon, \{i,j\} \subseteq S^{\mu^+}_\mu \cap E_3$ and
$\bigwedge\limits_{\xi<\varepsilon} f^{3,\bar\zeta,\xi}(i) = f^{3,\bar\zeta,\xi}(j)$ (which holds if $\bigwedge\limits_{\xi<\varepsilon} f^4_\xi(i) = f^4_\xi(j)$) and
$r \in P^*_{b_0}$ is an upper bound of $\{p^\xi_i \restriction b_0, q^\xi_i \restriction b_0 : \xi < \zeta\}$ <u>then</u>
$\{p^{3,\bar\zeta,\xi}_i, p^{3,\bar\zeta,\xi}_j : \xi < \varepsilon\} \cup \{r\}$ has an upper bound.

By clause (e) in the demand on St_3, for some club E_4 of μ^+

$(**)_4$ if $\{i,j\} \subseteq S^{\mu^+}_\mu \cap E_4$ and $\bigwedge\limits_{\xi<\varepsilon} f^4_\xi(i) = f^4_\xi(j)$ and r is an upper bound of $\{p^\xi_i \restriction b_0, p^\xi_j \restriction b_0 : \xi < \varepsilon\}$ then $\{p^\xi_i \restriction \Upsilon(i), p^\xi_j \restriction \Upsilon(j) : \xi < \varepsilon\} \cup \{r\}$ has an upper bound in $p^*_{\alpha_0}$ (even $p^*_{\alpha_{\text{Max}\{\Upsilon(i),\Upsilon(j)\}}}$).

Last

$(**)_5$ E is a club of μ^+ included in $E_1 \cap E_2 \cap E_3 \cap E_4$ such that:
$i < j \in E \Rightarrow \text{ Dom}(p^\xi_i \restriction c_3) \cup \text{ Dom}(q^\xi_i \restriction c_3) \subseteq \alpha_{\Upsilon(j)}$.

The rest is as in [Sh 276, §2]. $\square_{1.12}$

1.21 Theorem. *We can in 1.19 replace "measurable", by (strongly) Mahlo.*

1.22 Remark. It is not straightforward; e.g. we may use the version of squared diamond given in Fact 1.24 below.

We first prove two claims.

1.23 Claim. *Suppose λ is a strongly inaccessible Mahlo cardinal, $\chi > \lambda > \theta = \theta^{<\sigma}$, $\mathfrak{C}$ an expansion of $(\mathscr{H}(\chi), \in, <^*_\chi)$ by $\leq \theta$ relations. Then for some club E of λ for every inaccessible $\kappa \in E$ we have:*

$(*)_\kappa$ *for every $x \in \mathscr{H}(\chi)$ there are $B \in [\kappa]^\kappa$ and N_s (for $s \in [B \cup \{\kappa\}]^2$), $N'_{\{i\}}$ (for $i \in B \cup \{\kappa\}$), $N_{\{i\}}$ (for $i \in B$) and $N_\emptyset$ (so $N_{\{\kappa\}}$ is meaningless) such that ($L_{\sigma,\sigma}$ is like the first order logic but with conjunctions and a string of existential quantifiers of any length $< \sigma$):*

- (a) $x \in N_s \prec_{L_{\sigma,\sigma}} \mathfrak{C}$ *and* $\theta \subseteq N_s$ *sn*
- (b) $x \in N'_{\{i\}} \prec_{L_{\sigma,\sigma}} \mathfrak{C}$ *and* $\theta \subseteq N'_{\{i\}} \subseteq N_{\{i\}}$
- (c) $s \subseteq B \Rightarrow N_s \cap \lambda \subseteq \kappa$ & $N'_s \cap \lambda \subseteq \kappa$ *(when defined)*
- (d) $N_\emptyset \prec_{L_{\sigma,\sigma}} N_{\{i\}}$ *and* $min(N_{\{i\}} \cap \lambda \backslash N_\emptyset) >$ *sup* $\left[\bigcup\{N_s \cap \lambda : s \subseteq [B \cap i]^{\leq 2}\}\right]$
- (e) *for* $j < i, N_{\{j,i\}}$ *is the* $L_{\sigma,\sigma}$*- Skolem hull of* $N_{\{j\}} \cup N'_{\{i\}}$ *inside* $\mathfrak{C}$
- (f) *for* $j < i, N_{\{j\}} \cap \lambda$ *is an initial segment of* $N_{\{j,i\}} \cap \lambda$
- (g) *for* $j < i$, $Min\big(N_{\{j,i\}} \cap \lambda \backslash N_{\{j\}}\big) > \sup\{N_{\{j_1,i_1\}} \cap \lambda : j_1 < i_1 < i\}$
- (h) N_s, N'_s *have cardinality* θ *when defined.*

