New non-free Whitehead groups (corrected version)

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Abstract

We answer an open problem in [1] by showing that it is consistent that there is a strongly $\aleph_1$-free $\aleph_1$-coseparable group of cardinality $\aleph_1$ which is not $\aleph_1$-separable.

0 INTRODUCTION

An abelian group A is called a Whitehead group, or W-group for short, if $\mathrm{Ext}(A, \mathbb{Z}) = 0$. For historical reasons, A is called an $\aleph_1$-coseparable group if $\mathrm{Ext}(A, \mathbb{Z}^{(\omega)}) = 0$, but for convenience we shall use non-standard terminology and say A is a W_ω-group when A is $\aleph_1$-coseparable. Obviously a W_ω-group is a W-group. In 1973–75, the second author proved that it is consistent with ZFC + GCH that every W-group is free and consistent with ZFC that there are non-free W_ω-groups of cardinality $\aleph_1$ ([8], [9]); he later showed that it is consistent with ZFC + GCH that there are non-free W_ω-groups of cardinality $\aleph_1$ ([10], [11]). Before 1973 it was known (in ZFC) that every W-group is $\aleph_1$-free, separable, and slender, and assuming CH, every W-group is strongly $\aleph_1$-free. (See, for example, [3, pp. 178–180].) These turned out, by the results of the second author, to be essentially all that could be proved without additional set-theoretic hypotheses.

However, new questions of what could be proved in ZFC arose, inspired by the consistency results and their proofs. One of the most intriguing was:

(0.1) Does every strongly $\aleph_1$-free W_ω-group of cardinality $\aleph_1$ satisfy the stronger property that it is $\aleph_1$-separable?

(See [1, p. 454, Problem 5]). As we shall explain below, not only was the answer to this question affirmative in every known model of ZFC, but the nature of the known constructions of non-free Whitehead groups was such as to lead to the suspicion that the answer might be affirmative (provably in ZFC). However, in this paper we show that it is consistent that the answer is negative.

*Thanks to Rutgers University for its support of this research through its funding of the authors' visits to Rutgers.

†Partially supported by Basic Research Fund, Israeli Academy of Sciences. Pub. No. 559a

First we recall the key definitions. An abelian group A is $\aleph_1$-*free* if every countable subgroup of A is free; A is *strongly* $\aleph_1$-*free* if every countable subset is contained in a countable free subgroup B such that A/B is $\aleph_1$-free. A is $\aleph_1$-*separable* if every countable subset is contained in a countable free subgroup B which is a direct summand of A; so an $\aleph_1$-separable group is strongly $\aleph_1$-free. It is a consequence of CH (or even of $2^{\aleph_0} < 2^{\aleph_1}$) that there are strongly $\aleph_1$-free groups of cardinality $\aleph_1$ which are not $\aleph_1$-separable (see [12]). However, the existence of such groups is not settled by the hypothesis $2^{\aleph_0} = 2^{\aleph_1}$; specifically, in a model of MA + $\neg$CH every strongly $\aleph_1$-free group of cardinality $\aleph_1$ *is* $\aleph_1$-separable; but the methods of [6] show that it is consistent with $2^{\aleph_0} = 2^{\aleph_1}$ that there are strongly $\aleph_1$-free groups of cardinality $\aleph_1$ which are *not* $\aleph_1$-separable.

Now suppose A is strongly $\aleph_1$-free and is a W_ω-group. Consider a countable subgroup B of A such that A/B is $\aleph_1$-free. We have a short exact sequence

$$0 \to B \to A \to A/B \to 0$$

where the map of B into A is inclusion. Since B is a free group of countable rank, if we knew that A/B were a W_ω-group, then we would have $\mathrm{Ext}(A/B, \mathbb{Z}^{(\omega)}) = \mathrm{Ext}(A/B, B) = 0$ and we could conclude that this sequence splits and hence B is a direct summand of A. In every previously known model where there are non-free W_ω-groups, the construction of a W_ω-group A is such that A/B shares the properties of A closely enough that A/B is also a W_ω-group — when B is a countable subgroup such that A/B is $\aleph_1$-free. (For example, if A is constructed as in [1, Prop. XII.3.6(iii), p. 371], using a ladder system with a uniformization property, then A/B shares the same properties, because it is constructed using essentially the same ladder system.) Thus in these models the answer to (0.1) is affirmative. This motivates question (0.1) as well as the related question

> (0.2) If a group A of cardinality $\aleph_1$ is strongly $\aleph_1$-free and a W_ω-group, and B is a countable subgroup of A such that A/B is $\aleph_1$-free, is A/B a W_ω-group?

