# A COMPLETE BOOLEAN ALGEBRA THAT HAS NO PROPER ATOMLESS COMPLETE SUBLAGEBRA

Thomas Jech and Saharon Shelah

The Pennsylvania State University,
The Hebrew University and Rutgers University

Abstract. There exists a complete atomless Boolean algebra that has no proper atomless complete subalgebra.

An atomless complete Boolean algebra $B$ is called *simple* [5] if it has no atomless complete subalgebra $A$ such that $A \neq B$. We prove below that such an algebra exists.

The question whether a simple algebra exists was first raised in [8] where it was proved that $B$ has no proper atomless complete subalgebra if and only if $B$ is *rigid* and *minimal*. For more on this problem, see [4], [5] and [1, p. 664].

Properties of complete Boolean algebras correspond to properties of generic models obtained by forcing with these algebras. (See [6], pp. 266–270; we also follow [6] for notation and terminology of forcing and generic models.) When in [7] McAloon constructed a generic model with all sets ordinally definable he noted that the corresponding complete Boolean algebra is *rigid*, i.e. admitting no nontrivial automorphisms. In [9] Sacks gave a forcing construction of a real number of minimal degree of constructibility. A complete Boolean algebra $B$ that adjoins a minimal

The first author was supported in part by an NSF grant DMS-9401275.

The second author was partially supported by the U.S.–Israel Binational Science Foundation. Publication No. 566

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

set (over the ground model) is *minimal* in the following sense:

$$(1) \qquad \text{If } A \text{ is a complete atomless subalgebra of } B \text{ then there exists a partition } W \text{ of } 1 \text{ such that for every } w \in W,\ A_w = B_w, \text{ where } A_w = \{a \cdot w : a \in A\}.$$

In [3], Jensen constructed, by forcing over $L$, a definable real number of minimal degree. Jensen's construction thus proves that in $L$ there exists rigid minimal complete Boolean algebra. This has been noted in [8] and observed that $B$ is rigid and minimal if and only if it has no proper atomless complete subalgebra. McAloon then asked whether such an algebra can be constructed without the assumption that $V = L$. In [5] simple complete algebras are studied systematically, giving examples (in $L$) for all possible cardinalities.

In [10] Shelah introduced the $(f, g)$-bounding property of forcing and in [2] developed a method that modifies Sacks' perfect tree forcing so that while one adjoins a minimal real, there remains enough freedom to control the $(f, g)$-bounding property. It is this method we use below to prove the following Theorem:

**Theorem.** *There is a forcing notion $\mathcal{P}$ that adjoins a real number $g$ minimal over $V$ and such that $B(\mathcal{P})$ is rigid.*

**Corollary.** *There exists a countably generated simple complete Boolean algebra.*

The forcing notion $\mathcal{P}$ consists of finitely branching perfect trees of height $\omega$. In order to control the growth of trees $T \in \mathcal{P}$, we introduce a *master tree* $\mathcal{T}$ such that every $T \in \mathcal{P}$ will be a subtree of $\mathcal{T}$. To define $\mathcal{T}$, we use the following fast growing sequences of integers $(P_k)_{k=0}^{\infty}$ and $(N_k)_{k=0}^{\infty}$:

$$(2) \qquad P_0 = N_0 = 1, \quad P_k = N_0 \cdot \ldots \cdot N_{k-1}, \quad N_k = 2^{P_k}$$

(Hence $N_k = 1, 2, 4, 256, 2^{2^{11}}, \ldots$).

**Definition.** The *master tree* $\mathcal{T}$ and the *index function* ind:

(3)(i) $\mathcal{T} \subset [\omega]^{<\omega}$,

(ii) ind is a one-to-one function of $\mathcal{T}$ onto $\omega$,

(iii) ind $(< >) = 0$,

(iv) if $s, t \in \mathcal{T}$ and length(s) $<$ length(t) then $\text{ind}(s) < \text{ind}(t)$,

(v) if $s, t \in \mathcal{T}$, length(s) $=$ length(t) and $s <_{lex} t$ then $\text{ind}(s) < \text{ind}(t)$,

(vi) if $s \in \mathcal{T}$ and $\text{ind}(s) = k$ then $s$ has exactly $N_k$ successors in $\mathcal{T}$, namely all $s^\frown i$, $i = 0, \ldots, N_k - 1$.