Proof. Let $\theta_1 = 2^\theta, \theta_2 = 2^{\theta_1}$. Let $\mathfrak{A}$ and κ be such that:

$$\kappa \text{ strongly inaccessible}$$

$$\mathfrak{A} \prec_{L_{\theta_2^+,\theta_2}} \mathfrak{C}$$

$$\mathfrak{A}^{<\kappa} \subseteq \mathfrak{A}$$

$$\mathfrak{A} \cap \lambda = \kappa.$$

(Clearly for some club E of λ, for every strongly inaccessible $\kappa \in E$ there is $\mathfrak{A}$ as above; so it is enough to prove $(*)_\kappa$). Without loss of generality, $\kappa > \theta$. Next choose $\mathfrak{B}_i \prec_{L_{\theta_2^+,\theta_2^+}} \mathfrak{C}$, increasing continuous in i for $i < \kappa, \langle \mathfrak{B}_i : i \leq j\rangle \in \mathfrak{B}_{j+1}, \|\mathfrak{B}_j\| < \kappa, \mathfrak{B}_i \cap \kappa$ an ordinal and $\{x, \lambda, \theta, \sigma, \kappa, \lambda, \mathfrak{A}\} \in \mathfrak{B}_0$.
Let $\mathfrak{B} = \mathfrak{B}_{\theta^+}$, and let f be a function from $\mathfrak{B}$ into $\mathfrak{A}$, which is an $\prec_{L_{\theta_1^+,\theta_2^+}}$ elementary mapping (for the model $\mathfrak{C}, \mathrm{Dom}(f) = \mathfrak{B}, \mathrm{Rang}(f) \subseteq \mathfrak{A}$).

Let $N \prec_{L_{\sigma,\sigma}} \mathfrak{C}$ be such that
$\{x, \mathfrak{A}, \mathfrak{B}, \langle \mathfrak{B}_i : i \leq \theta^+\rangle, f, \sigma, \theta, \lambda, \kappa\} \in N, \theta + 1 \subseteq N, \|N\| = \theta, N^{<\sigma} \subseteq N$.

Let N^+ be the $L_{\sigma,\sigma}$-Skolem hull of $N \cup f(N)$ in $\mathfrak{C}$.
Let $N_\emptyset$ be $N^+ \cap \mathfrak{A} \cap \mathfrak{B}$, as $\|N_\emptyset\| \leq \theta$ we have $N_\emptyset \in \mathfrak{A} \cap \mathfrak{B}$. Let $N_{\{0\}} = N^+ \cap \mathfrak{A}$ (so $N_\emptyset = N_{\{0\}} \cap \mathfrak{B}$, and $N_\emptyset \cap \lambda (\subseteq \kappa)$ is an initial segment of $N_{\{0\}} \cap \lambda (\subseteq \kappa)$, let $N'_{\{\kappa\}} = N^+ \cap \mathfrak{B}$ and $N'_{\{0\}} = f(N'_{\{\kappa\}})$, so $N'_{\{0\}} \prec N_{\{0\}}$. Let $\alpha_0 = f(\kappa)$. Now we choose by induction on $i < \kappa, \alpha_i, N'_{\{i\}}, N_{\{i\}}, g_i$ and $N_{\{i,j\}}$ for $j < i$ such that:

(1) g_i is an $\prec_{L_{\sigma,\sigma}}$ elementary mapping from $N_{\{0\}}$ into $\mathfrak{A}, g_0 = \mathrm{id}_{N_{\{0\}}}$