By what we have just remarked, a positive answer to (0.2) implies a positive answer to (0.1). The converse holds as well: if A and B are as in the hypotheses of (0.2) and A is $\aleph_1$-separable, $A = F \oplus A'$ where F is countable and contains B; then A/B is a W_ω-group because $A/B = F/B \oplus A'$ and F/B is free by hypothesis on B.

We shall give a model of ZFC $+\neg$CH where the answer to (0.1) and (0.2) is negative.[1]

1 THE PROOF

Our main theorem is:

THEOREM 1 *There is a strongly $\aleph_1$-free W_ω-group A of cardinality $\aleph_1$ with a countable subgroup B of A such that A/B is $\aleph_1$-free but B is not a direct summand of A.*

Throughout, E will be a stationary subset of ω_1 consisting of limit ordinals, with (for technical reasons) $\omega \notin E$. We begin with a general construction of a group. Let π_n be the nth prime.

[1] We do not know if Theorem 8 of the original paper is correct, or if the answer to question (0.3) is "no".

Definition 2 *For each $\delta \in E$ let η_δ be a ladder on δ, that is, a strictly increasing function $\eta_\delta : \omega \to \delta$ whose range approaches δ. Let φ be a function from $E \times \omega$ to ω. Let F be the free abelian group with basis $\{x_\nu : \nu \in \omega_1\} \cup \{z_{\delta,n} : \delta \in E, n \in \omega\}$ and let K be the subgroup of F generated by $\{w_{\delta,n} : \delta \in E, n \in \omega\}$ where*

$$w_{\delta,n} = \pi_n z_{\delta,n+1} - z_{\delta,0} - x_{\eta_\delta(n)} - x_{\varphi(\delta,n)}. \tag{1}$$

Let $A = F/K$.

Clearly A is an abelian group of cardinality $\aleph_1$. Notice that because the right-hand side of (1) is 0 in A, we have for each $\delta \in E$ and $n \in \omega$ the following relation in A:

$$\pi_n z_{\delta,n+1} = z_{\delta,0} + x_{\eta_\delta(n)} + x_{\varphi(\delta,n)} \tag{2}$$

where, in an abuse of notation, we write, for example, $z_{\delta,n+1}$ instead of $z_{\delta,n+1} + K$. If we let

$$A_\alpha = \langle \{x_\nu : \nu < \alpha\} \cup \{z_{\delta,n} : \delta \in E \cap \alpha,\, n \in \omega\} \rangle . \tag{3}$$

for each $\alpha < \omega_1$, then for each $\delta \in E$, $z_{\delta,0} + A_\delta$ is non-zero and divisible in $A_{\delta+1}/A_\delta$ by π_n for all $n \in \omega$. Thus $A_{\delta+1}/A_\delta$ is not free and hence A is not free. (In fact $\Gamma(A) \supseteq \tilde{E}$; see [1, pp. 85f].) Moreover, A is strongly $\aleph_1$-free; in fact, for every $\alpha < \omega_1$, using Pontryagin's Criterion we can show that A/A_α is $\aleph_1$-free whenever $\alpha \notin E$.

We now define the model of ZFC where A is defined and has the desired properties. We begin with a model V of ZFC where GCH holds, choose $E \in V$, and define the group A in a generic extension V^{Q_0} using generic ladders η_δ, and generic φ. Specifically:

Definition 3 *Let Q_0 be the set of all finite functions q such that dom(q) is a finite subset of E and for all $\gamma \in$ dom(q), $q(\gamma)$ is a pair $(\eta^q_\gamma, \varphi^q_\gamma)$ where for some $r^q_\gamma \in \omega$:*

- *η^q_γ is a strictly increasing function: $r^q_\gamma \to \gamma$;*
- *$\varphi^q_\gamma : \{\gamma\} \times r^q_\gamma \to \omega$.*

Clearly Q_0 is c.c.c. We now do an iterated forcing to make A a W_ω-group. We begin by defining the basic forcing that we will iterate.