The forcing notion $\mathcal{P}$ is defined as follows:

**Definition.** $\mathcal{P}$ is the set of all subtrees $T$ of $\mathcal{T}$ that satisfy the following:

$$\text{for every } s \in T \text{ and every } m \text{ there exists some } t \in T,\ t \supset s, \tag{4}$$
$$\text{such that } t \text{ has at least } {P_{\text{ind}(t)}}^m \text{ successors in } T.$$

(We remark that $\mathcal{T} \in \mathcal{P}$ because for every $m$ there is a $K$ such that for all $k \geq K$, ${P_k}^m \leq 2^{P_k} = N_k$.)

When we need to verify that some $T$ is in $\mathcal{P}$ we find it convenient to replace (4) by an equivalent property:

**Lemma.** *A tree $T \subseteq \mathcal{T}$ satisfies* (4) *if and only if*

(5)(i) *every $s \in T$ has at least one successor in $T$,*

(ii) *for every $n$, if ind$(s) = n$ and $s \in T$ then there exists a $k$ such that if ind$(t) = k$ then $t \in T$, $t \supset s$ and $t$ has at least ${P_k}^n$ successors in $T$.*

*Proof.* To see that (5) is sufficient, let $s \in T$ and let $m$ be arbitrary. Find some $\bar{s} \in T$ such that $\bar{s} \supset s$ and $\text{ind}(\bar{s}) \geq m$, and apply (5ii). $\square$

The forcing notion $\mathcal{P}$ is partially ordered by inclusion. A standard forcing argument shows that if $G$ is a generic subset of $\mathcal{P}$ then $V[G] = V[g]$ where $g$ is the *generic branch*, i.e. the unique function $g : \omega \to \omega$ whose initial segments belong to all $T \in G$. We shall prove that the generic branch is minimal over $V$, and that the complete Boolean algebra $B(\mathcal{P})$ admits no nontrivial automorphisms.

First we introduce some notation needed in the proof:

$$\text{For every } k,\ s_k \text{ is the unique } s \in \mathcal{T} \text{ such that } \text{ind}(s) = k. \tag{6}$$

(7) If $T$ is a tree then $s \in \operatorname{trunk}(T)$ if for all $t \in T$, either $s \subseteq t$ or $t \subseteq s$.

(8) If $T$ is a tree and $a \in T$ then $(T)_a = \{s \in T : s \subseteq a \text{ or } a \subseteq s\}$.

Note that if $T \in \mathcal{P}$ and $a \in T$ then $(T)_a \in \mathcal{P}$. We shall use repeatedly the following technique:

**Lemma.** *Let $T \in \mathcal{P}$ and, let $l$ be an integer and let $U = T \cap \omega^l$ (the $l^{\text{th}}$ level of $T$). Let $\dot{x}$ be a name for some set in $V$. For each $a \in U$ let $T_a \subseteq (T)_a$ and $x_a$ be such that $T_a \in \mathcal{P}$ and $T_a \Vdash \dot{x} = x_a$.*

*Then $T' = \bigcup\{T_a : a \in U\}$ is in $\mathcal{P}$, $T' \subseteq T$, $T' \cap \omega^l = T \cap \omega^l = U$, and $T' \Vdash \dot{x} \in \{x_a : a \in U\}$.* $\square$

We shall combine this with *fusion*, in the form stated below:

**Lemma.** *Let $(T_n)_{n=0}^\infty$ and $(l_n)_{n=0}^\infty$ be such that each $T_n$ is in $\mathcal{P}$, $T_0 \supseteq T_1 \supseteq \cdots \supseteq T_n \supseteq \ldots$, $l_0 < l_1 < \cdots < l_n < \ldots$, $T_{n+1} \cap \omega^{l_n} = T_n \cap \omega^{l_n}$, and such that*