(2) $g_i(\alpha_0) = \alpha_i$

(3) for $j < i, N_{\{j,i\}}$ is the $L_{\sigma,\sigma}$-Skolem hull of $N_{\{j\}} \cup N'_{\{i\}}$ (in $\mathfrak{C}$)

(4) $N_{\{i,\kappa\}}$ is the $L_{\sigma,\sigma}$-Skolem hull of $N_{\{i\}} \cup N'_{\{\kappa\}}$

(5) $N_{\{i,\kappa\}}, N_{\{0,\kappa\}}$ are isomorphic, in fact there is an ismorphism from $N_{\{0,\kappa\}}$ onto $N_{\{i,\kappa\}}$ extending $g_i \cup \mathrm{id}_{N'_{\{\kappa\}}}$

(6) for $j < i$ there is an isomorphism from $N_{\{j,i\}}$ onto $N_{\{j,\kappa\}}$ extending $\mathrm{id}_{N_{\{j\}}} \cup (f^{-1} \circ g_i^{-1}) \restriction N'_{\{i\}}$

(7) $N_{\{j\}} \cap \lambda$ is an initial segment of $N_{\{j,i\}} \cap \lambda$ for $j < i$.

This is possible and gives the desired result. $\square_{1.23}$

1.24 Fact

Let χ be strongly inaccessible $(k+1)$-Mahlo, $\kappa < \chi$ are regular. By a forcing with a P which is κ^+-complete of cardinality χ, not collapsing cardinals nor cofinalities nor changing cardinal arithmetic we can get:

$(*)^{\kappa,k}_{\chi}$ there is $\bar{A} = \langle A_\alpha : \alpha < \chi \rangle$ and $\bar{C} = \langle C_\alpha : \alpha \in S \rangle$ such that:

(a) $S \subseteq \{\delta < \chi : \delta > \kappa \text{ and } \mathrm{cf}(\delta) \le \kappa\}$ and $\{\delta \in S : \mathrm{otp}(C_\delta) = \kappa\}$ is a stationary subset of χ

(b) $C_\alpha \subseteq \alpha \cap S, [\beta \in C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta]$, $\mathrm{otp}(C_\alpha) \le \kappa, C_\alpha$ a closed subset of α and $[\sup(C_\alpha) = \alpha \Leftrightarrow C_\alpha$ has no last element]

(c) $A_\alpha \subseteq \alpha$

(d) $\beta \in C_\alpha \Rightarrow A_\beta = A_\alpha \cap \beta$

(e) $\big\{\lambda < \chi : \lambda$ inaccessible, and for every $X \subseteq \lambda$ the set we have $\{\alpha < \lambda : \mathrm{otp}(C_\alpha) = \kappa, X \cap \alpha = A_\alpha\}$ is a stationary subset of $\lambda\big\}$ is not only stationary but is a k-Mahlo subset, moreover we actually get:

$(e)^+$ for every strongly inaccessible $\lambda \in (\theta, \chi), \langle (A_\alpha, C_\alpha) : \alpha \in S \cap \lambda \rangle$ is a club guessing squared diamond, that is clauses (a)-(d) hold with $\lambda, S \cap \lambda$ and: for every club E of λ and $X \subseteq \lambda$ for some $\delta \in S$ we have $C_\delta \cup \{\delta\} \subseteq E$ and $\mathrm{otp}(C_\delta) = \kappa$ and $\alpha \in C_\delta \cup \{\delta\} \Rightarrow A_\alpha = X \cap \alpha$.

Proof. This can be obtained e.g. by iteration with Easton support, in which for each strongly inaccessible $\lambda \in (\kappa, \chi]$ we add $\bar{A}, \bar{C}$ satisfying $(a) - (d)$ above, each condition being an initial segment.