Definition 4 *Given a homomorphism $\psi : K \to \mathbb{Z}^{(\omega)}$, let Q_ψ be the poset of all finite functions q into $\mathbb{Z}^{(\omega)}$ satisfying:*

There are $\delta_0 < \delta_1 < ... < \delta_m$ in E and $\{r_\ell : \ell \leq m\} \subseteq \omega$ such that dom(q) =

$$\{z_{\delta_\ell,n} : \ell \leq m, n \leq r_\ell\} \cup \{x_\nu : \nu \in I_q\}$$

where $I_q \subset \omega_1$ is finite and is such that for all $\ell \leq m$

$$\eta_{\delta_\ell}(n) \in I_q \Leftrightarrow n < r_\ell \tag{4}$$

and for all $\ell \leq m$ and $n < r_\ell$,

$$\psi(w_{\delta_\ell,n}) = \pi_n q(z_{\delta_\ell,n+1}) - q(z_{\delta_\ell,0}) - q(x_{\eta_{\delta_\ell}(n)}) - q(x_{\varphi(\delta,n)}). \tag{5}$$

Moreover, we require of q that for all $\ell \neq j$ in $\{0, ..., m\}$,

$$\eta_{\delta_j}(k) \neq \eta_{\delta_\ell}(i) \text{ for all } k \geq r_j \text{ and } i \in \omega. \tag{6}$$

We will denote $\{\delta_0, ..., \delta_m\}$ by $\mathrm{cont}(q)$ and r_ℓ by $\mathrm{num}(q, \delta_\ell)$. The partial ordering on Q_ψ is inclusion. Standard methods prove that Q_ψ is c.c.c.

Let $P = \left\langle P_i, \dot{Q}_i : 0 \le i < \omega_2 \right\rangle$ be a finite support iteration of length ω_2 so that for every $i \ge 1 \Vdash_{P_i} \dot{Q}_i = Q_{\dot{\psi}_i}$ where $\Vdash_{P_i} \dot{\psi}_i$ is a homomorphism: $K \to \mathbb{Z}^{(\omega)}$, where the enumeration of names $\{\dot{\psi}_i : 1 \le i < \omega_2\}$ is chosen so that if G is P-generic and $\psi \in V[G]$ is a homomorphism: $K \to \mathbb{Z}^{(\omega)}$, then for some $i \ge 1$, $\dot{\psi}_i$ is a name for ψ in V^{P_i}. Then P is c.c.c. and in $V[G]$ every homomorphism from K to $\mathbb{Z}^{(\omega)}$ extends to one from F to $\mathbb{Z}^{(\omega)}$. This means that $\mathrm{Ext}(A, \mathbb{Z}^{(\omega)}) = 0$, that is, A is a W_ω-group (see, for example, [1, p.8]).

Let $B = A_\omega$, i.e., the subgroup of A generated by $\{x_\ell : \ell \in \omega\}$. It is easy to check that A/B is $\aleph_1$-free. Now, aiming for a contradiction, suppose that in $V[G]$ there is a projection $h : A \to B$ (i.e., $h \restriction B$ is the identity). Then there is a condition $p_o \in G$ such that

$$p_o \Vdash \text{“}\dot{h} : A \to B \text{ is a projection”}$$

where $\dot{h}$ is a name for h.

For each ordinal $\xi \in \omega_1 - E$, choose a condition $p_\xi \ge p_o$ such that there is a $y_\xi \in V$ such that

$$p_\xi \Vdash \dot{h}(x_{\xi+1}) = y_\xi.$$

(That is, y_ξ is an element of $B \cap V$, and not just a name.)

We can assume that

> (†) $0 \in \mathrm{dom}(p_\xi)$; for each $j \in \mathrm{dom}(p_\xi)$, $p_\xi(j)$ is a function in V and not just a name; $r_\gamma^{p_\xi(0)} (= r_\xi)$ is independent of $\gamma \in \mathrm{dom}(p_\xi(0))$; if $j \in \mathrm{dom}(p_\xi) \setminus \{0\}$, $\gamma \in \mathrm{cont}(p_\xi(j))$ implies $\gamma \in \mathrm{dom}(p_\xi(0))$ and $\mathrm{num}(p_\xi(j), \gamma)$ $(= r'_{\xi,j})$ is $\le r_\xi$ and independent of γ. Moreover, if $\gamma > \xi$, then $\eta_\gamma(r'_{\xi,j} - 1) > \xi$.

When we say that “ν occurs in p” we mean that $\nu \in \mathrm{dom}(p) \cup \mathrm{dom}(p(0)) \cup \bigcup \{\{\eta_\gamma(n), \varphi(\delta, n)\} : \gamma \in \mathrm{dom}(p(0), n < r_\gamma^{p(0)}\}$ or x_ν belongs to the domain of some $p(j)$.