(9) *for every $n$, if $s_n \in T_n$ then there exists some $t \in T_{n+1}$, $t \supset s_n$, with length$(t) < l_{n+1}$, such that $t$ has at least ${P_{\operatorname{ind}(t)}}^n$ successors in $T_{n+1}$.*

*Then $T = \bigcap_{n=0}^\infty T_n \in \mathcal{P}$.*

*Proof.* To see that $T$ satisfies (5), note that if $s_n \in T$ then $s_n \in T_n$, and the node $t$ found by (9) belongs to $T$. $\square$

We shall now prove that the generic branch is minimal over $V$:

**Lemma.** *If $X \in V[G]$ is a set of ordinals, then either $X \in V$ or $g \in V[X]$.*

*Proof.* The proof is very much like the proof for Sacks' forcing. Let $\dot{X}$ be a name for $X$ and let $T_0 \in \mathcal{P}$ force that $\dot{X}$ is not in the ground model. Hence for every $T \leq T_0$ there exist $T'$, $T'' \leq T$ and an ordinal $\alpha$ such that $T' \Vdash \alpha \in \dot{X}$ and $T'' \Vdash \alpha \notin \dot{X}$. Consequently, for any $T_1 \leq T$ and $T_2 \leq T$ there exist $T_1' \leq T_1$ and $T_2' \leq T_2$ and an $\alpha$ such that both $T_1'$ and $T_2'$ decide "$\alpha \in \dot{X}$" and $T_1' \Vdash \alpha \in \dot{X}$ if and only if $T_2' \Vdash \alpha \notin \dot{X}$.

Inductively, we construct $(T_n)_{n=0}^\infty$, $(l_n)_{n=0}^\infty$, $U_n = T_n \cap \omega^{l_n}$, and ordinals $\alpha(a,b)$ for all $a, b \in U_n$, $a \neq b$, such that

(10)(i) $T_n \in \mathcal{P}$ and $T_0 \supseteq T_1 \supseteq \cdots \supseteq T_n \supseteq \ldots$,

(ii) $l_0 < l_1 < \cdots < l_n < \cdots$,

(iii) $T_{n+1} \cap \omega^{l_n} = T_n \cap \omega^{l_n} = U_n$,

(iv) for every $n$, if $s_n \in T_n$ then there exists some $t \in T_{n+1}$, $t \supset s_n$, with length$(t) < l_{n+1}$, such that $t$ has at least ${P_{\mathrm{ind}(t)}}^n$ successors in $T_{n+1}$,

(v) for every $n$, for all $a, b \in U_n$, if $a \neq b$ then both $(T_n)_a$ and $(T_n)_b$ decide "$\alpha(a,b) \in \dot{X}$" and $(T_n)_a \Vdash \alpha(a,b) \in \dot{X}$ if and only if $(T_n)_b \Vdash \alpha(a,b) \in \dot{X}$.

When such a sequence has been constructed, we let $T = \bigcap_{n=0}^\infty T_n$. As (9) is satisfied, we have $T \in \mathcal{P}$ and $T \leq T_0$. If $G$ is a generic such that $T \in G$ and if $X$ is the $G$-interpretation of $\dot{X}$ then the generic branch $g$ is in $V[X]$: for every $n$, $g \restriction l_n$ is the unique $a \in U_n$ with the property that for every $b \in U_n$, $b \neq a$, $\alpha(a,b) \in X$ if and only if $(T)_a \Vdash \alpha(a,b) \in \dot{X}$.

To construct $(T_n)_{n=0}^\infty$, $(l_n)_{n=0}^\infty$ and $\alpha(a,b)$, we let $l_0 = 0$ (hence $U_0 = \{s_0\}$) and proceed by induction. Having constructed $T_n$ and $l_n$, we first find $l_{n+1} > l_n$ as follows: If $s_n \in T_n$, we find $t \in T_n$, $t \supset s_n$, such that $t$ has at least ${P_{\mathrm{ind}(t)}}^n$ successors in $T_n$. Let $l_{n+1} =$ length $(t) + 1$. (If $s_n \notin T_n$, let $l_{n+1} = l_n + 1$.) Let $U_{n+1} = T_n \cap \omega^{l_{n+1}}$.