More specifically, we define and prove by induction on $\alpha \le \chi$

(1) [Definition] $P_\alpha = \Big\{ (a, \bar C, \bar A) : (a)\, a \subseteq \alpha \backslash \kappa^+,$

(b) for every strongly inaccessible $\lambda \in (\kappa, \chi]$ we have $\lambda > \sup(a \cap \lambda)$

$(c)\, \bar C = \langle C_\alpha : \alpha \in a \rangle$

$(d)\, C_\alpha \neq \emptyset \Rightarrow \ \mathrm{cf}(\alpha) \leq \kappa \ \& \ \ \mathrm{otp}(C_\alpha) \leq \kappa$

$(e)\, \beta \in C_\alpha \Rightarrow \beta \in a \ \& \ C_\beta = C_\alpha \cap \beta$

$(f)\, C_\alpha \neq \emptyset \Rightarrow C_\alpha$ closed

$(g)\, \bar A = \langle \underset{\sim}{A}_\alpha : \alpha \in a \rangle$

$(h)\, \underset{\sim}{A}_\alpha$ is a P_α-name of a subset of α

$(i)\, \beta \in C_\alpha \Rightarrow \Vdash_\alpha$ "$\underset{\sim}{A}_\alpha \cap \beta = \underset{\sim}{A}_\beta \Big\}$

<u>order</u> $p \leq q$ iff $a^p \subseteq a^q, \bar C^p = \bar C^q \restriction a^p, \bar A^p = \bar A^q \restriction a^p$.

2) [Claim]: $\beta < \alpha \Rightarrow P_\beta \lessdot P_\alpha$.
3) [Claim]: If $p \in P_\alpha, \beta < \alpha$, then $p \restriction \beta = (a^p \cap \beta, \bar C \restriction (a \cap \beta), \bar A \restriction (a \cap \beta))$ belongs to P_β and: if $p \restriction \beta \leq q \in P_\beta$ then p, q are compatible in a simple way: $p \ \& \ q$ is a lub of $\{p, q\}$.
4) [Claim]: If λ is strongly inaccessible $\leq \chi$ and $> \kappa$ then $P_\lambda = \bigcup\limits_{\alpha<\lambda} P_\alpha$. If in addition λ is Mahlo, then P_λ satisfies the λ-c.c.

Let $\underset{\sim}{c}_\alpha = c^p_\alpha, \underset{\sim}{A}_\alpha = A^p_\alpha$ for every large enough $p \in \underset{\sim}{G}_{P_\chi}$. The point is that for every strongly inaccessible $\lambda \in (\theta, \chi], P_\chi / P_\lambda$ does not add any subset of λ, and so $\langle (\underset{\sim}{C}_i, \underset{\sim}{A}_i[G]) : i < \lambda \rangle$ is as required. $\square_{1.24}$

1.25 Conclusion

Let $\theta = \theta^{<\sigma} < \lambda, \lambda$ a strongly inaccessible Mahlo cardinal, <u>then</u> for some θ^+-complete, λ-c.c. forcing notion of cardinality λ not collapsing cardinals not changing cofinalities nor changing cardinal arithmetic, in V^P we get:

$(**)^{\theta,2}_\lambda$ there are $\langle (B_\alpha, \bar M^\alpha, C_\alpha) : \alpha \in S \rangle$ such that:

- (a) $S \subseteq \{\delta < \chi : \mathrm{cf}(\delta) \leq \theta\}$ and $\{\delta \in S : \mathrm{otp}(C_\delta) = \theta\}$ is a stationary subset of χ and even of any strongly inaccessible $\lambda \in (\theta, \chi)$
- (b) $C_\alpha \subseteq \alpha \cap S, [\beta \in C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta],\ \mathrm{otp}(C_\alpha) \leq \theta, C_\alpha$ a closed subset of α so $[\sup(C_\alpha) = \alpha \Leftrightarrow C_\alpha$ has no last element)
- (c) $B_\alpha \subseteq \alpha,\ \mathrm{otp}(B_\alpha) = \omega \times\ \mathrm{otp}(C_\alpha), \beta \in C_\alpha \Rightarrow B_\beta = B_\alpha \cap \beta$
- (d) each $\langle M^\alpha_s : s \in [B_\alpha]^{\leq 2} \rangle$ is as in 1.23 (and $B_\alpha \subseteq B$) and $\beta \in C_\alpha \ \& \ s \in [B_\beta]^{\leq 2} \Rightarrow M^\alpha_s = M^\beta_s$

(e) diamond property: if $\mathfrak{B}$ is an expansion of $(\mathscr{H}(\chi), \in, <^*_\chi)$ by $\leq \theta$ relations, $B \in [\chi]^\chi$ then for a club E of χ for every strong inaccessible $\lambda \in \text{acc}(E)$ for stationarily many $\delta \in S \cap \lambda$ we have $\text{otp}(C_\delta) = \kappa, C_\delta \subseteq E$ and $B_\delta \subseteq B$ and $s \in [B_\delta]^{\leq 2} \Rightarrow M^\delta_s \prec \mathfrak{B}$.