Without loss of generality we can assume (passing to a subset $S \subseteq \omega_1 - E$) by Fodor's Lemma and the Δ-system lemma that

> (††) $\{\mathrm{dom}(p_\xi) : \xi \in S\}$ forms a Δ-system, whose root we denote C (i.e., $\mathrm{dom}(p_{\xi_1}) \cap \mathrm{dom}(p_{\xi_2}) = C = \{0, \mu_1, ..., \mu_d\}$ for all $\xi_1 \ne \xi_2$ in S); r_ξ $(= r)$ and $r'_{\xi,j}$ $(= r'_j)$ are independent of ξ; $\mathrm{dom}(p_\xi(0)) \cap \xi$ is independent of ξ; there is m such that for all $\xi \in S$, $\mathrm{dom}(p_\xi(0)) - \xi = \{\gamma_{\xi,0} < ... < \gamma_{\xi,m}\}$and for each $\ell \le m$ there is $t_\ell \le r - 1$ such that $\eta_{\gamma_{\xi,\ell}}(t_\ell - 1) < \xi \le \eta_{\gamma_{\xi,\ell}}(t_\ell)$ and for $i \le t_\ell - 1$, $\eta_{\gamma_{\xi,\ell}}(i)$ is independent of ξ. Moreover, for every $j \in C$, $\{\mathrm{dom}(p_\xi(j)) : \xi \in S\}$ forms a Δ-system and for all $\xi_1 \ne \xi_2$ in S, $p_{\xi_1}(j)$ and $p_{\xi_2}(j)$ agree on $\mathrm{dom}(p_{\xi_1}(j)) \cap \mathrm{dom}(p_{\xi_2}(j))$.

Let p^* denote the “heart” of the Δ-system; that is, $\mathrm{dom}(p^*) = C$ and for all $\mu \in C$, $\mathrm{dom}(p^*(\mu)) = \mathrm{dom}(p_{\xi_1}(\mu)) \cap \mathrm{dom}(p_{\xi_2}(\mu))$ $(= C_\mu$, say) for $\xi_1 \ne \xi_2 \in S$; and $p^*(\mu) \restriction C_\mu = p_{\xi_1}(\mu) \restriction C_\mu$.

We can assume that every ordinal which occurs in p^* is $< \xi$ for every $\xi \in S$, and that for every $\xi_1 < \xi_2$ in S, every ν which occurs in ξ_1 is $< \xi_2$. We can find $\delta \in E$ which is the limit of a strictly increasing sequence $\{\xi_\ell : \ell \in \omega\} \subseteq S$. Notice that no ordinal $\ge \delta$ occurs in any

p_{ξ_ℓ}. There is a condition $\tilde{p} \geq p^*$ which forces a value to $\dot{h}(z_{\delta,0})$, i.e., there is $y \in V$ such that $\tilde{p} \Vdash \dot{h}(z_{\delta,0}) = y$. We can assume that $\tilde{p}$ is as in (†); let $r^* = r_\delta^{\tilde{p}(0)}$. Fix $n \in \omega$ sufficiently large so that every ordinal which occurs in $\tilde{p}$ but not in p^* does not occur in p_{ξ_n}. Fix $k \in \omega$ such that $k > \sup\{j \in \omega : j$ occurs in $\tilde{p}$ or $p_{\xi_n}\}$ and such that $y, y_{\xi_n} \in \{x_j : j < k\}$. There is a condition q_0 extending $\tilde{p}(0)$ and $p_{\xi_n}(0)$ such that q_0 forces

$$\eta_\delta(r^*) = \xi_n + 1;\ \eta_\delta(r^* + 1) = \xi_{n+1};\ \text{and}\ \varphi(\delta, r^*) = k.$$

Then there is a condition $q \in P$ extending $\tilde{p}$ and p_{ξ_n} such that $q(0) \geq q_0$. (The only possible difficulty in defining $q(j)$ for $j > 0$ is in defining $q(j)(z_{\delta,r^*+1})$, $q(j)(\eta_\delta(r^*))$ and $q(j)(x_{\varphi(\delta,r^*)})$ to satisfy (5), but this can be done even though $q(j)(\eta_\delta(r^*)) = q(j)(x_{\xi_n+1})$ may be determined by $p_{\xi_n}(j)$, because x_k is new.)

Now consider a generic extension $V[G']$ such that $q \in G'$. In this model, by (2), we have that in A, π_n divides $z_{\delta,0} - x_{\eta_\delta(r^*)} - x_{\varphi(\delta,r^*)} = z_{\delta,0} - x_{\xi_n+1} - x_k$. Hence in B, π_n divides $h(z_{\delta,0} - x_{\xi_n+1} - x_k) = y - y_{\xi_n} - x_k$. But this is impossible by choice of k.

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