Next we consider, in succession, all pairs $\{a, b\}$ of district elements of $U_{n+1}$, eventually constructing conditions $T_a$, $a \in U_{n+1}$, and ordinals $\alpha(a,b)$, $a, b \in U_{n+1}$, such that for all $a$, $T_a \leq (T_n)_a$ and if $a \neq b$ then either $T_a \Vdash \alpha(a,b) \in \dot{X}$ and $T_b \Vdash \alpha(a,b) \notin \dot{X}$, or $T_a \Vdash \alpha(a,b) \notin \dot{X}$ and $T_b \Vdash \alpha(a,b) \in \dot{X}$. Finally, we let $T_{n+1} = \bigcup\{T_a : a \in U_{n+1}\}$.

It follows that $(T_n)_{n=0}^\infty$, $(l_n)_{n=0}^\infty$ and $\alpha(a,b)$ satisfy (10). $\square$

Let $B$ be the complete Boolean algebra $B(\mathcal{P})$. We shall prove that $B$ is rigid. Toward a contradiction, assume that there exists an automorphism $\pi$ of $B$ that is not the identity. First, there is some $u \in B$ such that $\pi(u) \cdot u = 0$. Let $p \in \mathcal{P}$ be such that $p \leq u$ and let $q \in \mathcal{P}$ be such that $q \leq \pi(p)$. Since $q \not\leq p$, there is some

$s \in q$ such that $s \notin p$. Let $T_0 = (q)_s$.

Note that for all $t \in T_0$, if $t \supseteq s$ then $t \notin p$. Let

$$A = \{\text{ind}(t) : t \in p\},$$

and consider the following property $\varphi(x)$ (with parameters in V):

(11)
$\varphi(x) \ \leftrightarrow$ if $x$ is a function from $A$ into $\omega$

such that $x(k) < N_k$ for all $k$, then there exists

a function $u$ on $A$ in the ground model $V$ such that the values of $u$ are finite sets of

integers and for every $k \in A$, $u(k) \subseteq \{0, ..., N_k - 1\}$ and $|u(k)| \le P_k$,

and $x(k) \in u(k)$. ∎

We will show that

$$p \Vdash \exists x \neg\varphi(x), \tag{12}$$

and

$$\text{there exists a } T \le T_0 \text{ such that } T \Vdash \forall x \varphi(x). \tag{13}$$

This will yield a contradiction: the Boolean value of the sentence $\exists x \neg\varphi(x)$ is preserved by $\pi$, and so

$$T_0 \le q \le \pi(p) \le \pi(\|\exists x \neg\varphi(x)\|) = \|\exists x \neg\varphi(x)\|,$$

contradicting (13).

In order to prove (12), consider the following (name for a) function $\dot{x} : A \to \omega$. For every $k \in A$, let

$$\dot{x}(k) = \dot{g}(\text{length}\,(s_k) + 1) \text{ if } s_k \subset \dot{g}, \text{ and } \dot{x}(k) = 0 \quad \text{otherwise.}$$

Now if $p_1 < p$ and $u \in V$ is a function on $A$ such that $u(k) \subseteq \{0, ..., N_k - 1\}$ and $|u(k)| \le P_k$ then there exist a $p_2 < p_1$ and some $k \in A$ such that $s_k \in p_2$ has at least $P_k^2$ successors, and there exist in turn a $p_3 < p_2$ and some $i \notin u(k)$ such that $s_k{}^\frown i \in \text{trunk}(p_3)$. Clearly, $p_3 \Vdash \dot{x}(k) \notin u(k)$.