Proof. By 1.24 + 1.23 (alternatively, force this directly: simpler than in 1.24.

Remark. In 1.24 we could force a stronger version.

Proof of 1.21. We repeat the main proof the one of Theorem 1.19, but using the diamond from 1.24 for $k = 0$. In fact the proof of 1.19 was written such that it can be read as a proof of 1.21, mainly in stage B we can get $(*)$ which is proved using measurability, but use only $(*)'$.

$\square_{1.21}$

Combining the above proof and [Sh 288] we get

1.26 Theorem. *Suppose*

(a) $\mu = \aleph_0$ *or* μ *is Laver indestructible supercompact (see [L]) or just* μ *as in [Sh 288, §4]*
(b) λ *is* n^**-Mahlo,* $\lambda > \theta > \mu$
(c) k_{n^*} *as in [Sh 228] (see below).*

Then for some μ^+*-c.c. forcing notion* P *of cardinality* λ *we have:*

$$\Vdash_P \text{“}2^\mu = \lambda \rightarrow [\theta]^{n^*+1}_{k_{n^*}+1}\text{”}, \text{ moreover for } \sigma < \mu, \\ \lambda \rightarrow [\theta]^{n^*+1}_{\sigma, k_{n^*}}.$$

1.27 Remark. 1) What is k_{n^*}?

Case 1: $\mu = \aleph_0$; define on $[{}^\omega 2]^{n^*}$ an equivalence relation E: if $w_1 = \{\eta_\ell : \ell < n^*\}, w_2 = \{\nu_\ell : \ell < n^*\}$ are members of $[{}^w 2]^{n^*}$ both listed in lexicographic increasing order, then $w_1 E w_2$ iff for any $\ell_1 < \ell_2 < n^*$ and $\ell_3 < \ell_4 < n^*$ we have

$$\ell g(\eta_{\ell_1} \cap \eta_{\ell_2}) < \ell g(\eta_{\ell_3} \cap \eta_{\ell_4}) \Leftrightarrow \ell g(\nu_{\ell_1} \cap \nu_{\ell_2}) < \ell g(\nu_{\ell_3} \cap \nu_{\ell_4}).$$

Lastly, k_{n^*} is the number of E-equivalence classes.

Case 2: $\mu > \aleph_0$.

Choose $<_\alpha$ be a well ordering of ${}^\alpha 2$ and let E be the following equivalence relation on $[{}^\mu 2]^{n^*}$: if $w_0 = \{\eta_\ell : \ell < n^*\}, w_2 = \{\nu_\ell : \ell < n^*\}$ are members of $[{}^\mu 2]^{n^*}$ both listed in lexicographic increasing order then: $w_1 E w_2$ iff for any $\ell_1 < \ell_2 < n^*$ and $\ell_3 < \ell_4 < n^*$ we have

(a) $\ell g(\eta_{\ell_1} \cap \eta_{\ell_2}) < \ell g(\eta_{\ell_3} \cap \eta_{\ell_4}) \Leftrightarrow \ell g(\nu_{\ell_1} \cap \nu_{\ell_2}) < \ell g(\nu_{\ell_3} \cap \nu_{\ell_4})$
(b) $\eta_{\ell_3} \restriction \ell g(\eta_{\ell_1} \cap \eta_{\ell_2}) <_{\ell g(\eta_{\ell_1} \cap \eta_{\ell_2})} \eta_{\ell_4} \restriction \ell g(\eta_{\ell_1} \cap \eta_{\ell_2}) \Leftrightarrow \nu_{\ell_3} \restriction \ell g(\nu_{\ell_1} \cap \nu_{\ell_2}) <_{\ell g(\nu_{\ell_1} \cap \nu_{\ell_2})} \nu_{\ell_4} \restriction \ell g(\nu_{\ell_1} \cap \nu_{\ell_1})$.

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