Property (13) will follow from this lemma:

**Lemma.** *Let $T_1 \leq T_0$ and $\dot{x}$ be such that $T_1$ forces that $\dot{x}$ is function from $A$ into $\omega$ such that $x(k) < N_k$ for all $k \in A$. There exist sequences $(T_n)_{n=1}^{\infty}$, $(l_n)_{n=1}^{\infty}$, $(j_n)_{n=1}^{\infty}$, $(U_n)_{n=1}^{\infty}$ and sets $z_a$, $a \in U_n$, such that*

(14)(i) $T_n \in \mathcal{P}$ *and* $T_1 \supseteq T_2 \supseteq \cdots \supseteq T_n \supseteq \ldots$,

(ii) $l_1 < l_2 < \cdots < l_n < \ldots$,

(iii) $T_{n+1} \cap \omega^{l_n} = T_n \cap \omega^{l_n} = U_n$,

(iv) *for every $n$, if $s_n \in T_n$ then there exists some $t \in T_{n+1}$, $t \supset s_n$, with length$(t) < l_{n+1}$, such that $t$ has at least ${P_{\mathrm{ind}(t)}}^n$ successors in $T_{n+1}$,*

(v) $j_1 < j_2 < \cdots < j_n < \ldots$,

(vi) *for every $a \in U_n$, $(T_n)_a \Vdash \langle \dot{x}(k) : k \in A \cap j_n \rangle = z_a$,*

(vii) *for every $k \in A$, if $k \geq j_n$ then $|U_n| < P_k$,*

(viii) *for every $k \in A$, if $k < j_n$ then $|\{z_a(k) : a \in U_n\}| \leq P_k$.*

Granted this lemma, (13) will follow: If we let $T = \bigcap_{n=1}^{\infty} T_n$, then $T \in \mathcal{P}$ and $T \leq T_1$ and for every $k \in A$, $T \Vdash \dot{x}(k) \in u(k)$ where $u(k) = \{z_a(k) : a \in U_n\}$ (for any and all $n > k$).

*Proof of Lemma.* We let $l_1 = j_1 = \text{length}(s)$, $U_1 = \{s\}$ and strengthen $T_1$ if necessary so that $T_1$ decides $\langle \dot{x}(k) : k \in A \cap j_1 \rangle$, and let $z_s$ be the decided value. We also assume that length$(s) \geq 2$ so that $|U_1| = 1 < P_k$ for every $k \in A$, $k \geq j_1$. Then we proceed by induction.

Having constructed $T_n$, $l_n$, $j_n$ etc., we first find $l_{n+1} > l_n$ and $j_{n+1} > j_n$ as follows: If $s_n \notin T_n$ (Case I), we let $l_{n+1} = l_n + 1$ and $j_{n+1} = j_n + 1$. Thus assume that $s_n \in T_n$ (Case II).

Since length$(s_n) \leq n \leq l_n$, we choose some $v_n \in U_n$ such that $s_n \subseteq v_n$. By (4) there exists some $t \in T_n$, $t \supset v_n$, so that if ind$(t) = m$ then $t$ has at least ${P_m}^{n+1}$ successors in $T_n$. Moreover we choose $t$ so that $m = \text{ind}(t)$ is big enough so that there is at least one $k \in A$ such that $j_n \leq k < m$. We let $l_{n+1} = \text{length}(t) + 1$ and $j_{n+1} = m = \text{ind}(t)$.

Next we construct $U_{n+1}, \{z_a : a \in U_{n+1}\}$ and $T_{n+1}$. In Case I, we choose for each $u \in U_n$ some successor $a(u)$ of $u$ and let $U_{n+1} = \{a(u) : u \in U_n\}$. For every

$a \in U_{n+1}$ we find some $T_a \subseteq (T_n)_a$ and $z_a$ so that $T_a \Vdash \langle \dot{x}(k) : k \in A \cap j_{n+1} \rangle = z_a$, and let $T_{n+1} = \bigcup\{T_a : a \in U_{n+1}\}$. In this case $|U_{n+1}| = |U_n|$ and so (vii) holds for $n+1$ as well, while (viii) for $n+1$ follow either from (viii) or from (vii) for $n$ (the latter if $j_n \in A$).

Thus consider Case II. For each $u \in U_n$ other than $v_n$ we choose some $a(u) \in T_n$ of length $l_{n+1}$ such that $a(u) \supset u$, and find some $T_{a(u)} \subseteq (T_n)_{a(u)}$ and $z_{a(u)}$ so that $T_{a(u)} \Vdash \langle \dot{x}(k) : k \in A \cap m \rangle = z_{a(u)}$.

Let $S$ be the set of all successors of $t$ (which has been chosen so that $|S| \geq {P_m}^{n+1}$ where $m = \mathrm{ind}(t)$); every $a \in S$ has length $l_{n+1}$. For each $a \in S$ we choose $T_a \subseteq (T_n)_a$ and $z_a$, so that $T_a \Vdash \langle \dot{x}(k) : k \in A \cap m \rangle = z_a$. If we denote $K = \max(A \cap m)$ then we have

$$|\{z_a : a \in S\}| \leq \prod_{i \in A \cap m} N_i \leq \prod_{i=0}^{K} N_i = P_{K+1} \leq P_m,$$

while $|S| \geq {P_m}^{n+1}$. Therefore there exists a set $U \subset S$ of size ${P_m}^n$ such that for every $a \in U$ the set $z_a$ is the same. Therefore if we let

$$U_{n+1} = U \cup \{a(u) : u \in U_n - \{v_n\}\},$$

and $T_{n+1} = \bigcup\{T_a : a \in U_{n+1}\}$, $T_{n+1}$ satisfies property (iv). It remains to verify that (vii) and (viii) hold.

To verify (vii), let $k \in A$ be such that $k \geq j_{n+1} = m$. Since $m = \mathrm{ind}(t)$, we have $m \notin A$ and so $k > m$. Let $K \in A$ be such that $j_n \leq K < m$. Since $|U_n| < P_K$, we have

$$|U_{n+1}| < |U_n| + |U| < P_K + N_m < P_m \cdot N_m = P_{m+1} \leq P_k.$$

To verify (viii), it suffices to consider only those $k \in A$ such that $j_n \leq k < m$. But then $|U_n| < P_k$ and we have

$$|\{z_a(k) : a \in U_{n+1}\}| \leq |\{z_a : a \in U_{n+1}\}| \leq |U_n| + 1 \leq P_k.$$

□

## References

[1] M. Bekkali and R. Bonnet, *Rigid Boolean algebras*, in "Handbook of Boolean algebras" (J. D. Monk, ed.), vol. 2, pp. 637–678.
[2] M. Goldstern and S. Shelah, *Many simple cardinal invariants*, Arch. Math. Logic **32** (1993), 203–221, (Publ. no. 448).
[3] R. B. Jensen, *Definable sets of minimal degree*, in "Mathematical Logic and Foundations of Set Theory" (Y.Bar-Hillel, ed.), pp. 122–128, North-Holland Publ. Co., 1970.
[4] T. Jech, *A propos d'algèbres de Boole rigides et minimales*, C. R. Acad. Sci. Paris Sér. A **274** (1972), 371–372.
[5] T. Jech, *Simple complete Boolean algebras*, Israel J. Math. **18** (1974), 1–10.
[6] T. Jech, *Set Theory*, Academic Press, New York 1978.
[7] K. McAloon, *Consistency results about ordinal definability*, Ann. Math. Logic **2** (1971), 449–467.
[8] K. McAloon, *Les algèbres de Boole rigides et minimales*, C. R. Acad. Sci. Paris Sér. A **272** (1971), 89–91.
[9] G. Sacks, *Forcing with perfect closed sets*, in "Axiomatic Set Theory" (D. Scott, ed.), Proc. Symp. Pure Math. 13, I, pp. 331–355, American Math. Society 1971.
[10] S. Shelah, *Vive la différence I: nonisomorphism of ultrapowers of countable models*, in "Set Theory of the Continuum" (H. Judah et al., eds.) MSRI vol. 26, pp.357–405, Springer-Verlag 1992.

Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

School of Mathematics, The Hebrew University, Jerusalem, Israel, and Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

*E-mail address*: jech@math.psu.edu, shelah@math.huji.ac.il